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On Robbins' example of a continued fraction expansion for a quartic power series over $\mathbb{F}_{13}$

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Abstract

The continued fraction expansion for a quartic power series over the finite field $\mathbb{F}_{13}$ was conjectured first in [W. Mills, D. Robbins, Continued fractions for certain algebraic power series, J. Number Theory 23 (1986) 388–404] and later in a more precise way in [W. Buck, D. Robbins, The continued fraction of an algebraic power series satisfying a quartic equation, J. Number Theory 50 (1995) 335–344]. Here this conjecture is proved by describing the continued fraction expansion for a large family of algebraic power series over a finite field.

MSC: 11J70; 11T55

Keywords: Continued fractions; Fields of power series; Finite fields

1. Introduction

In this note we discuss continued fractions in the function fields case. For a general presentation of this subject, the reader may consult [S]. The examples which are considered here belong to a particular class of algebraic power series called hyperquadratic. More references concerning this class of elements as well as comments on the particular quartic equation considered first by Mills and Robbins and then by Buck and Robbins can be found in [BL]. In this note we consider power series over a prime field $\mathbb{F}_p$ where p is an odd prime. We consider an indeterminate T, the ring of polynomials $\mathbb{F}_p[T]$ and the field of rational functions $\mathbb{F}_p(T)$. If $|T|$ is a fixed

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real number greater than one, we consider the ultrametric absolute value defined on $\mathbb{F}_p(T)$ by $|P/Q| = |T|^{\deg(P)-\deg(Q)}$. The completion of this field is the field of power series in $1/T$ over $\mathbb{F}_p$, which will here be denoted by $\mathbb{F}(p)$. If $\alpha \in \mathbb{F}(p)$ and $\alpha \neq 0$, we have

$$\alpha = \sum_{k \leqslant k_0} u_k T^k, \quad \text{where } k_0 \in \mathbb{Z},\ u_k \in \mathbb{F}_p,\ u_{k_0} \neq 0 \text{ and } |\alpha| = |T|^{k_0}.$$

We introduce the following subset of $\mathbb{F}(p)$:

$$\mathbb{F}(p)_+ = \{\alpha \in \mathbb{F}(p) \text{ with } |\alpha| \geqslant |T|\}.$$

We also know that each irrational element α of $\mathbb{F}(p)$ can be expanded as an infinite continued fraction. This will be denoted by $\alpha = [a_1, \ldots, a_n, \ldots]$, where the $a_i \in \mathbb{F}_p[T]$ are the partial quotients. As usual the tail of the expansion, $[a_n, a_{n+1}, \ldots]$, called the complete quotient, is denoted by α_n ($\alpha_1 = \alpha$). Finally the numerator and the denominator of the convergent $[a_1, a_2, \ldots, a_n]$ are denoted by x_n and y_n. These polynomials, called continuants, are both defined by the same recursive relation: $K_n = a_n K_{n-1} + K_{n-2}$ for $n \geqslant 2$, with the initials conditions $x_0 = 1$ and $x_1 = a_1$ for the numerator, while the initial conditions are $y_0 = 0$ and $y_1 = 1$ for the denominator.

2. Results

The main result, Theorem 1 below, is based upon the study developed in [L], Section 4. We recall the notations which were introduced there and which we have slightly modified here. For integers $k \geqslant 1$ and $1 \leqslant i \leqslant 2k$ we introduce the rational numbers

$$u_{i,k} = \prod_{1 \leqslant j < i/2} (2j)(2k-2j) \Big/ \prod_{1 \leqslant j < (i+1)/2} (2j-1)(2k-2j+1)$$

and also

$$\theta_k = (-1)^k \prod_{1 \leqslant j \leqslant k} (1 - 1/2j).$$

Given the integer k, we fix a prime number p with $p \geqslant 2k+1$. Thus it is clear that the rational numbers $u_{i,k}$ and θ_k can be reduced modulo p. Therefore in this note these numbers will also be considered as elements of $\mathbb{F}_p^*$. We introduce the following pair of polynomials:

$$P_k(T) = (T^2 - 1)^k \quad \text{and} \quad Q_k(T) = \int_0^T (x^2 - 1)^{k-1}\, dx.$$

Again these polynomials can either be considered as elements of $\mathbb{Q}[T]$ or, by reduction modulo p, as elements of $\mathbb{F}_p[T]$. Note that the existence of $Q_k(T)$ is ensured by the condition $p \geqslant 2k+1$. We recall that the origin of the numbers $u_{i,k}$ is to be found in the following continued fraction expansion,

$$P_k(T)/Q_k(T) = \left[(2k-1)T, \ldots, (2k-2i+1)u_{i,k}^{(-1)^i} T, \ldots, (-2k+1)u_{2k,k}T\right],$$

which holds in $\mathbb{Q}(T)$ as well as in $\mathbb{F}_p(T)$ by reduction modulo p. Note also the link between θ_k and Q_k which is given by the formula: $2k\theta_k Q_k(1) = -1$.

For an integer $l \geqslant 1$ and an integer $n \geqslant 1$, we define $f(n) = (2k+1)n + l - 2k$. We define the sequence $(i(n))_{n\geqslant 1}$ in the following way:

$$i(n) = 1 \quad \text{if } n \notin f\big(\mathbb{N}^*\big) \quad \text{and} \quad i\big(f(n)\big) = i(n) + 1.$$

Finally we introduce the sequence $(A_i)_{i\geqslant 1}$ of polynomials in $\mathbb{F}_p[T]$ defined recursively by

$$A_1 = T \quad \text{and} \quad A_{i+1} = \big[A_i^p / P_k\big] \quad \text{for } i \geqslant 1$$

(here the square brackets denote the integer part, i.e., the polynomial part). We have the following result.

Theorem 1. *Let p be an odd prime number. Let $k \geqslant 1$ be an integer with $2k < p$. Let $l \geqslant 1$ be an integer. Let $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a l-tuple in $(\mathbb{F}_p^*)^l$. Let $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_p^*)^2$. Let α be the infinite continued fraction $\alpha = [a_1, \ldots, a_l, \alpha_{l+1}] \in \mathbb{F}(p)$ defined by*

$$a_i(T) = \lambda_i T \quad \textit{for } 1 \leqslant i \leqslant l \quad \textit{and} \quad \alpha^p = \epsilon_1 P_k \alpha_{l+1} + \epsilon_2 Q_k.$$

Then α is the unique root in $\mathbb{F}(p)_+$ of the algebraic equation

$$y_l x^{p+1} - x_l x^p + (\epsilon_1 P_k y_{l-1} - \epsilon_2 Q_k y_l)x - \epsilon_1 P_k x_{l-1} + \epsilon_2 Q_k x_l = 0. \qquad (E)$$

Set formally $\delta_i = [2k\theta_k\lambda_i, \ldots, 2k\theta_k\lambda_1, \epsilon_2^{-1}]$ for $1 \leqslant i \leqslant l$. We assume that

$$\delta_i \in \mathbb{F}_p^* \quad \textit{for } 1 \leqslant i \leqslant l-1 \qquad (H_1)$$

and

$$\delta_l = \epsilon_1(\theta_k u_{2k,k}\epsilon_2)^{-1}. \qquad (H_2)$$

Then we have

$$a_n = \lambda_n A_{i(n)}, \quad \textit{where } \lambda_n \in \mathbb{F}_p^* \quad \textit{for } n \geqslant 1. \qquad (I)$$

Let $(\delta_n)_{n\geqslant 1}$ be the sequence defined from the l-tuple $(\delta_1, \ldots, \delta_l)$ by the recurrence relations

$$\delta_{f(n)} = \epsilon_1^{(-1)^n}\theta_k\delta_n \quad \textit{for } n \geqslant 1 \qquad (D_0)$$

and (D_i), for $1 \leqslant i \leqslant 2k$,

$$\delta_{f(n)+i} = \epsilon_1^{(-1)^{n+i}} 2k\theta_k i (u_{i,k}\delta_n)^{(-1)^i} \quad \textit{for } n \geqslant 1.$$

Then the sequence $(\lambda_n)_{n\geqslant 1}$ is defined from the l-tuple $(\lambda_1, \ldots, \lambda_l)$ by the recurrence relation

$$\lambda_{f(n)} = \epsilon_1^{(-1)^n}\lambda_n \quad \textit{for } n \geqslant 1 \qquad (L_0)$$

and (L_i)*, for* $1 \leqslant i \leqslant 2k$*,*

$$\lambda_{f(n)+i} = -\epsilon_1^{(-1)^{n+i}}(2k-2i+1)(u_{i,k}\delta_n)^{(-1)^i} \quad \textit{for } n \geqslant 1.$$

Remark. It is interesting to observe that, in the extremal case $p = 2k+1$, the sequence of polynomials A_i is constant and we have $A_i(T) = T$ for $i \geqslant 1$. Consequently when the conditions (H_1) and (H_2) are satisfied all the partial quotients of the expansion are linear. A first example was given in [MR], p. 400, and such continued fractions have been studied in a different approach in [LR1] and [LR2]. There we started from the algebraic equation (E) but we imposed a special form to this equation, this was apparently artificial and constrained us to take $l \geqslant p$.

Now we can turn to the conjecture made by Buck, Mills and Robbins [MR, p. 404], [BR, p. 342].

Theorem 2. *Let* α *be the unique root in* $\mathbb{F}(13)$ *of the algebraic equation*

$$x^4 + x^2 - Tx + 1 = 0.$$

Let $(A_i)_{i\geqslant 1}$ *be the sequence of polynomials in* $\mathbb{F}_{13}[T]$ *defined recursively by*

$$A_1 = T \quad \textit{and} \quad A_{i+1} = \left[A_i^{13}/(T^2-1)^4\right] \quad \textit{for } i \geqslant 1.$$

Let U *and* V *be the following vectors*:

$$U = (u_1, \dots, u_8) = (7, 10, 5, 12, 9, 11, 1, 5) \in (\mathbb{F}_{13})^8$$

and

$$V = (v_0, \dots, v_8) = (11, 10, 1, 11, 12, 5, 1, 12, 6) \in (\mathbb{F}_{13})^9.$$

Let $u \in \mathbb{F}_{169}$ *with* $u^2 = 8$*. Then we have the continued fraction expansion*

$$\alpha = [0, a_1, \dots, a_n, \dots]$$

with

$$a_n = u^{(-1)^{n+1}}\lambda_n A_{v_9(8n-2)+1}(uT) \quad \textit{for } n \geqslant 1, \textit{ where } \lambda_n \in \mathbb{F}_{13}^*$$

(*here* $v_9(m)$ *denotes the largest power of* 9 *dividing* m).

If $(\delta_n)_{n\geqslant 1}$ *is the sequence in* $\mathbb{F}_{13}^*$ *defined by the recurrence relations*

$$\delta_{9n-2+i} = v_i\delta_n^{(-1)^i} \quad \textit{for } 0 \leqslant i \leqslant 8 \textit{ and } n \geqslant 1$$

with the initial conditions $(\delta_1, \dots, \delta_6) = (11, 12, 5, 1, 12, 6)$*, then the sequence* $(\lambda_n)_{n\geqslant 1}$ *is defined by the recurrence relation*

$$\lambda_{9n-2} = -\lambda_n \quad \textit{for } n \geqslant 1$$

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with the initial conditions $(\lambda_1, \dots, \lambda_6) = (5, 12, 9, 11, 1, 5)$ *and by*

$$\lambda_{9n-2+i} = u_i \delta_n^{(-1)^i} \quad \text{for } 1 \leqslant i \leqslant 8 \text{ and } n \geqslant 1.$$

3. Proofs

Proof of Theorem 1. The existence of the continued fraction $\alpha \in \mathbb{F}(p)_+$ satisfying the given definition and the fact that it is the only root of equation (E) follows from Theorem 1 [L]. Now we use Proposition 4.6 of [L]. Hence we know that there exists $N \in \mathbb{N}^* \cup \{\infty\}$ depending on $(\lambda_1, \dots, \lambda_l)$ and (ϵ_1, ϵ_2) such that

$$a_n = \lambda_n A_{i(n)} \quad \text{with } \lambda_n \in \mathbb{F}_p^* \text{ for } 1 \leqslant n \leqslant f(N). \tag{I}$$

The sequence $(\lambda_n)_{1 \leqslant n \leqslant f(N)}$ is defined in the following way:

$$\lambda_{f(n)} = \epsilon_1^{(-1)^n} \lambda_n \quad \text{for } 1 \leqslant n \leqslant N \tag{L_0}$$

and (L_i), for $1 \leqslant i \leqslant 2k$,

$$\lambda_{f(n)+i} = -\epsilon_1^{(-1)^{n+i}} (2k - 2i + 1)(u_{i,k}\delta_n)^{(-1)^i} \quad \text{for } 1 \leqslant n \leqslant N - 1,$$

where the sequence $(\delta_n)_{1 \leqslant n \leqslant N}$ is defined recursively by

$$\delta_n = 2k\theta_k^{i(n)} \lambda_n + (\delta_{n-1} u_{2k,k})^{-1} \quad \text{for } 1 \leqslant n \leqslant N \tag{DL}$$

with the initial condition $\delta_0 = (u_{2k,k}\epsilon_2)^{-1}$. Note that the frame of Proposition 4.6 is more general than here. Indeed here the base field is supposed to be prime, it follows that the Frobenius isomorphism is the identity on $\mathbb{F}_p$ and this implies a simplification in (L_0) and (DL). Moreover the formulas given here are adapted to our new notations. Both sequences (λ_n) and (δ_n) are depending on each other and the process of their definition can be carried on as long as $\delta_n \neq 0$. If this process terminates then $N \in \mathbb{N}^*$, we have $\delta_N = 0$ and the continued fraction expansion is described by (I) but only up to a certain rank $f(N)$. This is what happens if the l-tuple $(\lambda_1, \dots, \lambda_l)$ is taken arbitrarily. We need to prove that under conditions (H_1) and (H_2) we have $N = \infty$ and also that the sequence (δ_n) satisfies the formulas (D_0) to (D_{2k}) stated in this theorem. First we observe that (H_1) simply says that $N > l$. Indeed for $1 \leqslant n \leqslant l$ we have $i(n) = 1$ and (DL) becomes $\delta_n = [2k\theta_k\lambda_n, \dots, 2k\theta_k\lambda_1, \epsilon_2^{-1}]$. We will prove that $\delta_n \neq 0$ for $n \geqslant l + 1$ by showing that the equalities (D_i) for $0 \leqslant i \leqslant 2k$ hold for $n \geqslant 1$. We introduce the following equalities

$$\delta_{f(n)-1} = \epsilon_1^{(-1)^{n-1}} \theta_k^{-1} \delta_{n-1} \quad \text{for } n \geqslant 1. \tag{$(H_2)_n$}$$

We observe that $(H_2)_1$ is simply (H_2) and thus it is assumed to be true. Now we shall prove that for $n \geqslant 1$ we have the following implications:

$$(H_2)_n \Longrightarrow (D_0)_n, \tag{1}$$

$$(D_i)_n \Longrightarrow (D_{i+1})_n \quad \text{for } 0 \leqslant i \leqslant 2k - 1, \tag{2}$$

$$(D_{2k})_n \Longrightarrow (H_2)_{n+1}. \tag{3}$$

This will prove by induction on n that, for $0 \leqslant i \leqslant 2k$, (D_i) hold for $n \geqslant 1$. To establish these implications we use the following equalities which are immediately derived from the definitions:

$$4k^2\theta_k^2 u_{2k,k} = 1 \tag{4}$$

and

$$u_{1,k} = 1, \quad u_{i+1,k} = u_{i,k}\big(i(2k-i)\big)^{(-1)^i} \quad \text{for } 1 \leqslant i \leqslant 2k-1. \tag{5}$$

To prove (1) we start from (DL) at the rank $f(n)$. Since $i(f(n)) = i(n)+1$, we have

$$\delta_{f(n)} = 2k\theta_k^{i(n)+1}\lambda_{f(n)} + (u_{2k,k}\delta_{f(n)-1})^{-1}$$

and, by (L_0) and (DL) at the rank n, this becomes

$$\delta_{f(n)} = \big(\delta_n - (\delta_{n-1}u_{2k,k})^{-1}\big)\epsilon_1^{(-1)^n}\theta_k + (u_{2k,k}\delta_{f(n)-1})^{-1}.$$

Now using $(H_2)_n$ we obtain immediately $\delta_{f(n)} = \delta_n\epsilon_1^{(-1)^n}\theta_k$ which is $(D_0)_n$. To prove (2) we start from (DL) at the rank $f(n)+i+1$ for $0 \leqslant i \leqslant 2k-1$. Since $i(f(n)+i+1) = 1$, we have

$$\delta_{f(n)+i+1} = 2k\theta_k\lambda_{f(n)+i+1} + (u_{2k,k}\delta_{f(n)+i})^{-1}.$$

Applying (L_{i+1}) and $(D_i)_n$ and using (4) and (5), we obtain without difficulties $(D_{i+1})_n$. Finally the last implication (3) is obtained very simply with (4)

$$\delta_{f(n+1)-1} = \delta_{f(n)+2k} = \epsilon_1^{(-1)^n} 4k^2\theta_k u_{2k,k}\delta_n = \epsilon_1^{(-1)^n}\theta_k^{-1}\delta_n.$$

So the proof is complete. □

Remark. It is clear that condition (H_1) is necessary to have $N = \infty$, but condition (H_2) may not be so. Indeed this condition implies a simple form for the sequence $(\delta_n)_{n\geqslant 1}$ showing that it never vanishes. We say that the expansion is perfect if (I) holds for $n \geqslant 1$ (i.e., $N = \infty$). So the expansion could perhaps be perfect without δ_n having the form given in the theorem. Indeed it is interesting to notice that more complex forms for this sequence have been pointed out when the base field is not prime and in the particular case $p = 2k+1$ (see [LR2], p. 562).

Now we turn to the second theorem.

Proof of Theorem 2. Let us consider the infinite continued fraction $\beta = [b_1, \ldots, b_6, \beta_7] \in \mathbb{F}(13)$ defined by

$$(b_1, \ldots, b_6) = (5T, 12T, 9T, 11T, T, 5T) \quad \text{and} \quad \beta^{13} = -P_4\beta_7 - 4Q_4.$$

This element β is the only solution in $\mathbb{F}(13)_+$ of the algebraic equation (E)

$$\big(T^5 + 3T^3 + 11T\big)x^{14} + \big(8T^6 + 12T^4 + 3T^2 + 1\big)x^{13} + \big(5T^2 + 1\big)x + 7T = 0.$$

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It is easy to check that conditions (H_1) and (H_2) of Theorem 1 are satisfied. Here $k = 4$, $\theta_4 = 2$ and $u_{8,4} = 3$. Moreover $l = 6$ and $(\epsilon_1, \epsilon_2) = (-1, -4)$. We have the resulting 6-tuple $(\delta_1, \dots, \delta_6) = (11, 12, 5, 1, 12, 6)$. Thus the expansion for β is perfect and, according to Theorem 1, we have

$$b_n = \lambda_n A_{i(n)}, \quad \text{where } \lambda_n \in \mathbb{F}_{13}^* \text{ for } n \geqslant 1.$$

Here we have $f(n) = 9n - 2$. It is elementary to check that the sequence $v_9(8n-2)+1$ satisfies the same relations as $i(n)$, and therefore here we have $i(n) = v_9(8n-2)+1$ for $n \geqslant 1$. By adapting the formulas given in Theorem 1 for the sequences $(\lambda_n)_{n\geqslant 1}$ and $(\delta_n)_{n\geqslant 1}$, we obtain immediately those which are stated in this theorem. Now we turn to the quartic equation which can be written as $x = 1/T + (x^2 + x^4)/T$. Hence by iteration, we see that it has a root α in $\mathbb{F}(13)$ with $|\alpha| = |T|^{-1}$. We put $\gamma(T) = u^{-1}\alpha^{-1}(u^{-1}T)$. Then it follows that γ is solution of the algebraic equation $B(x) = x^4 - 5Tx^3 + 5x^2 - 1 = 0$ and $\gamma \in \mathbb{F}(13)_+$. In order to have the continued fraction expansion for α as described in the theorem , we only need to prove that $1/\alpha(T) = u\beta(uT)$ or equivalently that $\gamma = \beta$. Thus it remains to prove that γ satisfies equation (E). This will be shown if the polynomial A on the left side of equation (E) is divisible by the polynomial B. A straightforward calculation shows that $A = BC$ with

$$\begin{aligned} C = {} & \big(T^5 + 3T^3 + 11T\big)x^{10} + \big(T^4 + 6T^2 + 1\big)x^9 + \big(2T^3 + 2T\big)x^8 \\ & + \big(5T^4 + 6T^2 + 8\big)x^7 + \big(10T^3 + 2T\big)x^6 + 12T^2x^5 + \big(12T^3 + 5T\big)x^4 \\ & + \big(10T^2 + 8\big)x^3 + 4Tx^2 + \big(8T^2 + 12\big)x + 6T. \end{aligned}$$

So the proof is complete. □

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