Let V be a n-dimensional Stein manifold, I be a closed ideal of holomorphic functions on V. It was proved by Roger Gay that, given an analytic functional T such that $hT = 0$ (as a functional) for any $h \in I$, one can find some (n,n) compactly supported current $\widetilde{T}$, such that $\widetilde{T}(\varphi) = 0$ for any $\varphi \in I\mathcal{E}^{0,0}(V)$ and $T(h) = \widetilde{T}(h)$ for any h analytic on V. In this paper, we give some explicit construction of $\widetilde{T}$ in terms of residual currents when I is defined as a complete intersection or is locally Cohen-Macaulay. Moreover, by means of integral representation formulas of the Andersson-Berndtsson-Passare type, we also study the non complete intersection case in order to represent analytic functionals orthogonal to the ideal in terms of currents annihilated (as currents) by some power (less than n) of the local integral closure of $I\mathcal{E}^{0,0}(V)$.

1. Introduction.

In [15], [16], one of us considered the following problem: give integral representations for entire solutions with exponential type in $\mathbf{C}^n$ to some particular system of convolution equations. Let $I\mathcal{E}^{0,0}(\mathbf{C}^n)$ be an ideal in $\mathcal{E}^{0,0}(\mathbf{C}^n) := \mathcal{C}^\infty(\mathbf{C}^n)$ generated (as an ideal in $\mathcal{E}^{0,0}(\mathbf{C}^n)$), by holomorphic functions which correspond to characteristic functions of the given equations. Using the techniques developed in the two papers quoted above, it is possible to prove the existence of (n,n) compactly supported currents T in $\mathbf{C}^n$, orthogonal (as currents) to such an ideal $I\mathcal{E}^{0,0}(\mathbf{C}^n)$; the integral representation of a solution h of the convolution equation (or the system of convolution equations) is then of the form

$$h(z) =< T(\zeta), exp(< \zeta, z >) >, \tag{1.1}$$

where $T = T_h$ is such a current. In this setting, this was a version of the so called *Ehrenpreis's fundamental Principle*. The results obtained in [15], [16], were, in some sense, optimum, as long as we limited ourselves to the use of cohomology without any growth conditions. Nevertheless, some natural questions were remaining unanswered. Among them, the following: given a compactly supported (n,n) current T orthogonal to some ideal $I\mathcal{E}^{0,0}(\mathbf{C}^n)$, is there any representation of T in terms of residual currents (as defined in Coleff-Herrera's work [8])? Or also this one: can one make T (in (1.1)) completely explicit (in terms of the characteristic functions of the convolution equations)?

The aim of this paper is to give a preliminary answer, which will be partial but positive and complete in some particular cases, to such questions. We will use two different kinds of techniques: one is cohomologic and is based on [15], [16]; the other one, more analytic in nature, is inspired by our approach of residue currents and division formulas via analytic continuation ([4], [5], [7],[36]).

The fundamental notion that appears in this context is the notion of *analytic functional* (on a complex analytic manifold V, which will usually be a Stein manifold) orthogonal to some closed ideal I of holomorphic functions on V. The set of all such functionals will be denoted by $I^\perp$. In our application, V is a convex open subset of $\mathbf{C}^n$. The exponential type functions, which are solutions of the system of convolution equations one considers, are the Fourier-Borel transforms of compactly supported distributions with support included in the zero set of the ideal I. In the one variable case, these distributions are linear combinations of Dirac masses and derivatives of Dirac masses at some finite set of points; via Fourier-Borel transform, the corresponding exponential type entire functions are in this case the exponential polynomials. In the multivariable case, the exponential type functions one obtains that way can be viewed as a natural generalization of exponential polynomials in higher dimension (such a generalization is certainly as natural as considering exponential polynomials in several variables, which correspond to zero dimensional ideals).

We will give, in Section 2, most of the background which is necessary in order to understand the cohomological method of [16]. Then, in Section 3, we will study in detail (in the particular case where I is given as a complete intersection) how residual currents play a role in this method (and how $I^\perp$ can be expressed in terms of them). In Section 4, we will apply cohomological methods to study situations more general than complete intersections. We will describe in detail what happens in the case when I is locally Cohen-Macaulay (for example, when I is a zero dimensional ideal). Finally, Section 5 will be devoted to the description of the analytic method.

For a general survey about convolution equations, we refer to [6].

2. Some preliminary background.

From now on, V will denote a complex analytic n-dimensional manifold, $n \geq 1$, which as usual we will assume to be countable at infinity. $\mathcal{O}(V)$ denotes the Fréchet space of holomorphic functions on V. Ω^p is the sheaf of holomorphic p-forms on V, $\mathcal{E}^{p,q}$ (resp. ${}'\mathcal{D}^{p,q}$) denotes the sheaf of (p,q) $\mathcal{C}^\infty$ differential forms on V (resp. (p,q) currents on V). As usual, $\mathcal{O}$ will be the sheaf of germs of holomorphic functions on V.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$ modules. One can associate to the functor (from the category of sheaves of $\mathcal{O}$-modules with base V into itself)

$$\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{M})$$

its successive derived functors (with values in the category of sheaves of $\mathcal{O}$-modules with base V)

$$\mathcal{M} \mapsto \mathcal{E}xt^q_{\mathcal{O}}(\mathcal{F}, \mathcal{M}),\ q \geq 0\,. \tag{2.1}$$

In the same vein, if c denotes the family of all compact subsets of V, one can associate to the functor (from the category of sheaves of $\mathcal{O}$-modules with base V into the category of abelian groups)

$$\mathcal{M} \mapsto \mathrm{Hom}_{c,\mathcal{O}}(\mathcal{F}, \mathcal{M}) := \Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{M}))$$

its successive derived functors (with values in the category of abelian groups)

$$\mathcal{M} \mapsto \mathrm{Ext}^q_{c,\mathcal{O}}(V; \mathcal{F}, \mathcal{M}),\ q \geq 0\,. \tag{2.2}$$

One can do the same for the family of all closed subsets of V; the derived functors (still with values in the category of abelian groups) will then be denoted as

$$\mathcal{M} \mapsto \mathrm{Ext}^q_{\mathcal{O}}(V; \mathcal{F}, \mathcal{M}),\ q \geq 0\,.$$

When $\mathcal{F}$ is coherent, one can compute these objects using local resolutions of $\mathcal{F}$.

For any nonempty compact subset $K \subset V$, let q_K be the semi-norm on $\mathcal{O}(V)$

$$f \mapsto q_K(f) = \sup\{|f(z)|;\ z \in K\}.$$

The standard topology on $\mathcal{O}(V)$ (that is that for which one has uniform convergence on any compact subset) is defined by the family of all semi-norms q_K, K being any compact of V. $\mathcal{O}(V)$, equipped with this topology, is a Frechet-Schwartz (FS) space. In particular, it is reflexive [23]. A linear form on $\mathcal{O}(V)$ is an *analytic functional* if and only if there exists $K \subset\subset V$, $c_K \geq 0$, such that, for any $f \in \mathcal{O}(V)$,

$$| < T, f > | \leq c_K q_K(f). \tag{2.3}$$

Such a functional is said to be *carried* by the compact subset K (which is then called a *carrier* for T) if and only if, for any relatively compact neighborhood Ω of K, there exists $C_\Omega \geq 0$, such that, for any $f \in \mathcal{O}(V)$,

$$| < T, f > | \leq C_\Omega q_{\overline{\Omega}}(f). \tag{2.4}$$

We will denote by $\mathcal{O}'(V)$ the topologic dual of $\mathcal{O}(V)$, equipped with the strong topology. From Hahn-Banach theorem, it follows that, if T is an analytic functional carried by some compact set K, then, for any open subset Ω of V containing K, there exists a (n,n) compactly supported current T_1, with support in Ω, such that T_1 represents T, that is

$$< T, f > = < T_1, f >,\ \ f \in \mathcal{O}(V). \tag{2.5}$$

We now recall Malgrange's version [27] of the duality theorem of J. P. Serre [35]. As usual, $\overline{\partial}$ is the $(0,1)$ component of the exterior derivative.

Let us consider the two exact sequences of sheaves of $\mathcal{O}$-modules with basis V:

$$0 \to \mathcal{O} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,0} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{0,n} \xrightarrow{\overline{\partial}} 0 \tag{2.6}$$

$$0 \to \Omega^n \overset{\bar\partial}{\to} {}^{\backprime}\mathcal{D}^{n,0} \overset{\bar\partial}{\to} \cdots \overset{\bar\partial}{\to} {}^{\backprime}\mathcal{D}^{n,n} \overset{\bar\partial}{\to} 0. \tag{2.7}$$

The sheaves involved in these two exact sequences satisfy interesting properties: for any $x \in V$, for any $p, q \in \mathbf{N}$ (in particular when $p = 0$ and q is arbitrary), $\mathcal{E}_x^{p,q}$ is a flat $\mathcal{O}_x$-module; similarly, for any such (p,q) (in particular when $p = n$ and q is arbitrary), ${}^{\backprime}\mathcal{D}_x^{p,q}$ is an injective $\mathcal{O}_x$-module (this is a consequence of the theorem of division for distributions [26]). From these two properties, one can get ([27]) the following version of Serre's duality theorem.

Theorem 2.1 [27]. *Let $\mathcal{F}$ be an analytic coherent sheaf of $\mathcal{O}$-modules on V. The sequences of $\mathcal{O}$-modules on V*

$$0 \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{E}^{0,0} \to \cdots \to \mathcal{F} \otimes \mathcal{E}^{0,n} \to 0 \tag{2.6$'$}$$

and

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \Omega^n) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,0}) \to \cdots \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,n}) \to 0 \tag{2.7$'$}$$

are in local duality; moreover (2.6)' is an exact sequence.

In fact one can say much more. Let $0 \le q \le n$. Following Serre's argument in [27], one can show the following. If

$$\Gamma(V, \mathcal{F} \otimes \mathcal{E}^{0,p}) \to \Gamma(V, \mathcal{F} \otimes \mathcal{E}^{0,p+1})$$

are homomorphisms (the two spaces being considered as topological vector spaces) for $p = q$ and $p = q-1$, then the cohomology space $H^q(V, \mathcal{F})$ (which can be identified to the q-cohomology group of the complex $(\Gamma(V, \mathcal{F} \otimes \mathcal{E}^{0,*}))$ since (2.6)' is a resolution with fine sheaves, (see [17], p. 181) is equipped with a canonical Frechet space structure; its topologic dual is isomorphic to the the the $n-q$ cohomology space of the complex

$$\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,*})),$$

that is to say,

$$\begin{aligned}&\mathrm{Ext}^{n-q}_{c,\mathcal{O}}(V; \mathcal{F}, \Omega^n) = \\ &= \frac{Ker\{\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,n-q})) \to \Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,n-q+1}))\}}{Im\{\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,n-q-1})) \to \Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,n-q}))\}}.\end{aligned}$$

For example, when V is a Stein manifold, it follows from the fact that $H^k(V, \mathcal{F}) = 0$ for $k > 0$ (Cartan's theorem B) that

$$\Gamma(V, \mathcal{F} \otimes \mathcal{E}^{0,0}) \to \Gamma(V, \mathcal{F} \otimes \mathcal{E}^{0,1})$$

and

$$\Gamma(V, \mathcal{F} \otimes \mathcal{E}^{0,-1}) := \Gamma(V, \mathcal{F}) \to \Gamma(V, \mathcal{F} \otimes \mathcal{E}^{0,0})$$

are homomorphisms between topological vector spaces. Therefore, one can apply our version of Serre's theorem for $q = 0$ and get that the space

$$\begin{aligned}&\mathrm{Ext}^{n}_{c,\mathcal{O}}(V; \mathcal{F}, \Omega^n) = \\ &= \frac{\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,n}))}{Im\ \{\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,n-1})) \to \Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, {}^{\backprime}\mathcal{D}^{n,n}))\}}\end{aligned}$$

is isomorphic (as a topological vector space) to the topologic dual space of $\Gamma(V, \mathcal{F}) = H^0(V, \mathcal{F})$. The duality is realized thanks to the bilinear form

$$\Phi: \ H^0(V, \mathcal{F}) \times \mathrm{Ext}^n_{c,\mathcal{O}}(V; \mathcal{F}, \Omega^n) \to \mathbf{C}$$

that can be expressed as follows. Since $H^0(V, \mathcal{F}) = \mathrm{Ext}^0_{\mathcal{O}}(V; \mathcal{O}, \mathcal{F})$, one can define, for $u \in H^0(V, \mathcal{F})$ and $v \in \mathrm{Ext}^n_{c,\mathcal{O}}(V; \mathcal{F}, \Omega^n)$,

$$\Psi(u, v) = u \bullet v,$$

where the $\bullet$ above is the composition product of Ext

$$\mathrm{Ext}^0_{\mathcal{O}}(V;\mathcal{O},\mathcal{F}) \times \mathrm{Ext}^n_{c,\mathcal{O}}(V;\mathcal{F},\Omega^n) \to \mathrm{Ext}^n_{c,\mathcal{O}}(V;\mathcal{O},\Omega^n).$$

Now

$$\Psi(u,v) \in \mathrm{Ext}^n_c(V;\mathcal{O},\Omega^n) = H^n_c(V;\mathcal{O},\Omega^n) := \frac{\Gamma_c(V,{}`\mathcal{D}^{n,n})}{\overline{\partial}\Gamma_c(V,{}`\mathcal{D}^{n,n-1})},$$

so that if t is the trace map which is induced (as one takes quotients) by

$$T \mapsto < T, 1 >,\ \Gamma_c(V,{}`\mathcal{D}^{n,n}) \to \mathbf{C},$$

one can naturally define

$$\Phi(u,v) = t(\Psi(u,v)),$$

which is the bilinear duality map.

Let V be an open subset of $\mathbf{C}^n$ and T some analytic functional which admits as a carrier some convex compact subset K of V. The function from $\mathbf{C}^n$ into $\mathbf{C}$

$$z \mapsto F(T)(z) := < T_\zeta, exp < \zeta, z >>$$

(where $< \zeta, z > := \sum_{j=1}^n \zeta_j z_j$) is called the *Fourier-Borel Transform* of T.

For any $\epsilon > 0$, there exists a constant $C_\epsilon \geq 0$ such that

$$|F(T)(z)| \leq C_\epsilon exp(H_K(z) + \epsilon\|z\|), \tag{2.8}$$

where $H_K : z \mapsto sup\{\Re < \zeta, z >; \zeta \in K\}$ is the support function of the compact subset K. The other way around, if f is an entire function of n variables satisfying (2.8) (for some convex compact K and for *any* $\epsilon > 0$, with some appropriate constant C_ϵ), there exists some analytic functional $T \in \mathcal{O}'(V)$, which admits K as a carrier, such that $F(T) = f$ ([20]).

We denote as $\mathrm{Exp}(K)$ the vector space of entire functions f in n complex variables, with exponential type, such that

$$\|f\|_K := sup\{|f(z)|exp(-H_K(z)); z \in \mathbf{C}^n\} < \infty.$$

This space, equipped with the norm $\|\cdot\|_K$, is a Banach space. The space

$$\mathrm{Exp}(V) := \varinjlim_{K \subset\subset V} \mathrm{Exp}(K)$$

will be equipped with the inductive limit topology, associated to some exhaustive sequence $(K_n)_{n\geq 1}$ of convex compact subsets of V. The topology one defines like that does not depend on the choice of the sequence $(K_n)_n$. It is immediate to check that

$$F : \mathcal{O}'(V) \to \mathrm{Exp}(V)$$

is an isomorphism between topological vector spaces. The space $\mathrm{Exp}(V)'$ is therefore isomorphic to the bidual $\mathcal{O}''(V)$ through the transpose ${}^tF$ of F. Since $\mathcal{O}(V)$ is reflexive, one may identify $\mathcal{O}''(V)$ and $\mathcal{O}(V)$. Therefore

$${}^tF(R)(z) = < R_\zeta, exp < \zeta, z >>,\ z \in V,\ R \in \mathrm{Exp}(V)'.$$

The convolution operation between $R \in \mathrm{Exp}(V)'$ and $f \in \mathrm{Exp}(V)$ is defined as usual by

$$(R * f)(z) = < R_u,\ f(z+u) >,\ z \in V.$$

This is also $F(\rho T)$, where $\rho :={}^t F(R)$ and $f = F(T)$. The convolution between elements in $\mathrm{Exp}(V)'$ is defined as

$$< R * S,\ f > := < R,\ S * f >,\ R,\ S \in \mathrm{Exp}(V)',\ f \in \mathrm{Exp}(V).$$

One has the relation

$$ {}^tF(R*S) = ({}^tF(R)) \cdot ({}^tF(S)), $$

which immediately implies that $\mathrm{Exp}(V)'$ is a convolution algebra isomorphic to $\mathcal{O}(V)$.

If one uses such a dictionary, solving a homogeneous convolution equation (resp. a system of homogeneous convolution equations) $R*f=0$ (resp. $R_1*f=\cdots=R_m*f=0$) is just the same thing than solving a division problem (resp. a system of division equations to solve simultaneously) $\rho T=0$ (resp. $\rho_1 T=\cdots=\rho_m T=0$), where ρ, $\rho_1\ldots,\rho_m\in\mathcal{O}(V)$ and $T\in\mathcal{O}'(V)$.

Solving the division equation $\rho T=0$ can be done easily on an n-dimensional Stein manifold. One can represent T with some (n,n) compactly supported current T_1 such that there exists a $(n,n-1)$ compactly supported current T_2 such that

$$ \rho T_1 = \overline{\partial} T_2. \tag{2.9} $$

The existence of T_2 is due to the following facts: ρT_1 (as a current) is orthogonal to $\mathcal{O}(V)$ and $\mathcal{O}'(V)$ can be identified with $\Gamma_c(V,{}'\mathcal{D}^{n,n})/\overline{\partial}\Gamma_c(V,{}'\mathcal{D}^{n,n-1})$. Then, from Malgrange's theorem about division of distributions, it follows that there exists $T_3\in{}'\mathcal{E}^{n,n-1}(V)$, with compact support, such that $\rho T_3=T_2$. Therefore, one has from (2.9)

$$ \rho(T_1-\overline{\partial}T_3)=0 $$

(as currents), which implies that $T_1-\overline{\partial}T_3$ is a (n,n) compactly supported current with support in $\mathcal{V}(\rho):=\{z\in V;\ \rho(z)=0\}$, which represents T.

In the case of a system $\rho_1T=\cdots=\rho_mT=0$, one would like any solution T to be represented by some (n,n) compactly supported current $\tilde{T}$, with support included in the analytic set

$$ \mathcal{V}(\rho_1,\cdots,\rho_m) := \bigcap_{1\le j\le m}\mathcal{V}(\rho_j), $$

the current $\tilde{T}$ being annihilated (*as a current*) by the functions ρ_j, $1\le j\le m$.

More generally, let W be an *invariant* subspace of $\mathrm{Exp}(V)$ (V being some open convex subset of $\mathbf{C}^n$). That is, W is closed and, for any $f\in W$, for any $u\in\mathbf{C}^n$, $z\mapsto f(z+u)$ is in W. One can remark that ${}^tF(W^\perp)$ is a closed ideal in $\mathcal{O}(V)$; moreover, the map $W\mapsto{}^tF(W^\perp)$ is a one-to-one map between invariant subspaces of $\mathrm{Exp}(V)$ and closed ideals in $\mathcal{O}(V)$. Given such an invariant subspace W, it would be interesting to give integral representations for the elements in W in terms of compactly supported (n,n) currents which are orthogonal (as currents) to the closed ideal $I={}^tF(W^\perp)$ associated to this invariant subspace.

In order to describe $I^\perp$ (that is, the orthogonal of I in $\mathcal{O}'(V)$), we use Malgrange's duality theorem [27]. We proceed as follows. The sheaf of ideals $\mathcal{I}$ which is generated by I is coherent, and such that $\Gamma(V,\mathcal{I})=I$ (V is a Stein manifold). From classical results, one knows that the sequence

$$ 0\to\Gamma(V,\mathcal{I})\to\Gamma(V,\mathcal{O})\to\Gamma(V,\frac{\mathcal{O}}{\mathcal{I}})\to 0 $$

is an exact sequence of Fréchet spaces. Then

$$ I^\perp=\Gamma(V,\mathcal{I})^\perp=\left(\Gamma\left(V,\frac{\mathcal{O}}{\mathcal{I}}\right)\right)' \tag{2.10} $$

and one can identify the dual space in (2.10) thanks to Lemma 2.2 that we are going to state and prove, in order to help the reader. We will denote as $\mathcal{V}(\mathcal{I})$ the support of the sheaf $\mathcal{O}/\mathcal{I}$. From now on, ${}'\mathcal{D}^{p,q}_I$ will be the sheaf of germs of (p,q) currents T, with support in $\mathcal{V}(\mathcal{I})$, such that, for any open subset $U\subset V$, for any $f\in\Gamma(U,\mathcal{I})$, the current $f\cdot T_{|U}$ is 0 on U. The space $\Gamma_c(V,{}'\mathcal{D}^{p,q}_I)$ is the set of sections with compact support of this sheaf.

Lemma 2.2. *Let $\mathcal{I}$ be a coherent subsheaf of the sheaf $\mathcal{O}$ on a Stein manifold. We have the three following assertions.*

(1) *The space* $\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, {}'\mathcal{D}^{n,n}))$ *can be identified with the space* $\Gamma_c(V, {}'\mathcal{D}_I^{n,n})$.
(2) *The space*

$$Im\{\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, {}'\mathcal{D}^{n,n-1})) \xrightarrow{\overline{\partial}} \Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, {}'\mathcal{D}^{n,n}))\}$$

can be identified to the subspace $\overline{\partial}[\Gamma_c(V, {}'\mathcal{D}_I^{n,n-1})]$ *of* $\Gamma_c(V, {}'\mathcal{D}_I^{n,n})$.
(3) *The following diagram is commutative*

$$\begin{array}{ccc}
\mathrm{Ext}^0_{\mathcal{O}}(V; \mathcal{O}, \mathcal{O}/\mathcal{I}) \times \mathrm{Ext}^n_{c,\mathcal{O}}(V; \mathcal{O}/\mathcal{I}, \Omega^n) & \xrightarrow{\bullet} & \mathrm{Ext}^n_{c,\mathcal{O}}(V; \mathcal{O}, \Omega^n) \\
\uparrow\ j & & \downarrow\ t \\
\frac{\Gamma(V,\mathcal{O})}{\Gamma(V,\mathcal{I})} \times \Big(\Gamma_c(V, {}'\mathcal{D}_I^{n,n})/\overline{\partial}[\Gamma_c(V, {}'\mathcal{D}_I^{n,n-1})]\Big) & \xrightarrow{\overline{d}} & \mathbf{C}
\end{array}$$

where $\bullet$ *denotes the composition law of* Ext, t *the trace map,* j *the identification map which results from* (1) *and* (2), *and* $\overline{d}$ *is obtained when one takes quotients from the standard duality* $d : (f, T) \mapsto < T, f >$ *between* $\mathcal{C}^\infty$ *functions and* (n, n) *compactly supported currents.*

Proof.

(1) An element h in $\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, `\mathcal{D}_I^{n,n}))$ is an $\mathcal{O}$-homomorphism with compact support, this support being included in the support of the sheaf $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, `\mathcal{D}_I^{n,n})$, which is contained in the support of $\mathcal{O}/\mathcal{I}$, that is $\mathcal{V}(\mathcal{I})$. This means that h can be considered as a collection of $\Gamma(U, \mathcal{O})$ homomorphisms

$$h_U : \Gamma(U, \mathcal{O})/\Gamma(U, \mathcal{I}) \to `\mathcal{D}^{n,n}(U)\,,$$

for any open subset $U \subset V$, all these homomorphisms being compatible with restrictions. Since $\Gamma(U, \mathcal{O})/\Gamma(U, \mathcal{I})$ is generated by a single element, h_U is determined by $T_U := h_U(\overline{1_U})$. Therefore, one can see immediately that the collection $(T_U)_U$ provides a (n, n) compactly supported current on V, $T_V = \Phi(h)$, which is in $\Gamma_c(V, `\mathcal{D}_I^{n,n})$ because of the definition of this space.

It is clear that the map $\Phi : h \mapsto \Phi(h) = T_V$ is a $\Gamma(V, \mathcal{O})$-linear homomorphism from $\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, `\mathcal{D}_I^{n,n}))$ into $\Gamma_c(V, `\mathcal{D}_I^{n,n})$.
This map is in fact one to one; in order to prove that, let us explicit its inverse. If T is a current in $\Gamma_c(V, `\mathcal{D}_I^{n,n}))$, let's associate to $T$ an element $\Psi(T)$ in $\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, `\mathcal{D}_I^{n,n}))$. Such an element is determined by the collection of homomorphisms $(\Psi(T)_U)_U$, U being any open subset of V, where

$$\Psi(T)_U : \Gamma(U, \mathcal{O})/\Gamma(U, \mathcal{I}) \to `\mathcal{D}^{n,n}(U)$$

is defined by

$$\Psi(T)_U(\dot{f}) = f.T_{|U},$$

$\dot{f}$ being the class of $f \in \Gamma(U, \mathcal{O})$ modulo $\Gamma(U, \mathcal{I})$. The map $\Psi : T \mapsto \Psi(T)$ is $\Gamma(V, \mathcal{O})$-linear and it is immediate to check that Φ and Ψ are inverse one to each other.

(2) The proof is analogous to the proof of point (1).

(3) We know that $\Gamma(V, \mathcal{O}/\mathcal{I}) = \mathrm{Ext}^0_{\mathcal{O}}(V, \mathcal{O}, \mathcal{O}/\mathcal{I})$. We also know that

$$\begin{aligned}&\mathrm{Ext}^n_{c,\mathcal{O}}(V; \mathcal{O}/\mathcal{I}, \Omega^n) = \\ &= \frac{\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, `\mathcal{D}^{n,n}))}{Im\ \{\Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, `\mathcal{D}^{n,n-1})) \to \Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, `\mathcal{D}^{n,n}))\}}\end{aligned}$$

from Serre-Malgrange duality theorem. The identification map j is obtained from

$$id_{\Gamma(V, \mathcal{O}/\mathcal{I})} \times \Phi$$

when one takes quotients. The fact that the diagram is commutative can be seen as follows: the composition product $\bullet$ of Ext associates to the pair $j(\dot{f}, \dot{T})$ the class of fT in $H^n_c(V; \mathcal{O}, \Omega^n) = \Gamma_c(V, `\mathcal{D}^{n,n})/\overline{\partial}[\Gamma_c(V, `\mathcal{D}^{n,n-1})]$.
The lemma is proved. $\diamondsuit$

Now, we know from lemma 2.2 that the orthogonal of any closed ideal I in $\mathcal{O}(V)$ can be identified with

$$\Gamma_c(V, `\mathcal{D}_I^{n,n}))/\overline{\partial}[\Gamma_c(V, `\mathcal{D}_I^{n,n-1})],$$

which is a representation of $\mathrm{Ext}^n_{c,\mathcal{O}}(V; \mathcal{O}/\mathcal{I}, \Omega^n)$.
In particular, if $\rho_1 T = \cdots = \rho_m T = 0$, there exists an (n, n) compactly supported current, with support in $\mathcal{V}(\rho_1, \ldots, \rho_m)$, which is orthogonal (as a current) to the ideal $(\rho_1, \ldots, \rho_m)\mathcal{E}^{0,0}(V)$ and represents T. Whenever V is a convex open set in $\mathbf{C}^n$, $R_1, \ldots, R_m$ are m elements in $\mathrm{Exp}'(V)$ (let us say, for example, m infinite order differential operators with constant coefficients), any solution f in $\mathrm{Exp}(V)$ of the system of convolution equations

$$R_1 * f = \cdots = R_m * f = 0$$

is of the form $f = F(T)$, where $T \in \mathcal{O}'(V)$ can be represented by some compactly supported (n, n) current T_1, with support included in $\mathcal{V}(\rho_1, \ldots, \rho_m)$ $(\rho_j =^t F(R_j))$ such that, for any open subset $U \subset V$, for any $1 \le j \le m$, $\rho_j \cdot (T_{1|U}) = 0$.

Remark 2.3. It is easy to extend the statements above to closed submodules in $\mathcal{O}^p(V)$. When V is a convex open subset in $\mathbf{C}^n$, one can apply these methods to study systems of convolution equations of the form

$$\sum_{j=1}^{p} R_{i,j} * \phi_j = 0,\ 1 \le j \le q$$

where $R_{i,j} \in \mathrm{Exp}'(V)$, $\phi_i \in \mathrm{Exp}(V)$ for $1 \le i \le q$, $1 \le j \le p$.

Remark 2.4. When V is a convex open subset in $\mathbf{C}^n$ and W an invariant subspace in $\mathrm{Exp}(V)$, there is a *unique* analytic subspace $\mathcal{V}(W) = \mathcal{V}(\mathcal{I})$ ($\mathcal{I}$ beeing the sheaf of ideals generated by $I^{\perp} :=^t F(W^{\perp})$), such that

$$W = F\left(\frac{\Gamma_c(V,\, `\mathcal{D}_I^{n,n}))}{\overline{\partial}[\Gamma_c(V,\, `\mathcal{D}_I^{n,n-1})]}\right).$$

Remark 2.5. For some given $T \in \mathcal{O}'(V)$, the ideal $I(T)$ of elements $\rho \in \mathcal{O}(V)$ such that $\rho T = 0$ is the largest closed ideal such that $T \in I^{\perp}$. When $I(T) \neq \{0\}$, one can see the analytic space defined by $I(T)$ as the *analytic carrier* of the functional T.

Remark 2.6. We would like to mention the beautiful result of L. Gruman, which generalizes, in some substantial way, Ritt's division theorem. Here is some simplified version of this statement:

Theorem [19]. *Let $(I_k)_{1\le k\le m}$ be a collection of finitely generated ideals in $\mathcal{O}(\mathbf{C}^n)$ and, for each k, T_k be some analytic functional in $I_k^{\perp}$. Assume that, for some collection of M elements $(\beta_i)_{1\le i\le M}$, for some collection of M complex numbers $(c_i)_{1\le i\le M}$, the function*

$$z \mapsto G(z) := \frac{\sum_{1\le k\le m} F(T_k)(z)}{\sum_{i=1}^{M} c_i exp < \beta_i, z >}$$

is an entire function in $\mathbf{C}^n$. Then, there are closed ideals J_i, $1 \le i \le N$, as well as compactly supported (n,n) currents T_i, $1 \le i \le N$, such that T_i is annihilated by J_i (as a current) for any i, and

$$G = \sum_{i=1}^{N} F(T_i).$$

We end this section with the definition of residue currents. We refer to [8], [10], [30], [36] for the properties of residue and multiple-residue principal value currents.

Let V be a complex manifold of dimension n and $\{Y_1, \ldots, Y_p\}$ a family of p hypersurfaces ($1 \le p \le n$). One can associate to any (r,q) semimeromorphic form having its poles on the union of the given hypersurfaces, a multiple residue current of bidegree $(r, p+q)$. In the particular case where each of the hypersurfaces Y_i is defined by the vanishing of a global holomorphic function $f_i \in \mathcal{O}(V)$ and the family is in complete intersection position, given any $\omega' \in \mathcal{E}^{n,n-p}(V)$, we can consider the semimeromorphic form $\omega := \frac{\omega'}{f_1 \cdots f_p}$, and the action of the associated residue (n,n)-current $\overline{\partial}\frac{1}{f_1} \wedge \ldots \wedge \overline{\partial}\frac{1}{f_p} \wedge \omega$ on a test function φ, may be defined as the limit along an admissible path $\varepsilon = \varepsilon(\delta)$ (in the sense of [8]), of the following integrals over semianalytic tubes

$$\overline{\partial}\frac{1}{f_1} \wedge \ldots \wedge \overline{\partial}\frac{1}{f_p} \wedge \omega(\varphi) = \lim_{\delta\to 0} \frac{1}{(2\pi i)^p} \int_{\{|f_1|=\varepsilon_1 \ldots |f_p|=\varepsilon_p\}} \frac{\varphi\ \omega'}{f_1 \cdots f_p}.$$

It is interesting to remark that the above limit does not exist in general if we just let $\varepsilon \in \mathbf{R}^p_{>0}$ tend to 0, even in the point case $p = n$, as it has been shown in [31].

The residue current may be equivalently defined by means of analytic continuation as follows: using Atiyah's theorem [2], one can consider the map

$$\lambda \mapsto I(\cdot;\lambda) = \frac{\lambda^p}{(2i\pi)^p} |f_1 \dots f_p|^{2(\lambda-1)} \bigwedge_{j=1}^{p} \overline{\partial f}_j$$

as a current-valued meromorphic map defined in the whole complex plane. One can show that this map has its poles in $\mathbf{Q}^-$, and is such that, for any $\omega \in \mathcal{E}^{n,n-p}(V)$, for any test function φ,

$$I(\omega\varphi;0) = \left\langle \overline{\partial}\frac{1}{f_1} \wedge \dots \wedge \overline{\partial}\frac{1}{f_p} \wedge \omega, \varphi \right\rangle .$$

The local behaviour of this residue current is the following:

Let $Y := \cap_i^p (f_i = 0)$ and (W, z) be a coordinate neighborhood such that for any $A \subseteq \{1, \dots, n\}$, $|A| = n-p$, there exists a sufficiently small polydisk $U_A \subseteq W$ with the property that $\Pi : U_A \cap Y \to \Pi(U_A \cap Y)$ is a ramified covering, where Π is the coordinate projection onto $\{z : z_j = 0 \,\forall j \notin A\}$. Denote $U := \cap_A U_A$. According to the fibered residue formula of Coleff–Herrera, there exist for each A, a positive integer m_A, a holomorphic function $\rho_A \in \mathcal{O}(U)$ such that $(\rho_A = 0) \cap Y$ contains $\mathrm{sing}(Y)$ and for any $r \in \mathbf{N}_0^p$, $|r| \leq m_A$, semimeromorphic functions $k_A[r]$ on Y with poles contained in $(\rho_A = 0)$ such that, for any test function φ with support contained in U:

$$\begin{aligned} &\overline{\partial}\frac{1}{f_1} \wedge \dots \wedge \overline{\partial}\frac{1}{f_p} \wedge \omega(\varphi) = \\ &= \sum_{|A|=n-p} P_{Y,\rho_A} \left(\sum_{|r| \leq m_A} \left(\frac{\partial^r}{\partial z_A^r} \right) (\varphi) \bigg|_Y . \, k_A[r] \, . \, dz_A \wedge d\overline{z}_A \right) \end{aligned} \tag{2.11}$$

(P_{Y,ρ_A} denotes the principal value current on Y associated to $(\rho_A = 0)$ and $\dfrac{\partial^r}{\partial z_A^r} = \dfrac{\partial^r}{\partial z_{i_1}^{r_1} \dots \partial z_{i_p}^{r_p}}$ if $A = \{i_1, \dots, i_p\}$ with $i_1 < i_2 < \dots < i_p$).

We observe that $\overline{\partial}\frac{1}{f_1} \wedge \dots \wedge \overline{\partial}\frac{1}{f_p} \wedge \omega(\varphi)$ is locally equal to a sum of principal values over Y of linear combinations (with semimeromorphic coefficients $k_A[r]$) of holomorphic derivatives transverse to reg Y of the test function φ. These principal values are reduced to integrations if the functions $k_A[r]$ have no singularities, or evaluations if $p = n$.

3. The representation of $\mathbf{I}^{\perp}$ in the complete intersection case; a cohomological approach.

In this section, we will characterize $I^{\perp}$ in terms of residue currents under the hypothesis that the ideal I is generated by a complete intersection $f_1, \ldots, f_p \in \mathcal{O}(V)$ (i.e. $V(I) := \cap_{i=1}^{p}(f_i = 0)$ has dimension $n-p$). For any $\omega \in \Gamma_c(V, \mathcal{E}^{n,n-p})$, the distribution ($(n,n)$ current) $\overline{\partial}\frac{1}{f_1} \wedge \ldots \wedge \overline{\partial}\frac{1}{f_p} \wedge \omega$, has compact support contained in $\mathrm{supp}(\omega) \cap V(I)$ and is annihilated by $f_1, \ldots, f_p$, defining an element in $I^{\perp}$. We will show that in fact, $I^{\perp}$ has $\overline{\partial}\frac{1}{f_1} \wedge \ldots \wedge \overline{\partial}\frac{1}{f_p}$ as a natural *generator*. Precisely, we will prove that $I^{\perp}$ can be identified in a natural way with the $\overline{\partial}$-cohomology space $H^{n-p}_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I}))$, and that the identification is realized via the morphism

$$\begin{array}{ccc} H^{n-p}_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I})) & \to & I^{\perp} \\ \overline{\varphi} & \mapsto & \left(\overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \varphi\right)_{|\mathcal{O}(V)} \end{array}.$$

We begin with a characterization of the above cohomology group.

Proposition 3.1. *Let V be a n-dimensional manifold and $f_1, \ldots, f_p \in \mathcal{O}(V)$ generating a (coherent) sheaf of ideals $\mathcal{I}$. Then, for any m,*

$$H^m_c(V; \Omega^n \otimes \mathcal{O}/\mathcal{I}) \simeq H^m_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I})) \simeq$$

$$\simeq \frac{\{\varphi \in \Gamma_c(V, \mathcal{E}^{n,m}) / \bar{\partial}\varphi = \sum f_i \psi_i, \text{ for some } \psi_i \in \Gamma_c(V, \mathcal{E}^{n,m+1})\}}{\bar{\partial}\Gamma_c(V, \mathcal{E}^{n,m-1}) + \{\sum f_i \varphi_i, \varphi_i \in \Gamma_c(V, \mathcal{E}^{n,m})\}}. \tag{3.1}$$

Proof. By arguments similar to those in Theorem 2.1, it follows that

$$0 \to \Omega^n \otimes \mathcal{O}/\mathcal{I} \overset{i \otimes id}{\to} \mathcal{E}^{n,0} \otimes \mathcal{O}/\mathcal{I} \overset{\bar{\partial} \otimes id}{\to} \mathcal{E}^{n,1} \otimes \mathcal{O}/\mathcal{I} \overset{\bar{\partial} \otimes id}{\to} \cdots \to 0$$

is a c-soft resolution of $\Omega^n \otimes \mathcal{O}/\mathcal{I}$, which proves the first isomorphism $H^m_c(V; \Omega^n \otimes \mathcal{O}/\mathcal{I}) \simeq H^m_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I}))$.

In order to prove the second isomorphism, note first that the quoted flatness of $\mathcal{E}^{\bullet,\bullet}$ over $\mathcal{O}$ implies that

$$0 \to \mathcal{E}^{n,\bullet} \otimes \mathcal{I} \to \mathcal{E}^{n,\bullet} \to \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I} \to 0$$

is exact.

As $H^1_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{I}) = 0$, we get the following commutative diagram with exact columns:

$$\begin{array}{ccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & \Gamma_c(V, \mathcal{E}^{n,m-1} \otimes \mathcal{I}) & \to & \Gamma_c(V, \mathcal{E}^{n,m} \otimes \mathcal{I}) & \to & \Gamma_c(V, \mathcal{E}^{n,m+1} \otimes \mathcal{I}) & \to \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \Gamma_c(V, \mathcal{E}^{n,m-1}) & \to & \Gamma_c(V, \mathcal{E}^{n,m}) & \to & \Gamma_c(V, \mathcal{E}^{n,m+1}) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & \Gamma_c(V, \mathcal{E}^{n,m-1} \otimes \frac{\mathcal{O}}{\mathcal{I}}) & \to & \Gamma_c(V, \mathcal{E}^{n,m} \otimes \frac{\mathcal{O}}{\mathcal{I}}) & \to & \Gamma_c(V, \mathcal{E}^{n,m+1} \otimes \frac{\mathcal{O}}{\mathcal{I}}) & \to \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

Moreover, the image of $\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{I})$ in $\Gamma_c(V, \mathcal{E}^{n,\bullet})$ is equal to $I \cdot \Gamma_c(V, \mathcal{E}^{n,\bullet})$, by collecting local representations of a section by means of a partition of unity.

Therefore, we can compute $\mathrm{Ker}(\bar{\partial} \otimes id)/\mathrm{Im}(\bar{\partial} \otimes id)$ as the stated quotient of global $\mathcal{C}^{\infty}$ compactly supported forms. $\diamondsuit$

We know from §2 that $I^\perp$ can be represented as $\mathrm{Ext}^n_{c,\mathcal{O}}(V;\, \mathcal{O}/\mathcal{I},\, \Omega^n)$. In the case of a complete intersection ideal of codimension p, this global Ext can be characterized as follows:

Proposition 3.2. *Let V be a n-dimensional manifold and let $\mathcal{I}$ be a coherent sheaf of ideals in $\mathcal{O}$. Suppose there exist $p \le n$ such that $\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)$ is nonzero only for $q = p$. Then,*

$$\mathrm{Ext}^n_{c,\mathcal{O}}(V;\, \mathcal{O}/\mathcal{I},\, \Omega^n) \simeq H^{n-p}_c(V, \mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)). \tag{3.2}$$

Moreover, if there exist $f_1, \dots, f_p$, p elements in $\mathcal{O}(V)$ in complete intersection position generating $\mathcal{I}$, one also has the following identification

$$\mathrm{Ext}^n_{c,\mathcal{O}}(V;\, \mathcal{O}/\mathcal{I},\, \Omega^n) \simeq H^{n-p}_c(V; \mathcal{O}/\mathcal{I} \otimes \Omega^n) \tag{3.3}$$

Proof. As $\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)$ is nonzero only for $q = p$, the spectral sequence of $\mathcal{E}xt$ degenerates at E_2 and one has for $q \ge p$ the following isomorphism

$$\mathrm{Ext}^q_{c,\mathcal{O}}(V;\, \mathcal{O}/\mathcal{I},\, \Omega^n) \simeq H^{q-p}_c(V, \mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)).$$

In particular, when $q = n$, we get (3.2). In order to prove this result, let us consider an injective resolution $\mathcal{J}^\bullet$ of Ω^n; one closes the double complex K as below

$$\begin{array}{cccccc}
 & 0 & & 0 & \\
 & \downarrow & & \downarrow & \\
0 \longrightarrow & \Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\frac{\mathcal{O}}{\mathcal{I}}, \Omega^n)) & \to \cdot\cdot & \Gamma_c(V, \mathcal{H}om_{\mathcal{O}}(\frac{\mathcal{O}}{\mathcal{I}}, \mathcal{J}^\bullet)) & \to \\
 & \downarrow & & \downarrow & \\
0 \to & \Gamma_c(V, \mathcal{C}^0(V, \mathcal{H}om_{\mathcal{O}}(\frac{\mathcal{O}}{\mathcal{I}}, \Omega^n))) & \to \cdot\cdot & \Gamma_c(V, \mathcal{C}^0(V, \mathcal{H}om_{\mathcal{O}}(\frac{\mathcal{O}}{\mathcal{I}}, \mathcal{J}^\bullet))) & \to \\
 & \downarrow & & \downarrow & \\
0 \to & \Gamma_c(V, \mathcal{C}^1(V, \mathcal{H}om_{\mathcal{O}}(\frac{\mathcal{O}}{\mathcal{I}}, \Omega^n))) & \to \cdot\cdot & \Gamma_c(V, \mathcal{C}^1(V, \mathcal{H}om_{\mathcal{O}}(\frac{\mathcal{O}}{\mathcal{I}}, \mathcal{J}^\bullet))) & \to \\
 & \downarrow & & \downarrow & \\
 & \vdots & & \vdots &
\end{array}$$

For this complex, one may compute

$$\begin{aligned}
{}'E_1^{p,q}(K) =& H^q(\mathcal{C}^p_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n))) \\
=& \mathcal{C}^p_c(V, \mathcal{H}^q(\mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \mathcal{J}^\bullet))) \\
=& \mathcal{C}^p_c(V, \mathcal{E}xt^q_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)),
\end{aligned}$$

since Godement's functor $\mathcal{F} \mapsto \mathcal{C}_c(V, \mathcal{F})$ is exact (see [17], p. 168). Therefore

$${}'E_2^{p,q}(K) = H^p_c(V, \mathcal{E}xt^q_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)).$$

Since we have, for any integer m

$${}'E_2^{m-r,r} = H^{m-r}_c(V, \mathcal{E}xt^r_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)) = 0$$

for $r \neq p$, it follows from Theorem 4.4.1, p. 81 in [17], that

$$H^n(K) =' E_2^{n-p,p}(K) = H^{n-p}_c(V, \mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)). \tag{3.4}$$

Since the sheaves $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \mathcal{J}^\bullet))$ are flabby, we have also, for any $q \ge 1$,

$${}''E_1^{p,q}(K) = H^q_c(\mathcal{C}^\bullet(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \mathcal{J}^p))) = H^q_c(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \mathcal{J}^p)) = 0. \tag{3.5}$$

Then, using again Godement's Theorem 4.4.1, we get from (3.5)

$$H^n(K) ='' E_2^{n,0}(K) = H_c^n(V, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \mathcal{J}^\bullet)) = \mathrm{Ext}^n_{c,\mathcal{O}}(V;\, \mathcal{O}/\mathcal{I},\, \Omega^n). \tag{3.6}$$

From (3.4) and (3.6), we deduce the isomorphism (3.2).

In order to prove (3.3), observe first that there exist an invertible matrix $A = (a_{ij}) \in GL(\mathbf{C}, p)$ such that the functions $g_i := \sum_{j=1}^p a_{ij}\ f_j$ form a regular sequence in $\mathcal{O}(V)$ generating I ([11, Prop. 1.2]). We can suppose then w.l.o.g. that the given generators $f_1, \ldots, f_p$ are a regular sequence and we consider the associated Koszul projective resolution of $\mathcal{O}/\mathcal{I}$:

$$0 \to \Lambda^p\mathcal{O}^p \xrightarrow{\beta^p} \Lambda^{p-1}\mathcal{O}^p \to \cdots \to \Lambda^1\mathcal{O}^p \xrightarrow{\beta^1} \mathcal{O} \to \mathcal{O}/\mathcal{I} \to 0\ .$$

The sheaves $\mathcal{E}xt^r_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)$ are the cohomology sheaves of the dual complex $(\mathcal{H}om_{\mathcal{O}}(\Lambda^\bullet\mathcal{O}^p, \Omega^n), {}^t\beta^\bullet)$, which again can be identified to a Koszul complex. Therefore,

a) for any $q \geq 1$, $q \neq p$, $\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n) = 0$, and so (3.2) is true.

b) $\mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n) \simeq \mathcal{O}/\mathcal{I} \otimes \Omega^n$, proving that

$$\mathrm{Ext}^n_{c,\mathcal{O}}(V;\, \mathcal{O}/\mathcal{I},\, \Omega^n) = H_c^{n-p}(V; \mathcal{O}/\mathcal{I} \otimes \Omega^n).$$

Finally, we deduce (3.3) from Proposition 3.1. $\diamond$

Corollary 3.3. *If I is generated by a complete intersection $f_1, \ldots, f_p \in \mathcal{O}(V)$,*

$$I^\perp \simeq H^{n-p}_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I}))\ .$$

We show next that every element in $I^\perp$ is represented by a residue current. The approach to prove this special case of the classical duality theorems ([33],[34]), is inspired by [28] (see also [13]).

Theorem 3.4. *Let V be an n-dimensional Stein manifold and $I \subseteq \mathcal{O}(V)$ generated by a complete intersection $f_1, \ldots, f_p \in \mathcal{O}(V)$. Let $\mathcal{I}$ denote the coherent sheaf of ideals associated to I. Then,*

$$\begin{array}{ccc} H^{n-p}_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I})) & \to & I^\perp \\ \overline{\varphi} & \mapsto & \left(\overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \varphi\right)_{|\mathcal{O}(V)} \end{array}$$

is an isomorphism.

Proof. By ([12, Prop. 1.2]), there exist an invertible matrix $A = (a_{ij}) \in GL(\mathbf{C}, p)$ such that the functions $g_i := \sum_{j=1}^p a_{ij}\ f_j$ are another set of p generators of I satisfying the additional hypotheses that, for any subset $J \subseteq \{1, \ldots, p\}$ with cardinal k, $\cap_{i\in J}\{g_i = 0\}$ is an analytic set of codimension k. Moreover, by the Transformation Law for residue currents (see for example [18] for the discrete case, [21], p. 171 and [12] for the complete intersection case), we have the equality of currents:

$$\overline{\partial}(1/g_1) \wedge \cdots \wedge \overline{\partial}(1/g_p) = \left(\frac{1}{\det(A)}\right) \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p)\ .$$

Therefore, we can suppose w.l.o.g. that the given generators $f_1, \ldots, f_p$ define a regular sequence for any order.

Let $(\mathcal{R}^\bullet, \rho)$ be the Koszul resolution of $\mathcal{O}/\mathcal{I}$:

$$\mathcal{O} \longrightarrow \Lambda^p\mathcal{O}^p \xrightarrow{\ \rho\ } \cdots \xrightarrow{\ \rho\ } \Lambda^1\mathcal{O}^p \xrightarrow{\ \rho\ } \Lambda^0\mathcal{O}^p \xrightarrow{\ \pi\ } \mathcal{O}/\mathcal{I} \longrightarrow 0\ ,$$

where $\rho(e_{i_1}\wedge\ldots\wedge e_{i_s})=\sum_{j=1}^{s}(-1)^{j-1}f_j\cdot e_{i_1}\wedge\ldots\wedge\widehat{e_{i_j}}\wedge\ldots\wedge e_{i_s}$.

Denote by $\mathcal{Z}_0$ the set

$$\{\omega\in\mathcal{H}om_{\mathcal{O}}(\mathcal{R}^\bullet,\mathcal{E}^{n,\bullet}):\omega(\Lambda^j\mathcal{O}^p)\subseteq\mathcal{E}^{n,n-j}, j=0,\ldots,p,\text{ and }\overline{\partial}\omega=\omega\rho\}$$

Following Palamodov ([28]), each $\omega\in\Gamma_c(V,\mathcal{Z}_0)$ is called a (closed) *peripherical form.*

Given $I\subseteq\{1,\ldots,p\}$, denote $e_I=e_{i_1}\wedge\ldots\wedge e_{i_s}$ if $I=\{i_1,\ldots,i_s\}$ with $i_1<i_2<\cdots<i_s$, and let $\omega_I=\omega(e_I)$. Then, each $\omega\in\Gamma_c(V,\mathcal{Z}_0)$ is determined by a collection $(\omega_I)_{I\subseteq\{1,\ldots,p\}}$ of $\mathcal{C}^\infty$ compactly supported forms verifying:

i) $\omega_I\in\Gamma_c(V,\mathcal{E}^{n,n-|I|})$, for all $I\subseteq\{1,\ldots,p\}$.

ii) $\overline{\partial}\omega_I=\sum_{j=1}^{|I|}(-1)^{j-1}f_j\cdot\omega_{I-\{i_j\}}$, for all I such that $|I|\geq 2$, and $\overline{\partial}\omega_{\{i\}}=f_i\cdot\omega(1)$ for all $i=1,\ldots,p$.

Consider

$$\mathcal{Z}_1:=\{\omega\in\mathcal{H}om_{\mathcal{O}}(\mathcal{R}^{\bullet},\mathcal{E}^{n,\bullet}):\omega(\Lambda^j\mathcal{O}^p)\subseteq\mathcal{E}^{n,n-j-1},\ j=0,\ldots,p\}$$

and let $\Delta:\mathcal{Z}_1\to\mathcal{Z}_0$ defined by

$$\Delta(\omega)(e_I)=\overline{\partial}\omega(e_I)+\omega(\rho(e_I)),$$

as in [28]. By Theorem 5.1 in [28],

$$\begin{array}{ccc}\frac{\Gamma_c(V,\mathcal{Z}_0)}{\Delta\Gamma_c(V,\mathcal{Z}_1)} & \xrightarrow{\alpha} & I^\perp\\ \overline{\omega} & \longmapsto & \int_V\omega(1)\ .\end{array}\tag{3.7}$$

is an isomorphism. By [25, Remarque 2.5],

$$\begin{array}{ccc}\frac{\Gamma_c(V,\mathcal{Z}_0)}{\Delta\Gamma_c(V,\mathcal{Z}_1)} & \xrightarrow{\beta} & H^{n-p}_{\overline{\partial}}(\Gamma_c(V,\mathcal{E}^{n,\bullet}\otimes\mathcal{O}/\mathcal{I})\\ \overline{\omega} & \longmapsto & \overline{\omega(e_1\wedge\ldots\wedge e_p)}\end{array}\tag{3.8}$$

is also an isomorphism.

We claim that for any closed peripherical form ω and any holomorphic function $h\in\mathcal{O}(V)$:

$$\int_V\omega(1)\cdot h=(-1)^p\Big\langle\overline{\partial}\frac{1}{f_1}\wedge\ldots\wedge\overline{\partial}\frac{1}{f_p}\wedge\omega_{\{1,\ldots,p\}},h\Big\rangle\ .\tag{3.9}$$

The proof of this fact is a consequence of properties 1.7.2(4), 1.7.6(2) and 1.7.7(3) of residue and residue-principal value currents shown in [8].

First,

$$\begin{aligned}\int_V\omega(1)\cdot h&=\Big([\frac{1}{f_1}]\wedge f_1\cdot\omega(1)\Big)(h)=\Big([\frac{1}{f_1}]\wedge\overline{\partial}\omega_{\{1\}}\Big)(h)\\&=-\Big(\overline{\partial}[\frac{1}{f_1}]\wedge\omega_{\{1\}}\Big)(h)\,.\end{aligned}$$

because $\overline{\partial}h=\overline{\partial}f_1=0$.

Similarly, for any $k\in\{1,\ldots,p\}$,

$$\begin{aligned}&\Big(\overline{\partial}\frac{1}{f_1}\wedge\ldots\wedge\overline{\partial}\frac{1}{f_{k-1}}\wedge\omega_{\{1,\ldots,k-1\}}\Big)(h)=\\&=\Big(\overline{\partial}\frac{1}{f_1}\wedge\ldots\wedge\overline{\partial}\frac{1}{f_{k-1}}\wedge\Big[\frac{1}{f_k}\Big]\wedge f_k\cdot\omega_{\{1,\ldots,k-1\}}\Big)(h)=\\&=\Big(\overline{\partial}\frac{1}{f_1}\wedge\ldots\wedge\overline{\partial}\frac{1}{f_{k-1}}\wedge\Big[\frac{1}{f_k}\Big]\wedge(-1)^{k-1}\overline{\partial}\omega_{\{1,\ldots,k\}}\Big)(h)+\\&+\sum_{j<k}\Big(\overline{\partial}\frac{1}{f_1}\wedge\ldots\wedge\overline{\partial}\frac{1}{f_{k-1}}\wedge\Big[\frac{1}{f_k}\Big]\wedge(-1)^{j+k+1}f_j\cdot\omega_{\{1,\ldots,\widehat{j},\ldots,k\}}\Big)(h)=\\&=(-1)^{k-1}(-1)^k\Big(\overline{\partial}\frac{1}{f_1}\wedge\ldots\wedge\overline{\partial}\frac{1}{f_k}\wedge\omega_{\{1,\ldots,k\}}\Big)(h)+0=\\&=-\Big(\overline{\partial}\frac{1}{f_1}\wedge\ldots\wedge\overline{\partial}\frac{1}{f_k}\wedge\omega_{\{1,\ldots,k\}}\Big)(h),\end{aligned}$$

which proves (3.9).

Taking into account (3.1), each class in $H^{n-p}_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I}))$ is represented by a form $\varphi \in \Gamma_c(V, \mathcal{E}^{n,n-p})$ with $\overline{\partial}\varphi = \sum f_i \psi_i$, for some $\underline{\psi_i \in \Gamma_c(V, \mathcal{E}^{n,n-p+1})}$. Let φ be any such form; by (3.8), there exists a peripherical form ω_φ such that $\overline{\varphi} = \omega_\varphi(e_1 \wedge \ldots \wedge e_p)$.

By (3.7), (3.8) and (3.9),

$$\begin{array}{ccc} H^{n-p}_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I})) & \to & I^\perp \\ \overline{\varphi} & \mapsto & \big(\overline{\partial}(1/f) \wedge \omega_\varphi(e_1 \wedge \ldots \wedge e_p)\big)_{|\mathcal{O}(V)} \end{array}$$

is an isomorphism.

On the other side, by (3.1), $\varphi - \omega_\varphi(e_1 \wedge \ldots \wedge e_p) = \overline{\partial}\psi + \sum f_i \varphi_i$, where $\psi \in \Gamma_c(V, \mathcal{E}^{n,n-p-1})$ and $\varphi_i \in \Gamma_c(V, \mathcal{E}^{n,n-p})$, $\forall i = 1, \ldots, p$. Now,

$$(a) \qquad \Big\langle \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \overline{\partial}\psi, h \Big\rangle = \pm \Big\langle \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \psi, \overline{\partial}h \Big\rangle = 0\,,$$

for any $h \in \mathcal{O}(V)$ and

$$(b) \qquad \Big\langle \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p), \sum f_i \varphi_i \Big\rangle = 0$$

since the residue current is annihilated by the ideal $(f_1, \ldots, f_p)$.

Finally, we deduce that

$$\begin{aligned} &\big(\overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \omega_\varphi(e_1 \wedge \ldots \wedge e_p)\big)_{|\mathcal{O}(V)} = \\ &= \big(\overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \varphi\big)_{|\mathcal{O}(V)}\,, \end{aligned}$$

which proves the theorem. $\diamondsuit$

Corollary 3.5. *Let V be an n-dimensional Stein manifold and $I \subseteq \mathcal{O}(V)$ generated by a complete intersection $f_1, \ldots, f_p \in \mathcal{O}(V)$. For any $T \in I^\perp$ and any open Stein neighborhood Ω of a carrier K of T, there exists a $\mathcal{C}^\infty$ $(n, n-p)$-form φ_T with compact support contained in Ω such that*

$$T(h) = \Big\langle \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \varphi_T, h \Big\rangle \quad \text{for all } h \in \mathcal{O}(V).$$

Moreover, φ_T can be chosen verifying:

$$\overline{\partial}\varphi_T = \sum_{i=1}^{p} f_i \cdot \varphi_i, \quad \text{for some } \varphi_i \in \Gamma_c(V, \mathcal{E}^{n,n-p+1})\,.$$

Proof. As T extends to an analytic functional over $\mathcal{O}(\Omega)$ by (2.5), the Corollary follows from Theorem 3.4 and the representation of

$$H^{n-p}_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I}))$$

given in Proposition 3.1. $\diamondsuit$

Remark 3.6. From the injectivity of the "residue" isomorphism

$$H^{n-p}_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I}) \longrightarrow I^\perp$$

in Theorem 3.4, we deduce that given $\varphi \in \Gamma_c(V, \mathcal{E}^{n,n-p})$ defining a cohomology class (i.e. such that $\overline{\partial}\varphi = \sum_{i=1}^p f_i \varphi_i$, for some $\varphi_i \in \Gamma_c(V, \mathcal{E}^{n,n-p+1})$, $i = 1, \ldots, p$), the restriction of the residue current $\overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \varphi$ to $\mathcal{O}(V)$ defines the zero functional iff there exist forms $\beta \in \Gamma_c(V, \mathcal{E}^{n,n-p-1})$, and $\beta_i \in \Gamma_c(V, \mathcal{E}^{n,n-p})$ $\forall i = 1, \ldots, p$, such that $\varphi = \overline{\partial}\beta + \sum_{i=1}^p f_i \beta_i$.

In fact,

$$\begin{array}{ccc} \mathcal{O}(V)/I \times H^{n-p}_{\overline{\partial}}(\Gamma_c(V, \mathcal{E}^{n,\bullet} \otimes \mathcal{O}/\mathcal{I}) & \longrightarrow & \mathbf{C} \\ \overline{h} \times \overline{\varphi} & \longmapsto & \Big(\overline{\partial}(1/f) \wedge \varphi\Big)(h) \end{array}$$

is a non-degenerated bilinear pairing.

We can also rediscover here the Duality Law for residue currents with holomorphic numerators ([11],[29]): Given $h \in \mathcal{O}(V)$, the condition $h \in I$ is equivalent to

$$\begin{aligned} \Big(h\overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p)\Big)(\varphi) &= \Big(\varphi\overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p)\Big)(h) \\ &= 0 \quad \forall \varphi \in \Gamma_c(V, \mathcal{E}^{n,n-p}) \end{aligned}$$

(or even for any such φ for which $\overline{\partial}\varphi$ is in the ideal generated by I in $\Gamma_c(V, \mathcal{E}^{n,n-p+1})$).

4. From complete intersections to locally Cohen–Macaulay ideals.

In this section, we extend the results of §3 to describe $I^\perp$ in terms of residue currents in case I is a closed ideal such that each local quotient $\mathcal{O}_x/\mathcal{I}_x$ is a Cohen–Macaulay ring of some fixed dimension. We recall that a local ring $\mathcal{O}_x/\mathcal{I}_x$ is said to be *Cohen–Macaulay* if it has a regular sequence of germs vanishing at x of length equal to $\dim(\mathcal{I}_x)$. In particular, if $\dim(\mathcal{I}_x) = 0$ or if $\mathcal{I}_x$ is a complete intersection, then $\mathcal{O}_x/\mathcal{I}_x$ is Cohen-Macaulay.

We give a simple proof for the case of dimension 0, using Taylor expansions of $\mathcal{C}^\infty$ functions in local coordinates around each point in $V(I)$. In the general Cohen-Macaulay case, the proof relies on cohomological methods.

The following lemma will be used to reduce this case to the complete intersection case previously studied.

Lemma 4.1. *Let V be a Stein manifold of dimension n and let $I \subseteq \mathcal{O}(V)$ be a proper closed ideal such that codim$V(I) = k \in \{1, \ldots, n\}$. There exist $f_1, \ldots, f_k \in I$ such that codim $\cap_{i=1}^k \{f_i = 0\} = k$.*

Proof. Let $f_1 \in I \setminus \{0\}$. For $k > 1$, given $1 \le j \le k-1$ and $f_1, \ldots, f_j \in I$ such that codim$\cap_{i=1}^j \{f_i = 0\} = j$, let N be a countable set such that $N \cap (Z_\nu \setminus \cap_{i=1}^j \{f_i = 0\}) \neq \emptyset$ for each irreducible component Z_ν of $\cap_{i=1}^j \{f_i = 0\}$. For any $x \in N$, $A_\nu := \{f \in I / f(x) \neq 0\}$ is an open dense subset of I, and then $A := \cap_\nu A_\nu$ in non void. For any $f_{j+1} \in A$, the sequence $\{f_1, \ldots, f_{j+1}\} \subseteq I$ is a complete intersection. $\diamondsuit$

We show in the next proposition that when dim$V(I) = 0$, each functional in $I^\perp$ is the restriction to $\mathcal{O}(V)$ of a residual current associated to any complete intersection ideal contained in I.

Proposition 4.2. *Let $I \subseteq \mathcal{O}(V)$ be a closed ideal such that dim$V(I) = 0$. For any $T \in I^\perp$ there exists a unique compactly supported residual current R_T such that:*

(i) $R_{T|\mathcal{O}(V)} = T$.

(ii) $h \cdot R_T = 0$ in $\Gamma_c(V, {}' \mathcal{D}^{n,n})$, for any $h \in I$.

Moreover, for any complete intersection ideal K contained in I with $V(K)$ discrete, generated by $f_1, \ldots, f_n$, there exists $\omega \in \Gamma_c(V, \mathcal{E}^{n,0})$ with $\overline{\partial}\omega \in K \cdot \Gamma_c(V, \mathcal{E}^{n,1})$ such that

$$R_T = \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_n) \wedge \omega \ .$$

Proof. By Lemma 4.1, there exists a complete intersection ideal $K \subseteq I$ of dimension 0 generated by a regular sequence $f_1, \dots, f_n$. Given a functional T orthogonal to I, T is also in $K^\perp$. By Theorem 3.4 (or Corollary 3.5), there exists $\omega \in \Gamma_c(V, \mathcal{E}^{n,0})$ with $\overline{\partial}\omega \in K \cdot \Gamma_c(V, \mathcal{E}^{n,1})$ such that the residual current $R_T := \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_n) \wedge \omega$ extends T and is annihilated by K as a current. It remains to show that R_T is also annihilated as a current by all elements in I.

Let $\varphi \in \Gamma_c(V, \mathcal{E}^{0,0})$ and $h \in I$. By a partition of unity argument, we may suppose that $\mathrm{supp}\,\varphi \cap \mathrm{V(K)} = \emptyset$ (and then $R_T(\varphi) = 0$), or that $\mathrm{supp}\,\varphi$ is contained in a small coordinate neighborhood Ω of a point $P_0 \in V(K)$ and $\mathrm{supp}\,\varphi \cap \mathrm{V(K)} = \{\mathrm{P_0}\}$. We assume then that there exist n global holomorphic functions $z_1, \dots, z_n$ defining a system of local coordinates in Ω and such that $z_i(P_0) = 0 \ \forall i = 1, \dots, n$. As R_T has compact support, it has finite order bounded by m. Consider the Taylor expansion of order m of $\varphi(z)$:

$$\varphi(z) = \varphi_1(z) + \varphi_2(z) + \varphi_3(z)$$

where

$$\begin{aligned}
\varphi_1(z) &= \sum_{\substack{\alpha \in \mathbf{N}^n \\ |\alpha| \leq m}} \frac{1}{\alpha!} \frac{\partial^\alpha \varphi}{\partial z^\alpha}(0)\ z^\alpha \\
\varphi_2(z) &= \sum_{j=1}^{n} \overline{z}_j\ \theta_j(z) \\
\varphi_3(z) &= \sum_{|\alpha| = m+1} K_\alpha(z)\ z^\alpha
\end{aligned} \tag{4.1}$$

for some θ_j and K_α in $\mathcal{E}^{0,0}(\Omega)$.

Now,

(i) $h \cdot R_T(\varphi_1) = R_T(h \cdot \varphi_1) = T(h \cdot \varphi_1) = 0$, because $\varphi_1 \in \mathcal{O}(V)$.

(ii) $h \cdot R_T(\varphi_2) = R_T(h \cdot \varphi_2) = 0$ because each function $\overline{z}_i$ is the conjugate of a holomorphic function vanishing at $\mathrm{supp} R_T \cap \Omega$ (cf. 1.7.5 (2) in [8]; it is in fact a consequence of (2.11)).

(iii) $h \cdot R_T(\varphi_3) = 0$ by the choice of m.

Then, $h \cdot R_T \equiv 0$, as wanted.

Suppose that $K' =< f_1', \dots, f_n' >$ is another complete intersection ideal of dimension 0 contained in I, and let $\omega' \in \Gamma_c(V, \mathcal{E}^{n,0})$ such that the residual current $R_T' := \overline{\partial}(1/f_1') \wedge \cdots \wedge \overline{\partial}(1/f_n') \wedge \omega'$ also coincides with T (and then with R_T) on $\mathcal{O}(V)$. Let $m \geq \max\{\mathrm{ord} R_T, \mathrm{ord} R_T'\}$. Again, by a partition of unity argument we may suppose that we are given a test $\mathcal{C}^\infty$ function φ such that $\mathrm{supp}\,\varphi \cap (\mathrm{V(K)} \cup \mathrm{V(K')}) = \emptyset$ (and then $R_T(\varphi) = 0 = R_T'(\varphi)$), or that $\mathrm{supp}\,\varphi$ is contained in a small coordinate neighborhood of a point in $V(K) \cup V(K')$. In this case, taking into account that $\mathrm{supp\,R_T} \subseteq \mathrm{V(K)}$, $\mathrm{supp\,R_T'} \subseteq \mathrm{V(K')}$, and expanding $\varphi = \varphi_1 + \varphi_2 + \varphi_3$ up to order m as in (4.1), we deduce by arguments similar to the previous one that $R_T(\varphi) = R_T'(\varphi)$. $\diamondsuit$

In the case of locally Cohen-Macaulay ideals with positive dimensional zero set, we still have a residue representative R_T for each $T \in I^\perp$ (see Theorem 4.4 below), but uniqueness is lost. This "non uniqueness" is already present in the complete intersection case. For example, consider $I =< z_1, \dots, z_{n-1} >$ in $V = \mathbf{C}^n$ and any $\varphi \in \Gamma_c(\mathbf{C}^n, \mathcal{E}^{n,0})$. Then, $\overline{\partial}(1/z_1) \wedge \cdots \wedge \overline{\partial}(1/z_{n-1}) \wedge \overline{\partial}\varphi$ is in general a non zero current, but its restriction to $\mathcal{O}(V)$ is the zero functional.

Theorem 4.4. *Let $I \subseteq \mathcal{O}(V)$ be a closed ideal such that each local quotient $\mathcal{O}_x/\mathcal{I}_x$ is a Cohen–Macaulay ring of some fixed dimension $n - p \in \{0, \dots, n\}$. For any $T \in I^\perp$ there exists a compactly supported residual current R_T such that:*

(i) $R_{T|\mathcal{O}(V)} = T$.

(ii) $h \cdot R_T = 0$ *in* $\Gamma_c(V,' \mathcal{D}^{n,n})$, *for any* $h \in I$.

Moreover, for any complete intersection ideal $K =< f_1, \ldots, f_p >$ contained in I, there exists $\omega \in \Gamma_c(V, \mathcal{E}^{n,n-p})$ with $\overline{\partial}\omega \in K \cdot \Gamma_c(V, \mathcal{E}^{n,n-p+1})$ such that the residue current $\overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \omega$ extends T.

Proof. By Lemma 4.1 there exists a complete intersection ideal K contained in I. Let $K =< f_1, \ldots, f_p >$ be any such ideal. As in particular $T \in K^{\perp}$, there exists by Corollary 3.5 a compactly supported form $\omega \in \Gamma_c(V, \mathcal{E}^{n,n-p})$ such that $\overline{\partial}\omega \in K \cdot \Gamma_c(V, \mathcal{E}^{n,n-p+1})$ and $R_{f,\omega} := \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge \omega$ coincides with T on $\mathcal{O}(V)$. This residue current $R_{f,\omega}$ is annihilated by K and we will show that in fact ω can be chosen in such a way that $h \cdot R_{f,\omega}(\varphi) = 0$, for any $h \in I$ and any test function φ. For any such ω, $R_{f,\omega}$ satisfies the required properties for R_T in the statement of the theorem.

Let $\mathcal{K}$ denote the coherent sheaf of (complete intersection) ideals associated to K. We know from Lemma 2.2 that $I^{\perp}$ (resp. $K^{\perp}$) can be naturally identified with $\mathrm{Ext}^n_{c,\mathcal{O}}(V; \mathcal{O}/\mathcal{I}, \Omega^n)$ (resp. $\mathrm{Ext}^n_{c,\mathcal{O}}(V; \mathcal{O}/\mathcal{K}, \Omega^n)$). The fact that each local quotient $\mathcal{O}_x/\mathcal{I}_x$ is Cohen–Macaulay of codimension p implies that $\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)_x = \mathrm{Ext}^q_{\mathcal{O}_x}(\mathcal{O}_x/\mathcal{I}_x, \Omega^n_x) = 0$ for $q \neq p, \forall x \in V$. We have then by (3.2) that $\mathrm{Ext}^n_{c,\mathcal{O}}(V;\, \mathcal{O}/\mathcal{I},\, \Omega^n) \simeq H^{n-p}_c(V, \mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n))$.

At the level of local rings, $\mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{K}, \Omega^n)_x$ can be identified with $(\mathcal{O}/\mathcal{K} \otimes \Omega^n)_x$ as was already observed in the proof of Proposition 3.2. The inclusion $\mathcal{K} \hookrightarrow \mathcal{I}$ induces an injection $\mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)_x \hookrightarrow \mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{K}, \Omega^n)_x$, and under the identification $\mathrm{Ext}^p_{\mathcal{O}_x}(\mathcal{O}_x/\mathcal{K}_x, \Omega^n_x) \simeq \mathcal{O}_x/\mathcal{K}_x \otimes \Omega^n_x$, the image of $\mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n)_x$ is just $[\mathcal{K}_x : \mathcal{I}_x]/\mathcal{K}_x \otimes \Omega^n_x$ (cf.[25], [32]), where as usual $[\mathcal{K}_x : \mathcal{I}_x]$ denotes the ideal quotient, i.e. all the germs sending any element in $\mathcal{I}_x$ into $\mathcal{K}_x$.

By the results in §3, $K^{\perp}$ may be regarded as $H^{n-p}_c(V; \mathcal{O}/\mathcal{K} \otimes \Omega^n)$ (cf. (3.3), and by the above paragraph, any element of $I^{\perp}$ (inside $K^{\perp}$) is in the image of $H^{n-p}_c(V; ([\mathcal{K} : \mathcal{I}]/\mathcal{K}) \otimes \Omega^n)$. By the identification in terms of global sections given in (3.1), any element in the image of $H^{n-p}_c(V; ([\mathcal{K} : \mathcal{I}]/\mathcal{K}) \otimes \Omega^n)$ is represented by a form $\omega \in [K : I] \cdot \Gamma_c(V, \mathcal{E}^{n,n-p})$. For any such ω , the residue current $R_{f,\omega}$ extends T by Theorem 3.4. Thus, given any $h \in I$, we know that $h \cdot \omega \in K \cdot \Gamma_c(V, \mathcal{E}^{n,n-p})$ and then,

$$h \cdot R_{f,\omega} = \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) \wedge (h \cdot \omega) = 0,$$

proving the theorem. $\diamondsuit$

5. An analytic description of the representation formulas.

In this section, we intend to make explicit some of the results proved with a cohomogical approach in the two preceeding sections. The first result we would like to illustrate that way is Corollary 3.5. We will start with some formulation of our results for V being some open pseudoconvex subset in $\mathbf{C}^n$. It is easy to extend these results to Stein manifolds. This can be done using Narasimhan-Forster result (see [14]), which allows us to consider the Stein manifold V as a submanifold in some $\mathbf{C}^N$, then using an holomorphic retraction on this manifold to extend holomorphic functions on V to the ambient space (see also [9].)

From now on V will be a pseudoconvex open subset of $\mathbf{C}^n$. We recall one basic consequence of Cartan's theorem A: given some holomorphic function f on V, one can find some holomorphic map

$$g = g_{[f]} = (g_{[f],1}, \ldots, g_{[f],n}) : V \times V \to \mathbf{C}^n$$

such that, for any $z \in V$, for any $\zeta \in V$,

$$f(z) - f(\zeta) = \sum_{k=1}^{n} g_{[f],k}(z, \zeta)(z_k - \zeta_k). \tag{5.1}$$

When V is convex, one can make explicit such a map $g_{[f]}$ thanks to Taylor integral formula

$$f(z) - f(\zeta) = \int_0^1 \left(\frac{d}{dt}[f(\zeta + t(z - \zeta))] \right) dt \; ;$$

when f is a polynomial, one can also use express $g_{[f]}$ as a polynomial map obtained with successive divided differences. For the sake of simplicity in our next computations, we will denote as $G_{[f]}$ the (1,0) differential form in $V \times V$

$$G_{[f]}(z,\zeta) := \sum_{k=1}^{n} g_{[f],k}(z,\zeta) d\zeta_k. \tag{5.2}$$

Such a form $G_{[f]}$ is called an *Hefer form* for f.

Let $(K_l)_{l=1}^{\infty}$ be some exhaustive sequence of compact subsets of V such that, for any $l \in \mathbf{N}^*$, $\widehat{[K_l]}_V \subset \overset{\circ}{K}_{l+1}$ and $\overset{\circ}{K}_l$ is a strictly pseudoconvex open subset with $\mathcal{C}^\infty$ boundary (for the existence of such a sequence $(K_l)_l$, see for example [22], corollary 1.5.11, here $\widehat{[K]}_V$ denotes the holomorphic hull of K). Let $l \in \mathbf{N}^*$; given K_l and its defining function ρ_l, one can construct a Henkin-Leiterer section

$$s^{[l]} : \overset{\circ}{K}_l \times \overset{\circ}{K}_l \to \mathbf{C}^n,$$

which is holomorphic in z, of arbitrary large regularity in ζ, and such that, for any compact $K \subset\subset K_l$, for any $z \in \overset{\circ}{K}$, for any $\zeta \in \overset{\circ}{K}$,

$$\Re(< s^{[l]}(z,\zeta), \zeta - z >) \geq \rho_l(\zeta) - \rho_l(z) + \delta(K)\|\zeta - z\|^2, \tag{5.3}$$

for some strictly positive constant $\delta(K)$. We will denote as $S^{[l]}$ the differential form on $\overset{\circ}{K}_l \times \overset{\circ}{K}_l$

$$S^{[l]}(z,\zeta) := \sum_{k=1}^{n} s_k^{[l]}(z,\zeta) d\zeta_k.$$

For each $l \in \mathbf{N}^*$, let $\varphi^{[l]}$ be an element in $\mathcal{D}(V)$, with support included in $\overset{\circ}{K}_{l+1}$, such that $\varphi^{[l]} \equiv 1$ in a neighborhood of $\widehat{[K_l]}_V$. We can further assume (from (5.3)) that for any z in some neighborhhood of $\widehat{[K_l]}_V$, for any ζ in the support of $\overline{\partial}\varphi^{[l]}$, $< s^{[l+1]}(z,\zeta), \zeta - z >$ does not vanish. We will also assume that $S^{[l+1]}$ is smooth for z in some neighborhood of $\widehat{[K_l]}_V$ and ζ in some neighborhood of the support of $\overline{\partial}\varphi^{[l]}$.

Finally, we remind in our setting Koppelman's representation formulas with weight factors, which are due to M. Andersson and B. Berndtsson ([1], [3]). Let

$$q : V \times V \mapsto \mathbf{C}^n, \ (z,\zeta) \mapsto (q_1(z,\zeta), \ldots, q_n(z,\zeta))$$

where the q_j are holomorphic in z and $\mathcal{C}^1$ in ζ. Let Φ be some entire function in one variable such that $\Phi(1) = 1$. For any positive integer m, let $\Gamma^{(m)}$ be the function on $V \times V$

$$\Gamma^{(m)} : \ (z,\zeta) \mapsto \left(\frac{d}{dt}\right)^m [\Phi]_{|t=<q(z,\zeta), z-\zeta>+1}.$$

Let also Q be the (1,0) differential form on $V \times V$ which is associated to q by

$$Q(z,\zeta) := \sum_{k=1}^{n} q_k(z,\zeta) d\zeta_k.$$

For any $h \in \mathcal{O}(V)$, for any $l \in \mathbf{N}^*$, one has the following representation formula for z in a neighborhood of $\widehat{[K_l]}_V$:

$$\begin{aligned} h(z) = h(z)\varphi^{[l]}(z) &= \frac{1}{n!(2i\pi)^n} \int_{\zeta \in V} h\varphi^{[l]}(\zeta)\Gamma^{(0)}(z,\zeta)(\overline{\partial}_\zeta Q)^n(z,\zeta) - \\ \frac{1}{(2i\pi)^n} \int_{\zeta\in V} h\overline{\partial}\varphi^{[l]}(\zeta) &\wedge \left(\sum_{m=0}^{n-1} \frac{\Gamma^{(m)} S^{[l+1]} \wedge (\overline{\partial}_\zeta S^{[l+1]})^{n-1-m} \wedge (\overline{\partial}_\zeta Q)^m(z,\zeta)}{m! < s^{[l+1]}(z,\zeta), \zeta - z >^{n-m}} \right) \end{aligned} \tag{5.4}$$

We need some additional notation. If U is some open subset of V, ω is an element in $\mathcal{E}^{p,q}(U \times U)$ of the form

$$\omega(z,\zeta) = \sum_{\substack{1\leq i_1<\cdots<i_p\leq n\\ 1\leq j_1<\cdots<j_q\leq n}} \omega_{I,J}(z,\zeta) d\zeta_I \wedge d\overline{\zeta}_J,$$

and $S \in {}'\mathcal{E}^{p,q}(U)$, we will denote as $< S_z, \omega(z,\cdot) >$ the differential form on U

$$< S_z, \omega(z,\cdot) >:\ \zeta \mapsto \sum_{\substack{1\leq i_1<\cdots<i_p\leq n\\ 1\leq j_1<\cdots<j_q\leq n}} < S_z, \omega_{I,J}(z,\zeta) > d\zeta_I \wedge d\overline{\zeta}_J.$$

In particular, when $S = \mu$, where μ is a measure with compact support in U,

$$\int_U \omega(z,\zeta) d\mu(z) := \sum_{\substack{1\leq i_1<\cdots<i_p\leq n\\ 1\leq j_1<\cdots<j_q\leq n}} \left(\int_U \omega_{I,J}(z,\zeta) d\mu(z) \right) d\zeta_I \wedge d\overline{\zeta}_J.$$

We now are ready to state our results about the description of the orthogonal of some ideal I in $\mathcal{O}(V)$ which is defined as a complete intersection. In our first statement, we deal with the zero dimensional case.

Proposition 5.1. *Let $f_1, \dots, f_n$ define a complete intersection in $\mathcal{O}(V)$. Let T be some element in $\mathcal{O}'(V)$ such that, for any $j \in \{1, \dots, n\}$, $f_j T \equiv 0$ (as an analytic functional). There exist $l_0 \in \mathbf{N}^*$, and a Radon measure μ_0 with support in K_{l_0}, such that, for any $h \in \mathcal{O}(V)$,*

$$T(h) =< \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_n), h\omega_T >, \tag{5.5}$$

where ω_T is the element in $\mathcal{D}^{n,0}(V)$ defined as

$$\omega_T(\zeta) = (-1)^{n(n-1)/2} \varphi^{[l_0]}(\zeta) \int_V \left(\bigwedge_{j=1}^n G_j(z,\zeta) \right) d\mu_0(z), \tag{5.6}$$

the $G_j = \sum_k g_{j,k} d\zeta_k$, $1 \leq j \leq n$, being Hefer forms respectively for $f_1, \dots, f_n$.

Before giving the proof of this statement, let us complete it with the following remark.

Remark 5.2. From our point of view, the functions $\varphi^{[l]}$, $l \in \mathbf{N}^*$, were fixed *a priori* (that is independently of the functional T one wants to represent). For a given T, we can replace $\varphi^{[l]}$ by any element $\varphi_T^{[l]}$ in $\mathcal{D}(\overset{\circ}{K}_{l+1})$ which coincides with $\varphi^{[l]}$ on $\widehat{[K_l]}_V$ and is identically equal to 0 near each common zero of $f_1, \dots, f_n$ in $K_{l+1} \setminus \widehat{[K_l]}_V$ (these zeroes are isolated). Formula (5.5) remains valid if in ω_T one replaces $\varphi^{[l_0]}$ by $\varphi_T^{[l_0]}$, which gives $\tilde{\omega}_T$ instead of ω_T. So we have also the representation formula

$$T(h) =< \overline{\partial}(1/f_1) \wedge \cdots \overline{\partial}(1/f_n), h\tilde{\omega}_T >, \tag{5.7}$$

where now $\tilde{\omega}_T$ satisfies $\overline{\partial}\tilde{\omega}_T \equiv 0$ in some neighborhood of $\{f_1 = \cdots = f_n = 0\}$ (since the G_j have holomorphic coefficients in z *and* ζ). Such a representation formula (5.7) (with the additional condition that $\overline{\partial}\omega_T$ is in the ideal generated by the f_j's in $\mathcal{E}^{n,1}(V)$, which is clearly realized here for $\tilde{\omega}_T$, since this form is $\overline{\partial}$ closed near $\{f_1 = \cdots = f_n = 0\}$), fits with Corollary 3.5 in the zero dimensional case.

Proof of Proposition 5.1. As we have already seen in the proof ot Theorem 3.4, we may assume that $f_1, \dots, f_n$ define a regular sequence in $\mathcal{O}(V)$ whatever the order is (that is, for any $1 \leq k \leq n$, for any subet $J \subset \{1, \cdots, n\}$ such that $|J| = k$, $\mathrm{codim}_V\{f_j = 0; j \in J\} = k$). Let $T \in \mathcal{O}'(V)$. Since the $(K_l)_{l=1}^\infty$ define an exhaustion of V, it follows from Hahn-Banach theorem that there exists some $l_0 \in \mathbf{N}^*$, some Radon measure μ_0 supported by K_{l_0} such that, for any $h \in \mathcal{O}(V)$,

$$< T, h >= \int_{K_{l_0}} h(z) d\mu_0(z). \tag{5.8}$$

We now represent any holomorphic function $h \in \mathcal{O}(V)$ using formula (5.4) for some adapted choice of q and Φ. This is inspired from M. Passare and B. Berndtsson works ([3], [7], [29], [36]). We take here the approach which was proposed in [4] (see also [5], chapter 3). In fact q depends on a complex parameter λ such that, at the beginning $\Re(\lambda) >> 0$. Instead of defining $q = q_\lambda$, it seems easier to plot the associated (1,0) differential form on $V \times V$, namely

$$Q(z,\zeta) = Q_\lambda(z,\zeta) := \frac{1}{n}\sum_{j=1}^n |f_j(\zeta)|^{2(\lambda-1)}\overline{f_j(\zeta)}\sum_{k=1}^n g_{j,k}(z,\zeta)d\zeta_k.$$

The key fact is that

$$< q_\lambda(z,\zeta), z-\zeta > +1 = \left(1 - \frac{1}{n}\sum_{j=1}^n |f_j(\zeta)|^{2\lambda}\right) + \frac{1}{n}\sum_{j=1}^n |f_j(\zeta)|^{2\lambda}\overline{f_j(\zeta)}f_j(z).$$

The entire function Φ we choose is the polynomial

$$\Phi(t) = \frac{1}{n!}\prod_{j=0}^{n-1}(nt-j).$$

For $z \in K_{l_0}$, one can represent $h(z)\varphi^{[l_0]}(z)$ thanks to formula (5.4). So

$$\begin{aligned} &(2i\pi)^n h(z) = (2i\pi)^n h(z)\varphi^{[l_0]}(z) = \\ &= \frac{1}{n!}\int_{\zeta\in V}\varphi^{[l_0]}h(\zeta)\Gamma^{(0)}(z,\zeta)(\overline{\partial}_\zeta Q_\lambda)^n(z,\zeta) - \\ &- \int_{\zeta\in V} h\overline{\partial}\varphi^{[l_0]} \wedge \left(\sum_{m=0}^{n-1}\frac{\Gamma^{(m)}S^{[l_0+1]}\wedge(\overline{\partial}_\zeta S^{[l_0+1]})^{n-1-m}\wedge(\overline{\partial}_\zeta Q_\lambda)^m(z,\zeta)}{m! < s^{[l_0+1]}(z,\zeta), \zeta - z >^{n-m}}\right) \end{aligned} \tag{5.9}$$

The first integral in the right handside of (5.9), as one can check immediately, equals

$$\begin{aligned} &\Theta_{h,l_0}(\lambda) = \\ &= (-1)^{n(n-1)/2}\lambda^n\int_V h\varphi^{[l_0]}|f_1\cdots f_n|^{2(\lambda-1)}\left(\bigwedge_{j=1}^n\overline{\partial f_j(\zeta)}\right)\wedge\left(\bigwedge_{j=1}^n G_j(z,\zeta)\right) \end{aligned} \tag{5.10}$$

As a function of the complex parameter λ, it is known (from [2]) that Θ_{h,l_0} can be continued up to the whole complex plane as a meromorphic function of λ; in fact, as shown in ([5], chapter 3, Proposition 3.6 and Theorem 3.18), all the poles of Θ_{h,l_0} are in $\{\Re(\lambda) < 0\}$ and the value at $\lambda = 0$ is

$$\Theta_{h,l_0}(0) = (-1)^{n(n-1)/2}\left\langle\overline{\partial}(1/f_1)\wedge\cdots\wedge\overline{\partial}(1/f_n), h\varphi^{[l_0]}\bigwedge_{j=1}^n G_j(z,\cdot)\right\rangle.$$

Inspired by this idea, one can take in (5.9), as in [5], the analytic continuation of both sides (as meromorphic functions of the complex parameter λ) and compare the values obtained at $\lambda = 0$. Thanks to the choice of Φ, the formula that one obtains at this stage (see [5], pages 74 to 79) is, for $z \in K_{l_0}$, of the form

$$h(z) = h(z)\varphi^{[l_0]}(z) = \sum_{j=1}^n f_j(z)h_j(z) + r(z), \tag{5.11}$$

where the h_j are holomorphic in some neighborhood of $[\widehat{K_{l_0}}]_V$ (since $< s^{[l_0]}(z,\zeta), \zeta - z >$ does not vanish for z in some neighborhood of $[\widehat{K_{l_0}}]_V$ and ζ in the support of $\overline{\partial}\varphi^{[l_0]}$) and

$$r(z) = (-1)^{n(n-1)/2} \Big\langle \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_n), h\varphi^{[l_0]} \bigwedge_{j=1}^{n} G_j(z,\cdot) \Big\rangle.$$

Since holomorphic functions in some neighborhood of $[\widehat{K_{l_0}}]_V$ can be uniformly approximated on K_{l_0} by elements in $\mathcal{O}(V)$ ([23], Theorem 2.6.11 and Lemma 4.3.1), the fact that T, as a functional, is orthogonal to $\sum f_j \mathcal{O}(V)$ implies

$$\sum_{j=1}^{n} \int_{K_{l_0}} f_j(z) h_j(z) d\mu_0(z) = 0.$$

Therefore, it follows from (5.11)

$$< T, h > = \int_{K_{l_0}} h(z) d\mu_0(z) = \int_{K_{l_0}} r(z) d\mu_0(z).$$

We finally use Fubini theorem in order to get

$$\begin{aligned} &\int_{K_{l_0}} r(z) d\mu_0(z) = \\ &= (-1)^{n(n-1)/2} \int_{K_{l_0}} \Big\langle \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_n), h\varphi^{[l_0]} \bigwedge_{j=1}^{n} G_j(z,\cdot) \Big\rangle d\mu_0(z) = \\ &= < \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_n), h\omega_T >, \end{aligned}$$

where

$$\omega_T(\zeta) = (-1)^{n(n-1)/2} \varphi^{[l_0]}(\zeta) \int_V \left(\bigwedge_{j=1}^{n} G_j(z,\zeta) \right) d\mu_0(z).$$

This concludes the proof of the proposition. $\diamondsuit$

For non zero dimensional complete intersections, we have the following result:

Proposition 5.3. *Let $f_1, \ldots, f_n$ define a complete intersection in $\mathcal{O}(V)$. Let T be some element in $\mathcal{O}'(V)$ such that, for any $j \in \{1, \ldots, p\}$, $f_j T \equiv 0$ (as an analytic functional). There exists $l_0 \in \mathbf{N}^*$, μ_0 a Radon measure with support in K_{l_0}, such that, for any $h \in \mathcal{O}(V)$, one has*

$$< T, h > = < \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p), h\eta_T >,$$

where η_T is the $(n, n-p)$ form

$$\begin{aligned} &\eta_T(\zeta) = \\ &= -C_p \overline{\partial}\varphi^{[l_0]}(\zeta) \wedge \int_{K_{l_0}} \left(\frac{S^{[l_0+1]} \wedge (\overline{\partial}_\zeta S^{[l_0+1]})^{n-p-1}(z,\zeta)}{< s^{[l_0+1]}(z,\zeta), \zeta - z >^{n-p}} \right) \wedge G(z,\zeta) d\mu_0(z) \end{aligned}$$

where

$$G(z,\zeta) := \bigwedge_{j=1}^{p} G_j(z,\zeta), \quad C_p := \frac{(-1)^{\frac{p(p-1)}{2}}}{(2i\pi)^{n-p}}$$

the $G_j = \sum_k g_{j,k} d\zeta_k$ being Hefer forms attached to $f_1, \ldots, f_p$. Moreover, there are p elements in $\mathcal{D}^{n,n-p+1}(V)$, $\eta_{T,1}, \ldots, \eta_{T,p}$, such that

$$\overline{\partial}\eta_T(\zeta) = \sum_{j=1}^{p} f_j(\zeta) \eta_{T,j}(\zeta). \tag{5.12}$$

Remark 5.4. This representation formula, with the additional property (5.12), fits with the statement in Corollary 3.5 as well as in the zero dimensional case.

Proof of Proposition 5.3. The proof follows the same lines of the proof of Proposition 5.1. Again, we may assume that $f_1, \dots, f_p$ define a regular sequence in $\mathcal{O}(V)$ whatever the order is (that is, for any $1 \leq k \leq p$, for any subet $J \subset \{1, \cdots, p\}$ such that $|J| = k$, $\mathrm{codim}_V\{f_j = 0; j \in J\} = k$). As before, there exist some $l_0 \in \mathbf{N}^*$, and some Radon measure μ_0 supported by K_{l_0} such that, for any $h \in \mathcal{O}(V)$,

$$< T, h >= \int_{K_{l_0}} h(z) d\mu_0(z).$$

We represent h in K_{l_0} using formula (5.4) for a convenient choice of q and Φ; q will at the beginning depend on a complex parameter λ such that $\Re(\lambda) >> 0$; let's express $q = q_\lambda$ by plotting its associated $(1,0)$ differential form

$$Q(z,\zeta) = Q_\lambda(z,\zeta) := \frac{1}{p} \sum_{j=1}^{p} |f_j(\zeta)|^{2(\lambda-1)} \overline{f_j(\zeta)} \sum_{k=1}^{n} g_{j,k}(z,\zeta) d\zeta_k.$$

We have, as before,

$$< q_\lambda(z,\zeta), z - \zeta > +1 = \left(1 - \frac{1}{p} \sum_{j=1}^{p} |f_j(\zeta)|^{2\lambda} \right) + \frac{1}{p} \sum_{j=1}^{p} |f_j(\zeta)|^{2\lambda} \overline{f_j(\zeta)} f_j(z).$$

The entire function Φ we now choose is the polynomial

$$\Phi(t) = \frac{1}{p!} \prod_{j=0}^{p-1} (pt - j).$$

For $z \in K_{l_0}$, one can represent $h(z)\varphi^{[l_0]}(z)$ thanks to formula (5.4). Since it is immediate that, for $m > p$ (and in particular, for $m = n$)

$$(\overline{\partial}_\zeta Q_\lambda)^m = \left(\frac{\lambda}{p}\right)^m \left(\sum_{j=1}^{p} |f_j(\zeta)|^{2(\lambda-1)} \overline{\partial f_j} \wedge G_j(z,\zeta) \right)^m = 0,$$

one has from (5.4), for z in some neighborhhod of $[\widehat{K_{l_0}}]_V$,

$$\begin{aligned} &(2i\pi)^n h(z) = \\ &- \int_{\zeta \in V} h \overline{\partial} \varphi^{[l_0]} \wedge \left(\sum_{m=0}^{p} \frac{\Gamma^{(m)} S^{[l_0+1]} \wedge (\overline{\partial}_\zeta S^{[l_0+1]})^{n-1-m} \wedge (\overline{\partial}_\zeta Q_\lambda)^m(z,\zeta)}{m! < s^{[l_0+1]}(z,\zeta), \zeta - z >^{n-m}} \right). \end{aligned}$$

This can be written as

$$\begin{aligned} &(2i\pi)^n h(z) = \\ &= -(-1)^{p(p-1)/2} \lambda^p \int_{\zeta \in V} h |f|^{2(\lambda-1)} \overline{\partial f} \wedge G(z,\zeta) \wedge \overline{\partial} \varphi^{[l_0]} \wedge \Psi_{l_0}(z,\zeta) - \\ &- \int_{\zeta \in V} h \overline{\partial} \varphi^{[l_0]} \wedge \left(\sum_{m=0}^{p-1} \frac{\Gamma^{(m)} S^{[l_0+1]} \wedge (\overline{\partial}_\zeta S^{[l_0+1]})^{n-1-m} \wedge (\overline{\partial}_\zeta Q_\lambda)^m(z,\zeta)}{m! < s^{[l_0+1]}(z,\zeta), \zeta - z >^{n-m}} \right) \end{aligned} \tag{5.14}$$

where

$$\Psi_{l_0}(z,\zeta) := \frac{S^{[l_0+1]}(z,\zeta) \wedge (\overline{\partial}_\zeta S^{[l_0+1]})^{n-1-p}}{< s^{[l_0+1]}(z,\zeta), \zeta - z >^{n-p}}.$$

and

$$\overline{\partial f} := \bigwedge_{j=1}^{p} \overline{\partial f_j}$$
$$|f| := |f_1 \cdots f_p|\,.$$

When z is fixed in some neighborhood of $[\widehat{K_{l_0}}]_V$, all the forms

$$\zeta \to \overline{\partial}\varphi^{[l_0]} \wedge \frac{S^{[l_0+1]} \wedge (\overline{\partial}_\zeta S^{[l_0+1]})^{n-1-m} \wedge (\overline{\partial}_\zeta Q_\lambda)^m(z,\zeta)}{m! < s^{[l_0+1]}(z,\zeta), \zeta - z >^{n-m}},\ m = 0,\dots,p$$

(among which is $\overline{\partial}\varphi^{[l_0]} \wedge \Psi_{l_0}(z,\cdot)$) have coefficients in $\mathcal{D}(V)$. This follows from the fact that (5.3) (with $l = l_0 + 1$) is satisfied for z in some neighborhood of $[\widehat{K_{l_0}}]_V$ and z in the support of $\overline{\partial}\varphi^{[l_0]}$. As in the zero dimensional case, it can be shown that the function

$$\lambda \overset{\Xi_{h,l_0}}{\mapsto} -(-1)^{p(p-1)/2} \frac{\lambda^p}{(2i\pi)^n} \int_{\zeta \in V} h|f|^{2(\lambda-1)} \overline{\partial f} \wedge \overline{\partial}\varphi^{[l_0]} \wedge \Psi_{l_0}(z,\zeta) \wedge G(z,\zeta)$$

can be continued as a meromorphic function in the whole complex plane; its poles (as shown in [5], chapter 3, Proposition 3.6 and Theorem 3.18), are in $\Re(\lambda) < 0$ and

$$\Xi_{h,l_0}(0) = -\frac{(-1)^{p(p-1)/2}}{(2i\pi)^{n-p}} \Big\langle \bigwedge_{j=1}^{p} \overline{\partial}(1/f_j), h(\cdot)\overline{\partial}\varphi^{[l_0]} \wedge \Psi_{l_0}(z,\cdot) \wedge G(z,\cdot) \Big\rangle\,.$$

It is clear again (from Atiyah's theorem [2]) that both sides of (5.14) (considered as functions of the complex parameter λ) have analytic continuations to $\mathbf{C}$ as meromorphic functions. We take the analytic continuations of both sides and identify their values at $\lambda = 0$. Thanks to the choice of Φ, one can show (exactly as in pages 74 to 79 in [5]) that one gets from (5.14), for z in some neighborhhood of $[\widehat{K_{l_0}}]_V$, a formula of the form

$$h(z) = \sum_{j=1}^{p} h_j(z) f_j(z) + r(z),$$

where the h_j are holomorphic in this neighborhood and

$$r(z) = -\frac{(-1)^{p(p-1)/2}}{(2i\pi)^{n-p}} \Big\langle \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p), h(\cdot)\overline{\partial}\varphi^{[l_0]} \wedge \Psi_{l_0}(z,\cdot) \wedge G(z,\zeta) \Big\rangle\,.$$

As in the preceeding proof, the hypothesis on T implies

$$< T, h >= \int_{K_{l_0}} h(z) d\mu_0(z) = \int_{K_{l_0}} r(z) d\mu_0(z).$$

Applying Fubini theorem as at the end of the proof of Proposition 5.1 leads to

$$< T, h >=< \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p), h\eta_T >,$$

where η_T is the $(n, n-p)$ form

$$\begin{aligned}
&\eta_T(\zeta) = \\
&= -C_p \overline{\partial}\varphi^{[l_0]}(\zeta) \wedge \int_{K_{l_0}} \left(\frac{S^{[l_0+1]} \wedge (\overline{\partial}_\zeta S^{[l_0+1]})^{n-p-1}(z,\zeta)}{< s^{[l_0+1]}(z,\zeta), \zeta - z >^{n-p}} \right) \wedge G(z,\zeta) d\mu_0(z)
\end{aligned}$$

We now are left with the proof of (5.12). For z in some neighborhhood U of $[\widehat{K_{l_0}}]_V$ and ζ in some neighborhood $\widetilde{U}$ of $Supp(\overline{\partial}\varphi^{[l_0]})$, one has

$$< s^{[l_0+1]}(z,\zeta), \zeta - z > \neq 0.$$

Let us write, on $U \times \widetilde{U}$,

$$\Sigma(z,\zeta) := \frac{S^{[l_0+1]}(z,\zeta)}{< s^{[l_0+1]}(z,\zeta), \zeta - z >} = \sum_{k=1}^{n} \sigma_k(z,\zeta) d\zeta_k.$$

It is clear than on $U \times \widetilde{U}$,

$$\sum_{k=1}^{n} \sigma_k(\zeta_k - z_k) \equiv 1,$$

which implies, for $(z,\zeta) \in U \times \widetilde{U}$,

$$\sum_{k=1}^{n} (\zeta_k - z_k) \overline{\partial}_\zeta \sigma_k(z,\zeta) \equiv 0. \tag{5.15}$$

Let us state now the following technical lemma.

Lemma 5.5. *Let $U \times \widetilde{U}$ a product of open sets in $\mathbf{C}^n \times \mathbf{C}^n$. Let $\sigma = (\sigma_1, \dots, \sigma_n)$ a $\mathcal{C}^\infty$ map from $U \times \widetilde{U}$ in $\mathbf{C}^n$ such that*

$$\sum_{k=1}^{n} (\zeta_k - z_k)\sigma_k(z,\zeta) \equiv 1, \ \ (z,\zeta) \in U \times \widetilde{U} \tag{5.16}$$

and the σ_k are holomorphic with respect to the z variables. Let $f_1, \dots, f_p$ be p functions holomorphic in U and $(g_{j,k})_{\substack{1\le j\le p\\ 1\le k\le n}}$ a matrix of holomorphic functions in $U \times \widetilde{U}$ such that, for any $j \in \{1, \dots, p\}$, one has

$$f_j(\zeta) - f_j(z) = \sum_{k=1}^{n} g_{j,k}(z,\zeta)(\zeta_k - z_k), \ \ (z,\zeta) \in U \times \widetilde{U}. \tag{5.17}$$

Let g_j, $j = 1, \dots, p$ the differential forms

$$g_j(z,\zeta) := \sum_{k=1}^{n} g_{j,k}(z,\zeta) d\zeta_k.$$

For any ordered subset $J \subset \{1, \dots, p\}$, such that $\#J = m$, $1 \le m \le p$, one can find a list of forms U_j^J, $j \in J$, $U_{j,l}^J$, $j, l \in J$, $j \neq l$, $U_{j,l,r}^J$, $j, l, r \in J$, $l \neq j$, $r \neq j, l$, and so on, with $\mathcal{C}^\infty$ coefficients holomorphic with respect to the z variables, such that

$$\left(\sum_{k=1}^{n} \overline{\partial}_\zeta \sigma_k(z,\zeta) \wedge d\zeta_k\right)^{n-m} \wedge \left(\bigwedge_{j\in J} g_j(z,\zeta)\right) = \sum_{j\in J} (f_j(\zeta) - f_j(z)) U_j^J$$

$$\overline{\partial}_\zeta U_j^J = \sum_{\substack{l\in J\\ l\neq j}} (f_l(\zeta) - f_l(z)) U_{j,l}^J(z,\zeta), \ \ j \in J$$

$$\overline{\partial}_\zeta U_{j,l}^J = \sum_{\substack{r\in J\\ r\neq j,l}} (f_r(\zeta) - f_r(z)) U_{j,l,r}^J(z,\zeta), \ \ j, l \in J, \ j \neq l, \cdots$$

Proof. The proof is a proof by induction on $m = \#J$. We start with $m = 1$. From the condition (5.16) on σ, it is clear that for any $z \in U$, for any $\zeta \in \widetilde{U}$, $z \neq \zeta$. For any point (z,ζ) in $U \times \widetilde{U}$, there exists $t \in \{1, \dots, n\}$

such that $z_t - \zeta_t$ remains different from zero in a neightborhood. Fix $t \in \{1, \dots, n\}$ and consider a point in $U \times \widetilde{U}$ such that, in a neighborhood of (z, ζ), $\zeta_t \neq z_t$ (for some $t \in \{1, \dots, n\}$). In thi neighborhood of (z, ζ), one can write

$$\begin{aligned}
&\left(\sum_{k=1}^{n} \overline{\partial}_\zeta \sigma_k \wedge d\zeta_k\right)^{n-1} \wedge g_1(z,\zeta) = \\
&= \Big(\sum_{\substack{1\le k\le n\\ k\neq t}} \overline{\partial}_\zeta \sigma_k \wedge \big(d\zeta_k - \frac{\zeta_k - z_k}{\zeta_t - z_t} d\zeta_t\big)\Big)^{n-1} \wedge g_1 \\
&= (-1)^{\frac{(n-1)(n-2)}{2}} (n-1) \Big(\bigwedge_{\substack{1\le k\le n\\ k\neq t}} \overline{\partial}_\zeta \sigma_k\Big) \wedge \Big(\bigwedge_{\substack{1\le k\le n\\ k\neq t}} (d\zeta_k - \frac{\zeta_k - z_k}{\zeta_t - z_t} d\zeta_t)\Big) \wedge g_1 \\
&= (-1)^{\frac{n-1)(n-2)}{2} + n - t} (n-1) \Big(\bigwedge_{\substack{1\le k\le n\\ k\neq t}} \overline{\partial}_\zeta \sigma_k\Big) \wedge \big(\Delta(z,\zeta) d\zeta_1 \wedge \cdots d\zeta_n\big)
\end{aligned}$$

where $\Delta(z, \zeta)$ is the determinant

$$\Delta(z,\zeta) := \begin{vmatrix}
1 & 0 & . & 0 & g_{1,1}(z,\zeta) & 0 & . & . & 0 \\
0 & 1 & . & 0 & g_{1,2}(z,\zeta) & 0 & . & . & 0 \\
0 & 0 & 1 & . & . & . & . & . & 0 \\
. & . & . & . & . & . & . & . & . \\
-\frac{\zeta_1 - z_1}{\zeta_t - z_t} & . & . & . & g_{1,t}(z,\zeta) & . & . & . & -\frac{\zeta_n - z_n}{\zeta_t - z_t} \\
0 & . & . & . & . & 1 & 0 & . & 0 \\
. & . & . & . & . & . & . & . & . \\
0 & . & . & 0 & g_{1,n-1}(z,\zeta) & 0 & . & 1 & 0 \\
0 & . & . & 0 & g_{1,n}(z,\zeta) & 0 & . & . & 1
\end{vmatrix}.$$

Note that this determinant is preserved if one adds to the row L_t with index t the linear combination

$$\sum_{\substack{1\le k\le n\\ k\neq t}} \frac{\zeta_k - z_k}{\zeta_t - z_t} L_k.$$

If one uses formula (5.17), one can see that in this new determinant, the line L_t is now replaced by the line

$$\widetilde{L}_t = (0, \cdots, 0, \frac{f_1(\zeta) - f_1(z)}{\zeta_t - z_t}, 0, \cdots, 0)$$

where the single nonzero term is at position t. Therefore, one has

$$(-1)^{n-t} \Delta(z,\zeta) \bigwedge_{1\le k\le n} d\zeta_k = \frac{f_1(\zeta) - f_1(z)}{\zeta_t - z_t} \Big(\bigwedge_{\substack{1\le k\le n\\ k\neq t}} d\zeta_k\Big) \wedge d\zeta_t\,,$$

so that

$$\begin{aligned}
&\Big(\sum_{\substack{1\le k\le n\\ k\neq t}} \overline{\partial}_\zeta \sigma_k \wedge \Big(d\zeta_k - \frac{\zeta_k - z_k}{\zeta_t - z_t} d\zeta_t\Big)\Big)^{n-1} \wedge g_1 = \\
&= \Big(\frac{f_1(\zeta) - f_1(z)}{\zeta_t - z_t}\Big) (-1)^{\frac{(n-1)(n-2)}{2}} (n-1) \Big(\bigwedge_{\substack{1\le k\le n\\ k\neq t}} \overline{\partial}_\zeta \sigma_k\Big) \wedge \Big(\bigwedge_{\substack{1\le k\le n\\ k\neq t}} d\zeta_k\Big) \wedge d\zeta_t \\
&= \Big(\frac{f_1(\zeta) - f_1(z)}{\zeta_t - z_t}\Big) \Big(\sum_{\substack{1\le k\le n\\ k\neq t}} \overline{\partial}_\zeta \sigma_k \wedge d\zeta_k\Big)^{n-1} \wedge d\zeta_t. \qquad (5.18)
\end{aligned}$$

This formula can be rewritten as

$$\left(\sum_{k=1}\overline{\partial}_\zeta\sigma_k\wedge d\zeta_k\right)^{n-1}\wedge g_1(z,\zeta)=\left(\frac{f_1(z)-f_1(\zeta)}{\zeta_t-z_t}\right)\Xi_t(z,\zeta). \tag{5.19}$$

Outside the set F_t where $(\zeta_t-z_t)\sigma_t(z,\zeta)=0$, we have

$$\left(\sum_{k=1}\overline{\partial}_\zeta\sigma_k\wedge d\zeta_k\right)^{n-1}\wedge g_1(z,\zeta)=\left(\frac{f_1(z)-f_1(\zeta)}{\sum_{t=1}^n(\zeta_t-z_t)\sigma_t(z,\zeta)}\right)\sigma_t(z,\zeta)\Xi_t(z,\zeta).$$

Using the standard rule about proportions that asserts that if

$$\frac{A}{B}=\frac{C}{D},$$

we have also

$$\frac{A}{B}=\frac{C}{D}=\frac{A+C}{B+D},$$

we get, combining formulas of the form (5.19) for all different values of t, that outside the union of the F_t, $t=1,\ldots,n$, we have

$$\begin{aligned}\left(\sum_{k=1}\overline{\partial}_\zeta\sigma_k\wedge d\zeta_k\right)^{n-1}\wedge g_1(z,\zeta)=&\frac{f_1(\zeta)-f_1(z)}{\sum_{t=1}^n(\zeta_t-z_t)\sigma_t(z,\zeta)}\left(\sum_{t=1}^n\sigma_t\Xi_t(z,\zeta)\right)\\ =&(f_1(\zeta)-f_1(z))\left(\sum_{t=1}^n\sigma_t\Xi_t(z,\zeta)\right).\end{aligned}$$

Such a formula holds everywhere by continuity. Now, one can see at once that

$$\begin{aligned}\overline{\partial}_\zeta\left[\sum_{t=1}^n\sigma_t\Xi_t\right]&=\sum_{t=1}^n(\overline{\partial}_\zeta\sigma_t\wedge d\zeta_t)\wedge\left(\sum_{\substack{1\le k\le n\\ k\neq t}}\overline{\partial}_\zeta\sigma_k\wedge d\zeta_k\right)^{n-1}\\ &=\left(\sum_{1\le k\le n}\overline{\partial}_\zeta\sigma_k\wedge d\zeta_k\right)^{n}=0.\end{aligned}$$

This is our statement in the case $m=1$. The proof of the inductive step from $m-1$ to m is exactly the same computation. Suppose the elements in J are in increasing order and that $\#J=m$. Let us denote as κ_n the constant

$$\kappa_n:=(-1)^{(n-m)(n-m-1)/2}\binom{n}{m}.$$

Then, one has

$$\begin{aligned}&\left(\sum_{k=1}^n\overline{\partial}_\zeta\sigma_k(z,\zeta)\wedge d\zeta_k\right)^{n-m}\wedge\left(\bigwedge_{j\in J}g_{j_l}(z,\zeta)\right)=\\ &=\kappa_n\sum_{\substack{1\le k_1<k_2<\cdots<k_{n-m}\le n\\ k_j\neq t}}\left(\bigwedge_{j=1}^{n-m}\overline{\partial}_\zeta\sigma_{k_j}\right)\wedge\left(\bigwedge_{j=1}^{n-m}\left(d\zeta_{k_j}-\frac{\zeta_{k_j}-z_{k_j}}{\zeta_t-z_t}d\zeta_t\right)\right)\wedge\\ &\qquad\wedge\left(\bigwedge_{l=1}^m g_{j_l}(z,\zeta)\right)\\ &=\kappa_n\sum_{\substack{1\le k_1<k_2<\cdots<k_{n-m}\le n\\ k_j\neq t}}\epsilon(k)\left(\bigwedge_{j=1}^{n-m}\overline{\partial}_\zeta\sigma_{k_j}\right)\wedge\Delta_{k,t}(z,\zeta)\bigwedge_{j=1}^n d\zeta_j\end{aligned}$$

where $\epsilon(k)$ is a convenient sign and $\Delta_{k,t}$ is the determinant constructed exactly like Δ except that the columns labelled with $q_1, \ldots, q_m$ ($\{q_1, \ldots, q_m\} = \{1, \ldots, n\} \setminus \{k_1, \ldots, k_{n-m}\}$) are respectively replaced by the columns

$$\begin{pmatrix} g_{j_l,1} \\ \cdot \\ \cdot \\ \cdot \\ g_{j_l,n} \end{pmatrix}, \; l = 1, \ldots, m\,.$$

Changing the determinant Δ_k just by adding to the t-th row the same linear combination of rows as before, one can rewrite

$$\epsilon(k)\Delta_{k,t}(z,\zeta) \bigwedge_{j=1}^{n} d\zeta_j = \sum_{j\in J} \frac{f_j(\zeta) - f_j(z)}{\zeta_t - z_t} \left(\bigwedge_{j=1}^{n-m} \overline{\partial}_\zeta \sigma_{k_j} \right) \wedge G_{j,t}(z,\zeta)$$

where $G_{j,t}(z,\zeta)$ is the form obtained from $\bigwedge_{l\in J} g_l(z,\zeta)$ by replacing g_j by $d\zeta_t$. Therefore, adding all these terms for all k and multiplying by the coefficient

$$(-1)^{(n-m)(n-m-1)/2} \binom{n}{m}\,,$$

we get

$$\Big(\sum_{k=1}^{n} \overline{\partial}_\zeta \sigma_k(z,\zeta) \wedge d\zeta_k \Big)^{n-m} \wedge \Big(\bigwedge_{j\in J} g_j(z,\zeta) \Big) = \sum_{j\in J} (f_j(\zeta) - f_j(z)) \Xi^J_{j,t}, \tag{5.20}$$

where

$$\Xi^J_{j,t} := \Big(\sum_{\substack{1\le k\le n \\ k\ne t}} \overline{\partial}_\zeta \sigma_k \wedge \big(d\zeta_k - \frac{\zeta_k - z_k}{\zeta_t - z_t} d\zeta_t \big) \Big)^{n-m} \wedge G_{j,t}\,.$$

Combining formulas of the form (5.20) for the different values of t, $1 \le t \le n$, exactly as in the case $m = 1$, we get

$$\Big(\sum_{k=1}^{n} \overline{\partial}_\zeta \sigma_k(z,\zeta) \wedge d\zeta_k \Big)^{n-m} \wedge \Big(\bigwedge_{j\in J} g_j(z,\zeta) \Big) = \sum_{j\in J} (f_j(\zeta) - f_j(z)) U^J_j\,,$$

where

$$\begin{aligned} U^J_j(z,\zeta) &= \sum_{t=1}^{n} \sigma_t \Big(\sum_{\substack{1\le k\le n \\ k\ne t}} \overline{\partial}_\zeta \sigma_k \wedge d\zeta_k \Big)^{n-m} \wedge G_{j,t}(z,\zeta) \\ &= \pm \sum_{t=1}^{n} \sigma_t d\zeta_t \wedge \Big(\sum_{1\le k\le n} \overline{\partial}_\zeta \sigma_k \wedge d\zeta_k \Big)^{n-m} \wedge \Big(\bigwedge_{\substack{l\in J \\ l\ne j}} g_l(z,\zeta) \Big) \end{aligned}$$

One can see immediately that, for $j \in J$,

$$\overline{\partial}_\zeta U^J_j(z,\zeta) = \pm \Big(\sum_{1\le k\le n} \overline{\partial}_\zeta \sigma_k \wedge d\zeta_k \Big)^{n-m+1} \wedge \Big(\bigwedge_{\substack{l\in J \\ l\ne j}} g_l(z,\zeta) \Big).$$

We now can use the inductive hypothesis to conclude to the proof of the lemma. $\diamondsuit$

We now go back to the proof of Proposition 5.3. Since $S^{[l_0+1]} \wedge S^{[l_0+1]} \equiv 0$, one can check that on $U \times \widetilde{U}$,

$$\Sigma \wedge (\overline{\partial}_\zeta \Sigma)^{n-p-1} \equiv \frac{S^{[l_0+1]} \wedge (\overline{\partial}_\zeta S^{[l_0+1]})^{n-p-1}(z,\zeta)}{< s^{[l_0+1]}(z,\zeta), \zeta - z >^{n-p}}\,.$$

Therefore, on $U \times \widetilde{U}$, one has

$$\overline{\partial}_\zeta \left[\left(\frac{S^{[l_0+1]} \wedge (\overline{\partial}_\zeta S^{[l_0+1]})^{n-p-1}(z,\zeta)}{< s^{[l_0+1]}(z,\zeta), \zeta - z >^{n-p}} \right) \wedge G(z,\zeta) \right] \equiv$$
$$\equiv (\overline{\partial}_\zeta \Sigma(z,\zeta))^{n-p} \wedge G(z,\zeta).$$

From Lemma 5.5, one can conclude that in $U \times \tilde{U}$

$$\overline{\partial}_\zeta [\Psi_{l_0}(z,\zeta) \wedge G(z,\zeta)] = \sum_{j=1}^{p} (f_j(\zeta) - f_j(z))\omega_j(z,\zeta),$$

where the forms ω_j are in $\mathcal{E}^{n,n-p}(U \times \widetilde{U})$ and have coefficients holomorphic in z. Since

$$\overline{\partial}\eta_T(\zeta) = -\frac{(-1)^{\frac{p(p-1)}{2}}}{(2i\pi)^{n-p}} \overline{\partial}\varphi^{[l_0]}(\zeta) \wedge \int_{K_{l_0}} \overline{\partial}_\zeta [\Psi_{l_0}(z,\zeta) \wedge G(z,\zeta)] d\mu_0(z)$$

and

$$\int_{K_{l_0}} f_j(z)\omega_j(z,\zeta) d\mu_0(z) = 0, \ j = 1, \ldots, p$$

(this is the hypothesis on T and the fact that one can approach uniformly on K_{l_0} any holomorphic function on U by elements in $\mathcal{O}(V)$), we get

$$\overline{\partial}\eta_T(\zeta) = \sum_{j=1}^{p} f_j(\zeta)\eta_{T,j}(\zeta) \tag{5.21}$$

for some differential forms $\eta_{T,j}$ in $\mathcal{D}^{n,n-p+1}(V)$. $\diamondsuit$

We now intend to give some representation formulas for analytic functionals orthogonal to some ideal (again in $\mathcal{O}(V)$, V being some pseudoconvex open subset in $\mathbf{C}^n$) in the non complete intersection case. The first result in this direction is related to the case when the number p of generators of the ideal is strictly smaller than the dimension n. The statement is inspired by the statement of Proposition 5.3. Before stating our first result, we need some preliminary notation. Let I be some ideal in $\mathcal{O}(V)$, $\sqrt{I}$ its radical (in $\mathcal{O}(V)$) and $\overline{I}$ its integral closure. We will denote as $\widetilde{\sqrt{I}}\mathcal{E}^{0,0}(V)$ the ideal generated in $\mathcal{E}^{0,0}(V)$ by the conjugates of all elements in $\sqrt{I}$.

Proposition 5.6. *Let $p < n$ and I be an ideal in $\mathcal{O}(V)$ generated by $f_1, \ldots, f_p$. For each $m \in \{1, \ldots, p\}$ there is a family of $\binom{p}{m}$ $(0,k)$ currents $T_{m,J}$, $J \subset \{1, \ldots, p\}$, $\#J = m$, which are annihilated (as currents) by the ideal $\overline{I}^m \mathcal{E}^{0,0}(V) + \widetilde{\sqrt{I}}\mathcal{E}^{0,0}(V)$, such that any functional T orthogonal to I can be represented as*

$$T(h) = \sum_{m=1}^{p} \sum_{J \subset \{1,\ldots,p\}} < T_{k,J}, h\eta_T^{(m,J)} >, \tag{5.22}$$

where, for any m, J, $\eta_T^{m,J} \in \mathcal{D}^{n,n-m}(V)$ satisfies

$$\overline{\partial}\eta_T^{(m,J)}(\zeta) = \sum_{j \in J} f_j(\zeta)\eta_{T,j}^{(m,J)}(\zeta) \tag{5.23}$$

for some elements $\eta_{T,j}^{(m,J)} \in \mathcal{D}^{n,n-m+1}(V)$.

Proof. The proof is similar to the proofs of Proposition 5.1 and 5.2. Let T be an analytic functional orthogonal to $I = (f_1, \ldots, f_p)$. There exists $l_0 \in \mathbf{N}^*$, some Radon measure μ_0 supported by K_{l_0}, such that, for any $h \in \mathcal{O}(V)$,

$$< T, h >= \int_{K_{l_0}} h(z) d\mu_0(z).$$

Once again, we represent h in K_{l_0} using formula (5.4) for a convenient choice of q and Φ; q will at the beginning depend on a complex parameter λ such that $\Re(\lambda) >> 0$; the (1,0) differential form associated to $q = q_\lambda$ is now

$$\widetilde{Q}(z,\zeta) = \widetilde{Q}_\lambda(z,\zeta) := \|f(\zeta)\|^{2\lambda} \left(\frac{\sum_{j=1}^p \overline{f_j(\zeta)} \sum_{k=1}^n g_{j,k}(z,\zeta) d\zeta_k}{\|f(\zeta\|^2} \right),$$

where

$$\|f(\zeta)\|^2 := \sum_{j=1}^p |f_j(\zeta)|^2.$$

Note that from an immediate computation, we have

$$1 + < \tilde{q}_\lambda(z,\zeta), z-\zeta >= 1 - \|f(\zeta)\|^{2\lambda} + \|f(\zeta)|^{2\lambda} \frac{\sum_{j=1}^p \overline{f_j(\zeta)} f_j(z)}{\|f(\zeta)\|^2}. \tag{5.24}$$

We take $\Phi(t) = t^p$. For z in some neighborhood of $[\widehat{K_{l_0}}]_V$, one has the following representation formula

$$h(z) = -\frac{1}{(2i\pi)^n} \int_{\zeta \in V} h \overline{\partial} \varphi^{[l_0]} \wedge \left(\sum_{m=0}^p \frac{\Gamma^{(m)}}{m!} \Sigma \wedge (\overline{\partial}_\zeta \Sigma)^{n-1-m} \wedge (\overline{\partial}_\zeta \widetilde{Q}_\lambda)^m (z,\zeta) \right) \tag{5.25}$$

where

$$\Sigma(z,\zeta) = \Sigma^{[l_0+1]}(z,\zeta) := \frac{S^{[l_0+1]}(z,\zeta)}{< s^{[l_0+1]}(z,\zeta), \zeta - z >}.$$

One can write the differential form which appears between brackets in the integrand (5.25) as

$$\sum_{m=0}^p \binom{p}{m} (1 - \|f(\zeta)\|^{2\lambda})^{p-m} \Sigma \wedge (\overline{\partial}_\zeta \Sigma)^{n-1-m} \wedge (\overline{\partial}_\zeta \widetilde{Q}_\lambda)^m (z,\zeta) +$$

$$+ \sum_{j=1}^p f_j(z) \theta_j(z,\zeta;\lambda),$$

where the forms θ_j have coefficients in $\mathcal{E}^{0,0}(U \times \widetilde{U})$, where U is a neighborhhod of $[\widehat{K_{l_0}}]_V$ and $\widetilde{U}$ a neighborhood of $Supp(\overline{\partial}\varphi^{[l_0]})$. Moreover, these coefficients are holomorphic in the z variables. Let $\widetilde{Q}_0$ be the differential form

$$\widetilde{Q}_0(z,\zeta) := \frac{\sum_{j=1}^p \overline{f_j(\zeta)} \sum_{k=1}^n g_{j,k}(z,\zeta) d\zeta_k}{\|f(\zeta\|^2}.$$

Clearly, for any $m \in \{1, \ldots, p-1\}$,

$$\begin{aligned} &[\overline{\partial}_\zeta \widetilde{Q}_\lambda(z,\zeta)]^m = \\ &= \|f(\zeta)\|^{2m\lambda} \left([\overline{\partial}_\zeta \widetilde{Q}_0(z,\zeta)]^m + m\lambda \frac{\overline{\partial}\|f\|^2}{\|f\|^2} \wedge \widetilde{Q}_0(z,\zeta) \wedge [\overline{\partial}_\zeta \widetilde{Q}_0(z,\zeta)]^{m-1} \right) \end{aligned} \tag{5.26}$$

and

$$[\overline{\partial}_\zeta \widetilde{Q}_\lambda(z,\zeta)]^p = p!(-1)^{p(p-1)/2} \lambda \|f(\zeta)\|^{2p(\lambda-1)} G(z,\zeta) \wedge \overline{\partial f(\zeta)} \tag{5.27}$$

where, as before

$$G(z,\zeta) := \bigwedge_{j=1}^p g_j(z,\zeta), \quad \overline{\partial f(\zeta)} := \bigwedge_{j=1}^p \overline{\partial f_j(\zeta)}.$$

Let us write, for $m \in \{1,\dots,p\}$,

$$\widetilde{Q}_0(z,\zeta) \wedge [\overline{\partial}_\zeta \widetilde{Q}_0(z,\zeta)]^{m-1} = \sum_{\substack{J\subset\{1,\dots,p\},\#J=m\\ j_1<j_2<\dots<j_m}} \Theta_{m,J}(\zeta) \wedge \Big(\bigwedge_{l=1}^{m} g_{j_l}(z,\zeta)\Big).$$

For any $m \in \{1,\dots,p\}$, for any $J \subset \{1,\dots,p\}$, the map

$$\lambda \mapsto \Lambda_{m,J}(\lambda) := \lambda \|f\|^{2\lambda} \frac{\overline{\partial}\|f\|^2}{\|f\|^2} \wedge \Theta_{m,J} \tag{5.28}$$

has a meromorphic continuation as a map from $\mathbf{C}$ to $'\mathcal{D}^{0,m}(V)$. This follows from Atiyah's theorem. We denote as $T_{m,J}$ the $(0,m)$ current defined as the coefficient of λ^0 in the Laurent expansion of (5.28) around $\lambda = 0$. We now take the meromorphic continuation of both sides in formula (5.25) and identify the Laurent coefficients at the origin. The formula one gets at this stage is

$$\begin{aligned}(2i\pi)^n h(z) &= \\ &= -\sum_{m=0}^{p} \sum_{\substack{J\subset\{1,\dots,p\}\\ \#J=m}} \Big\langle \gamma_m T_{m,J}, h\overline{\partial}\varphi^{[l_0]} \wedge \Big[\Sigma \wedge (\overline{\partial}_\zeta \Sigma)^{n-1-m} \wedge \big(\bigwedge_{l=1}^{m} g_{j_l}\big)\Big](z,\cdot)\Big\rangle + \\ &\quad + \sum_{j=1}^{p} h_j(z) f_j(z),\end{aligned} \tag{5.29}$$

where the h_j are holomorphic functions in some neighborhood of $[\widehat{K_{l_0}}]_V$ and

$$\gamma_m := m\int_0^1 u^{m-1}\Phi^{(m)}(1-u)du = m\binom{p}{m}\int_0^1 u^{m-1}(1-u)^{p-m}du = 1$$

for this particular choice of Φ. We now use Fubini theorem and the fact that the functional T is orthogonal to the ideal, so that one can write, for $h \in \mathcal{O}(V)$,

$$T(h) = \sum_{m=1}^{p} \sum_{J\subset\{1,\dots,p\}} < T_{m,J}, h\eta_T^{(m,J)} >,$$

where

$$\begin{aligned}(2i\pi)^n \eta_T^{(m,J)}(\cdot) &:= \\ &= -\overline{\partial}\varphi^{[l_0]} \wedge \int_{K_{l_0}} \Big[\Sigma^{[l_0+1]} \wedge (\overline{\partial}_\zeta \Sigma^{[l_0+1]})^{n-1-m} \wedge \big(\bigwedge_{l=1}^{m} g_{j_l}\big)\Big](z,\cdot)d\mu_0(z).\end{aligned}$$

As in the proof of Proposition 5.3,

$$\begin{aligned}\overline{\partial}\eta_T^{(m,J)}(\zeta) &= \\ &= \frac{1}{(2i\pi)^n}\overline{\partial}\varphi^{[l_0]}(\zeta) \wedge \int_{K_{l_0}} \Big[(\overline{\partial}_\zeta \Sigma^{[l_0+1]})^{n-m} \wedge \big(\bigwedge_{l=1}^{m} g_{j_l}\big)\Big](z,\zeta)d\mu_0(z).\end{aligned}$$

One can use again Lemma 5.5 (with the functions f_j, $j \in J$), in order to prove the existence of smooth differential forms $\eta_T^{(m,J)}$ such that, for any $m \in \{1,\dots,p\}$, for any $J \subset \{1,\dots,p\}$,

$$\overline{\partial}\eta_T^{(m,J)}(\zeta) = \sum_{j\in J} f_j(\zeta)\eta_{T,j}^{(m,J)}(\zeta).$$

In order to conclude the proof of the proposition, we need to prove that the currents $T_{m,J}$ are orthogonal (as currents) to $\overline{I}^m\mathcal{E}^{0,0}(V)+\widetilde{\sqrt{I}}\mathcal{E}^{0,0}(V)$.

This can be done with the use of resolution of singularities, as in the proof of Theorem 3.25 in [5] (from which the whole proof of Proposition 5.7 is inspired). Clearly, the question is a local one. Let $f := f_1 \cdots f_p$. Let $z^{(0)} \in V$. There exists some neighborhood $\Omega(z^{(0)})$, some n-dimensional analytic manifold $\mathcal{X}$, together with a proper map $\pi : \mathcal{X} \mapsto \Omega(z^{(0)})$ such that the restriction of π to $\mathcal{X} \setminus \pi^* f^{-1}(0)$ is a biholomorphic map over $\Omega(z^{(0)}) \setminus f^{-1}(0)$ and such that, for each $x_0 \in \mathcal{X}$, there are local coordinates w centered at x_0, so that, in a local chart around x_0, one has, for each $j \in \{1,\dots,p\}$,

$$\pi^* f_j(w) = u_j(w) w^{\alpha_j},$$

where $\alpha_j \in \mathbf{N}^n$ and u_j is a non-vanishing holomorphic function in the local chart. We next introduce, for such a local chart, the Newton polyedron $\Gamma^+(\alpha_1,\dots,\alpha_p)$, defined as the convex hull of

$$\bigcup_{j=1}^{p} \{\alpha_j + \overline{(\mathbf{R}^+)}^n\},$$

and the toric manifold $\widetilde{\mathcal{X}}_\alpha = \mathcal{X}(\Gamma^+(\alpha_1,\dots,\alpha_p))$ associated to it (see [24]). The projection map $\tilde{\pi}_\alpha$ is a proper map from $\widetilde{\mathcal{X}}_\alpha$ to $\mathbf{C}^n$; furthermore, this map is invertible from $\widetilde{\mathcal{X}}_\alpha \backslash \tilde{\pi}_\alpha(\{w_1 \cdots w_n = 0\})$ to $\mathbf{C}^n \backslash \{w_1 \cdots w_n = 0\}$. In some local chart $\mathcal{U}$ in $\widetilde{\mathcal{X}}_\alpha$, all monomials $\tilde{\pi}_\alpha^*(w_j^\alpha)$, $j=1,\dots,p$, are multiples of a distinguished one among them, that we will denote M. So in such a chart, one can write, for $j=1,\dots,p$,

$$(\tilde{\pi}_\alpha^* \circ \pi)\left[\frac{\overline{f}_j}{\|f\|^2}\right] = \frac{\theta}{M},$$

where θ is a smooth function. Similarly

$$(\tilde{\pi}_\alpha^* \circ \pi)\left[\frac{\overline{\partial}\|f\|^2}{\|f\|^2}\right] = \eta_1 \frac{\overline{\partial M}}{\overline{M}} + \eta_2,$$

where η_1 is a smooth function (in the local chart) and η_2 a smooth $(0,1)$ form. So that, for any $J \subset \{1,\dots,p\}$, one can write, for $\Re(\lambda) >> 0$,

$$(\tilde{\pi}_\alpha^* \circ \pi)(\Lambda_{m,J}) = \lambda \frac{|M|^{2m\lambda}|\nu|^{2m\lambda}}{M^m}\left(\sigma_{m,J} \wedge \frac{\overline{\partial M}}{\overline{M}} + \tau_{m,J}\right),$$

where again $\sigma_{m,J}$ is a smooth $(0,p-1)$ form (in the local chart), $\tau_{m,J}$ a smooth $(0,p)$ form, and ν a non-vanishing smooth function in the local chart. Now, if $h \in \overline{I}^m\mathcal{E}^{0,0}(V)$, one has locally

$$|h(\zeta)| \leq C\|f(\zeta)\|^m,$$

so that, as it can be seen immediately, M^m divides $(\tilde{\pi}_\alpha^* \circ \pi)(h)$ in the local chart. Since a function of the form

$$\lambda \mapsto \lambda \int_{\mathbf{C}}^{n} \frac{|M|^{2\lambda}|\nu|^{2\lambda}\psi(t)\overline{dt} \wedge dt}{\overline{t}_j},$$

(where $\psi \in \mathcal{D}(\mathbf{C}^n)$, M is a monomial, ν a smooth function non-vanishing on $Supp(\psi)$) has a meromorphic continuation which is holomorphic and vanishes at the origin, it is clear that any current $T_{m,J}$ is annihilated by h. If now $h \in \widetilde{\sqrt{I}}\mathcal{E}^{0,0}(V)$, any coordinate function t_l which appears in the expression of M is such that $\overline{t}_l$ divides $(\tilde{\pi}_\alpha^* \circ \pi)(h)$. Since a function of the form

$$\lambda \mapsto \lambda \int_{\mathbf{C}}^{n} \frac{|M|^{2\lambda}|\nu|^{2\lambda}\psi(t)\overline{dt} \wedge dt}{M^m},$$

(where $\psi \in \mathcal{D}(\mathbf{C}^n)$, M is a monomial, ν a smooth function non-vanishing on $Supp(\psi)$) has a meromorphic continuation which is holomorphic and vanishes at the origin, it is clear that any current $T_{m,J}$ is also annihilated by h. $\Diamond$

The currents involved in this representation formula are not annihilated by the ideal I (as currents). So we do not recover, with such a formula, the existence of a (n,n) compactly supported current S, *which is annihilated by the ideal as a current*, such that, for any $h \in \mathcal{O}(V)$, $T(h) = S(h)$. This was the result stated (and proved with cohomological methods) in section 2. Nevertheless, it seems interesting to remark that it is possible, by a slight change in the method of proof of Proposition 5.6, to give another explicit representation formula for the unique residual current extending a given functional orthogonal to an ideal with isolated zeroes (see Proposition 4.2).

Proposition 5.7. *Let I be some ideal in $\mathcal{O}(V)$ generated by $f_1, \ldots, f_p$ ($p \geq n$) and such that $V(I)$ is discrete. Any analytic functional T which is orthogonal to the ideal (as a functional) can be represented with a compactly supported (n,n) current S which is orthogonal to the ideal as a current and can be described in terms of the analytic continuation of $\lambda \mapsto \|f\|^{2\lambda}$.*

Proof. We follow exactly the same proof as in Proposition 5.6, except that we take $\Phi(t) = t^{n+1}$ instead of $\Phi(t) = t^p$. We keep the same notations than in the proof of the mentioned proposition. Formula (5.29) is now replaced by the following. For $z \in [\widehat{K_{l_0}}]$, we have

$$\begin{aligned}
&(2i\pi)^n h(z) = \\
&= -\sum_{m=0}^{n-1} \sum_{\substack{J\subset\{1,\ldots,p\}\\ \#J=m}} \Big\langle T_{m,J}, h\overline{\partial}\varphi^{[l_0]} \wedge \Big[\Sigma \wedge (\overline{\partial}_\zeta \Sigma)^{n-1-m} \wedge \big(\bigwedge_{l=1}^{m} g_{j_l}\big)\Big](z,\cdot)\Big\rangle + \\
&+ \sum_{\substack{J\subset\{1,\ldots,p\}\\ \#J=n}} < T_{n,J}, h\varphi^{[l_0]} \bigwedge_{l=1}^{n} g_{j_l} > + \sum_{j=1}^{p} h_j(z) f_j(z),
\end{aligned} \tag{5.30}$$

where the h_j are holomorphic functions in some neighborhhood of $[\widehat{K_{l_0}}]_V$. We have then the following representation for T

$$T(h) = \sum_{m=1}^{n} \sum_{J\subset\{1,\ldots,p\}} < T_{m,J}, h\eta_T^{(m,J)} >,$$

where

$$\begin{aligned}
&(2i\pi)^n \eta_T^{(m,J)}(\cdot) := \\
&= -\overline{\partial}\varphi^{[l_0]} \wedge \int_{K_{l_0}} \Big[\Sigma^{[l_0+1]} \wedge (\overline{\partial}_\zeta \Sigma^{[l_0+1]})^{n-1-m} \wedge \big(\bigwedge_{l=1}^{m} g_{j_l}\big)\Big](z,\cdot) d\mu_0(z)
\end{aligned}$$

for $1 \leq m \leq n-1$, $J \subset \{1, \ldots, p\}$, $\#J = m$ and

$$\eta_T^{(n,J)}(\cdot) := \frac{1}{(2i\pi)^n} \varphi^{[l_0]}(\zeta) \int_{K_{l_0}} \left(\bigwedge_{l=1}^{n} g_{j_l}\right)(z,\cdot) d\mu_0(z)$$

for $J \subset \{1, \ldots, p\}$, $\#J = n$. Currents $T_{m,J}$ for $m \in \{1, \ldots, n\}$ are orthogonal to $\widetilde{\sqrt{I}}\mathcal{E}^{0,0}(V)$ (as currents). So is the current

$$S = \sum_{m=1}^{n} \sum_{J\subset\{1,\ldots,p\}} T_{m,J} \wedge \eta_T^{(m,J)}.$$

Since T is orthogonal to I as a functional and T and S coincide on $\mathcal{O}(V)$, $S(h) = 0$ for any $h \in I$. Moreover, as $\widetilde{\sqrt{I}} \cdot S = 0$, it follows that $\mathrm{supp}(S)$ is a finite set contained in $V(I)$. Then, for m sufficiently big, $S(\varphi) = 0$

for any $\varphi \in \mathcal{E}^{0,0}(V)$ vanishing on $V(I)$ up to order m. Now, the proof of $S(h\ \varphi) = 0$ for any $h \in I$ and $\varphi \in \mathcal{E}^{0,0}(V)$, follows by the same arguments as in the proof of Proposition 4.2. Namely, by means of a partition of unity, me may assume that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(S)$ is a single point P_0. Let $z_1, \ldots, z_n$ and the decomposition $\varphi(z) = \varphi_1(z) + \varphi_2(z) + \varphi_3(z)$ be as in (4.1). Then, $S(h\ \varphi_1) = 0$ because $\varphi_1 \in \mathcal{O}(V)$; $S(h\ \varphi_2) = 0$ because $\varphi_2 \in \sqrt{I}\mathcal{E}^{0,0}(V)$, and $S(h\ \varphi_3) = 0$ by the choice of m. Therefore, we deduce that S is orthogonal as a current to $I\mathcal{E}^{0,0}(V)$ and the proposition is proved. $\diamondsuit$

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6. References.

[1] M. Andersson and B. Berndtsson, Henkin-Ramirez formulas with weight factors, Ann. Inst. Fourier **32**, 91-110 (1982)
[2] M. F. Atiyah, Resolution of singularities and division of distributions, Comm. Pure Appl. Math. **23**, 145-150 (1970)
[3] B. Berndtsson, A formula for interpolation and division in $\mathbf{C}^n$, Math. Ann. **263**, 399-418 (1983)
[4] C. A. Berenstein, R. Gay and A. Yger, Analytic continuation of currents and division problems, Forum Math. **1**, 15-51 (1989)
[5] C. A. Berenstein, R. Gay, A. Vidras and A. Yger, *Residue currents and Bézout identities*, Progress in Mathematics, 114, Birkäuser, 1993.
[6] C. A. Berenstein and D. C. Struppa, Convolution equations and complex analysis, *Encyclopaedia of Mathematical Sciences*, vol 54, Springer-Verlag, 1991.
[7] C. A. Berenstein and A. Yger, Analytic Bezout identities, Advances in Appl. Math. **10**, 51-74 (1989)
[8] N. Coleff and M. Herrera, *Les Courants Résiduels Associés à une Forme Méromorphe*, Springer L.N. Math. 633, 1978.
[9] J. P. Demailly and C. Laurent, Formules intégrales pour les formes différentielles de type (p, q) dans les variétés de Stein, Ann. Sci. Ec. Norm. Sup. Paris **20**, 579-598 (1987)
[10] J. P. Demailly and M. Passare, Courants résiduels et classe fondamentale, to appear in: L'Enseignement Mathématique.
[11] A. Dickenstein and C. Sessa, Canonical Representative in Moderate Cohomology, Inventiones math. **80**, 417-434 (1985)
[12] A. Dickenstein and C. Sessa, Résidus de formes méromorphes et cohomologie modérée, Actes du Colloque de géométrie Analytique, Paris 7, Juin 1992, *à paraitre*, Hermann, Paris.
[13] A. Dickenstein and C. Sessa, Peripherical integrals and residual currents, C. R. Acad. Sci. Paris, t.**320**, Série I, p. 929-933 (1995)
[14] O. Forster, Plongements de variétés de Stein, Comment. Math. Helv. **45**, vol 2, 170-184 (1970)
[15] R. Gay, Sur un problème de division des fonctionnelles analytiques. Application aux fonctions moyenne-périodiques, C.R. Acad. Sc. Paris, t. **283**, Série A, p. 835-838 (1976)
[16] R. Gay, Division des fonctionnelles analytiques. Application aux fonctions entières de type exponentiel moyenne-périodiques, Springer Lecture Notes in Math. 822, 77-89.
[17] R. Godement, *Topologie Algébrique et théorie des faisceaux*, Actualités Scientifiques et Industrielles, Hermann, Paris, 1964.
[18] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.
[19] L. Gruman, Solutions of difference equations with non constant coefficients, Springer Lecture Notes in Math. 1094, 84-138.
[20] L. Gruman and P. Lelong, *Entire Functions of Several complex variables*, Springer-Verlag, New York, 1986.
[21] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. 20, Springer-Verlag, 1966.
[22] G. Henkin and J. Leiterer, *Theory of Functions on Complex Manifolds*, Monographs in Mathematics 79, Birkäuser, 1984.

[23] L. Hörmander, *An Introduction to Complex analysis in Several complex variables*, North Holland, Amsterdam, 1973.
[24] A. Khovanskii, Newton polyedra and toroidal varieties, Func. Anal. Appl. **11**, 289-295 (1978)
[25] M. Lejeune-Jalabert, Liaison et résidu, Lecture Notes in Math. 961, 1982, 233-240.
[26] B. Malgrange, *Ideals of differentiable functions*, Tata Institute, Bombay, 1966.
[27] B. Malgrange, Systèmes différentiels à coefficients constants, Séminaire Bourbaki 15, no 246 (1962-63)
[28] V. P. Palamodov, Intégration sur des espaces analytiques complexes et dualité locale. C.R. Acad. Sci. Paris, t. **315**, Série I, p. 401-406 (1992)
[29] M. Passare, Residues, currents, and their relation to ideals of holomorphic functions, Math. Scand. **62**, 75-152 (1988)
[30] M. Passare, A Calculus for Meromorphic Currents, Journal für reine und angewandte Mathematik **392**, p. 37-56 (1988)
[31] M. Passare and A. Tsikh, Defining the residue current of a complete intersection, to appear in *Complex Analysis, Harmonic Analysis and Applications*, Pitman Research Notes in Mathematics, 347, Longman Scientific and Technical, Harlow, UK, 1996.
[32] C. Peskine and L. Szpiro, Liaison des variétés algébriques, Inventiones math. **26**, 271-302 (1974)
[33] J.-P. Ramis and G. Ruget, Complexe dualisant et théorèmes de dualité en Géométrie Analytique Complexe, Inst. Hautes Etudes Sci. Publ. Math. **38**, 77-91 (1970)
[34] J.-P. Ramis, G. Ruget and J. L. Verdier, Dualité relative en géométrie analytique complexe, Inventiones math. **13**, 261-273 (1971)
[35] J. P. Serre, Un théorème de dualité, Comm. Math. Helv. **29**, 9-26 (1955)
[36] A. Yger, Formules de Division et Prolongement Méromorphe, dans "Séminaire Lelong-Dolbeault-Skoda", Springer Lecture Notes Math. 1295, 226-283 (1987)