Analytic residues along algebraic cycles*†

by

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*The invited lecture delivered by the first author at $FoCM02$ touched upon the crucial role that the theory of residues plays in complexity problems and effective constructions in Algebra. The keystone of this interaction between Analysis and Algebra is the Abel-Jacobi vanishing theorem. In this paper we develop further these ideas and give additional applications to Commutative Algebra and Algebraic Geometry. We would like to thank Teresa Krick and her co-organizers for their kind invitation. The authors' research on this subject has been supported by NSA and NSF Grants.

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Abstract

Let W be a q-dimensional irreducible algebraic subvariety in the affine space $\mathbf{A}^n_{\mathbf{C}}$, $P_1, ..., P_m$ m elements in $\mathbf{C}[X_1, ..., X_n]$, and $V(P)$ the set of common zeros of the P_j's in $\mathbf{C}^n$. Assuming that $|W|$ is not included in $V(P)$, one can attach to P a family of non trivial W-restricted residual currents in ${}'\mathcal{D}^{0,k}(\mathbf{C}^n)$, $1 \leq k \leq \min(m, q)$, with support on $|W|$. These currents (constructed following an analytic approach) inherit most of the properties that are fulfillled in the case $q = n$. When the set $|W| \cap V(P)$ is discrete and $m = q$, we prove that for every point $\alpha \in |W| \cap V(P)$ the W-restricted analytic residue of a $(q, 0)$-form $R d\zeta_I$, $R \in \mathbf{C}[X_1, ..., X_n]$, at the point α is the same as the residue on $\mathcal{W}$ (completion of W in $\mathrm{Proj}\,\mathbf{C}[X_0, ..., X_n]$) at the point α in the sense of Serre ($q = 1$) or Kunz-Lipman ($1 < q < n$) of the q-differential form $(R/P_1 \cdots P_q) d\zeta_I$. We will present a restricted affine version of Jacobi's residue formula and applications of this formula to higher dimensional analogues of Reiss (or Wood) relations, corresponding to situations where the Zariski closures of $|W|$ and $V(P)$ intersect at infinity in an arbitrary way.

1 Introduction

Let us recall first two questions about effective constructions in Commutative Algebra and Algebraic Geometry where residue currents play a central role, both as a discovery tool and in the proofs. These are the Hilbert's Nullstellensatz and the construction of Arakelov measures.

The main idea of our work on this subject was to use the analytic theory of multidimensional residues and residue currents to find key identities and several explicit constructions. What we used repeatedly was the fact that residues could also be computed by analytic continuation of associated zeta functions. We refer to the short monograph [BGVY] for the details. We recall here just a few points.

Let us assume that $f_1, \ldots, f_n, g$ are holomorphic functions near the origin of $\mathbf{C}^n$, and assume that $\{f_1 = \cdots = f_n = 0\} = \{0\}$, then the residue of the meromorphic function $g/f_1 \cdots f_n$ at $z = 0$, as defined by Poincaré, is given by

$$\mathrm{Res}\,(\frac{g}{f_1 \cdots f_n}, 0) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi i)^n} \int_{|f|=\varepsilon} \frac{g(\zeta)}{f_1(\zeta) \cdots f_n(\zeta)} d\zeta, \tag{1.1}$$

$d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$, where $|f| = \varepsilon$, for $\varepsilon = (\varepsilon_1 \ldots \varepsilon_n)$, denotes the cycle $\{|f_1| = \varepsilon_1, \ldots, |f_n| = \varepsilon_n\}$. When the Jacobian J of the f_j does not vanish at 0 we have

$$\mathrm{Res}\,(\frac{g}{f_1 \cdots f_n}, 0) = \frac{g(0)}{J(0)} \tag{1.2}$$

as expected. This definition of the residue in several variables was introduced by Jacobi at least for polynomials in [Ja2]. It has been extended in [CH] to define residue currents : namely, if we replace $gd\zeta$ by a smooth compactly supported $(n, 0)$ differential form φ, then the limit in (1.1) still exists provided $\epsilon_1, ..., \epsilon_n$ approach 0 in an admissible way, and we may define a $(0, n)$ current $\overline{\partial}(1/f)$ as

$$\langle \bar{\partial}\frac{1}{f}, \varphi\rangle \ := \lim_{\varepsilon \to 0} \frac{1}{(2\pi i)^n} \int\limits_{|f|=\varepsilon} \frac{\varphi}{f_1 \cdots f_n}\,. \tag{1.3}$$

What one does next is to relate this current $\bar{\partial}(1/f)$ to the current-valued holomorphic map in $\mathbf{C}$

$$\lambda \mapsto |f_1 \cdots f_n|^{2\lambda} \overline{\partial f_1} \wedge \ldots \wedge \overline{\partial f_n} = |f|^{2\lambda} \overline{\partial f}, \qquad \mathrm{Re}\,\lambda >> 0$$

Using the Bernstein–Sato functional equation one sees that the holomorphic function defined for Re $\lambda >> 0$,

$$\lambda \mapsto \lambda \int\limits_{\mathbf{C}^n} |f|^{2(\lambda-1} \overline{\partial f} \wedge \varphi,$$

has an analytic continuation to $\lambda = 0$ and it satisfies

$$\left[\lambda \int\limits_{\mathbf{C}^n} |f|^{2(\lambda-1} \overline{\partial f} \wedge \varphi \right]_{\lambda=0} = c_n \langle \bar{\partial}\frac{1}{f}, \varphi\rangle, \tag{1.4}$$

where $c_n \neq 0$ is an absolute constant.

The way residues help in finding identities depend on the Abel-Jacobi vanishing theorem and its generalizations. The following result is due to Jacobi [Ja2] : let $P_1, \ldots, P_n$ be polynomials in $\mathbf{C}^n$ without any common zeros at infinity and let Q be another polynomial satisfying the inequality

$$\deg Q \leq \deg P_1 + \cdots + \deg P_n - n - 1\ ; \tag{1.5}$$

then

$$\langle \overline{\partial}\frac{1}{P}, Qdz\rangle = 0\,, \tag{1.6}$$

where

$$\langle \overline{\partial}\frac{1}{P}, Qdz\rangle$$

denotes the total sum of local residues

$$\text{Res }(\frac{Q}{P_1\cdots P_n}, \alpha)$$

at all common zeroes α of $P_1, ..., P_n$ in the affine space $\mathbf{C}^n$. In other words, the sum of all the residues of the meromorphic function $Q/P_1\cdots P_n$ vanishes. When all the zeros are simple we obtain Jacobi's original statement

$$\sum_\alpha \frac{Q(\alpha)}{J(\alpha)} = 0.$$

We refer to [Gr], [Ku2], and [EGH] for interesting geometric applications of this theorem.

A problem one often finds in trying to apply Jacobi's theorem is that given an ideal I in $\mathbf{C}[X_1, ..., X_n]$ defining a zero-dimensional variety in $\mathbf{C}^n$, it may not be possible to find $P_1, \ldots P_n$ in I without common zeros at infinity. What we can do is construct in some elementary way (that is essentially without using elimination theory) polynomials $P_1, \ldots P_n$ in such an ideal I so that the map

$$\zeta \mapsto P(\zeta) = (P_1(\zeta), \ldots, P_n(\zeta))$$

is a proper map of $\mathbf{C}^n$ into itself. The properness condition is equivalent to a Lojasiewicz type inequality: there are constants $K > 0$, $\gamma > 0$, and $\delta > 0$ with the property that if $|z| > K$ then

$$\|P(\zeta)\| \geq \gamma \|\zeta\|^\delta, \tag{1.7}$$

Such a δ is called a Lojasiewicz exponent for P.

In the situation of Hilbert's Nullstellensatz, where we have a collection of polynomials $p_1, \ldots, p_m$ in $\mathbf{C}[X_1, ..., X_n]$ without common zeros, we can find polynomials $P_1, \ldots P_n$ in the ideal generated by them that define a proper map and also have their degrees and size of their coefficients *controlled* by degrees and size of the coefficients of the original polynomials p_j. In particular, it follows that there is an effective generalization of the vanishing theorem of Abel–Jacobi. Namely, there is a proper affine function $\theta\ :\mathbb{N}^n \to \mathbf{R}^+$ such that for any polynomial Q and any $m \in \mathbb{N}^n$ such that

$$\deg Q \le \theta(m) \tag{1.8}$$

one has

$$\langle \overline{\partial} \frac{1}{P^{m+\underline{1}}}, Q d\zeta \rangle = 0, \tag{1.9}$$

where $P^{m+\underline{1}} = (P_1^{m_1+1}, \dots, P_n^{m_n+1})$. Note that a proper map usually has zeroes at ∞, this is the point that makes this statement a strong generalization of the Abel-Jacobi theorem. The proofs of (1.9) given in [BY1], [BGVY] depend very heavily on the properties of residue currents.

In [BY1] we used the method just sketched to compute residues in $\mathbf{C}^n$ and the generalized Abel-Jacobi theorem to obtain effective estimates on the solvability of the Nullstellensatz for polynomials $p_1, \dots, p_M \in \mathbf{Z}[X_1, ..., X_n]$ without common zeros in $\mathbf{C}^n$. This was based on the previous work of Brownawell [Br1,Br2], J. Heintz and its collaborators [CGH1], J. Kollár [Ko], who proved that in the above situation there exist polynomials $q_1, \dots, q_M \in \mathbf{Z}[X_1, ..., X_n]$ and $r_0 \in \mathbf{Z}\backslash\{0\}$ such that the equation

$$p_1 q_1 + \cdots + p_M q_M = r_0 \tag{1.10}$$

is satisfied while

$$\max_j \deg q_j = O\left((\max_j \deg p_j)^n \right).$$

By itself, this bound does not produce sufficiently good estimates on the complexity of deciding whether the Bézout equation is solvable. One needs to obtain also *a priori* estimates on the (logarithmic) size of an "optimal" solution $q_1, \dots, q_M, r_0$. In fact, it was shown in [BY1] that, given M polynomials $p_1, ..., p_M$ in $\mathbf{Z}[X_1, ..., X_n]$ (with degrees d_j in decreasing order and absolute values of coefficients bounded by e^h), one can solve (1.10) with $r_0 \in \mathbf{Z}$, $q_1, ..., q_M \in \mathbf{Z}[X_1, ..., X_n]$ satisfying the following estimates :

$$\begin{cases} \max_j \deg q_j \le (3/2)^7 n(2n+1) d_1 \dots d_\mu \\ \max(\log r_0, \max_j h(q_j)) \le K(n) d_1^4 (d_1 \dots d_\mu)^8 (h + \log M + d_1 \log d_1), \end{cases} \tag{1.11}$$

where $K(n)$ is a computable constant and $h(q_j)$ denotes the maximum of the logarithms of the absolute values of the coefficients of q_j. One can also replace $\mathbf{Z}$ by an arbitrary integral domain $\mathbf{A}$ equipped with a size function, irrespective of the characteristic of the corresponding quotient field $\mathbf{K}$ and corresponding algebraic closure $\overline{\mathbf{K}}$ [BY4]. Though one may substitute algebraic tools from residue calculus [BY3, BY4] to analytic tools, such analytic methods provided [BY6] some insight respect to the following result recently obtained by M. Hickel [Hi] : let I be an ideal in $\mathbf{C}[X_1, ..., X_n]$, $\overline{I}$ its integral closure in this ring, $p_1, \ldots, p_m$ a system of generators of I, and let $d_j = \deg p_j$, such that $d_1 \leq d_2 \cdots \leq d_m = d$; then one has the following alternative :

- if $m \leq n$, every $p \in \overline{I}$ is such one can express p^m as
$$p^m = \sum_{1 \leq j \leq m} q_j p_j$$
with $\max_j \deg(q_j p_j) \leq m \deg p + m d_1 \ldots d_m$;
- if $n > m$, for every $p \in \overline{I}$ there exist q_j such that
$$p^{n+1} = \sum_{1 \leq j \leq m} q_j p_j$$
satisfying
$$\max_j \deg(q_j p_j) \leq (n+1) \deg p (n+1) \min \left\{ d^n, \frac{d_1 \ldots d_m}{d_1^{m-n}} \right\} .$$

The use of (1.4) to compute residues plays also a role in a subject that is of interest in Algebraic Geometry, Mathematical Physics, and Number Theory. We refer for example to the construction and the properties derived thereby of the so-called Green currents [BY8, BY5]. These currents appear in the work of Arakelov on intersection theory and its applications to Physics [Ar], as well in work of Faltings [Fa], Bost, Gillet, and Soulé [BGS], and the very interesting lecture of McMullen [Mc] relating them to the Fermat's last theorem.

As we pointed it out in this introduction, Abel-Jacobi vanshing theorem, together with various connected tools from residue calculus in $\mathbf{C}^n$, plays a major role in our approach towards algebraic intersection or division problems. What we do in the body of the paper is to extend this theorem to the case where the underlying space is not $\mathbf{C}^n$. In fact, we replace $\mathbf{C}^n$ by a q-dimensional irreducible algebraic subvariety W and, assuming that W is not included in the variety $V(P)$ of common zeros of a family P of m polynomials $P_1, ..., P_m$ in $\mathbf{C}[X_1, ..., X_n]$, we shall attach to this polynomial family P a family of non-trivial W-restricted residual currents in ${}'\mathcal{D}^{0,k}(\mathbf{C}^n)$, $1 \leq k \leq \min(m, n)$, with support on $|W|$. These currents (constructed using analytic ideas) inherit most of the properties that are fulfilled in the case $W = \mathbf{C}^n$. When the set $|W| \cap V(P)$ is discrete and $m = q$, we prove that, for every point $\alpha \in |W| \cap V(P)$, the W-restricted analytic residue of a $(q, 0)$-form $R d\zeta_I$, $R \in \mathbf{C}[X_1, ..., X_n]$, at the point α is the same as the residue on $\mathcal{W}$ (completion of W in $\mathrm{Proj}\, \mathbf{C}[X_0, ..., X_n]$) at the point α in the sense of Serre ($q = 1$) or Kunz-Lipman ($1 < q < n$) of the q-differential form $(R/P_1 \cdots P_q) d\zeta_I$. We present a restricted version of the affine version of Jacobi's residue formula (1.6) and obtain applications of this formula to higher dimensional analogues of the Reiss-Wood relations, corresponding to situations where the Zariski closures of $|W|$ and $V(P)$ intersect at infinity in an arbitrary way. We expect this extended Jacobi residue to have as many useful applications to effective constructions in algebraic varieties as our previous work had for the Nullstellensatz.

2 Preliminaries

Let Γ be a complete integral curve embedded as a closed subscheme in $\mathrm{Proj}\, \mathbf{C}[X_0, ..., X_n]$ and $\mathbf{C}(\Gamma)$ its function field. Following the exposition of Hübl and Kunz of the Serre's approach [HK2], the residue of a meromorphic $(1, 0)$-differential form $\omega \in \Omega^1_{\mathbf{C}(\Gamma)/\mathbf{C}}$ at the point $\alpha \in \Gamma$ is defined as follows : let $\mathcal{M}_1, \ldots, \mathcal{M}_d$ be the minimal prime ideals of the completion $\widehat{\mathcal{O}}_{\Gamma,\alpha}$ of the local ring of Γ at α and let $\overline{R}_j$, $j = 1, ..., d$, be the integral closures of the "branches" $R_j = \widehat{\mathcal{O}}_{\Gamma,\alpha}/\mathcal{M}_j$, $j = 1, \ldots, d$, of the curve Γ at the point α. Then $\overline{R}_j$ is isomorphic to an algebra of formal power series $\mathbf{C}[[t_j]]$ and in $\mathbf{C}((t_j))$ the differential $(1, 0)$-form ω can be written as

$$\omega = \sum_{k \geq k_j} a_k^j \, t_j^k \, ,$$

where $a_k^j \in \mathbf{C}$, $k \geq k_j$, are complex numbers which are independent of the parameters t_j. Define

$$\mathrm{Res}_{\Gamma,\alpha,\overline{R}_j}\, \omega := a_{-1}^j\,, \quad \mathrm{Res}_{\Gamma,\alpha}\, \omega := \sum_{j=1}^d a_{-1}^j\,.$$

It was pointed by G. Biernat in [Bi] that, if $f_1, ..., f_n$ are n germs of holomorphic functions in n variables (with Jacobian determinant $J_f \in \mathcal{O}_n$) such that $(f_1, ..., f_{n-1})$ define a germ of curve γ (with branches parametrized respectively by $\varphi_1, ..., \varphi_d$) and $\dim[\gamma \cap \{f_n J_f = 0\}] = 0$, then, for any $h \in \mathcal{O}_n$, the Grothendieck residue

$$\mathrm{Res}_0 \Big[\frac{h d\zeta_1 \wedge \cdots \wedge d\zeta_n}{f_1 \cdots f_n}\Big] := \frac{1}{(2i\pi)^n} \int\limits_{\substack{|f_1|=\epsilon_1 \\ \vdots \\ |f_n|=\epsilon_n}} \frac{h d\zeta_1 \wedge \cdots \wedge d\zeta_n}{f_1 \cdots f_n}$$

(with the orientation for the cycle $\{|f_1| = \epsilon_1\,, \ldots\,, |f_n| = \epsilon_n\}$ that ensures the positivity of the differential form $d\arg f_1 \wedge \cdots \wedge d\arg f_n$ on it) equals

$$\sum_{j=1}^d \mathrm{Res}_{t=0} \left[\frac{(f_n \circ \varphi_j)'(t) h(\varphi_j(t))}{f_n(\varphi_j(t)) J_f(\varphi_j(t))}\, dt\right].$$

In particular, if ω denotes the $(1,0)$-meromorphic differential form

$$\omega := \frac{g d\zeta_\alpha}{f_n}\,, \quad g \in \mathcal{O}_n\,,\ \alpha \in \{1, ..., n\}\,,$$

then

$$\mathrm{Res}_0 \Big[\frac{df_1 \wedge \cdots \wedge df_{n-1}}{f_1 \cdots f_{n-1}} \wedge \omega\Big]$$

equals the sum

$$\sum_{j=1}^d \nu_j \mathrm{Res}_{\gamma_j,0}\,[\omega]\,, \tag{2.1}$$

where $\gamma_1, ..., \gamma_d$ correspond to the irreducible germs of curves attached to the isolated primes in the decomposition of $(f_1, ..., f_{n-1})$, the meromorphic form ω is considered as restricted on the germs of curves $\gamma_1, ..., \gamma_d$, and $\mathrm{Res}_{\gamma_j,0}\,[\omega]$ is defined on the model of the Kunz-Hübl residue, this notion being transposed from the algebraic context to the analytic one (see also [Lej]). This suggests a natural relation between the approaches developed by Serre-Hübl-Kunz and the analytic residue approach developed by Coleff-Herrera [CH] (which precisely allows the transposition of the definition of the Grothendieck residue in the complete intersection case to the setting of currents).

The analytic approach we use to define restricted residual currents on a q-dimensional reduced analytic space $\mathcal{Y} \subset U$, where U is an open subset of $\mathbf{C}^n$, will be described in section 3 as follows : if $f_1, ..., f_m$ are m functions holomorphic in U, then the map

$$\lambda \mapsto \Phi_{Y,f}(\lambda) := \|f\|^{2\lambda} [Y] \, ,$$

where $[Y]$ denotes the integration current on $Y = |\mathcal{Y}|$, can be meromorphically continued as a ${}'\mathcal{D}^{(n-q,n-q)}(U)$-map. Moreover, for any $k \in \{1, ..., m\}$, for any ordered subset $\mathcal{I} \subset \{1, ..., m\}$ with cardinal $k \leq \min(q, m)$, the analytic continuation of

$$\lambda \mapsto \lambda c_k \Phi_{Y,f}(\lambda - k - 1) \wedge \overline{\partial} \|f\|^2 \wedge \Big(\sum_{l=1}^{k} (-1)^{l-1} \overline{f_{i_l}} \bigwedge_{\substack{j=1 \\ j \neq l}}^{k} \overline{df_{i_j}} \Big) \, ,$$

where

$$c_k := \frac{(-1)^{k(k-1)/2}(k-1)!}{(2i\pi)^k}$$

is holomorphic at the origin. Its value at 0 defines, up to a multiplicative constant, a residual regular holonomic $(n-q, n-q+k)$-current which is supported by $Y \cap V(f)$; regular holonomiticity is here understood in the sense of Björk ([Bj2], chapter 9). Properties of such currents are similar to those introduced above in the case $q = n$. Proposition 3.1 will summarize the different properties of such restricted residual currents. The main case of interest for us will be the case where $m \leq q$ and $\dim (Y \cap V(f)) \leq q - m$, that is, when $f_1, ..., f_m$ define a complete intersection in $\mathcal{Y}$. In this case, the restricted residue current corresponding to $\mathcal{I} := \{1, ..., m\}$ is the Coleff-Herrera current on Y

$$\Big(\bigwedge_{j=1}^{m} \overline{\partial} \frac{1}{f_j} \Big) \wedge [Y]$$

introduced in [CH]. It is not surprising that residual restricted currents in such a complete intersection setting obey the transformation law for multidimensional residue calculus ([BGVY], chapter 6), which we will prove (and use next) in the case $m = q$. If $f_1 = P_1, ..., f_q = P_q$ are polynomials and W is an affine q-dimensional algebraic subvariety of the affine scheme $\mathbf{A}^n_{\mathbf{C}}$ such that $\dim (V(P) \cap |W|) = 0$, we will prove in section 4 that the total sum of restricted residues

$$\mathrm{Res} \begin{bmatrix} [W] \wedge Q dX_{i_1} \wedge \cdots \wedge dX_{i_q} \\ \\ P_1, ..., P_q \end{bmatrix}$$

vanishes as soon as the degree of Q is sufficiently small, under a properness assumption on the restriction of $(P_1, ..., P_q)$ to $|W|$. We will thus transpose to the restricted case an Abel-Jacobi formula proved in the case $q = n$ and $W = \mathbf{A}^n_{\mathbf{C}}$ in [VY].

Let again W be a q-dimensional irreducible algebraic subvariety in the affine scheme $\mathbf{A}^n_{\mathbf{C}}$ and $P_1, ..., P_q$, q polynomials in $\mathbf{C}[X_1, ..., X_n]$ such that $|W| \cap V(P)$ is a discrete (hence finite) algebraic set in $\mathbf{C}^n$. Let $Q \in \mathbf{C}[X_1, ..., X_n]$ and $\mathcal{I}$ a subset in $\{1, ..., n\}$ with cardinal q. The meromorphic differential form

$$\omega := \frac{Q d\zeta_{i_1} \wedge \cdots \wedge d\zeta_{i_q}}{P_1 \cdots P_q}$$

induces an element in $\Omega^q_{\mathbf{C}(\mathcal{W})/\mathbf{C}}$, where $\mathcal{W}$ denotes the completion of W in $\mathrm{Proj}\, \mathbf{C}[X_0, ..., X_n]$. We will prove in section 5, thanks to the algebraic residue theorem in [L2] and the properties of restricted residual currents that were pointed out in previous sections, that the residue at a closed point α in $|W| \cap V(P)$ (in the sense of Hübl or Lipman [L1]) of the differential form ω (viewed as an element in $\Omega^q_{\mathbf{C}(\mathcal{W})/\mathbf{C}}$) equals

$$\mathrm{Res}_{W,\alpha}\, [\omega] := \Big\langle \Big(\bigwedge_{j=1}^{q} \overline{\partial} \frac{1}{P_j} \Big) \wedge [W] \, , \, \psi Q d\zeta_{i_1} \wedge \cdots \wedge d\zeta_{i_q} \Big\rangle \, ,$$

where ψ denotes a test-function with compact support in some small neighborhood of α, such that $\psi \equiv 1$ near α. The result is clear when α is a smooth point of W, it will follow from the algebraic residue formula combined with a perturbation argument in the case α is a singular point of W. As a consequence of the fact that the analytic and algebraic approaches lead to the same restricted residual objects, we will extend in section 5 (with an algebraic formulation) to such a restricted context the affine Jacobi's theorem obtained in the non-restricted case $W = \mathbf{A}^n_{\mathbf{C}}$ in [VY].

Theorem 2.1 *Let W be a q-dimensional irreducible affine algebraic subvariety in $\mathbf{A}^n_{\mathbf{C}}$ $(0 < q < n)$ and $P_1, ..., P_q$ be q polynomials in $\mathbf{C}[X_1, ..., X_n]$ such that there exist strictly positive rational numbers $\delta_1, ..., \delta_q$ and two constants $K > 0$, $\kappa > 0$ such that :*

$$\zeta \in |W| \, , \ \|\zeta\| \geq K \implies \sum_{j=1}^{q} \frac{|P_j(\zeta)|}{\|\zeta\|^{\delta_j}} \geq \kappa \, ; \tag{2.2}$$

then, for any $Q \in \mathbf{C}[X_1, ..., X_n]$ such that $\deg Q < \delta_1 + \cdots + \delta_q - q$, *for any multi-index $(i_1, ..., i_q)$ in $\{1, ..., n\}^q$,*

$$\sum_{\alpha \in |W| \cap V(P)} \operatorname{Res}_{W,\alpha} \Big[\frac{Q d\zeta_{i_1} \wedge \cdots \wedge d\zeta_{i_q}}{P_1 \cdots P_q} \Big] = 0 \, . \tag{2.3}$$

We will derive (in sections 5 and 6) some consequences of this result in the spirit of Cayley-Bacharach's theorem and Wood's results [W]. The key point here (compare to the framework of [HK2] or [Ku2]) is that the properness assumption along $|W|$ (2.2) which is satisfied by the polynomial map $P := (P_1, ..., P_q)$ does not imply that the Zariski closures of $|W|$ and $V(P_1, ..., P_q)$ in $\mathbb{P}^n(\mathbf{C})$ have an empty common intersection on the hyperplane at infinity.

3 Restricted residual currents

We begin this section by recalling some basic facts about currents on analytic manifolds, especially integration currents on analytic sets or Coleff-Herrera currents and their "multiplication" with integration currents. We inspire ourselves on [Bj1], [BY8], [BY5], and [Meo].

We start with some basic facts about integration on a q-dimensional irreducible analytic subset Y in $U \subset \mathbf{C}^n$ [Le]. The subset Y_{reg} of regular points of Y is a q-dimensional complex manifold. The set of singular points Y_{sing} is an analytic subset of U with complex dimension $\dim Y_{\mathrm{sing}} < q$. Therefore for any smooth (q, q) test form $\phi_{(q,q)} \in \mathcal{D}^{(q,q)}(U)$, one can define the action of the integration current $[Y]$ on $\phi_{(q,q)}$ as

$$\begin{aligned} \langle [Y] \, , \, \phi_{(q,q)} \rangle &= \int_Y \phi_{(q,q)}(\zeta, \bar{\zeta}) = \int_{Y_{\mathrm{reg}}} \phi_{(q,q)}(\zeta, \bar{\zeta}) + \int_{Y_{\mathrm{sing}}} \phi_{(q,q)}(\zeta, \bar{\zeta}) \\ &= \int_{Y_{\mathrm{reg}}} \phi_{(q,q)}(\zeta, \bar{\zeta}) \, . \end{aligned}$$

For $\operatorname{Re}\lambda > 0$ and $f_1, ..., f_m$ holomorphic in U, one can define the (q,q)-current $\|f\|^{2\lambda}[Y]$ by

$$\langle \|f\|^{2\lambda}[Y]\,,\,\phi_{(q,q)}\rangle := \int_{Y_{\rm reg}} \|f\|^{2\lambda}\,\phi_{(q,q)}\,.$$

It is known ([Bj1],[Bj2]) that this current $[Y]$ is a regular holonomic current, which implies, for each point z_0 in $U\cap Y$, for any distribution coefficient $T_{[Y]}$ of the integration current $[Y]$, the existence of a Bernstein-Sato relation

$$\mathcal{Q}_{T_{[Y]},z_0}\Big(\lambda,\zeta,\bar\zeta,\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\bar\zeta}\Big)\,\Big[\|f\|^{2(\lambda+1)}\otimes T_{[Y]}\Big] = b_{z_0}(\lambda)\,(\|f\|^{2\lambda}\otimes T_{[Y]}) \tag{3.1}$$

($b_{z_0}\in \mathbf{C}[\lambda]$) valid in a neighborhood of z_0. In fact, this does not follow directly from Theorem 3.2.6 in [Bj1] since $\|f\|^2$ is a real analytic function (and not a holomorphic one). Nevertheless, the existence of Bernstein-Sato relations of the form (3.1) remains valid here since $\|f\|^2$ has the particular form

$$\|f(\zeta)\|^2 = \sum_{j=1}^m f_j(\zeta)\overline{f_j(\zeta)}$$

and the integration current on $Y=\{g_1=\cdots=g_N=0\}$ admits a Siu decomposition

$$[Y] = \sum_{1\le i_1<\cdots<i_{n-q}\le N} T_{i_1,\dots,i_{n-q}}\wedge\bigwedge_{l=1}^{n-q} dg_{i_l}\,,$$

where the $T_{i_1,\dots,i_{n-q}}$ are $(0,n-q)$ currents which are regular holonomic because of Coleff-Herrera type ([BY5, Meo, Bj1]). One can then proceed in

$$\mathbf{U} := \{(\zeta,\overline{\zeta})\,:\,\zeta\in U\}\subset \mathbf{C}^{2n}$$

with blocks of variables $(\zeta,\overline{\zeta})$ and profit from the fact that formally ∂_ζ and $\partial_{\overline{\zeta}}$ can be considered as derivations respect to independent sets of variables. Consider then the function of one complex variable defined by

$$\lambda\mapsto \Phi_{Y,f}(\lambda) := \|f\|^{2\lambda}[Y]\,. \tag{3.2}$$

This function (which is a ${}'\mathcal{D}^{(n-q,n-q)}(U)$-current valued function) is well defined and holomorphic in $\{\lambda\in\mathbf{C}\,;\,\operatorname{Re}\lambda>0\}$. Thanks to the Bernstein-Sato relations (3.1), it can be continued to the whole complex plane as a meromorphic function. The poles of this meromorphic extension are strictly negative rational numbers. Furthermore, there is a true pole at any point $\lambda = -k$, $k \in \mathbb{N}^*$.

In fact, we will need a more precise result, where the construction of the meromorphic continuation of (3.2) play a role. What we need is formulated in the following proposition.

Proposition 3.1 *Let Y be an irreducible q-dimensional analytic subset of $U \subset \mathbf{C}^n$ and $f_1, ..., f_m$ m functions holomorphic in U. For any $k \in \{1, ..., m\}$ and for any ordered subset $\mathcal{I} \subset \{1, ..., m\}$ with cardinal $k \le \min(q, m)$, the ${}'\mathcal{D}^{(n-q,n-q+k)}$-valued map*

$$\lambda \mapsto \lambda c_k \|f\|^{2(\lambda-k-1)} [Y] \wedge \overline{\partial} \|f\|^2 \wedge \Big(\sum_{l=1}^{k} (-1)^{l-1} \overline{f_{i_l}} \bigwedge_{\substack{j=1 \\ j \neq l}}^{k} \overline{df_{i_j}} \Big)$$

(*which is holomorphic in* $\mathrm{Re}\,\lambda > k+1$) *can be continued as a meromorphic map to the whole complex plane, with no pole at $\lambda = 0$. Its value at $\lambda = 0$ defines a residual $(n-q, n-q+k)$-current which is supported by the analytic set $Y \cap \{f_1 = \cdots = f_m = 0\} = Y \cap V(f)$ and denoted as*

$$\varphi \in \mathcal{D}^{(q,q-k)} \mapsto \mathrm{Res} \begin{bmatrix} [Y] \wedge (\cdot) \\ f_{i_1}, ..., f_{i_k} \\ f_1, ..., f_m \end{bmatrix} (\varphi) = \mathrm{Res} \begin{bmatrix} [Y] \wedge \varphi \\ f_{i_1}, ..., f_{i_k} \\ f_1, ..., f_m \end{bmatrix}. \tag{3.3}$$

Proof. Assume that Y is defined (in U) by the equations $g_1 = \cdots = g_N = 0$ and that ν is the (Hilbert-Samuel) multiplicity of the ideal $\mathcal{O}_{U,y}$ generated by $g_1, ..., g_N$ at a generic point $y \in Y$. Let $d = n - q$. One can conclude from [Meo] that $[Y]$ coincides with the value at $\mu = 0$ of the meromorphic ${}'\mathcal{D}^{(d,d)}(U)$-valued map Ψ_g

$$\mu \overset{\Psi_g}{\mapsto} \frac{\mu (d-1)!}{(2i\pi)^d \, \nu} \|g\|^{2\mu} \, \overline{\partial} \log \|g\|^2 \wedge \partial \log \|g\|^2 \wedge \sum_{\substack{j_1 < \cdots < j_{d-1} \\ 1 \le j_j \le N}} \bigwedge_{l=1}^{d-1} \Big(\frac{\overline{\partial g_{j_l}} \wedge \partial g_{j_l}}{\|g\|^2} \Big).$$

In fact, in the general situation where $(g_1, ..., g_N)$ define a q-purely dimensional cycle $\mathcal{Z}$ (non necessarily irreducible) in U, the integration current (with multiplicities) on $\mathcal{Z}$ can be expressed as the value at $\lambda = 0$ of some meromorphic ${}'\mathcal{D}^{(d,d)}(U)$-valued function which can be made explicit in terms of $g_1, ..., g_N$ (see Theorem 3.1 in [BY5] for a proof in the algebraic case). Let $\mathcal{I} \subset \{1, ..., m\}$ with cardinal $k \leq \min(q, m)$ and, for $\operatorname{Re} \lambda > k+1$,

$$\Theta_{f,\mathcal{I}}(\lambda) := \lambda \|f\|^{2(\lambda-k-1)} \overline{\partial} \|f\|^2 \wedge \Big(\sum_{l=1}^{k} (-1)^{l-1} \overline{f_{i_l}} \bigwedge_{\substack{j=1 \\ j \neq l}}^{k} \overline{df_{i_j}} \Big) .$$

In order to prove the proposition, we can localize the problem and assume that the origin belongs to $Y \cap V(f)$. As in our previous work (see for example [BY8], pages 32-33, or [BY5], page 208) we construct an analytic n-dimensional manifold $\mathcal{X}$, a neighborhood V of 0 in U, a proper map $\pi : \mathcal{X} \to V$ which realizes a local isomorphism between $V \setminus \{f_1 \cdots f_m \, g_1 \cdots g_N = 0\}$ and $\mathcal{X} \setminus \pi^{-1}(\{f_1 \cdots f_m \, g_1 \cdots g_N = 0\})$, such that in local coordinates on $\mathcal{X}$ (centered at a point x), one has, in the corresponding local chart $\mathcal{U}_x$ around x,

$$\begin{aligned} f_j \circ \pi(t) &= u_j(t) \, t_1^{\alpha_{j1}} \cdots t_n^{\alpha_{jn}} = u_j(t) \, t^{\alpha_j}, \ j = 1, \ldots, m \\ g_k \circ \pi(t) &= v_k(t) \, t_j^{\beta_{k1}} \cdots t_n^{\beta_{kn}} = v_k(t) \, t^{\beta_k}, \ k = 1, ..., N \end{aligned}$$

where the u_j, $j = 1, ..., m$ and the v_k, $k = 1, ..., N$, are non vanishing holomorphic functions in $\mathcal{U}_x$, at least one of the monomials t^{α_j}, $j = 1, ..., m$ divides all of them (we will denote this monomial as t^{α}), and at least one of the monomials t^{β_k}, $k = 1, ..., N$ divides all of them (we will denote this monomial as t^{β}).
When φ is a $(q, q-k)$-test form with support in V, one has, for $\operatorname{Re} \lambda >> 0$,

$$\int_{V \cap Y} \Theta_{f,\mathcal{I}}(\lambda) \wedge \varphi = \Big[\int_V \Psi_g(\mu) \wedge \Theta_{f,\mathcal{I}}(\lambda) \wedge \varphi \Big]_{\mu=0}$$

(the right hand side being continued as a meromorphic function of μ which has no pole at $\mu = 0$). For λ fixed with $\operatorname{Re} \lambda >> 0$, one can rewrite for $\operatorname{Re} \mu >> 0$ the integral

$$\int_V \Psi_g(\mu) \wedge \Theta_{f,\mathcal{I}}(\lambda) \wedge \varphi$$

as a sum of integrals of the form

$$\int_{\mathcal{U}_x} \pi^*[\Psi_g](\mu) \wedge \pi^*[\Theta_{f,\mathcal{I}}(\lambda)] \wedge \rho \pi^*(\varphi) \, , \tag{3.4}$$

where ρ is a test-function in $\mathcal{U}_x$ which corresponds to a partition of unity for $\pi^*(\operatorname{Supp}\varphi)$. We know from Lemmas 2.1 and 2.2 in [BY5] that

$$\Big[\pi^*[\Psi_g(\mu)]\Big]_{\mu=0} = \Big[\Psi_{g\circ\pi}(\mu)\Big]_{\mu=0}$$

is a positive ∂ and $\overline{\partial}$-closed current $\theta_{\mathcal{U}_x}$ in $\mathcal{U}_x$, which implies that, as soon as $\operatorname{Re}\lambda >> 0$,

$$\Big[\int_{\mathcal{U}_x} \pi^*[\Psi_g](\mu) \wedge \pi^*[\Theta_{f,\mathcal{I}}(\lambda)] \wedge \rho\pi^*(\varphi)\Big]_{\mu=0} = \int_{\mathcal{U}_x} \theta_{\mathcal{U}_x} \wedge \pi^*[\Theta_{f,\mathcal{I}}(\lambda)] \wedge \rho\pi^*(\varphi)\,.$$

On the other hand, in $\mathcal{U}_x$ and for $\operatorname{Re}\lambda >> 0$, a straightforward computation leads to

$$\pi^*\Big[\Theta_{f,\mathcal{I}}\Big](\lambda) = \lambda\frac{a^{2\lambda}|t^\alpha|^{2\lambda}}{t^{k\alpha}}\Big(\vartheta + \varpi \wedge \frac{\overline{dt^\alpha}}{\overline{t^\alpha}}\Big)\,,$$

where ϑ and ϖ are smooth differential forms in $\mathcal{U}_x$ (with respective types $(0,k)$ and $(0,k-1)$) and a is a strictly positive real analytic function in $\mathcal{U}_x$. It follows from Stokes' theorem that

$$\int_{\mathcal{U}_x} \pi^*[\Theta_{f,\mathcal{I}}](\lambda) \wedge \theta_{\mathcal{U}_x} \wedge \rho\pi^*(\varphi) = \int_{\mathcal{U}_x} \frac{|t^\alpha|^{2\lambda}}{t^{k\alpha}}\,\theta_{\mathcal{U}_x} \wedge \xi_\varphi(\rho\,;\,t,\lambda)\,, \tag{3.5}$$

where $(t,\lambda) \mapsto \xi_\varphi(\rho\,;\,t,\lambda)$ is a $(n-q,n-q)$-differential form with smooth coefficients (in t) depending holomorphically in λ.
One can see also that, for $\operatorname{Re}\mu >> 0$,

$$\pi^*[\Psi_g](\mu) = \mu\, b^{2\mu}\, |t^\beta|^{2\mu}\,\Big(\frac{\overline{dt^\beta}}{\overline{t^\beta}} + \eta_{(0,1)}\Big) \wedge \Big(\frac{dt^\beta}{t^\beta} + \eta_{(1,0)}\Big) \wedge \upsilon\,,$$

where b is a strictly positive real analytic function in $\mathcal{U}_x$, $\eta_{(0,1)}$, $\eta_{(1,0)}$, υ are smooth differential forms in $\mathcal{U}_x$ with respective types $(0,1)$, $(1,0)$ and $(d-1,d-1)$. This implies that, if $t_{i_1},...,t_{i_s}$ are the coordinates that appear in t^β,

$$\theta_{\mathcal{U}_x} = \sum_{l=1}^{s} [t_{i_l} = 0]\,\wedge \omega_{i_l}\,,$$

where ω_{i_l} is a smooth $(d-1,d-1)$-form in $\mathcal{U}_x$ and $[t_i = 0]$ denotes the integration current (without multiplicities) on $\{t_i = 0\}$. Therefore, for $\operatorname{Re}\lambda >> 0$,

$$\int_{\mathcal{U}_x} \pi^*[\Theta_{f,\mathcal{I}}](\lambda) \wedge \theta_{\mathcal{U}_x} \wedge \rho\pi^*(\varphi) = \sum_{\substack{l=1\\ (t_{i_l},t^\alpha)=1}}^{s} \int_{\{t_{i_l}=0\}\cap\mathcal{U}_x} \frac{|t^\alpha|^{2\lambda}}{t^{k\alpha}}\omega_{i_l}(t) \wedge \xi_\varphi(\rho\,;\,t,\lambda).$$

Such a function of λ can be continued to a meromorphic function in the whole complex plane, with no pole at $\lambda = 0$ (using Stokes' theorem). The assertion of the proposition follows, since for $\mathrm{Re}\,\lambda >> 0$,

$$\int_V \Psi_g(\mu) \wedge \Theta_{f,\mathcal{I}}(\lambda) \wedge \varphi$$

is a sum of integrals of the form (3.4). $\diamondsuit$

Keeping the above notation one obtains the following corollary.

Corollary 3.1 *Under the conditions of Proposition 3.1, the residual current defined by* (3.3) *has the following properties*
1) *For any* $h \in H(U)$ *such that*

$$\forall K \subset\subset U \cap Y\,, \exists C_K > 0\,,\ |h| \leq C_K \|f\| \ \text{on}\ K\,, \tag{3.6}$$

one has

$$\mathrm{Res}\begin{bmatrix} h^k[Y] \wedge (\cdot) \\ f_{i_1}, ..., f_{i_k} \\ f_1, ..., f_m \end{bmatrix} \equiv 0$$

2) *If* $h \in H(U)$ *and*

$$h(z) = 0,\ \forall z \in Y \cap V(f)\,, \tag{3.7}$$

then one has

$$\mathrm{Res}\begin{bmatrix} \overline{h}[Y] \wedge (\cdot) \\ f_{i_1}, ..., f_{i_k} \\ f_1, ..., f_m \end{bmatrix} \equiv 0$$

Proof. Let us now suppose that h satisfies (3.6). If we do not perform integration by parts as in (3.5), we have, for $\mathrm{Re}\,\lambda >> 0$,

$$\begin{aligned} &\int_{\mathcal{U}_x} \pi^*[\Theta_{f,\mathcal{I}}](\lambda) \wedge \theta_{\mathcal{U}_x} \wedge \rho\pi^*(h^k\varphi) \\ &= \lambda \sum_{\substack{l=1 \\ (t_{i_l}, t^\alpha)=1}}^{s} \int_{\{t_{i_l}=0\}\cap\mathcal{U}_x} \frac{a^{2\lambda}|t^\alpha|^{2\lambda}}{t^{k\alpha}} \Big(\vartheta + \varpi \wedge \frac{\overline{dt^\alpha}}{\overline{t^\alpha}}\Big) \wedge \omega_{i_l}(t) \wedge \rho\pi^*(h^k\varphi)\,. \end{aligned}$$

Condition (3.6) implies that there exists some positive constant κ such that, for any $l = 1, ..., s$ with t_{i_l} coprime with t^α,

$$|\pi^* h(t_1, ..., \overset{i_l}{0}, ..., t_n)| \leq \kappa |t^\alpha|\,, \quad t \in \mathrm{Supp}\,\rho\,,$$

which implies that $t^{k\alpha}$ divides $(\pi^* h^k)_{\{|t_{i_l}=0\}}$ on the support of ρ. This implies that for such h,

$$\left[\int_{\mathcal{U}_x} \pi^*[\Theta_{f,\mathcal{I}}](\lambda) \wedge \theta_{\mathcal{U}_x} \wedge \rho\pi^*(h^k\varphi)\right]_{\lambda=0} = 0\,,$$

which gives the first assertion of the corollary since

$$\int_V \Psi_g(\mu) \wedge \Theta_{f,\mathcal{I}}(\lambda) \wedge \varphi$$

is a sum of integrals of the form (3.4).
If h vanishes on $Y \cap V(f)$, then, for any $l = 1, ..., s$ such that t_{i_l} is coprime with t^α, any coordinate which divides t^α also divides $(\pi^* h)_{|\{t_{i_l}=0\}}$ on the support of ρ. This implies that any expression of the form

$$\int_{\{t_{i_l}=0\}\cap\mathcal{U}_x} \frac{a^{2\lambda}|t^\alpha|^{2\lambda}}{t^{k\alpha}} \Big(\vartheta + \varpi \wedge \frac{\overline{dt^\alpha}}{\overline{t^\alpha}}\Big) \wedge \omega_{i_l}(t) \wedge \rho\pi^*(\overline{h}\varphi)$$

has in fact no antiholomorphic singularity (therefore has a meromorphic extension which is polefree at the origin). It follows that for such h, one has again

$$\left[\int_{\mathcal{U}_x} \pi^*[\Theta_{f,\mathcal{I}}](\lambda) \wedge \theta_{\mathcal{U}_x} \wedge \rho\pi^*(\overline{h}\varphi)\right]_{\lambda=0} = 0\,,$$

which proves the remaining assertion of the corollary since again

$$\int_V \Psi_g(\mu) \wedge \Theta_{f,\mathcal{I}}(\lambda) \wedge \varphi$$

is a sum of integrals of the form (3.4). $\diamondsuit$

When $k = m \leq q$, we will use the simplified notation

$$\mathrm{Res}\begin{bmatrix} [Y] \wedge (\cdot) \\ f_1, ..., f_m \end{bmatrix}(\varphi) := \mathrm{Res}\begin{bmatrix} [Y] \wedge (\cdot) \\ f_1, ..., f_m \\ f_1, ..., f_m \end{bmatrix}.$$

The transformation law for residual currents can be transposed to the case of restricted residual currents. Since we deal in this paper with restricted residual currents supported by discrete sets, we state the transformation law in this particular setting. One has the following proposition :

Proposition 3.2 *Let Y be an irreducible q-dimensional analytic subset of $U \subset \mathbf{C}^n$ and $f_1, ..., f_q$, $g_1, ..., g_q$, $2q$ functions holomorphic in U such that $Y \cap V(f)$ and $Y \cap V(g)$ are discrete analytic sets. Assume that there exist q^2 holomorphic functions in U, a_{kl}, $1 \leq k, l \leq q$, such that*

$$g_k(\zeta) = \sum_{l=1}^{q} a_{kl}(\zeta)\, f_l(\zeta)\,,\ \ k = 1, ..., q\,, \quad \zeta \in Y$$

Then, one has the following equality between restricted residual currents :

$$\mathrm{Res} \begin{bmatrix} [Y] \wedge (\cdot) \\ f_1, ..., f_q \end{bmatrix} = \mathrm{Res} \begin{bmatrix} \Delta\, [Y] \wedge (\cdot) \\ g_1, ..., g_q \end{bmatrix}, \tag{3.8}$$

where $\Delta := \det[a_{kl}]_{1 \leq k, l \leq q}$.

Proof. In order to prove this equality, we just need to prove it when U is a neighborhood V of a point $\alpha \in Y \cap (V(f) \cup V(g))$ such that α is the only point of $Y \cap (V(f) \cup V(g))$ which lies in this neighborhood. Thanks to the first assertion in Corollary 3.1, it is enough to test the two currents involved in (3.8) on test forms in $\mathcal{D}^{(q,0)}(V)$ whose coefficients are holomorphic in a neighborhood of α. Let φ be such a test form. Since

$$\begin{aligned} &\overline{\partial}\Big[\|f\|^{2(\lambda-q)}\,[Y] \wedge \Big(\sum_{j=1}^{q}(-1)^{j-1}\overline{f}_j \bigwedge_{\substack{l=1\\ l\neq j}}^{q} \overline{df_j}\Big)\Big] \\ &= \lambda\|f\|^{2(\lambda-q)}\,[Y]\ \wedge \bigwedge_{j=1}^{q} \overline{df_j} \\ &= \lambda\|f\|^{2(\lambda-q-1)}\,[Y] \wedge \overline{\partial}\|f\|^2 \wedge \Big(\sum_{j=1}^{q}(-1)^{j-1}\overline{f}_j \bigwedge_{\substack{l=1\\ l\neq j}}^{q} \overline{df_l}\Big) \end{aligned}$$

for $\mathrm{Re}\,\lambda >> 0$ and

$$\sum_{j=1}^{q} s_j(\zeta) f_j(\zeta) = 1\,, \quad \forall\, \zeta \in (V \cap Y_{\mathrm{reg}}) \setminus \{\alpha\},$$

where

$$s_j := \frac{\overline{f}_j}{\|f\|^2}\,, \quad j = 1, ..., q,$$

one has, by Stokes' theorem, that

$$\operatorname{Res}\left[\begin{array}{c}[Y]\wedge\varphi\\ f_1,\dots,f_q\end{array}\right] = (-1)^q\omega_q\int_{Y_{\rm reg}}\frac{\sum\limits_{j=1}^{q}(-1)^{j-1}\overline{f}_j\bigwedge\limits_{\substack{l=1\\ l\neq j}}^{q}\overline{df_j}}{\|f\|^{2q}}\wedge\overline{\partial}\varphi$$

$$= (-1)^q\omega_q\int_{Y_{\rm reg}}\Big(\sum_{j=1}^{q}(-1)^{j-1}s_j\bigwedge_{\substack{l=1\\ l\neq j}}^{q}ds_l\Big)\wedge\overline{\partial}\varphi\,. \quad (3.9)$$

Similarly, if we introduce

$$t_j := \frac{\overline{g}_j}{\|g\|^2}\,,\quad j=1,\dots,q\,,$$

and

$$\widetilde{s}_j := \sum_{l=1}^{q}a_{lj}t_l\,,\quad j=1,\dots,q\,,$$

one has also

$$\sum_{j=1}^{q}\widetilde{s}_j(\zeta)f_j(\zeta)=1\,,\quad \forall\,\zeta\in(V\cap Y_{\rm reg})\setminus\{\alpha\}\,.$$

Let, for $\xi\in[0,1]$ and $j=1,\dots,q$,

$$s_j^{(\xi)} = (1-\xi)\,s_j+\xi\,\widetilde{s}_j\,.$$

Note that we have

$$\sum_{j=1}^{q}s_j^{(\xi)}(\zeta)f_j(\zeta)=1\,,\quad \forall\xi\in[0,1]\,,\quad \forall\zeta\in(V\cap Y_{\rm reg})\setminus\{\alpha\}\,.$$

Therefore, one has, since

$$\bigwedge_{j=1}^{q}\overline{\partial}_\zeta s_j^{(\xi)}\equiv 0$$

on $(V\cap Y_{\rm reg})\setminus\{\alpha\}$,

$$\frac{d}{d\xi}\left[\int_{W_{\rm reg}}\Big(\sum_{j=1}^{q}(-1)^{j-1}s_j^{(\xi)}\bigwedge_{\substack{l=1\\ l\neq j}}^{q}ds_l^{(\xi)}\Big)\wedge\overline{\partial}\varphi\right]\equiv 0$$

on $[0,1]$. It follows from (3.9) that

$$\begin{aligned}
\mathrm{Res}\begin{bmatrix} [Y]\wedge\varphi \\ f_1,....,f_q \end{bmatrix} &= (-1)^q\omega_q\int_{Y_{\mathrm{reg}}}\Big(\sum_{j=1}^q(-1)^{j-1}\widetilde{s}_j\bigwedge_{\substack{l=1\\ l\neq j}}^q\overline{\partial}\widetilde{s}_l\Big)\wedge\overline{\partial}\varphi \\
&= (-1)^q\omega_q\int_{Y_{\mathrm{reg}}}\Delta\,\frac{\sum_{j=1}^q(-1)^{j-1}\overline{g}_j\bigwedge_{\substack{l=1\\ l\neq j}}^q\overline{dg_j}}{\|g\|^{2q}}\wedge\overline{\partial}\varphi \\
&= \mathrm{Res}\begin{bmatrix} \Delta\,[Y]\wedge\varphi \\ g_1,....,g_q \end{bmatrix}.
\end{aligned}$$

this concludes the proof of the proposition. $\diamondsuit$

As a consequence of this result, we will state in the algebraic context the following analogue of the global transformation law. We need first some additional of notation. Assume that W is a q-dimensional irreducible algebraic subvariety in the affine space $\mathbf{A}^n_{\mathbf{C}}$ (the integration current on $|W|$ without multiplicities taken into account being denoted as $[W]$) and that $P_1,...,P_q$ are q elements in $\mathbf{C}[X_1,...,X_n]$ such that $|W|\cap V(P_1,...,P_q)$ is a discrete (hence finite) algebraic subset of $\mathbf{C}^n$. For any $Q\in\mathbf{C}[X_1,...,X_n]$, any ordered subset $\{i_1,...,i_q\}$ of $\{1,...,n\}$, we will denote as

$$\mathrm{Res}\begin{bmatrix} [W]\wedge Q\bigwedge_{l=1}^q dX_{i_l} \\ \\ P_1,...,P_q \end{bmatrix}$$

the result of the action of the W-restricted current

$$\varphi\mapsto\mathrm{Res}\begin{bmatrix} [W]\wedge\varphi \\ P_1,...,P_q \end{bmatrix}$$

on the $(q,0)$-test form $Q(\zeta)\psi(\zeta)\bigwedge_{l=1}^q d\zeta_{i_l}$, where ψ is any test-function in $\mathcal{D}(\mathbf{C}^n)$ which equals 1 in a neighborhood of $|W|\cap V(P)$. If

$$\Gamma=\sum_{j=1}^M\nu_jW_j$$

(where $W_1,...,W_M$ are M irreducible algebraic subsets in $\mathbf{C}^n$ and $\nu_j\in\mathbb{N}^*$, $j=1,...,M$) is an effective q-dimensional algebraic cycle in the affine space

$\mathbf{C}^n$ and $P_1, ..., P_q$ are q polynomials such that $W_j \cap V(P)$ is discrete for any $j = 1, ..., M$, we will also denote as

$$\mathrm{Res}\begin{bmatrix} [\Gamma] \wedge Q \bigwedge\limits_{l=1}^{q} dX_{i_l} \\ \\ P_1, ..., P_q \end{bmatrix}$$

the weighted sum

$$\sum_{j=1}^{M} \nu_j \mathrm{Res}\begin{bmatrix} [W_j] \wedge Q \bigwedge\limits_{l=1}^{q} dX_{i_l} \\ \\ P_1, ..., P_q \end{bmatrix}$$

Corollary 3.2 *Let Γ be an effective q-dimensional algebraic cycle in the affine space $\mathbf{C}^n$ and $P_1, ..., P_q, R_1, ..., R_q$ be $2q$ polynomials such that* $\mathrm{Supp}\,\Gamma \cap V(P_1, ..., P_q)$ *and* $\mathrm{Supp}\,\Gamma \cap V(R_1, ..., R_q)$ *are discrete (hence finite) algebraic subsets of $\mathbf{C}^n$. Assume that there is a (q, q)-matrix of polynomials $[A_{k,l}]_{1 \le k,l \le q}$ such that*

$$R_k = \sum_{l=1}^{q} A_{kl} P_l \quad \textit{on} \quad \mathrm{Supp}\,\Gamma \quad k = 1, .., q\,.$$

Then, for any $Q \in \mathbf{C}[X_1, ..., X_n]$, any ordered subset $\{i_1, ..., i_q\}$ of $\{1, ..., n\}$, one has

$$\mathrm{Res}\begin{bmatrix} [\Gamma] \wedge Q \bigwedge\limits_{l=1}^{q} dX_{i_l} \\ \\ P_1, ..., P_q \end{bmatrix} = \mathrm{Res}\begin{bmatrix} [\Gamma] \wedge \Delta\, Q \bigwedge\limits_{l=1}^{q} dX_{i_l} \\ \\ R_1, ..., R_q \end{bmatrix}, \qquad (3.10)$$

where Δ denotes the determinant of the matrix $[A_{k,l}]_{1 \le k,l \le q}$.

Another key point about the restricted residual current in the discrete context is the following annihilator property :

Proposition 3.3 *Let Y be an irreducible q-dimensional analytic subset of $U \subset \mathbf{C}^n$ and $f_1, ..., f_q$ be q functions holomorphic in U such that $Y \cap V(f)$ is a discrete analytic set. Then one has, for $k = 1, ..., q$,*

$$\mathrm{Res}\begin{bmatrix} f_k [Y] \wedge (\cdot) \\ f_1, ..., f_q \end{bmatrix} = 0 \qquad (3.11)$$

Proof. We give here a self-contained proof of the above proposition. Actually, because of the properties quoted in Corollary 3.1, it is enough to show that if $\alpha \in V(P) \cap Y$ and φ is a test-function with support arbitrarily small about α with $\varphi = 1$ in some neighborhood v_α of α, then, for any function $h \in C^\infty(U)$ which is holomorphic on v_α, for any ordered subset $\mathcal{I} = \{i_1, ..., i_q\} \subset \{1, ..., n\}$, one has, for $j = 1, ..., q$,

$$\operatorname{Res}\begin{bmatrix} f_j[Y] \wedge h\varphi \, d\zeta_{\mathcal{I}} \\ f_1, ..., f_q \end{bmatrix} = 0\,.$$

One can use Stokes' formula (as in the proof of Proposition 3.2) and write

$$\operatorname{Res}\begin{bmatrix} f_k[Y] \wedge h\varphi \, d\zeta_{\mathcal{I}} \\ f_1, ..., f_q \end{bmatrix} = (-1)^q \omega_q \int_{Y_{\mathrm{reg}}} h f_k \Big(\sum_{j=1}^q (-1)^{j-1} s_j \bigwedge_{\substack{l=1 \\ l \neq j}}^q ds_l\Big) \wedge \overline{\partial}\varphi \wedge d\zeta_{\mathcal{I}}\,,$$

where $s_j := \overline{f_j}/\|f\|^2$, $j = 1, ..., q$. One can see at once that

$$\begin{aligned} f_k \Big(\sum_{j=1}^q (-1)^{j-1} s_j \bigwedge_{\substack{l=1 \\ l \neq j}}^q ds_l\Big) \wedge d\zeta_{\mathcal{I}} \wedge [Y] &= \Big(\bigwedge_{\substack{l=1 \\ l \neq k}}^q ds_l\Big) \wedge \overline{\partial}\varphi \wedge d\zeta_{\mathcal{I}} \wedge [Y] \\ &= \pm d\left[s_{k'} \Big(\bigwedge_{\substack{l=1 \\ l \neq k, k'}}^q ds_l\Big) \wedge \overline{\partial}\varphi \wedge d\zeta_{\mathcal{I}} \wedge [Y]\right] \end{aligned}$$

for $k' \neq k$, since $s_1 f_1 + \cdots + s_q f_q \equiv 1$ on $Y \cap \operatorname{Supp} \overline{\partial}\varphi$, which shows that

$$\int_{Y_{\mathrm{reg}}} h f_k \Big(\sum_{j=1}^q (-1)^{j-1} s_j \bigwedge_{\substack{l=1 \\ l \neq j}}^q ds_l\Big) \wedge \overline{\partial}\varphi \wedge d\zeta_{\mathcal{I}} = 0$$

as a consequence of Stokes' formula on Y. $\diamondsuit$

We remark here that there is an alternate proof of the last proposition. In fact, when $m \leq q$ and $f_1, ..., f_m$ define a complete intersection on Y, one can show that the restricted residual current

$$\operatorname{Res}\begin{bmatrix} [Y] \wedge (\cdot) \\ f_1, ..., f_m \end{bmatrix}$$

coincides with the Coleff-Herrera current $\Big(\bigwedge_{j=1}^m \overline{\partial}(1/f_j)\Big) \wedge [Y]$ as it is defined in [CH]. The proof of this claim can be carried out as it was done in the non restricted case in [PTY], section 5. Since the proof of this fact is rather tedious, we will not give it here. A consequence of this result is that, when $f_1, ..., f_m$ ($m \leq q$) define a complete intersection on Y, one has for $k = 1, ..., m$,

$$\operatorname{Res}\left[\begin{array}{c} f_k[Y] \wedge (\cdot) \\ f_1, ..., f_m \end{array}\right] = f_k\Big(\bigwedge_{j=1}^{m} \overline{\partial}\frac{1}{f_j}\Big) \wedge [Y] = 0$$

(see [CH]). This implies the proposition when $m = q$.
Note moreover that Proposition 3.2 also holds when $m < q$: namely, if $(f_1, ..., f_m)$ and $(g_1, ..., g_m)$ define complete intersections on Y and are such that there exist holomorphic functions a_{kl}, $1 \leq k, l \leq m$ in the ambient space U satisfying

$$g_k(\zeta) = \sum_{l=1}^{m} a_{kl}(\zeta) f_l(\zeta)\,,\ k = 1, ..., m\,,\ \zeta \in Y\,,$$

then formula (3.8) remains valid with m instead of q.

4 An Abel-Jacobi formula in the restricted case (analytic approach)

One of the key facts about restricted residual currents (as defined through the analytic approach described in section 3) is that they satisfy (in the 0-dimensional complete intersection setting) Abel-Jacobi's formula, exactly as in the non-restricted case (see [VY]). Such a result will be, together with the validity of the transformation law in the restricted context) a crucial fact in order to compare our analytic approach and the algebraic one.

Proposition 4.1 *Let W be a q-dimensional irreducible affine algebraic subvariety of the affine scheme $\mathbf{A}^n_{\mathbf{C}}$ $(0 < q < n)$ and $P_1, ..., P_q$ be q polynomials in $\mathbf{C}[X_1, ..., X_n]$ such that there exist strictly positive rational numbers $\delta_1, ..., \delta_q$ and two constants $K > 0$, $\kappa > 0$ with :*

$$\zeta \in |W|\,,\ \|\zeta\| \geq K \implies \sum_{j=1}^{q} \frac{|P_j(\zeta)|}{\|\zeta\|^{\delta_j}} \geq \kappa\,. \tag{4.1}$$

Then, for any $Q \in \mathbf{C}[X_1, ..., X_n]$ *such that* $\deg Q < \delta_1 + \cdots + \delta_q - q$*, for any multi-index* $(i_1, ..., i_q)$ *in* $\{1, ..., n\}^q$*,*

$$\operatorname{Res}\begin{bmatrix} [W] \wedge Q \bigwedge\limits_{l=1}^{q} dX_{i_l} \\ \\ P_1, ..., P_q \end{bmatrix} = 0\,. \tag{4.2}$$

Before we give the proof of this result, let us state an important corollary :

Corollary 4.1 *Let* W *be a* q*-dimensional irreducible algebraic subvariety in the affine scheme* $\mathbf{A}^n_{\mathbf{C}}$ *and* $\mathcal{W}$ *be its completion in* $\operatorname{Proj}\mathbf{C}[X_0, ..., X_n]$*. Let* $P_1, ..., P_q$ *be* q *elements in* $\mathbf{C}[X_1, ..., X_n]$*, with respective degrees* $D_1, ..., D_q$*, such that*

$$|\mathcal{W}| \cap \Big\{[\zeta_0 : ... : \zeta_n] \in \mathbb{P}^n(\mathbf{C})\,;\, {}^hP_j(\zeta_0, ..., \zeta_n) = 0,\ j = 1, ..., q\Big\} \subset \mathbf{C}^n\,, \tag{4.3}$$

where ${}^hP_j$*,* $j = 1, ..., q$*, denotes the homogeneization of the polynomial* P_j*. Then, for any* $Q \in \mathbf{C}[X_1, ..., X_n]$ *such that* $\deg Q < D_1 + \cdots + D_q - q$*, for any multi-index* $(i_1, ..., i_q)$ *in* $\{1, ..., n\}^q$*,*

$$\operatorname{Res}\begin{bmatrix} [W] \wedge Q \bigwedge\limits_{l=1}^{q} dX_{i_l} \\ \\ P_1, ..., P_q \end{bmatrix} = 0 \tag{4.4}$$

Proof of Corollary 4.1. Assume that

$$|\mathcal{W}| = \Big\{[\zeta_0 : ... : \zeta_n] \in \mathbb{P}^n(\mathbf{C})\,;\, {}^h\mathcal{G}_j(\zeta_0, ..., \zeta_n) = 0,\ j = 1, ..., N\Big\}\,,$$

where $\mathcal{G}_1, ..., \mathcal{G}_N$ are homogeneous polynomials in $\tilde{\zeta} = (\zeta_0, ..., \zeta_n)$. Condition (4.3) implies that

$$|\mathcal{W}| \cap \Big\{[\zeta_0 : ... : \zeta_n] \in \mathbb{P}^n(\mathbf{C})\,;\, {}^hP_j(\zeta_0, ..., \zeta_n) = 0,\ j = 1, ..., q\Big\}$$

is a finite set in $\mathbf{C}^n$; this implies (through a compacity argument) that there exists $K, \kappa > 0$ such that, for any $(\zeta_0, ..., \zeta_n) \in \mathbf{C}^{n+1} \setminus \{(0, ..., 0)\}$ such that

$$\Big(|\zeta_1|^2 + \cdots + |\zeta_n|^2\Big)^{1/2} \geq K|\zeta_0|\,,$$

one has

$$\sum_{j=1}^{q} \frac{|{}^h P_j(\widetilde{\zeta})|}{\|\widetilde{\zeta}\|^{D_j}} + \sum_{l=1}^{M} \frac{|\mathcal{G}_l(\widetilde{\zeta})|}{\|\widetilde{\zeta}\|^{\deg \mathcal{G}_l}} \geq \kappa \, .$$

Condition (4.1) with $\delta_j = D_j$, $j = 1, ..., q$, holds if we restrict to the affine space $\mathbf{C}^n$. The statement (4.4) follows then from (4.2). $\diamondsuit$

We remark that a proposition similar to Proposition 4.1 was proved in the non restricted case ($W = \mathbf{A}^n_{\mathbf{C}}$) in [VY]. Unfortunately, the proof which is given there (and depends heavily on resolution of singularities on the analytic manifold $\mathbb{P}^n(\mathbf{C})$) cannot immediately be transposed to the restricted case (since the Zariski closure $|\mathcal{W}|$ of $|W|$ in $\mathbb{P}^n(\mathbf{C})$ is not a smooth manifold anymore). Instead, we will follow an alternative approach (applicable also for the case $q = n$), based on an argument in the affine space (and not in its compactification $\mathbb{P}^n(\mathbf{C})$), which was proposed by Haï Zhang in [Z]. Our task has been to adapt this argument to the restricted case.

Note that, if $z = Aw$ is a linear change of variables in $\mathbf{C}^n$, one has, for any element in $\mathcal{D}^{(q,0)}(\mathbf{C}^n)$

$$\mathrm{Res} \begin{bmatrix} [W] \wedge \varphi \\ f_1, ..., f_q \end{bmatrix} = \mathrm{Res} \begin{bmatrix} [A^{-1}(W)] \wedge A^* \varphi \\ f_1 \circ A, ..., f_q \circ A \end{bmatrix} .$$

Therefore, we do not loose generality is we assume that $\mathcal{I} = \{1, ..., q\}$ and that the projection

$$\Pi \ : (\zeta_1, ..., \zeta_n) \mapsto (\zeta_1, ..., \zeta_q)$$

is a proper map from $|W|$ to $\mathbf{C}^q$ (coordinates can be choosen in such a way that Noether normalization theorem applies respect to any $(q, n-q)$ splitting $\zeta = (\zeta', \zeta'')$ of the set of variables $(\zeta_1, ..., \zeta_n)$, see for example [Fo, Ru]).

For δ_i, $i = 1, \ldots, q$ which appear in the statement of Proposition 4.1 we choose a positive integer N large enough so that

$$N \prod_{\substack{l=1 \\ l \neq j}}^{q} \delta_l > 2, \quad j = 1, \ldots, q \, . \tag{4.5}$$

Then, let

$$\delta^{[j]} := N \prod_{\substack{l=1 \\ l \neq j}}^{q} \delta_l \, , \quad j = 1, ..., q \, ,$$

and

$$\delta := N\delta_1 \cdots \delta_q = \delta_j \delta^{[j]}\,, \quad j = 1, ..., q\,.$$

Similarly, for the polynomials $P_1, \dots, P_q$, one can define, in the affine open set $\mathbf{C}^n \setminus \{P_1 \cdots P_q = 0\}$, the C^∞ functions

$$\widetilde{s}_j := \frac{|P_j|^{\delta^{[j]}}}{P_j \sum\limits_{l=1}^{q} |P_l|^{\delta^{[l]}}}\,, \quad j = 1, ..., q\,.$$

These functions $\widetilde{s}_j$, $j = 1, ..., q$, extend (provided $N >> 1$) to C^1 functions in $\mathbf{C}^n \setminus V(P)$, satisfying

$$\sum_{j=1}^{q} \widetilde{s}_j(\zeta) P_j(\zeta) = 1\,, \quad \zeta \in \mathbf{C}^n \setminus V(P)\,.$$

Let finally

$$u_j := |P_j|^{\delta^{[j]}/2}\,, \quad j = 1, ..., q$$

and

$$S := \sum_{j=1}^{q} u_j^2 = \|u\|^2\,.$$

At this point we return to the

Proof of Proposition 4.1. One can suppose without any loss of generality that $\{i_1, ..., i_q\} = \{1, ..., q\}$ and that the projection Π is a proper map from $|W|$ to $\mathbf{C}^q$. Condition (4.1) implies the existence of a strictly positive constant κ_N such that

$$S(\zeta) \geq \kappa_N \|\zeta\|^\delta\,, \quad \zeta \in |W|\,,\ \|\zeta\| \geq K\,. \tag{4.6}$$

Let

$$\theta \in \mathcal{D}(] - 3\kappa_N/4, 3\kappa_N/4[$$

such that $\theta \equiv 1$ on $[-\kappa_N/4, \kappa_N/4]$; for any $R > 0$, let the element φ_R in $C^1(\mathbf{C}^n)$ defined as

$$\varphi_R\ : \zeta \mapsto \theta(S(\zeta)/R^\delta)\,.$$

Since the restriction $S_{|_{|W|}}$ is a proper map (all δ_j's, $j = 1, ..., q$, being strictly positive) and $V(P) \cap |W|$ is a discrete (hence finite) algebraic subset of $\mathbf{C}^n$

(this follows also from (4.1)), there exists R_0 such that for $R > R_0$, $\varphi_R \equiv 1$ in a neighborhood of $|W| \cap V(P)$. Therefore, if

$$s_j := \frac{\overline{P_j}}{\|P\|^2}$$

one has (see for example formula (3.9))

$$\operatorname{Res}\begin{bmatrix} [W] \wedge Q \bigwedge_{l=1}^{q} dX_l \\ \\ P_1, ..., P_q \end{bmatrix} = c_q \int_{|W|_{\mathrm{reg}}} \Big(\sum_{j=1}^{q} (-1)^{j-1} s_j \bigwedge_{\substack{l=1 \\ l \neq j}}^{q} ds_l \Big) \wedge Q \, d\zeta' \wedge \overline{\partial} \varphi_R \tag{4.7}$$

for any $R > R_0$, where $d\zeta' = \bigwedge_{l=1}^{q} d\zeta_l$. It follows from an homotopy argument similar to the one which is developed in the proof of Proposition 3.2 that

$$\begin{aligned} \operatorname{Res}\begin{bmatrix} [W] \wedge Q \bigwedge_{l=1}^{q} dX_l \\ \\ P_1, ..., P_q \end{bmatrix} &= c_q \int_{|W|_{\mathrm{reg}}} \Big(\sum_{j=1}^{q} (-1)^{j-1} \widetilde{s}_j \bigwedge_{\substack{l=1 \\ l \neq j}}^{q} d\widetilde{s}_l \Big) \wedge Q \, d\zeta' \wedge \overline{\partial} \varphi_R \\ &= c_q \int_{|W|_{\mathrm{reg}}} \Big(\sum_{j=1}^{q} (-1)^{j-1} \widetilde{s}_j \bigwedge_{\substack{l=1 \\ l \neq j}}^{q} \overline{\partial} \widetilde{s}_l \Big) \wedge Q \, d\zeta' \wedge \overline{\partial} \varphi_R \end{aligned} \tag{4.8}$$

for any $R > R_0$. Since $P_j \widetilde{s}_j = u_j^2/S$, $j = 1, ..., q$, one can rewrite (4.8) as

$$\operatorname{Res}\begin{bmatrix} [W] \wedge Q \bigwedge_{l=1}^{q} dX_l \\ \\ P_1, ..., P_q \end{bmatrix}$$

$$= c_q \, 2^{q-1} \int_{|W|_{\mathrm{reg}}} \Big(\prod_{j=1}^{q} \frac{|P_j|}{P_j} u_j^{1 - \frac{2}{\delta[j]}} \Big) \frac{\sum_{j=1}^{q} (-1)^{j-1} u_j \bigwedge_{\substack{l=1 \\ l \neq j}}^{q} du_l}{\|u\|^{2q}} \wedge Q \, d\zeta' \wedge \overline{\partial} \varphi_R$$

$$= \frac{(-1)^q\, c_q\, 2^{q-1}}{R^\delta} \int_{|W|_{\rm reg}} \Big(\prod_{j=1}^q \frac{|P_j|}{P_j}\, u_j^{1-\frac{2}{\delta[j]}}\Big) \frac{\bigwedge_{l=1}^q du_l}{\|u\|^{2(q-1)}} \wedge \theta'\Big(\frac{\|u\|^2}{R^\delta}\Big)\, Q\, d\zeta'. \tag{4.9}$$

For any order $\mathcal{J} \subset \{1, ..., q\}$, let

$$\omega_{\mathcal{J}} = \bigwedge_{j=1}^q d\alpha_{\mathcal{J},l}$$

where

$$\alpha_{\mathcal{J},l}(\zeta_1, ..., \zeta_q) := \begin{cases} \operatorname{Re}\zeta_j \ \text{ if } j \in \mathcal{J} \\ \operatorname{Im}\zeta_j \ \text{ if } j \notin \mathcal{J}\,; \end{cases}$$

then one can write

$$d\zeta' = \bigwedge_{l=1}^q d\zeta_l = \sum_{\mathcal{J} \subset \{1,...,q\}} i^{q-\#\mathcal{J}}\, d\omega_{\mathcal{J}}\,.$$

In order to prove formula (4.4), it is enough to prove that for any $\mathcal{J} \subset \{1, ..., q\}$, one has, as soon as $\deg Q < \delta_1 + \cdots + \delta_q - q$,

$$\lim_{R\to+\infty} \left[\frac{1}{R^\delta} \int_{|W|_{\rm reg}} \Big(\prod_{j=1}^q \frac{|P_j|}{P_j}\, u_j^{1-\frac{2}{\delta[j]}}\Big) \frac{\bigwedge_{l=1}^q du_l}{\|u\|^{2(q-1)}} \wedge \theta'\Big(\frac{\|u\|^2}{R^\delta}\Big)\, Q\, \omega_{\mathcal{J}}(\zeta')\right] = 0\,. \tag{4.10}$$

Since the restriction of $P = (P_1, ..., P_q)$ to each connected sheet $\mathcal{F}$ (above the ζ'-space) of the $2q$-dimensional real manifold $|W|_{\rm reg}$ is proper, the map

$$F_{\mathcal{J},\mathcal{F}} \ : \ \zeta \in \mathcal{F} \mapsto (u_1, ..., u_q, \alpha_{\mathcal{J},1}, ..., \alpha_{\mathcal{J},q})$$

is a $\mathbf{R}^{2q}$-valued proper map, with topological degree $d_{\mathcal{J},\mathcal{F}}$. Moreover, condition (4.6) implies that, for $R > K$,

$$\operatorname{Supp}\Big(\theta(S/R^\delta)\Big) \subset \{\zeta \in \mathbf{C}^n \ : \ \|\zeta\| < R\}\,.$$

Actually, for $\|\zeta\| \geq R > K$, one has

$$S(\zeta) \geq \kappa_N \|\zeta\|^\delta \geq \kappa_N R^\delta > (3\kappa_N/4) R^\delta\,.$$

For such R, one has

$$\Big\| \prod_{j=1}^{q} \frac{|P_j|}{P_j} \, Q \, \theta'(S/R^{\delta}) \Big\|_{\infty} \leq C \, R^{\deg Q} \,,$$

where $C = C(\theta, Q)$ is a positive constant. It follows then from the properness of all maps $F_{\mathcal{J},\mathcal{F}}$ and from the positivity of the differential form

$$\Big(\prod_{j=1}^{q} u_j^{1 - \frac{2}{\delta[j]}} \Big) \bigwedge_{l=1}^{q} du_l$$

in $]0, \infty[^q$ that

$$\begin{aligned}
&\frac{1}{R^{\delta}} \Big| \int_{|W|_{\rm reg}} \Big(\prod_{j=1}^{q} \frac{|P_j|}{P_j} u_j^{1 - \frac{2}{\delta[j]}} \Big) \frac{\bigwedge_{l=1}^{q} du_l}{\|u\|^{2(q-1)}} \wedge \theta' \left(\frac{\|u\|^2}{R^{\delta}} \right) Q \, \omega_{\mathcal{J}}(\zeta') \Big| \\
&\quad \leq \frac{(\sum_{\mathcal{F}} d_{\mathcal{J},\mathcal{F}}) \, C \, R^{\deg Q}}{R^{\delta}} \left(\int_{\frac{\kappa_N R^{\delta}}{4} \leq \|u\|^2 \leq \frac{3\kappa_N R^{\delta}}{4}} \Big(\prod_{j=1}^{q} u_j^{1 - \frac{2}{\delta[j]}} \Big) \frac{\bigwedge_{l=1}^{q} du_l}{\|u\|^{2(q-1)}} \right) \\
&\quad\quad \times \left(\int_{\|t\| < R} dt_1 \wedge \cdots \wedge dt_q \right) \\
&\leq \frac{(\sum_{\mathcal{F}} d_{\mathcal{J},\mathcal{F}}) \; C_N \, R^{\deg Q + q}}{R^{q\delta}} \left(\int_{\frac{\kappa_N R^{\delta}}{4} \leq \|u\|^2 \leq \frac{3\kappa_N R^{\delta}}{4}} \Big(\prod_{j=1}^{q} u_j^{1 - \frac{2}{\delta[j]}} \Big) \bigwedge_{l=1}^{q} du_l \right) \\
&\leq \frac{(\sum_{\mathcal{F}} d_{\mathcal{J},\mathcal{F}}) \; \widetilde{C}_{N,\vec{\delta}} \, R^{\deg Q + q}}{R^{q\delta}} \, R^{\frac{\delta}{2} \sum_{j=1}^{q} \left(1 - \frac{1}{\delta[j]}\right) + q \frac{\delta}{2}} \\
&\leq (\sum_{\mathcal{F}} d_{\mathcal{J},\mathcal{F}}) \; \widetilde{C}_{N,\vec{\delta}} \, R^{\deg Q + q - \delta_1 - \cdots - \delta_q} = \mathbf{o}(1) \,,
\end{aligned}$$

which proves the conclusion (4.10) we need. The proof of Proposition 4.1 is therefore completed. $\diamondsuit$

5 Analytic versus algebraic approach

Let $\mathcal{X}$ be an integral $\mathbf{C}$-variety of dimension q and $\mathcal{D}_1$,...,$\mathcal{D}_q$ be q Cartier divisors on $\mathcal{X}$ such that $|\mathcal{D}_1| \cap \cdots \cap |\mathcal{D}_q|$ is finite. If ω is a meromorphic form in $\Omega^q_{\mathbf{C}(\mathcal{X})/\mathbf{C}}$ which has a simple pole along $\mathcal{D}_1 + \cdots + \mathcal{D}_q$, one may define (see

[Hu], page 621) the local residue of ω at any closed point α in $|\mathcal{D}_1| \cap \cdots \cap |\mathcal{D}_q|$. That is, if

$$\omega = \frac{\eta}{f_1 \cdots f_q} \,,$$

where $\eta \in \omega^q_{\mathbf{C}(\mathcal{X})/\mathbf{C},\alpha}$ and $f_j = 0$, $j = 1, ..., q$, is a local equation for $\mathcal{D}_j$ at α then

$$\mathrm{Res}_{\mathcal{X};\mathcal{D}_1,...,\mathcal{D}_q,\alpha}(\omega) = \mathrm{Res}_{\mathbf{C}(\mathcal{X})/\mathbf{C},\alpha}\left(\begin{bmatrix} \eta \\ f_1, ..., f_q \end{bmatrix}\right).$$

When $\mathcal{X}$ is smooth, this definition agrees with the definition in [GH], chapter 5, section 1. (See [L1], Appendix A). Adding the hypothesis that $\mathcal{X}$ is $\mathbf{C}$-complete, one has (see Proposition 12.2, page 108, in [L2])

$$\sum_{\alpha \in |\mathcal{D}_1| \cap \cdots \cap |\mathcal{D}_q|} \mathrm{Res}_{\mathcal{X};\mathcal{D}_1,...,\mathcal{D}_q,\alpha}(\omega) = 0 \,,$$

which is known as residue theorem on $\mathcal{X}$ (it extends the classical residue theorem on a complete integral curve in its algebraic formulation, see [Se]).

Such a residue theorem holds in our analytic setting (and is essentially a consequence of Stokes' formula). Namely, if W is an integral algebraic q-dimensional subscheme in $\mathbf{A}^n_{\mathbf{C}}$ (with completion $\mathcal{W}$ in $\mathrm{Proj}\,\mathbf{C}[X_0, ..., X_n]$) and $P_1, ..., P_q$ are q polynomials in n variables such that $|\mathcal{W}| \cap \{{}^hP_1 = \cdots = {}^hP_q = 0\}$ is finite and included in $\mathbf{C}^n$, then ($[W]$ being understood as the integration current free of multiplicities),

$$\mathrm{Res}\begin{bmatrix} [W] \wedge Q(X) dX_{i_1} \wedge \cdots \wedge dX_{i_q} \\ P_1, ..., P_q \end{bmatrix} = 0$$

when $\deg Q \leq \sum_{j=1}^q \deg P_j - q - 1$ (corollary 4.1) for any ordered subset $\{i_1, ..., i_q\} \subset \{1, ..., n\}$.

On the other hand, the transformation law holds for our analytic restricted residue (see Corollary 3.2). Such a transformation law remains valid (in its local formulation) for restricted residue symbols defined through the algebraic approach (see Theorem 2.4 in [HK1]).

Finally, the local residue symbol

$$\mathrm{Res}_{\mathcal{W};\mathcal{D}_1,...,\mathcal{D}_q,\alpha}(\omega) = \mathrm{Res}_{\mathbf{C}(\mathcal{X})/\mathbf{C},\alpha}\left(\begin{bmatrix} \eta \\ f_1, ..., f_q \end{bmatrix}\right),$$

where

$$\omega = \frac{\eta}{f_1 \cdots f_q}\,,$$

$\eta \in \omega^q_{\mathbf{C}(\mathcal{X})/\mathbf{C},\alpha}$ and $f_j = 0$, $j = 1, ..., q$, is a local equation for $\mathcal{D}_j$ at α, equals to 0 as soon as $\eta = f_j \widetilde{\eta}$ for some $\widetilde{\eta} \in \omega^q_{\mathbf{C}(\mathcal{X})/\mathbf{C},\alpha}$ (see also [HK1], section 2). The same annihilation property is satisfied by the restricted residual current (Proposition 3.3).

Our goal in this section is to profit from the fact that both restricted residual objects (defined through the algebraic or analytic approach) satisfy the transformation law, the residue formula, the annihilation property, in order to show that they coincide. Therefore, we are able to give an algebraic formulation of the Proposition 4.1, which is the Theorem 2.1 stated in our preliminaries section.

In order to do that, we will need the following technical lemma :

Lemma 5.1 *Let $|W|$ be an irreducible q-dimensional algebraic set in $\mathbf{C}^n$ and $|\mathcal{W}|$ its Zariski closure in $\mathbb{P}^n(\mathbf{C})$. Let also $P_1, ..., P_q$ be q polynomials in $\mathbf{C}[X_1, ..., X_n]$ such that $V(P) \cap |W|$ is a discrete (hence finite) algebraic subset of $\mathbf{C}^n$, with $0 \in V(P) \cap |W|$. Then, there exists $N_0 > 0$ such that, for any integer $N \geq N_0$, one can find $qn + 1$ complex parameters u_{jk}, $j = 1, \ldots, q$, $k = 1, \ldots, n$, $t \in \mathbf{C}^*$, so that, if*

$$\widetilde{P}_j^{(N,u,t)}(X) := tP_j(X) + \Big(\sum_{k=1}^{n} u_{jk} X_k\Big)^N\,, \quad j = 1, ..., q\,,$$

one has :

- *any point $\alpha \in |W| \cap V(\widetilde{P}^{(N,u,t)})$ but 0 belongs to $|W|_{\mathrm{reg}}$;*
- *the set*

  $$|\mathcal{W}| \cap \{[\zeta_0 : ... : \zeta_n] \in \mathbb{P}^n(\mathbf{C}) \,:\, {}^h P_j^{(N,u,t)}(\zeta_0, ..., \zeta_n) = 0\,,\ j = 1, ..., q\}$$

  is contained in $\mathbf{C}^n$.

Proof. Since $|W|$ is irreducible and q-dimensional, one has $\dim |W|_{\mathrm{sing}} < q$; one can find an algebraic affine hypersurface $H := \{\zeta \in \mathbf{C}^n\,;\, H(\zeta) = 0\}$ (with Zariski closure $|\mathcal{H}|$) such that $|W|_{\mathrm{sing}} \subset H$ and $\dim(|\mathcal{W}| \cap |\mathcal{H}|) < q$.

Let $N_0 > \deg P_j$, $j = 1, ..., q$, and $N \geq N_0$. Assume also that $N \geq \rho_{P,W}(0)$, where $\rho_{P,W}(0)$ is the order of vanishing of P at the origin (along $|W|$).
Let $u = [u_{jk}]$, $j = 1, ..., q$, $k = 1, ..., n$, be a (q, n) matrix with generic complex entries,

$$M_u := \{\zeta \in \mathbf{C}^n \ : \ u_{j1}\zeta_1 + \cdots + u_{jn}\zeta_n = 0\,,\ j = 1, ..., q\}$$

and $|\mathcal{M}_u|$ its Zariski closure in $\mathbb{P}^n(\mathbf{C})$. Since $\dim |\mathcal{W}| = q$ and $\dim(|\mathcal{W}| \cap |\mathcal{H}|) < q$, $|\mathcal{M}_u| \cap |\mathcal{W}| \subset \mathbf{C}^n$ and $|\mathcal{M}_u| \cap |\mathcal{W}| \cap |\mathcal{H}| = \{0\}$ for u generic. Therefore, for such a generic choice of u ($u = u^0$) (this choice will be refined later), for any $t \in \mathbf{C}^*$, the polynomials

$$tP_j(X) + (u^0_{j1}X_1 + \cdots + u^0_{jn}X_n)^N\,, \quad j = 1, ..., q\,,$$

define in $\mathbf{C}^n$ an algebraic set $Z^{(N,u^0,t)}$ whose closure $\mathcal{Z}^{(N,u^0,t)}$ in $\mathbb{P}^n(\mathbf{C})$ intersects $|\mathcal{W}|$ only at points in $\mathbf{C}^n$ (note that 0 is one of these points). The algebraic set $|W| \cap Z^{(N,u^0,t)}$ can be described as

$$|W| \cap Z^{(N,u^0,t)} = \{\zeta^{(N,1)}(u^0, t), ..., \zeta^{(N,m)}(u^0, t)\} \cup \{0\}\,,$$

where m is fixed (depending on N and $|\mathcal{W}|$) and the $t \mapsto \zeta^{(N,j)}(u^0, t)$, $j = 1, ..., m$, are algebraic $\mathbf{C}^n$-valued functions of t which are not identically 0 and can be classified in two classes, depending on their behavior when $|t|$ tends to zero. A branch $t \mapsto \zeta^{(N,j)}(u^0, t)$ will be in the first class if $\zeta^{(N,j)}(u^0, t)$ tends to zero when $|t|$ tends to 0. It will be in the second class if $\zeta^{(N,j)}(u^0, t)$ tends to a point in $|W| \cap M_{u^0}$ which is distinct from 0 when $|t|$ goes to 0. It follows then from $M_{u^0} \cap |W| \cap H = \{0\}$ that none of the functions

$$t \mapsto H(\zeta^{(N,j)}(u^0, t))$$

where $t \mapsto \zeta^{(N,j)}(u^0, t)$ belongs to the second category, can be identically equal to 0. The behavior of branches of the first category can now be studied when $|t|$ goes to infinity. The assumption on N ensures us that such branches either approach points in $(|W| \cap V(P)) \setminus \{0\}$, either satisfy

$$\lim_{|t| \to \infty} |\zeta^{(N,j)}(u^0, t)| = +\infty$$

in the second alternative. The hypothesis on u^0 implies that the function $t \mapsto H(\zeta^{(N,j)}(u^0, t))$ is not identically 0 if we are in the second alternative.

If u^0 is conveniently choosen (in terms of the Taylor developments at the first order for $P_1, ..., P_q$ at the points in $(|W| \cap V(P)) \setminus \{0\}$, the assertion $t \mapsto H(\zeta^{(N,j)}(u^0, t)) \not\equiv 0$ also holds for branches concerned by the first alternative. Finally, for any branch $t \mapsto \zeta^{(N,j)}(u^0, t)$, one has $H(\zeta^{(N,j)}(u^0, t) \not\equiv 0$. Therefore, once u^0 has been conveniently chosen, one can pick up $t \neq 0$ such that the map $\widetilde{P}^{(N,u^0,t)}$ satisfies the assertions of the lemma. $\diamondsuit$

We can now relate the analytic and algebraic approaches for restricted residual symbols.

Proposition 5.1 *Let $\mathcal{W}$ be a complete integral $\mathbf{C}$-variety of dimension q, embedded in the projective scheme* $\mathrm{Proj}\,\mathbf{C}[X_0, ..., X_n]$*, α be a closed point in $|\mathcal{W}|$ such that $\alpha \in \mathbf{C}^n$ and $\mathcal{D}_1$,...,$\mathcal{D}_q$ be q Cartier divisors on $\mathcal{W}$ so that the intersection $|\mathcal{D}_1| \cap \cdots \cap |\mathcal{D}_q|$ defines a zero-dimensional scheme on $\mathcal{W}$ in a neighborhood of α. If*

$$\omega = \frac{\eta}{P_1 \cdots P_q},$$

where $\eta = Q\, dX_{i_1} \wedge \cdots \wedge dX_{i_q}$, $Q \in \mathbf{C}[X_1, ..., X_n]$ induces an element in $\omega^q_{\mathbf{C}(\mathcal{X})/\mathbf{C},\alpha}$ and $P_1, ..., P_q$ are elements in $\mathbf{C}[X_1, ..., X_n]$ such that P_j, $j = 1, ..., q$, is a local equation for $\mathcal{D}_j$ at α, then, for any function $\varphi \in \mathcal{D}(\mathbf{C}^n)$ with arbitrary small support around α satisfying $\varphi \equiv 1$ in a neighborhood of α, one has

$$\mathrm{Res}_{\mathcal{W};\mathcal{D}_1,...,\mathcal{D}_q,\alpha}(\omega) = \mathrm{Res}\begin{bmatrix} [W] \wedge \varphi\, \eta \\ P_1, ..., P_q \end{bmatrix}. \tag{5.1}$$

Proof. One can assume for the sake of simplicity that $\alpha = 0$. Let $\mathcal{M}$ be the maximal ideal $(X_1, ..., X_n)$ in the local algebra $\mathcal{O}_{\mathbf{C}[X_1,...,X_n],0}$ and $(I(W))_0$ the localization at 0 of the radical ideal

$$I(W) := \{g \in \mathbf{C}[X_1, ..., X_n]\,;\, g(\zeta) = 0\ \forall \zeta \in |\mathcal{W}| \cap \mathbf{C}^n\}.$$

Choose $p \in \mathbb{N}^*$ such that

$$\mathcal{M}^p \in \Big([(P_1, ..., P_q)_0]^2, I(W)_0\Big).$$

It follows from the validity of the transformation law and the annihilating property in the algebraic context that, if

$$\widetilde{P}_j(X) := P_j(X) + \Big(\sum_{k=1}^{n} u_{jk} X_k\Big)^p,$$

then one has, for any $\eta = Q\, dX_{i_1} \wedge \cdots \wedge dX_{i_q}$, $Q \in \mathbf{C}[X_1, ..., X_n]$, that

$$\mathrm{Res}_{\mathcal{W};\mathcal{D}_1,...,\mathcal{D}_q,0}\Big(\frac{\eta}{P_1\cdots P_q}\Big) = \mathrm{Res}_{\mathcal{W};\widetilde{\mathcal{D}}_1,...,\widetilde{\mathcal{D}}_q,0}\Big(\frac{\eta}{\widetilde{P}_1\cdots \widetilde{P}_q}\Big)\,, \tag{5.2}$$

where $\widetilde{\mathcal{D}}_j$, $j = 1, ..., q$, is the Cartier divisor on $\mathcal{W}$ with local equation $\widetilde{P}_j$ in a neighborhood of the origin. On the other hand, it follows from Proposition 3.2 and Proposition 3.3 that, for any test-function φ with arbitrary small support around the origin, one has also

$$\mathrm{Res}\begin{bmatrix} [W] \wedge \varphi\, \eta \\ P_1, ..., P_q \end{bmatrix} = \mathrm{Res}\begin{bmatrix} [W] \wedge \varphi\, \eta \\ \widetilde{P}_1, ..., \widetilde{P}_q \end{bmatrix}\,. \tag{5.3}$$

If the u_{jk}, $j = 1, ..., q$, $k = 1, ..., n$ are generic (see for example the construction in the proof of Lemma 5.1), the algebraic set $V(\widetilde{P}) \cap |\mathcal{W}| \cap \mathbf{C}^n$ is discrete (hence finite). We can then conclude from (5.2) and (5.3) that in order to prove (5.1), it is not restrictive to assume that the algebraic set $V(P) \cap |\mathcal{W}| \cap \mathbf{C}^n$ is finite, what we will do from now on.

The same argument as above shows that, in order to prove (5.1), one can replace P_j, $j = 1, ..., q$, by the polynomial

$$\frac{1}{t}\widetilde{P}_j^{(N,u,t)}$$

constructed in Lemma 5.1 (N being choosen sufficiently large, certainly such that $N \geq \max\deg P_j$, $\mathcal{M}^N \subset (I(P)_0, I(W)_0))$ and $\deg Q < q(N-1)$), and this is what we do (preserving the notations P_j and $\mathcal{D}_j$). As a consequence of the residue formula in the algebraic context (which we recalled at the beginning of this section) and of Corollary 4.1, one has

$$\sum_{\alpha \in V(P)\cap\mathcal{W}(\mathbf{C})} \mathrm{Res}_{\mathcal{W};\mathcal{D}_1,...,\mathcal{D}_q,\alpha}\Big(\frac{\eta}{P_1\cdots P_q}\Big) = \sum_{\alpha \in V(P)\cap\mathcal{W}(\mathbf{C})} \mathrm{Res}\begin{bmatrix} [W] \wedge \varphi\, \eta \\ P_1, ..., P_q \end{bmatrix} \tag{5.4}$$

whenever φ is a test-function in $\mathcal{D}(\mathbf{C}^n)$ with arbitrary small support around the points $\alpha \in V(P) \cap |\mathcal{W}|$, such that $\varphi \equiv 1$ in a neighborhood of each of these points (φ_α will denote next $\varphi\,\theta_\alpha$, where θ_α is a test-function with support arbitrary small around α and $\theta_\alpha \equiv 1$ in a neighborhood of α). If α

is any point in $V(P) \cap |\mathcal{W}|$ distinct from 0, $\mathcal{W}$ is smooth about α (Lemma 5.1, first assertion) and we know in this case that

$$\mathrm{Res}_{\mathcal{W};\mathcal{D}_1,\dots,\mathcal{D}_q,\alpha}\Big(\frac{\eta}{P_1\cdots P_q}\Big) \quad = \quad \mathrm{Res}\begin{bmatrix} [W]\wedge\varphi_\alpha\,\eta \\ P_1,\dots,P_q \end{bmatrix}, \tag{5.5}$$

since the construction of our restricted residual currents corresponds to the construction proposed in [GH], chapter 5, section 1 (this is a consequence of the classical relation between Bochner-Martinelli and Cauchy kernels), which is known to fit with the algebraic approach in the smooth case (as it was recalled at the beginning of this section). Formula (5.1) follows then from (5.4) and from the identifications (5.5). $\diamondsuit$

Proof of Theorem 2.1. We may now transpose to the algebraic context the analytic result stated in Proposition 4.1. This gives the statement of the Theorem 2.1 of our introduction, provided we remember that we have

$$\mathrm{Res}_{\mathcal{W};\mathcal{D}_1,\dots,\mathcal{D}_q,\alpha}(\omega) = \mathrm{Res}_{\mathbf{C}(\mathcal{W})/\mathbf{C},\alpha}\left(\begin{bmatrix} \eta \\ f_1,\dots,f_q \end{bmatrix}\right)$$

for any point α in $|\mathcal{W}| \cap |\mathcal{D}_1| \cap \cdots \cap |\mathcal{D}_q| \cap \mathbf{C}^n$ (here we just assume that $\mathcal{D}_1$,...,$\mathcal{D}_n$ define a 0-dimensional scheme on W, there is no assumption about what happens on $|\mathcal{W}| \setminus |W|$) and any ω in $\Omega^q_{\mathbf{C}(\mathcal{W})/\mathbf{C}}$ with simple poles (in W) along $\mathcal{D}_1 + \cdots + \mathcal{D}_q$ ($\eta = f_1 \cdots f_q\,\omega$, where f_j denotes a local equation for the Cartier divisor $\mathcal{D}_j$). Since the reference to the divisors $\mathcal{D}_1,\dots,\mathcal{D}_q$ was implicit in the expression of the element in $\Omega^q_{\mathbf{C}(\mathcal{W})/\mathbf{C}}$, we used the abridged notation $\mathrm{Res}_{W,\alpha}[\,]$ instead of $\mathrm{Res}_{\mathcal{W};\mathcal{D}_1,\dots,\mathcal{D}_q,\alpha}$ in order to formulate the statement in this theorem. $\diamondsuit$

As a direct consequence we formulate the restricted version of the Cayley-Bacharach Theorem.

Corollary 5.1 *Let W be a q-dimensional irreducible affine algebraic subvariety in $\mathbf{A}^n_{\mathbf{C}}$ $(0 < q < n)$ and $P_1,\dots,P_q$ be q polynomials in $\mathbf{C}[X_1,\dots,X_n]$ satisfying the condition* (2.2). *Assume also that $V(P)$ and $|W|$ intersect transversally at any of the k distinct points which constitute $V(P) \cap |W|$. Then any algebraic hypersurface $\{Q = 0\}$, $Q \in \mathbf{C}[X_1,\dots,X_n]$, such that $\deg Q < \delta_1 + \cdots + \delta_q - q$, which passes through any $k-1$ points of the set $V(P) \cap |W|$ passes through the last one also.*

6 An affine version of Wood's theorem.

Let γ_1, ...,γ_d be d pieces of manifold in $\mathbb{P}^n(\mathbf{C})$ and $|\mathcal{L}_{0,0}|$ be a line in $\mathbb{P}^n(\mathbf{C})$ which intersects each of the γ_j transversally respectively at distinct points p_{j0}, $j = 1, ..., d$. Assume that affine coordinates are such that the support $|\mathcal{L}_{0,0}|$ is the line $\zeta_1 = \cdots = \zeta_{n-1} = 0$. Then, for $(\alpha, \beta) \in (\mathbf{C}^{n-1})^2$ close to $(0, 0)$, the projective line

$$|\mathcal{L}_{\alpha,\beta}| := \{[\zeta_0 : \, ... \, : \zeta_n] \in \mathbb{P}^n(\mathbf{C})\, ; \, \zeta_k = \alpha_k\, \zeta_n + \beta_k\, \zeta_0\, , \ k = 1, ..., n-1\}$$

intersects transversally γ_1,...,γ_d at the respective points $p_1(\alpha, \beta)$,..., $p_d(\alpha, \beta)$ ($p_j(\alpha, \beta)$ being close to p_{j0}). In [W], J. Wood gave a simple criterion for the local germs of manifold $\gamma_1, ..., \gamma_d$ to be germs of a global algebraic hypersurface (with degree d) $|\mathcal{H}|$ in $\mathbb{P}^n(\mathbf{C})$ satisfying the relation such that

$$|\mathcal{H}| \cap |\mathcal{L}_{\alpha,\beta}| = \{p_1(\alpha, \beta), ..., p_d(\alpha, \beta)\}$$

for (α, β) close to $(0, 0)$. The (necessary and sufficient) condition he gave can be formulated as follows :

$$\sum_{j=1}^{d} \zeta_n[p_j(\alpha, \beta)] = h_0(\alpha) + \sum_{k=1}^{n-1} h_k(\alpha)\, \beta_k\, , \tag{6.6}$$

where $h_0, ..., h_{n-1}$ are germs of holomorphic functions in α at the origin (here $\zeta_n[p]$, where p denotes a point in $\mathbf{C}^n$, means the n-th affine coordinate of p). Note that the algebraic hypersurface $|\mathcal{H}|$ (in $\mathbb{P}^n(\mathbf{C})$) which interpolates $\gamma_1, ..., \gamma_d$ is such that its intersection at infinity with any line $|\mathcal{L}_{\alpha,\beta}|$, with (α, β) close to $(0, 0)$, is empty. What we would like to state here is an affine analog of this result, $\mathbf{P}^n(\mathbf{C})$ being replaced by some irreducible q-dimensional affine algebraic subvariety of $\mathbf{C}^n$ ($q = 2, ..., n$).

Let us first state the following easy consequence of our Theorem 2.1.

Proposition 6.1 *Let W be an algebraic irreducible q-dimensional subvariety of the affine scheme $\mathbf{A}^n_{\mathbf{C}}$ (with $2 \leq q \leq n$), m be a positive integer strictly between 0 and q, and γ_1,...,γ_d be d disjoint pieces of $q-m$-dimensional analytic manifold such that γ_j lies in $|W|_{\rm reg}$ for $j = 1, ..., d$. Furthermore, assume that the affine $n + m - q$-dimensional subspace*

$$L_{0,0} := \{\zeta \in \mathbf{C}^n\, ; \, \zeta_k = 0\, , \ k = 1, ..., q - m\}$$

intersects each γ_j transversally respectively at points p_{j0}, $j = 1, ..., d$. Suppose that there are strictly positive rational numbers $\delta_1, ..., \delta_m$ and polynomials $P_1, ..., P_m$ with $\deg P_j = d_j \geq \delta_j$, $j = 1, ..., m$, such that

- *$|W| \cap V(P)$ is a $q - m$-dimensional variety in $\mathbf{C}^n$ which interpolates the pieces γ_j and is such that $|W| \cap V(P) \cap L_{0,0} = \{p_{10}, ..., p_{d0}\}$;*
- *there exists strictly positive constants κ, K such that*

$$\zeta \in |W|\,,\ \|\zeta\| \geq K \Longrightarrow \sum_{j=1}^{m} \frac{|P_j(\zeta)|}{\|\zeta\|^{\delta_j}} + \sum_{k=1}^{q-m} \frac{|\zeta_k|}{\|\zeta\|} \geq \kappa\,. \tag{6.7}$$

Then, for (α, β) close to $(0, 0)$ in $(\mathbf{C}^{n+m-q})^{q-m} \times \mathbf{C}^{q-m}$, the affine $n+m-q$-dimensional subspace

$$L_{\alpha,\beta} := \Big\{\zeta \in \mathbf{C}^n\,;\, \zeta_k = \sum_{r=1}^{n+m-q} \alpha_{k,r}\zeta_{q-m+r} + \beta_k\,,\ k = 1, ..., q - m\Big\}$$

intersects each γ_j transversally respectively at the points $p_j(\alpha, \beta)$, $j = 1, ..., d$ (necessarily distinct and close to the p_{j0}) and one has

$$\sum_{j=1}^{d} \zeta_l[p_j(\alpha, \beta)] = \sum_{\substack{\underline{k} \in \mathbf{N}^{q-m} \\ |\underline{k}| \leq \rho+1}} h^{(l)}_{\underline{k}}(\alpha)\, \beta_1^{k_1} \cdots \beta_{q-m}^{k_{q-m}}\,,\ l = q - m + 1, ..., n\,, \tag{6.8}$$

where the $h^{(l)}_{\underline{k}}$ are germs of holomorphic functions in α about the origin and

$$\rho := \sum_{j=1}^{m} (d_j - \delta_j)$$

Proof. Let, for $k = 1, ..., q - m$,

$$\Lambda_{\alpha,\beta,k}(\zeta) := \zeta_k - \sum_{r=1}^{n+m-q} \alpha_{k,r}\, \zeta_{q-m+r} - \beta_k\,,\ \zeta \in \mathbf{C}^n\,;$$

condition (6.7) implies that, when (α, β) is sufficiently close to $(0, 0)$, one has

$$\zeta \in |W|\,,\ \|\zeta\| \geq K \Longrightarrow \sum_{j=1}^{m} \frac{|P_j(\zeta)|}{\|\zeta\|^{\delta_j}} + \sum_{k=1}^{q-m} \frac{|\Lambda_{\alpha,\beta,k}(\zeta)|}{\|\zeta\|} \geq \frac{\kappa}{2}\,. \tag{6.9}$$

This shows that for (α,β) close to $(0,0)$, the only points in $L_{\alpha,\beta}\cap|W|\cap V(P)$ are d points $p_j(\alpha,\beta)$, $j=1,...,d$ which approach the points $p_{10},...,p_{d0}$ (about each of these points, one can use the implicit function theorem in order to describe the intersection $\gamma_j\cap L_{\alpha,\beta}$). This proves the first assertion of the proposition.

It follows from Proposition 4.1 that, as soon as the multi-index $\underline{k}\in\mathbb{N}^{q-m}$ is such that

$$\sum_{j=1}^{m}(d_j-1)+1<\sum_{j=1}^{m}\delta_j+\sum_{l=1}^{q-m}(k_l+1)-q=\sum_{j=1}^{m}\delta_j+|\underline{k}|-m\,,$$

then, for $l=q-m+1,...,n$, for any finite ordered subset $\{i_1,...,i_{q-m}\}\subset\{1,...,n\}$,

$$\operatorname{Res}\left[\begin{array}{c}[W]\wedge X_l\Big(\bigwedge\limits_{j=1}^{m}dP_j\Big)\wedge\Big(\bigwedge\limits_{l=1}^{q-m}dX_{i_l}\Big)\\ \\ P_1,...,P_m,(\Lambda_{\alpha,\beta,1})^{k_1+1},...,(\Lambda_{\alpha,\beta,q-m})^{k_{q-m}+1}\end{array}\right]=0$$

for (α,β) such that (6.9) holds. It is immediate to check (use for example formula (4.7)) that for such (α,β), one has, for any multi-index $\underline{k}=(k_1,...,k_{q-m})\in\mathbb{N}^{q-m}$,

$$\begin{aligned}&\frac{\partial^{|\underline{k}|}}{\partial\beta_1^{k_1}\cdots\partial\beta_{q-m}^{k_{q-m}}}\operatorname{Res}\left[\begin{array}{c}[W]\wedge X_l\Big(\bigwedge\limits_{j=1}^{m}dP_j\Big)\wedge\Big(\bigwedge\limits_{l=1}^{q-m}dX_{i_l}\Big)\\ \\ P_1,...,P_m,\Lambda_{\alpha,\beta,1},...,\Lambda_{\alpha,\beta,q-m}\end{array}\right]\\ &\quad=\pm\operatorname{Res}\left[\begin{array}{c}[W]\wedge X_l\Big(\bigwedge\limits_{j=1}^{m}dP_j\Big)\wedge\Big(\bigwedge\limits_{l=1}^{q-m}dX_{i_l}\Big)\\ \\ P_1,...,P_m,(\Lambda_{\alpha,\beta,1})^{k_1+1},...,(\Lambda_{\alpha,\beta,q-m})^{k_{q-m}+1}\end{array}\right].\end{aligned}\tag{6.10}$$

Then it follows from (6.9) that the right-hand side of (6.10) (hence the left-hand side) equals identically 0 when

$$|\underline{k}|>\sum_{j=1}^{m}(d_j-\delta_j)+1=\rho+1\,.$$

This proves that, when (α,β) is close to $(0,0)$ and $l=m-q+1,...,n$,

$$\sum_{j=1}^{d}\zeta_l[p_j(\alpha,\beta)]\equiv\operatorname{Res}\left[\begin{array}{c}[W]\wedge\zeta_l\Big(\bigwedge\limits_{j=1}^{m}dP_j\Big)\wedge\Big(\bigwedge\limits_{l=1}^{q-m}d\Lambda_{\alpha,\beta,l}\Big)\\ \\ P_1,...,P_m,\Lambda_{\alpha,\beta,1},...,\Lambda_{\alpha,\beta,q-m}\end{array}\right]$$

is a polynomial expression in $\beta = (\beta_1, ..., \beta_{q-m})$ with total degree at most $\rho+1$ (the coefficients being holomorphic functions in α). The second assertion of the proposition is proved. $\diamondsuit$

Remark. Note that we recover here as a particular case the necessity of Wood's condition in the case $W = \mathbf{A}^n_{\mathbf{C}}$, $m = 1$, $\delta_1 = d_1 = d$, which means precisely that in this case we also impose the restriction

$$\{\widetilde{\zeta} \in \mathbb{P}^n(\mathbf{C})\,;\, {}^h P_1(\widetilde{\zeta}) = 0\} \cap |\mathcal{L}_{0,0}| = \{p_{10}, ..., p_{d0}\}\,.$$

Furthermore, one can state the following proposition, which appears as a weak converse of Proposition 6.1 in the affine setting.

Proposition 6.2 *Let γ_1,...,γ_d be d disjoint pieces of $n-m$-dimensional analytic manifold ($1 \leq m < n$) in the affine space $\mathbf{C}^n$. Suppose that for any $(\alpha, \beta) \in (\mathbf{C}^m)^{n-m} \times \mathbf{C}^{n-m}$ close to $(0,0)$, the affine m-dimensional subspace*

$$L_{\alpha,\beta} := \Big\{\zeta \in \mathbf{C}^n\,;\, \zeta_k = \sum_{r=1}^{m} \alpha_{k,r}\zeta_{n-m+r} + \beta_k\,,\ k = 1, ..., n-m\Big\}$$

intersects transversally γ_1,...,γ_d respectively at points $p_1(\alpha,\beta), ..., p_d(\alpha,\beta)$. Assume that there exists $D \in \mathbb{N}$ and analytic functions $h^{(l)}_{\underline{k}}$, $|\underline{k}| \leq D+1$, $l = n-m+1, ..., n$, in a neighborhood of 0 in $(\mathbf{C}^m)^{n-m}$ such that for (α,β) close to $(0,0)$ in $(\mathbf{C}^m)^{n-m} \times \mathbf{C}^{n-m}$, for any $l = n-m+1, ..., n$,

$$\sum_{j=1}^{d} \zeta_l[p_j(\alpha,\beta)] = \sum_{\substack{\underline{k}\in\mathbf{N}^{n-m}\\ |\underline{k}|\leq D+1}} h^{(l)}_{\underline{k}}(\alpha)\, \beta_1^{k_1}\cdots\beta_{q-m}^{k_{n-m}}\,. \tag{6.11}$$

Then, one can find a collection of polynomials $(P_\iota)_{\iota\in\mathcal{J}}$ with degree at most $d+D$ which define an affine algebraic variety $V(P)$ such that for some convenient constants $\epsilon > 0$, $\kappa > 0$, $K > 0$, one has :

- *if*

$$\Gamma_\epsilon := \bigcup_{\substack{(\alpha,\beta)\in(\mathbf{C}^m)^{n-m}\times\mathbf{C}^{n-m}\\ \max(\|\alpha\|,\|\beta\|)<\epsilon}} L_{\alpha,\beta}\,,$$

  then

$$\zeta \in \Gamma_\epsilon\,,\ \|\zeta\| \geq K \implies \max_{\iota\in\mathcal{J}} |P_\iota(\zeta)| \geq \kappa\|\zeta\|^d\,; \tag{6.12}$$

- *for* $\max(\|\alpha\|, \|\beta\|) < \epsilon$, *one has*

$$L_{\alpha,\beta} \cap V(P) = \{p_1(\alpha,\beta), ..., p_d(\alpha,\beta)\}\,. \tag{6.13}$$

The proof of this proposition is directly inspired on [W] (page 237, proof of the sufficiency). First we observe, as in Wood's argument, that conditions (6.11) imply that for any integer $\sigma \in \mathbb{N}^*$, for any $l = n-m+1, ..., n$, one has, for (α,β) close to $(0,0)$ in $(\mathbf{C}^m)^{n-m} \times \mathbf{C}^{n-m}$,

$$\sum_{j=1}^{d} (\zeta_l[p_j(\alpha,\beta)])^\sigma = \sum_{\substack{\underline{k}\in\mathbb{N}^{n-m}\\ |\underline{k}|\le D+\sigma}} h^{(l)}_{\sigma,\underline{k}}(\alpha)\, \beta_1^{k_1}\cdots\beta_{n-m}^{k_{n-m}}\,, \tag{6.14}$$

where the $h^{(l)}_{\sigma,\underline{k}}$ are analytic functions in α in a neighborhood of 0. One can then define, for any $l = n-m+1, ..., n$, the polynomial A_l in the variable X_l (with coefficients analytic in α and polynomial in β) as

$$\begin{aligned} A_l(X_l,\alpha\,;\,\beta) &= \prod_{j=1}^{d}(X_l - \zeta_l[p_j(\alpha,\beta)]) \\ &= X_l^d - A_{l1}(\alpha,\beta)X_l^{d-1} + \cdots + (-1)^d A_{ld}(\alpha,\beta) \end{aligned} \tag{6.15}$$

(α and β close to 0 in their respective spaces). For any such α, denote as $P_{l,\alpha}$ the element in $\mathbf{C}[X_1, ..., X_n]$ defined as

$$P_{l,\alpha}(X) = A_l\Big(X_l,\alpha\,;\,X_1 - \sum_{r=1}^{m}\alpha_{1,r}X_{n-m+r}, ..., X_{n-m} - \sum_{r=1}^{m}\alpha_{n-m,r}X_{n-m+r}\Big)\,.$$

For each α close to 0 and each $l \in \{n-m+1, ..., n\}$, $P_{l,\alpha}$ is a polynomial in variables $(X_1, ..., X_n)$ with total degree less than $d+D$, such that all pieces of manifold $\gamma_1, ..., \gamma_d$ lie in $V(P_{n-m+1,\alpha}, ..., P_{n,\alpha})$ for any α close to 0 in $(\mathbf{C}^m)^{n-m}$. Let now $\mathcal{F}$ be the finite subset in $L_{0,0}$ defined as

$$\zeta \in \mathcal{F} \Longleftrightarrow \forall\, l = n-m+1, ..., n\,,\ \exists j \in \{1, ..., d\}\,,\ \zeta_l = \zeta_l[p_{j0}]$$

and $\mathcal{F}' := \mathcal{F} \setminus \{p_{10}, ..., p_{d0}\}$, and Λ be an affine form in $X_{n-m+1}, ..., X_n$ such that for any $\zeta \in \mathcal{F}'$,

$$\Lambda(\zeta) \ne \Lambda(p_{j0}),\ j = 1, ..., d\,;$$

if

$$B(X_{n-m+1},...,X_n\,;\,\beta) := \prod_{j=1}^{d}(\Lambda(X_{n-m+1},...,X_n)) - \Lambda[p_j(0,\beta)])$$

and $Q(X) := B(X_{n-m+1},...,X_n\,;\,X_1,...,X_{n-m})$, one can check that the collection of all polynomials $P_{l,\alpha}$, $l = n-m+1,...,n$, together with the polynomial Q fits with the assertions (6.12) and (6.13). $\diamondsuit$

In the particular case $m = 1$, one can be more precise and repeat Wood's argument in order to obtain the following :

Proposition 6.3 *Let γ_1,...,γ_d be d disjoint pieces of smooth analytic hypersurface in the affine space $\mathbf{C}^n$. Suppose that for any $(\alpha,\beta) \in (\mathbf{C}^{n-1})^2$ close to $(0,0)$, the affine line*

$$L_{\alpha,\beta} := \Big\{\zeta \in \mathbf{C}^n\,;\,\zeta_k = \alpha_k\zeta_n + \beta_k\,,\ k = 1,...,n-1\Big\}$$

intersects transversally γ_1,...,γ_d respectively at points $p_1(\alpha,\beta),...,p_d(\alpha,\beta)$. Assume that there exists $D \in \mathbb{N}$ and analytic functions $h_{\underline{k}}$, $|\underline{k}| \leq D+1$, in a neighborhood of 0 in $\mathbf{C}^{n-1}$ such that for (α,β) close to $(0,0)$ in $(\mathbf{C}^{n-1})^2$,

$$\sum_{j=1}^{d}\zeta_n[p_j(\alpha,\beta)] = \sum_{\substack{\underline{k}\in\mathbf{N}^{n-m}\\|\underline{k}|\leq D+1}} h_{\underline{k}}(\alpha)\,\beta_1^{k_1}\cdots\beta_{q-m}^{k_{n-m}}\,.$$

(one from the $h_{\underline{k}}$ for $|\underline{k}| = D+1$ being non identically zero). Then, one can find a polynomial P with degree $d+D$ which defines an affine algebraic variety $V(P)$ such that for some convenient constants $\epsilon > 0$, $\kappa > 0$, $K > 0$, one has :

- *if*

$$\Gamma_\epsilon := \bigcup_{\substack{(\alpha,\beta)\in(\mathbf{C}^m)^{n-m}\times\mathbf{C}^{n-m}\\ \max(\|\alpha\|,\|\beta\|)<\epsilon}} L_{\alpha,\beta}\,,$$

  then

$$\zeta \in \Gamma_\epsilon\,,\ \|\zeta\| \geq K \implies |P(\zeta)| \geq \kappa\|\zeta\|^d\,;$$

- *for* $\max(\|\alpha\|,\|\beta\|) < \epsilon$, *one has*

$$L_{\alpha,\beta}\cap V(P) = \{p_1(\alpha,\beta),...,p_d(\alpha,\beta)\}\,.$$

Remark. In the particular case $W = \mathbf{A}^n_{\mathbf{C}}$, $m = 1$, Proposition 6.3 appears as the reciprocal assertion to Proposition 6.1. The difficulty in the more general case $W = \mathbf{A}^n_{\mathbf{C}}$, $m > 1$, is to be able to interpolate the germs $\gamma_1, ..., \gamma_d$ by an algebraic complete intersection $V(P_1, ..., P_m)$. It does not seem possible when $m > 1$ even if conditions (6.10) are satisfied with $D = 0$ (which would mean that the projective variety $\{{}^hP_1 = ... = {}^hP_m = 0\}$ corresponding to the complete intersection $V(P)$ that interpolates the pieces γ_j does not hit $|\mathcal{H}_\infty| \cap |\mathcal{L}_{0,0}|$). We do not have the answer to that question yet. Nevertheless, Proposition 6.2 can be seen as an attempt to settle a converse to Proposition 6.1 in general.

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