Notations used in the problem are those used all through the notes.

1. Consider a polynomial \( F \in \mathbb{C}[X_1, X_2] \) with support the set
   \[ \{(0,0), (1,0), (2,0), (3,0), (0,3), (1,1), (2,2)\} \]
   Draw its Newton diagram \( \Delta \) (on a figure called from now on figure 1) and compute its 2-dimensional euclidean volume.
   Let \( \mathcal{A}_{\text{AVT}}(F) \) be the archimedean amoeba of the polynomial \( F \).

2. Show that the number of unbounded connected components of \( \mathbb{R}^2 \setminus \mathcal{A}_{\text{AVT}}(F) \) cannot exceed 8.

3. How many (exactly) unbounded components of \( \mathbb{R}^2 \setminus \mathcal{A}_{\text{AVT}}(F) \) among those (eventual) 8 mentionned in 2 do have a 2-dimensional cone as recession cone ?
   Draw on a separate figure (figure 2) the 2-dimensional cones corresponding to such components and picture roughly on the same figure the corresponding unbounded connected components of \( \mathbb{R}^2 \setminus \mathcal{A}_{\text{AVT}}(F) \).

4. What are the possible 1-recession cones for the other (eventual) unbounded components of \( \mathbb{R}^2 \setminus \mathcal{A}_{\text{AVT}}(F) \) ? How many such components at most do share each of these 1-dimensional recession cones ?

5. Why does the number of bounded connected components of \( \mathbb{R}^2 \setminus \mathcal{A}_{\text{AVT}}(F) \) does not exceed 3 ? Assuming that \( \mathbb{R}^2 \setminus \mathcal{A}_{\text{AVT}}(F) \) has exactly the maximal number of connected components, that is 11, complete figure 2 as a new figure (figure 3) where you sketch the drawing of the 11 connected components of \( \mathbb{R}^2 \setminus \mathcal{A}_{\text{AVT}}(F) \).

6. Let
   \[ R_F : (x_1, x_2) \in \mathbb{R}^2 \mapsto \frac{1}{4\pi^2} \int_{[0,2\pi]^2} \log |F(e^{x_1+it_1}, e^{x_2+it_2})| \, dt_1 \, dt_2 \]
   be the Ronkin fonction \( R_F \) and
   \[ p_{R_F} : \mathbb{R}^2 = (\text{Trop} \setminus \{-\infty\})^2 \mapsto \mathbb{R} \]
   be the evaluation of the tropical polynomial \( p_{R_F} \) (recall how such \( p_{R_F} \) is deduced from \( R_F \)). Explain why the following inequality holds :
   \[ \forall (x_1, x_2) \in \mathbb{R}^2, \ p_{R_F}(x_1, x_2) \leq R_F(x_1, x_2) \]
   PTO
7. In the particular case where $\mathbb{R}^2 \setminus \mathcal{A}_{V_\Sigma(F)}$ has exactly 4 connected components, compute the Monge-Ampère real measure $\mu[p_{R_F}, \ldots, p_{R_F}]$ attached to the convex function $p_{R_F}$. In the general case (where the number of connected components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_\Sigma(F)}$ lies between 4 and 11), compute $\int \int_{\mathbb{R}^2} d\mu[p_{R_F}, \ldots, p_{R_F}]$ and compare it to $\int \int_{\mathbb{R}^2} d\mu[R_F, \ldots, R_F]$; what is the support of $\mu[R_F, \ldots, R_F]$? What is the relation between the number of bounded 2-dimensional faces in the roof of $p_{R_F}$ and the number of nodes in the tropical deformation $V_{\text{trop}}(p_{R_F})$ of the Ronkin function $R_F$? On which condition on the roof of $p_{R_F}$ does $\mathbb{R}^2 \setminus \mathcal{A}_{V_\Sigma(F)}$ have exactly 4 connected components? Why does the number of edges in the tropical deformation $V_{\text{trop}}(p_{R_F})$ equal the number of bounded edges of the roof of $p_{R_F}$?

8. Suppose that $F(X_1, X_2) = 27 + 4X_1^3 - 4X_2^3 + 18X_1X_2 - X_1^2X_2^2$, that is the Sylvester resultant of the polynomial $X^3 + X_1X^2 + X_2X - 1$, considered as a polynomial in $X$, and its derivative with respect to $X$, namely $3X^2 + 2X_1X + X_2$. What is the degree of the logarithmic Gauss map $\gamma_F$, from the Zariski closure of $V_\Sigma(F)$ (in the toric variety $\mathcal{X}(\Delta)$) into $\mathbb{P}^1(\mathbb{C})$? Is the toric variety $\mathcal{X}(\Delta)$ simplicial? Is it a 2-dimensional complex manifold?

9. We admit that, under the hypothesis in 10, the number of connected components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_\Sigma(F)}$ equals exactly 4. Starting from figure 1 (featuring $\Delta$), draw a picture of the compactified amoeba of $F$ in that particular case.

10. Take $F$ as in 1. Consider the toric variety $\mathcal{X}(\Delta)$ and its algebraic moment map $\mu : \mathcal{X}(\Delta) \rightarrow \Delta$. Suppose that the Zariski closure of $V_\Sigma(F)$ in $\mathcal{X}(\Delta)$ hits transversally each of the 4 toric curves corresponding to the 4 rays in the rational fan $\Sigma(\Delta)$ (dual to $\Delta$), and that the images of such intersection points by $\mu$ are all distinct. How many connected components does $\mathbb{R}^2 \setminus \mathcal{A}_{V_\Sigma(F)}$ has in that case? Starting from figure 1 again (featuring $\Delta$), draw a picture of the compactified amoeba of $F$ in that situation.

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