Exercise 1. Let us consider a Poisson process $(N_t)_{t\geqslant 1}$ with intensity λ. Prove that $(N_t)_{t\geqslant 1}$ is a jump Markov process with values in $\mathbb{N}$ and with transition matrices

$$P_{xy}(t) = e^{-\lambda t}\frac{(\lambda t)^{y-x}}{(y-x)!}\mathbb{1}_{\{y\geqslant x\}}.$$

Give its infinitesimal generator.

Exercise 2. Let us consider a Poisson process $(N_t)_{t\geqslant 1}$ with intensity λ and X_0 a random variable independent with $(N_t)_{t\geqslant 1}$ and with values in $E = \{-1, 1\}$. We define, for all $t \geqslant 0$,

$$X_t = (-1)^{N_t} X_0.$$

Prove that $(X_t)_{t\geqslant 1}$ is a jump Markov process and calculate its transition matrices. Give its infinitesimal generator. This process is called "Telegraph process".

Exercise 3. [Revisions] For all the following matrices:

1. Show that it is a stochastic matrix, or complete missing values to obtain a stochastic matrix.
2. Draw the relative graph.
3. Give communicating classes.
4. Give closed classes, absorbing states, recurrent and transient states. Indicate if the Markov chain is irreducible or not.

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} ? & 0 & 1/12 & 1/4 & 1/3 \\ 0 & ? & 0 & 0 & 1/4 \\ 1/3 & 1/12 & ? & 0 & 1/12 \\ 0 & 1/6 & 1/4 & ? & 1/6 \\ 0 & 2/3 & 0 & 0 & ? \end{pmatrix}, \qquad D = \begin{pmatrix} ? & 0 & 0 & 0 & 0 & 0 \\ 1/2 & ? & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & ? & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & ? & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & ? & 1/2 \\ 0 & 0 & 0 & 0 & 0 & ? \end{pmatrix}.$$

Exercise 4. For all the following matrices:

1. Prove that it is an infinitesimal generator.
2. Draw the relative graph and give communicating classes and their properties.

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & 0 & 1 & 0 & 1 \\ 1 & -3 & 0 & 1 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 1 & 1 & 0 & -2 & 0 \\ 1 & 0 & 1 & 1 & -3 \end{pmatrix},$$

$$C = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 5. Let a jump Markov process $(X_t)_{t\geqslant 0}$ and $(\mathcal{F}_t^X)$ his natural filtration. Show that its jump times T_n and the time of the first return to state x, denoted R_x, are stopping times.

Exercise 6. Let us consider two independent Poisson processes $(N_t)_{t\geqslant 0}$ and $(N'_t)_{t\geqslant 0}$ with intensity λ and μ. We denote (T_n) jump times of the process $(N_t)_{t\geqslant 0}$. We set $M_n = N'_{T_n} - N'_{T_{n-1}}$ for all $n \geqslant 1$.

1. Draw one draw of trajectories of $(N_t)_{t\geqslant 0}$ and $(N'_t)_{t\geqslant 0}$, and for a fixed n give the relative value of M_n.
2. M_n has values in which set?
3. Show that $\mathbb{P}(M_{n+1} = k) = \mathbb{P}(M_1 = k)$ and that the law of M_n does not depend on n.
4. Let us consider a random variable $X \sim \mathcal{E}(\alpha)$. Show that for all integer k we have $\mathbb{E}[X^k] = \alpha^{-k} k!$.
5. Deduce the law of M_1.