

On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions

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- 1 Recalls and basic results on BSDEs
 - What is a BSDE ?
 - What we know about quadratic BSDEs
- 2 Uniqueness result
 - Framework and tools
 - sketch of the proof
 - Construction of the control problem
- 3 Links with PDEs

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(W_t)_{t \in \mathbb{R}^+}$ be a Brownian motion in \mathbb{R}^d , $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ be his augmented natural filtration, T be a nonnegative real number, ξ a real \mathcal{F}_T -measurable random variable, $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$.

$$Y_t = \xi - \int_t^T g(r, Y_r, Z_r) dr + \int_t^T Z_r dW_r, \quad 0 \leq t \leq T. \quad (1.1)$$

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Definition

A solution to (1.1) is a pair of processes $(Y_t, Z_t)_{0 \leq t \leq T}$ such that :

- 1 (Y, Z) is a predictable process with values in $\mathbb{R} \times \mathbb{R}^{1 \times d}$,
- 2 \mathbb{P} - a.s. $t \mapsto Y_t$ is continuous and $\int_0^T |g(r, Y_r, Z_r)| + \|Z_r\|^2 dr < \infty$
- 3 (Y, Z) verifies (1.1).

Simple example

We take $g = 0$.

- 1 A natural idea is to consider $Y_t := \mathbb{E}[\xi | \mathcal{F}_t]$.
- 2 The martingale representation theorem gives us $Y_t = \mathbb{E}[\xi] + \int_0^t Z_s dW_s$.
- 3 By a simple calculus we obtain

$$Y_t = \xi - \int_t^T Z_s dW_s.$$

Existence and unicity results

- Existence and uniqueness of BSDEs when g is Lipschitz with respect to y and z : E. Pardoux et S. Peng (1990).

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- Existence of solutions to quadratic BSDEs with ξ unbounded : P. Briand et Y. Hu (2006).
- Uniqueness of solutions to quadratic BSDEs with g a convex function with respect to z and ξ unbounded : P. Briand et Y. Hu (2008).

framework

Assumptions : There exist three constants $\bar{\beta} \geq 0$, $\bar{\gamma} > 0$ and $r \geq 0$ together with two progressively measurable nonnegative stochastic processes $(\bar{\alpha}_t)_{0 \leq t \leq T}$ and $(\underline{\alpha}_t)_{0 \leq t \leq T}$ such that, \mathbb{P} -a.s.,

- 1 $z \mapsto g(t, y, z)$ is a convex function $\forall (t, y) \in [0, T] \times \mathbb{R}$;
- 2 $\forall (t, z) \in [0, T] \times \mathbb{R}^{1 \times d}$,

$$|g(t, y, z) - g(t, y', z)| \leq \bar{\beta} |y - y'|, \quad \forall (y, y') \in \mathbb{R}^2;$$

- 3 growth condition : $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}$,

$$-\underline{\alpha}_t - r(|y| + |z|) \leq g(t, y, z) \leq \bar{\alpha}_t + \bar{\beta} |y| + \frac{\bar{\gamma}}{2} |z|^2.$$

Existence result

Theorem

If there exists $p > 1$ such that

$$\mathbb{E} \left[\exp \left(\gamma e^{\beta T} \left(\xi^- + \int_0^T \bar{\alpha}_t dt \right) \right) + (\xi^+)^p + \left(\int_0^T \underline{\alpha}_t dt \right)^p \right] < +\infty$$

then the BSDE (1.1) has a solution (Y, Z) .

Fenchel-Legendre transform

Since $g(t, y, \cdot)$ is a convex function, we can define the Fenchel-Legendre transform of g :

$$f(t, y, q) = \sup_z (zq - g(t, y, z)), \quad \forall t \in [0, T], q \in \mathbb{R}^d, y \in \mathbb{R}.$$

f is a function with value in $\mathbb{R} \cup \{+\infty\}$.

Proposition

- $\forall (t, y, y', q) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ such that $f(t, y, q) < +\infty$,

$$f(t, y', q) < +\infty \text{ et } |f(t, y, z) - f(t, y', z)| \leq \bar{\beta} |y - y'|.$$

- f is a convex function with respect to q ,
- The Fenchel-Legendre transform of f is g .

sketch of the proof for the uniqueness result

What happened when g does not depend on y ?

$$Y_t = \xi - \int_t^T g(s, Z_s) ds + \int_t^T Z_s dW_s.$$

We have $g(s, Z_s) = \sup_{q_s} (Z_s q_s - f(s, q_s)) = Z_s q_s^* - f(s, q_s^*)$.

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$$Y_t = \xi + \int_t^T f(s, q_s^*) ds + \int_t^T Z_s (dW_s - q_s^* ds) \quad (2.1)$$

$$\leq \xi + \int_t^T f(s, q_s) ds + \int_t^T Z_s (dW_s - q_s ds). \quad (2.2)$$

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Finally,

$$Y_t = \operatorname{ess\,inf}_{q \in \mathcal{A}} \mathbb{E}^{\mathbb{Q}} \left[\xi + \int_t^T f(s, q_s) ds \mid \mathcal{F}_t \right].$$

Questions

- Are we allowed to apply Girsanov ?
- Which admissible control set \mathcal{A} could we choice ?
- What happened when f depends on y ?

Construction of the control problem (1/4)

$$\mathcal{A} := \left\{ (q_s)_{s \in [0, T]}, \int_0^T |q_s|^2 ds < +\infty \text{ } \mathbb{P} - \text{a.s.}, \right.$$

$(M_t)_{t \in [0, T]}$ is a martingale,

$$\mathbb{E}^{\mathbb{Q}} \left[|\xi| + \int_0^T |f(s, 0, q_s)| ds \right] < +\infty,$$

with $M_t := \exp \left(\int_0^t q_s dW_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right)$ and $\frac{d\mathbb{Q}}{d\mathbb{P}} := M_T$ }

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There exists a solution to the BSDE

$$Y_t^q = \xi + \int_t^T f(s, Y_s^q, q_s) ds + \int_t^T Z_s^q dW_s^q, \quad 0 \leq t \leq T.$$

with $dW_t^q := dW_t - q_t dt$.

Construction of the control problem (2/4)

We have $Y \leq \text{ess inf}_{q \in \mathbb{A}} Y^q$. We must show that $q^* \in \mathcal{A}$. If T is small enough and if there exist some given exponential moments for $\sup_{0 \leq t \leq T} Y_t^- + \int_0^t \bar{\alpha}_s ds$ and $\sup_{0 \leq t \leq T} Y_t^+$ then we are able to show that M_T^* is a martingale.

In the general case we divide $[0, T]$ into subsets. Thus, for N big enough we define $t_i := \frac{iT}{N}$ with $i \in \{0, \dots, N\}$.

Construction of the control problem (3/4)

$$\mathcal{A}_{t_i, t_{i+1}}(\eta) := \left\{ (q_s)_{s \in [t_i, t_{i+1}]}, \int_{t_i}^{t_{i+1}} |q_s|^2 ds < +\infty \text{ } \mathbb{P} - \text{a.s.}, \right.$$

$$(M_t^i)_{t \in [t_i, t_{i+1}]} \text{ is a martingale, } \mathbb{E}^{\mathbb{Q}^i} \left[|\eta| + \int_{t_i}^{t_{i+1}} |f(s, 0, q_s)| ds \right] < +\infty,$$

$$\left. \text{with } M_t^i := \exp \left(\int_{t_i}^t q_s dW_s - \frac{1}{2} \int_{t_i}^t |q_s|^2 ds \right) \text{ and } \frac{d\mathbb{Q}^i}{d\mathbb{P}} := M_{t_{i+1}}^i \right\}.$$

There exists a solution to the BSDE

$$Y_t^{\eta, q} = \eta + \int_t^{t_{i+1}} f(s, Y_s^{\eta, q}, q_s) ds + \int_t^{t_{i+1}} Z_s^{\eta, q} dW_s^q, \quad t_i \leq t \leq t_{i+1}.$$

Construction of the control problem (4/4)

$$\mathcal{A} := \left\{ (q_s)_{s \in [0, T]}, \quad q_{|[t_{N-1}, T]} \in \mathcal{A}_{t_{N-1}, T}(\xi), \right. \\ \left. \forall i \in \{N-2, \dots, 0\}, \quad q_{|[t_i, t_{i+1}]} \in \mathcal{A}_{t_i, t_{i+1}} \left(Y_{t_{i+1}}^q \right) \right. \\ \left. \text{with } Y_{t_{i+1}}^q := Y_{t_{i+1}}^{Y_{t_{i+2}}^q, q_{|[t_{i+1}, t_{i+2}]}} \text{ and } Y_T^q := \xi \right\}.$$

We can define our cost functional

$$\forall i \in \{N-1, \dots, 0\}, \forall t \in [t_i, t_{i+1}], \quad Y_t^q := Y_t^{Y_{t_{i+1}}^q, q_{|[t_i, t_{i+1}]}}.$$

Result

Theorem

We suppose that there exists a solution (Y, Z) of the BSDE (1.1) verifying $\exists p > \bar{\gamma}, \exists \varepsilon > 0$,

$$\mathbb{E} \left[\exp \left(p \sup_{0 \leq t \leq T} \left(Y_t^- + \int_0^t \bar{\alpha}_s ds \right) \right) + \exp \left(\varepsilon \sup_{0 \leq t \leq T} Y_t^+ \right) \right] < +\infty,$$

Then we have $Y = \text{ess inf}_{q \in \mathcal{A}} Y^q$, and there exists $q^* \in \mathcal{A}$ such that $Y = Y^{q^*}$. Moreover, this implies that the solution (Y, Z) is unique among solutions verifying such assumption.

Remarks :

- 1 To obtain the existence of a solution (Y, Z) that verifies such assumption it is sufficient to suppose that $\xi^- + \int_0^T \bar{\alpha}_t dt$ have an exponential moment of order $qe^{\bar{\beta}}$ avec $q > \bar{\gamma}$ and $\xi^+ + \int_0^T \underline{\alpha}_t dt$ have an exponential moment of order $\varepsilon > 0$.
- 2 When g does not depend on y , we do not have to divide $[0, T]$. We have

$$Y_t = \operatorname{ess\,inf}_{q \in \mathcal{A}_{0,T}(\xi)} \mathbb{E}^{\mathbb{Q}} \left[\xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right], \quad \forall t \in [0, T].$$

Link with PDEs

Let us consider the following semi-linear PDE

$$\partial_t u(t, x) + \mathcal{L}u(t, x) - g(t, x, u(t, x), -\sigma^* \nabla_x u(t, x)) = 0, \quad u(T, \cdot) = h, \quad (3.1)$$

with \mathcal{L} is the infinitesimal generator of the diffusion

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r) dW_r, \quad t \leq s \leq T, \text{ and } X_s^{t,x} = x, \quad s \leq t, \quad (3.2)$$

and the BSDE

$$Y_t^{t,x} = h(X_T^{t,x}) - \int_t^T g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds - \int_t^T Z_s^{t,x} dW_s, \quad 0 \leq t \leq T, \quad (3.3)$$

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and the BSDE

$$Y_t^{t,x} = h(X_T^{t,x}) - \int_t^T g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds - \int_t^T Z_s^{t,x} dW_s, \quad 0 \leq t \leq T, \quad (3.3)$$

The nonlinear Feynman-Kac formula consists in proving that the function defined by the formula

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad u(t, x) := Y_t^{t,x} \quad (3.4)$$

is a viscosity solution to the PDE (3.1).

Exponential Moments

We suppose σ continuous and b K -Lipschitz in x . We have

Lemma

$$\forall \lambda \in \left[0, \frac{1}{2e^{2KT} \|\sigma\|_\infty^2 T} \right], \exists C \geq 0, \quad \mathbb{E} \left[\sup_{0 \leq t \leq T} e^{\lambda |X_t^{t_0, x_0}|^2} \right] \leq C e^{C|x_0|^2}.$$

Assumptions

We suppose that $g : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $h : \mathbb{R}^d \rightarrow \mathbb{R}$ are continuous and that there exist five constants $r \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\alpha \geq 0$ et $\alpha' \geq 0$ such that :

- 1 $|g(t, x, y, z) - g(t, x, y', z)| \leq \beta |y - y'|$;
- 2 $z \mapsto g(t, x, y, z)$ is a convex function on $\mathbb{R}^{1 \times d}$;
- 3 $-r(1 + |x|^2 + |y| + |z|) \leq g(t, x, y, z) \leq r + \alpha |x|^2 + \beta |y| + \frac{\gamma}{2} |z|^2$,
 $-r - \alpha' |x|^2 \leq h(x) \leq r(1 + |x|^2)$;
- 4 $|g(t, x, y, z) - g(t, x', y, z)| \leq r(1 + |x| + |x'|) |x - x'|$,
 $|h(x) - h(x')| \leq r(1 + |x| + |x'|) |x - x'|$;

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$$\alpha' + T\alpha < \frac{1}{2\gamma e^{3\beta T} \|\sigma\|_\infty^2 T}.$$

Results





Proposition

The function u defined by (3.4) is continuous on $[0, T] \times \mathbb{R}^d$ and satisfies

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |u(t, x)| \leq C(1 + |x|^2).$$

Moreover, u is a viscosity solution to the PDE (3.1).

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