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Hyperquadratic power series in $\mathbb{F}_3((T^{-1}))$ with partial quotients of degree 1

Domingo Gómez-Pérez · Alain Lasjaunias

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Abstract In this note, we describe a large family of nonquadratic continued fractions in the field $\mathbb{F}_3((T^{-1}))$ of power series over the finite field $\mathbb{F}_3$. These continued fractions are remarkable for two reasons: first, they satisfy an algebraic equation with coefficients in $\mathbb{F}_3[T]$ given explicitly, and, second, all the partial quotients in the expansion are polynomials of degree 1. In 1986, in a basic article in this area of research, Mills and Robbins (J. Number Theory 23:388–404, 1986) gave the first example of an element belonging to this family.

Keywords Finite fields · Fields of power series · Continued fractions

Mathematics Subject Classification (2000) 11J70 · 11J61 · 11T55

1 Introduction

We are concerned with power series in $1/T$ over a finite field, where T is an indeterminate. If the base field is $\mathbb{F}_q$, the finite field of characteristic p with q elements, these power series belong to the field $\mathbb{F}_q((T^{-1}))$, which here will be denoted by $\mathbb{F}(q)$. Thus a nonzero element of $\mathbb{F}(q)$ is represented by

$$\alpha = \sum_{k \le k_0} u_k T^k \quad \text{where } k_0 \in \mathbb{Z}, u_k \in \mathbb{F}_q \text{ and } u_{k_0} \neq 0.$$

D. Gómez-Pérez
Universidad de Cantabria, 39005 Santander, Spain
e-mail: domingo.gomez@unican.es
url: http://personales.unican.es/gomezd

A. Lasjaunias (✉)
Université Bordeaux 1, C.N.R.S.-UMR 5251, 33405 Talence, France
e-mail: Alain.Lasjaunias@math.u-bordeaux1.fr
url: http://www.math.u-bordeaux1.fr/~lasjauni/

The absolute value on this field is defined by $|\alpha| = |T|^{k_0}$ where $|T| > 1$ is a fixed real number. We also denote by $\mathbb{F}(q)^+$ the subset of power series α such that $|\alpha| > 1$. We know that each irrational element $\alpha \in \mathbb{F}(q)^+$ can be expanded as an infinite continued fraction. This is denoted

$$\alpha = [a_1, a_2, \dots, a_n, \dots] \quad \text{where } a_i \in \mathbb{F}_q[T] \text{ and } \deg(a_i) > 0 \text{ for } i \geq 1.$$

By truncating this expansion, we obtain a rational element, called a convergent to α and denoted by x_n/y_n for $n \geq 1$. The polynomials $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$, called continuants, are both defined by the same recursion formula: $K_n = a_n K_{n-1} + K_{n-2}$ for $n \geq 2$, with the initial conditions $x_0 = 1$ and $x_1 = a_1$ or $y_0 = 0$ and $y_1 = 1$. The polynomials a_i are called the partial quotients of the expansion. For $n \geq 1$, we denote $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \dots]$, calling it the complete quotient, and we have

$$\alpha = [a_1, a_2, \dots, a_n, \alpha_{n+1}] = (x_n\alpha_{n+1} + x_{n-1})/(y_n\alpha_{n+1} + y_{n-1}).$$

The reader may consult [5] for a general account on continued fractions in power series fields and also [6] for a wider presentation of Diophantine approximation in function fields and more references.

In 1986, Mills and Robbins [4], by developing the pioneering work by Baum and Sweet [1], introduced a particular subset of algebraic power series. These power series are irrational elements $\alpha \in \mathbb{F}(q)$ satisfying an equation $\alpha = f(\alpha^r)$ where r is a power of the characteristic p of the base field and f is a linear fractional transformation with integer (polynomials in $\mathbb{F}_q[T]$) coefficients. The subset of such elements is denoted by $\mathbb{H}_r(q)$ and its elements are called hyperquadratic.

Throughout this note, the base field is $\mathbb{F}_3$, i.e., $q = 3$. We are concerned with elements in $\mathbb{H}_3(3)$ which are not quadratic and have all partial quotients of degree 1 in their continued fraction expansion. A first example of such power series appeared in [4, pp. 401–402].

2 Results

In [2], the second named author of this note investigated the existence of elements in $\mathbb{H}_3(3)$ with all partial quotients of degree 1. The theorem which we present here is an extended version of the one presented in [2]. However, the proof given here is based on a different method. This method used to obtain other continued fraction expansions of hyperquadratic power series was developed in [3]. We have the following:

Theorem 1 *Let* $m \in \mathbb{N}^*$, $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_m) \in (\mathbb{F}_3^*)^m$ *where* $\eta_m = (-1)^{m-1}$ *and* $\mathbf{k} = (k_1, k_2, \dots, k_m) \in \mathbb{N}^m$ *where* $k_1 \geq 2$ *and* $k_{i+1} - k_i \geq 2$ *for* $i = 1, \dots, m-1$. *We define the following integers*:

$$t_{i,n} = k_m\big(3^n - 1\big)/2 + k_i 3^n \quad \textit{for } 1 \leq i \leq m \textit{ and } n \geq 0.$$

We observe that we have $t_{i,n} < t_{i+1,n}$ *for all* (i, n) *and* $t_{m,n} < t_{1,n+1}$. *Also* $t_{i,n} \neq t_{j,n'} + 1$. *Accordingly, we can define two sequences* $(\lambda_t)_{t\geq 1}$ *and* $(\mu_t)_{t\geq 1}$ *in* $\mathbb{F}_3$. *For*

$n \geq 0$, *we have*

$$\lambda_t = \begin{cases} 1 & \text{if } 1 \leq t \leq t_{1,0}, \\ (-1)^{mn+i} & \text{if } t_{i,n} < t \leq t_{i+1,n} \text{ for } 1 \leq i < m, \\ (-1)^{m(n+1)} & \text{if } t_{m,n} < t \leq t_{1,n+1}. \end{cases}$$

Also $\mu_1 = 1$ *and for* $n \geq 0$, $1 \leq i \leq m$ *and* $t > 1$

$$\mu_t = \begin{cases} (-1)^{n(m+1)}\eta_i & \text{if } t = t_{i,n} \text{ or } t = t_{i,n} + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\omega(m, \boldsymbol{\eta}, \mathbf{k}) \in \mathbb{F}(3)$ *be defined by the infinite continued fraction expansion*

$$\omega(m, \boldsymbol{\eta}, \mathbf{k}) = [a_1, a_2, \ldots, a_n, \ldots] \quad \text{where } a_n = \lambda_n T + \mu_n \text{ for } n \geq 1.$$

We consider the two usual sequences $(x_n)_{n\geq 1}$ *and* $(y_n)_{n\geq 1}$ *as being the numerators and denominators of the convergents to* $\omega(m, \boldsymbol{\eta}, \mathbf{k})$.

Then $\omega(m, \boldsymbol{\eta}, \mathbf{k})$ *is the unique root in* $\mathbb{F}(3)^+$ *of the quartic equation*

$$X = \frac{x_l X^3 + (-1)^{m-1} x_{l-3}}{y_l X^3 + (-1)^{m-1} y_{l-3}},$$

where $l = 1 + k_m$.

Remark The case $m = 1$ and thus $\boldsymbol{\eta} = (1)$, $\mathbf{k} = (k_1)$ of this theorem is proved in [2]. The case $m = 2$, $\boldsymbol{\eta} = (-1, -1)$ and $\mathbf{k} = (3, 6)$ corresponds to the example introduced by Mills and Robbins [4].

The generality of this theorem is underlined by the following conjecture based on extensive computer checking.

Conjecture *Let* $\alpha \in \mathbb{H}_3(3)$ *be an element which is not quadratic. Then* α *has all its partial quotients of degree* 1, *from a certain rank, if and only if there exist a linear fractional transformation* f, *with coefficients in* $\mathbb{F}_3[T]$ *and determinant in* $\mathbb{F}_3^*$, *a triple* $(m, \boldsymbol{\eta}, \mathbf{k})$, *and a pair* $(\lambda, \mu) \in \mathbb{F}_3^* \times \mathbb{F}_3$ *such that* $\alpha(T) = f(\omega(m, \boldsymbol{\eta}, \mathbf{k})(\lambda T + \mu))$.

3 Proofs

The proof of the theorem stated above will be divided into three steps.

First step of the proof According to [3, Theorem 1, p. 332], there exists a unique infinite continued fraction $\beta = [a_1, \ldots, a_l, \beta_{l+1}] \in \mathbb{F}(3)$, satisfying

$$\beta^3 = (-1)^m (T^2 + 1)\beta_{l+1} + T + 1 \quad \text{and}$$

$$a_i = \lambda_i T + \mu_i, \text{ for } 1 \leq i \leq l,$$

where λ_i, μ_i are the elements defined in the theorem. We know that this element is hyperquadratic and that it is the unique root in $\mathbb{F}(3)^+$ of the algebraic equation $X = (x_l X^3 + B)/(y_l X^3 + D)$ where

$$B = (-1)^m\big(T^2+1\big)x_{l-1} - (T+1)x_l \quad \text{and} \quad D = (-1)^m\big(T^2+1\big)y_{l-1} - (T+1)y_l.$$

We need to transform B and D. Using the recursive formulas for the continuants, we can write

$$K_{l-3} = (a_l a_{l-1} + 1)K_{l-1} - a_{l-1}K_l. \tag{1}$$

The l first partial quotients of β are given from the hypothesis of the theorem, and we have

$$a_{l-1} = (-1)^{m-1}(T+1) \quad \text{and} \quad a_l = (-1)^{m-1}(-T+1). \tag{2}$$

Combining (1), applied to both sequences x and y, and (2), we get

$$B = (-1)^{m-1}x_{l-3} \quad \text{and} \quad D = (-1)^{m-1}y_{l-3}.$$

Hence we see that β is the unique root in $\mathbb{F}(3)^+$ of the quartic equation stated in the theorem.

Second step of the proof In this section, $l \geq 1$ is a given integer. We consider all the infinite continued fractions $\alpha \in \mathbb{F}(3)$ defined by $\alpha = [a_1, \ldots, a_l, \alpha_{l+1}]$ where $\alpha_{l+1} \in \mathbb{F}(3)$ and

$$a_i = \lambda_i T + \mu_i \quad \text{with } (\lambda_i, \mu_i) \in \mathbb{F}_3^* \times \mathbb{F}_3, \text{ for } 1 \leq i \leq l \quad \text{and} \tag{3}$$

$$\alpha^3 = \epsilon\big(T^2+1\big)\alpha_{l+1} + \epsilon' T + \nu_0 \quad \text{with } \big(\epsilon, \epsilon', \nu_0\big) \in \mathbb{F}_3^* \times \mathbb{F}_3^* \times \mathbb{F}_3. \tag{4}$$

See [3, Theorem 1, p. 332], for the existence and uniqueness of $\alpha \in \mathbb{F}(3)$ defined by the above relations. Our aim is to show that these continued fraction expansions can be explicitly described, under particular conditions on the parameters $(\lambda_i, \mu_i)_{1\leq i\leq l}$ and $(\epsilon, \epsilon', \nu_0)$. Following the same method as in [3], we first prove:

Lemma 2 *Let $(\lambda, \epsilon, \epsilon') \in (\mathbb{F}_3^*)^3$ and $\nu \in \mathbb{F}_3$. We set $U = \lambda T^3 - \epsilon' T + \nu$, and $V = \epsilon(T^2+1)$. We set $\delta = \lambda + \epsilon'$ and assume that $\delta \neq 0$. We define $\epsilon^* = 1$ if $\nu = 0$ and $\epsilon^* = -1$ if $\nu \neq 0$. Then the continued fraction expansion for U/V is given by*

$$U/V = \big[\epsilon\lambda T, -\epsilon(\delta T + \nu), -\epsilon\big(\epsilon^*\delta T + \nu\big)\big].$$

Moreover, by setting $U/V = [u_1, u_2, u_3]$, for $X \in \mathbb{F}(3)$ we have

$$[U/V, X] = \left[u_1, u_2, u_3, \frac{X}{(T^2+1)^2} + \frac{\epsilon^*\epsilon(\delta T + \nu)}{T^2+1}\right].$$

Proof Since $\epsilon^2 = 1$ and $\delta^2 = 1$, we can write

$$U = \epsilon\lambda T V - \delta T + \nu \quad \text{and} \quad V = \epsilon(\delta T + \nu)(\delta T - \nu) + \epsilon\big(1 + \nu^2\big). \tag{5}$$

Clearly, (5) implies the following continued fraction expansion:

$$U/V = \left[\epsilon\lambda T, -\epsilon(\delta T + \nu), \epsilon\left(1+\nu^2\right)(-\delta T + \nu)\right]. \tag{6}$$

Finally, observing that $\epsilon(1+\nu^2) = \epsilon^*\epsilon$ and $\epsilon^*\epsilon\nu = -\epsilon\nu$, we see that (6) is the expansion stated in the lemma. The last formula is obtained from [3, Lemma 3.1 p. 336]. According to this lemma, we have

$$[U/V, X] = \left[u_1, u_2, u_3, X'\right] \quad \text{where } X' = X(u_2u_3+1)^{-2} - u_2(u_2u_3+1)^{-1}.$$

We check that $u_2u_3 = T^2$ if $\nu = 0$ and $u_2u_3 = \nu^2 - T^2$ if $\nu \neq 0$; therefore, we have $u_2u_3 + 1 = \epsilon^*(T^2+1)$, and this implies the desired equality. □

We shall prove now another lemma. In the sequel, we define $f(n)$ as $3n + l - 2$ for $n \geq 1$. We have the following:

Lemma 3 *Let $\alpha = [a_1, \ldots, a_n, \ldots]$ be an irrational element of $\mathbb{F}(3)$. We assume that for an index $n \geq 1$ we have $a_n = \lambda_n T + \mu_n$ with $(\lambda_n, \mu_n) \in \mathbb{F}_3^* \times \mathbb{F}_3$ and*

$$\alpha_n^3 = \epsilon\left(T^2+1\right)\alpha_{f(n)} + z_n T + \nu_{n-1} \quad \textit{where } (\epsilon, z_n, \nu_{n-1}) \in \left(\mathbb{F}_3^*\right)^2 \times \mathbb{F}_3.$$

We set $\nu_n = \mu_n - \nu_{n-1}$ and $\epsilon_n^* = 1$ if $\nu_n = 0$ or $\epsilon_n^* = -1$ if $\nu_n \neq 0$. We set $\delta_n = \lambda_n + z_n$, and $z_{n+1} = -\epsilon_n^*\delta_n$. We assume that $\delta_n \neq 0$. Then we have

$$(a_{f(n)}, a_{f(n)+1}, a_{f(n)+2}) = \left(\epsilon\lambda_n T, -\epsilon(\delta_n T + \nu_n), -\epsilon\left(\epsilon_n^*\delta_n T + \nu_n\right)\right)$$

and

$$\alpha_{n+1}^3 = \epsilon\left(T^2+1\right)\alpha_{f(n+1)} + z_{n+1}T + \nu_n.$$

Proof We can write $\alpha_n^3 = [a_n^3, \alpha_{n+1}^3] = [\lambda_n T^3 + \mu_n, \alpha_{n+1}^3]$. Consequently,

$$\alpha_n^3 = \epsilon\left(T^2+1\right)\alpha_{f(n)} + z_n T + \nu_{n-1}$$

is equivalent to

$$\left[\left(\lambda_n T^3 + \mu_n - z_n T - \nu_{n-1}\right)/\left(\epsilon\left(T^2+1\right)\right), \epsilon\left(T^2+1\right)\alpha_{n+1}^3\right] = \alpha_{f(n)}. \tag{7}$$

Now we apply Lemma 2 with $U = \lambda_n T^3 - z_n T + \nu_n$ and $X = \epsilon(T^2+1)\alpha_{n+1}^3$. Consequently, (7) can be written as

$$\left[\epsilon\lambda_n T, -\epsilon(\delta_n T + \nu_n), -\epsilon\left(\epsilon_n^*\delta_n T + \nu_n\right), X'\right] = \alpha_{f(n)}, \tag{8}$$

where

$$X' = \left(\epsilon\alpha_{n+1}^3 + \epsilon\epsilon_n^*(\delta_n T + \nu_n)\right)/\left(T^2+1\right). \tag{9}$$

Moreover, we have $|\alpha_{n+1}^3| \geq |T^3|$, and consequently $|X'| > 1$. Thus (8) implies that the three partial quotients $a_{f(n)}$, $a_{f(n)+1}$ and $a_{f(n)+2}$ are as stated in this lemma, and also that we have $X' = \alpha_{f(n+1)}$. Combining this last equality with (9), and observing that $-\epsilon_n^* \nu_n = \nu_n$, we obtain the result. □

Applying Lemma 2, we see that, for a continued fraction defined by (3) and (4), the partial quotients, from the rank $l+1$ onward, can be given explicitly three by three, as long as the quantity δ_n is not zero. This is taken up in the following proposition:

Proposition 4 *Let $\alpha \in \mathbb{F}(3)$ be an infinite continued fraction expansion defined by (3) and (4). Then there exists $N \in \mathbb{N}^* \cup \{\infty\}$ satisfying the following conditions:*

1. *For $1 \le n < f(N)$, we have $a_n = \lambda_n T + \mu_n$ where $(\lambda_n, \mu_n) \in \mathbb{F}_3^* \times \mathbb{F}_3$.*
2. *For $1 \le n < f(N)$, define $\nu_n = \sum_{1 \le i \le n} (-1)^{n-i} \mu_i + (-1)^n \nu_0$.*

Then we have

$$\mu_{f(n)} = 0 \quad \textit{and} \quad \mu_{f(n)+1} = \mu_{f(n)+2} = -\epsilon \nu_n \quad \textit{for } 1 \le n < N.$$

3. *For $1 \le n < N$, define $\epsilon_n^* = 1$ if $\nu_n = 0$ or $\epsilon_n^* = -1$ if $\nu_n \neq 0$.*

Let $(\delta_n)_{1 \le n \le N}$ be the sequence defined recursively by

$$\delta_1 = \lambda_1 + \epsilon' \quad \textit{and} \quad \delta_n = \lambda_n - \epsilon_{n-1}^* \delta_{n-1} \quad \textit{for } 2 \le n \le N.$$

Then, for $1 \le n < N$, we have

$$\lambda_{f(n)} = \epsilon \lambda_n, \quad \lambda_{f(n)+1} = -\epsilon \delta_n \quad \textit{and} \quad \lambda_{f(n)+2} = -\epsilon \epsilon_n^* \delta_n.$$

Proof Starting from (4), since $f(1) = l+1$, setting $\epsilon' = z_1$ and observing that all the partial quotients are of degree 1, we can apply repeatedly Lemma 3 as long as we have $\delta_n \neq 0$. If δ_n happens to vanish, the process is stopped and we denote by N the first index such that $\delta_N = 0$, otherwise N is ∞. The formula $\nu_n = \mu_n - \nu_{n-1}$ clearly implies the equality for ν_n. From the formulas $\delta_n = \lambda_n + z_n$ and $z_{n+1} = -\epsilon_n^* \delta_n$ for $n \ge 1$, we obtain the recursive formulas for the sequence δ. Finally, the formulas concerning μ and λ are directly derived from the three partial quotients $a_{f(n)}, a_{f(n)+1}$ and $a_{f(n)+2}$ given in Lemma 3. □

Last step of the proof We start from the element $\beta \in \mathbb{F}(3)$, introduced in the first step of the proof, defined by its l first partial quotients, where $l = k_m + 1$, and by (4) with $(\epsilon, \epsilon', \nu_0) = ((-1)^m, 1, 1)$. According to the first step of the proof, we need to show that $\beta = \omega(m, \boldsymbol{\eta}, \mathbf{k})$. To do so, we apply Proposition 4 to β, and we show that $N = \infty$ and that the resulting sequences $(\lambda_n)_{n \ge 1}$ and $(\mu_n)_{n \ge 1}$ are those which are described in the theorem.

From the definition of the l-tuple $(\mu_1, \ldots, \mu_l)$ and $\nu_o = 1$, we obtain

$$\nu_t = \eta_i \quad \text{if } t = t_{i,0} \quad \text{and} \quad \nu_t = 0 \quad \text{otherwise, for } 1 \le t \le l. \tag{10}$$

Since $\mu_{f(n)+1} = \mu_{f(n)+2}$, we have $\nu_{f(n)+2} = \nu_{f(n)}$. Since $\mu_{f(n)} = 0$, we also have $\nu_{f(n)} = -\nu_{f(n)-1} = -\nu_{f(n-1)+2}$. This implies $\nu_{f(n)+2} = (-1)^{n-1} \nu_{f(1)+2}$. Since

$\nu_{f(1)+2} = \nu_{f(1)} = -\nu_{f(1)-1} = -\nu_l = 0$, we obtain

$$\nu_{f(n)} = \nu_{f(n)+2} = 0 \quad \text{for } 1 \leq n < N. \tag{11}$$

Moreover, from $\nu_{f(n)+1} = \mu_{f(n)+1} - \nu_{f(n)}$ and (11), we also get

$$\nu_{f(n)+1} = -\epsilon\nu_n \quad \text{for } 1 \leq n < N. \tag{12}$$

Now, it is easy to check that we have $f(t_{i,n}) + 1 = t_{i,n+1}$. Since $\epsilon = (-1)^m$, (12) implies $\nu_{t_{i,n}} = (-1)^{m+1}\nu_{t_{i,n-1}}$ if $t_{i,n} < f(N)$. By induction from (10), with (11) and (12), we obtain

$$\nu_{t_{i,n}} = (-1)^{(m+1)n}\eta_i \quad \text{and} \quad \nu_t = 0 \quad \text{if } t \neq t_{i,n}, \text{ for } 1 \leq t < f(N). \tag{13}$$

Since we have $\mu_n = \nu_n + \nu_{n-1}$, from (11) and $\nu_0 = 1$, we see that μ_n satisfies the formulas given in the theorem, for $1 \leq n < f(N)$. Moreover, (13) clearly implies the following:

$$\epsilon_t^* = \begin{cases} -1 & \text{if } t = t_{i,n}, \\ 1 & \text{otherwise,} \end{cases} \quad \text{for } 1 \leq t < f(N). \tag{14}$$

Now we turn to the definition of the sequence $(\lambda_n)_{n\geq 1}$ given in the theorem, corresponding to the element ω. With our notations and according to (14), we observe that this definition can be translated into the following formulas:

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_n = \epsilon_{n-1}^*\lambda_{n-1} \quad \text{for } 2 \leq n < f(N). \tag{15}$$

Consequently, to complete the proof, we need to establish that $N = \infty$ and that (15) holds. The recurrence relation binding the sequences δ and λ, introduced in Proposition 4, can be written as

$$\delta_n + \lambda_n = -\epsilon_{n-1}^*(\delta_{n-1} + \lambda_{n-1}) + \epsilon_{n-1}^*\lambda_{n-1} - \lambda_n \quad \text{for } 2 \leq n \leq N. \tag{16}$$

Comparing (15) and (16), we see that $\delta_n + \lambda_n = 0$, for $n \geq 1$, will imply that δ_n never vanishes, i.e., $N = \infty$, and that the sequence $(\lambda_n)_{n\geq 1}$ is the one which is described in the theorem. So we only need to prove that $\delta = -\lambda$. Since β and ω have the same first partial quotients, (15) holds for $2 \leq n \leq l$. Since $\delta_1 = \lambda_1 + \epsilon' = -1 = -\lambda_1$, combining (15) and (16), we obtain $\delta_n = -\lambda_n$ for $1 \leq n \leq l$. We also have, by Proposition 4, $\lambda_{l+1} = \lambda_{f(1)} = \epsilon\lambda_1 = (-1)^m = \lambda_l$, and therefore we get $\delta_{l+1} = \lambda_{l+1} - \epsilon_l^*\delta_l = \lambda_{l+1} + \lambda_l = -\lambda_{l+1}$. By induction, we shall now prove that $\delta_t = -\lambda_t$ for $t = f(n) + 1$, $f(n) + 2$ and $f(n+1)$ with $n \geq 1$. From (11) and (12), we have $\epsilon_{f(n)}^* = \epsilon_{f(n)+2}^* = 1$ and $\epsilon_{f(n)+1}^* = \epsilon_n^*$. Thus we get, using Proposition 4:

$$\delta_{f(n)+1} = \lambda_{f(n)+1} - \epsilon_{f(n)}^*\delta_{f(n)} = \lambda_{f(n)+1} + \lambda_{f(n)} = -\epsilon\delta_n + \epsilon\lambda_n = -\lambda_{f(n)+1},$$

$$\delta_{f(n)+2} = \lambda_{f(n)+2} - \epsilon_{f(n)+1}^*\delta_{f(n)+1} = \lambda_{f(n)+2} + \epsilon_n^*\lambda_{f(n)+1} = -\lambda_{f(n)+2},$$

$$\begin{aligned}\delta_{f(n+1)} &= \lambda_{f(n+1)} - \epsilon_{f(n)+2}^*\delta_{f(n)+2} = \epsilon\lambda_{n+1} + \lambda_{f(n)+2} = \epsilon\left(\lambda_{n+1} - \epsilon_n^*\delta_n\right) \\ &= \epsilon\delta_{n+1} = -\epsilon\lambda_{n+1} = -\lambda_{f(n+1)}.\end{aligned}$$

So the proof of the theorem is complete.

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