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Ideals Generated by Exponential-Polynomials

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INTRODUCTION

In several problems in Harmonic Analysis, and in Number Theory as well, lemmas on small values of holomorphic functions play an important role.

Let us give first an example from Harmonic Analysis. Let $\mu_1,..., \mu_m$ be m distributions with compact support in $\mathbb{R}^n$ whose Fourier transforms satisfy, in $\mathbb{C}^n$, a lower estimate of the form

$$\sum |\hat{\mu}_j(\zeta)| \geqslant \frac{c}{(1+\|\zeta\|)^k} e^{-k\|\operatorname{Im}\zeta\|}. \tag{0.1}$$

One can then solve Bezout's equation

$$\mu_1 * \nu_1 \cdots + \mu_m * \nu_m = \delta \tag{0.2}$$

with $\nu_1,..., \nu_m$ also distributions with compact support [31, 33].

In many examples (0.1) cannot be verified, even if one knows that the functions $\hat{\mu}_j$ have no common zeroes in $\mathbb{C}^n$, without recourse to deep results in number theory (see, e.g., [12]). On the other hand, using again examples of a number theoretical nature one can find simple examples showing

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that the fact the $\hat{\mu}_j$ have no common zeroes does not imply (0.1), for instance, let us consider on the real line the two measures

$$\mu_1 = \delta_1 + \delta_{-1}, \qquad \mu_2 = \delta_\lambda + \delta_{-\lambda},$$

where λ is an irrational. One sees that (0.1) is satisfied if and only if λ is not a Liouville number [20].

In the same vein, in the techniques used to estimate degrees of transcendency, it seems that knowledge about zeroes must be replaced by knowledge about the small values of the auxiliary functions involved [39, 40, 49].

The lemmas on small values that we consider here relate to a very particular class of entire holomorphic functions in $\mathbb{C}^n$, a class which appears in Harmonic Analysis as Fourier transforms of distributions with finite support in $\mathbb{R}^n$ (i.e., difference-differential operators), and also has a role in Number Theory, namely, being the exponential-polynomials with real frequencies. The class of exponential-polynomials with complex frequencies can also be studied using the methods we present here.

The main question we consider is the following: given $F_1, ..., F_m$ m exponential-polynomials with real frequencies and such that the set of common zeroes in $\mathbb{C}^n$ is either discrete or empty, is it possible to estimate the size of the connected components of the set where the inequality (0.1) is not satisfied? This would correspond to a refined type of transcendency of the exponential functions with respect to the algebraic functions.

In the case of two variables we have shown elsewhere [11] that for two exponential-polynomials with rational frequencies the answer to the above question is positive. We will see here that this is still the case when we deal with m exponential-polynomials in two variables. We will also show that this is not always true, even for rational frequencies, when the number of variables n is bigger than two. Nevertheless we give here a general method to attack this kind of problem (Theorem 3.1), which together with techniques from the solution of Schanuel's conjecture for formal power series [3] and a method of "geometric duality," which we develop in Section 8, allows us, for instance, to study exhaustively the case $n = 3$.

The same kind of tools allow us to study certain problems in Harmonic Analysis which had not been tractable by known methods. In order to give an idea of the type of questions we have in mind, let us describe an example introduced by Delsarte [18]. Consider two distributions of finite support in $\mathbb{R}^2$ of the form

$$\begin{aligned} \mu_1 &= a_1\delta_{(0,0)} + b_1\delta_{(1,0)} + c_1\delta_{(1,1)} + d_1\delta_{(0,1)} + \nu_1 \\ \mu_2 &= a_2\delta_{(0,0)} + b_2\delta_{(1,0)} + c_2\delta_{(1,1)} + d_2\delta_{(0,1)} + \nu_2, \end{aligned} \tag{0.3}$$

where the supports of ν_1 and ν_2 lie in the open square $(0, 1) \times (0, 1)$ and the coefficients of the Dirac measures satisfy the condition

$$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 c_1 \\ b_2 c_2 \end{vmatrix} \cdot \begin{vmatrix} c_1 d_1 \\ c_2 d_2 \end{vmatrix} \cdot \begin{vmatrix} d_1 a_1 \\ d_2 a_2 \end{vmatrix} \neq 0.$$

We show here that every solution f of the system of convolution equations

$$\mu_1 * f = \mu_2 * f = 0$$

can be represented in terms of elementary solutions of the form $P(x, y)\, e^{i(\alpha x + \beta y)}$, where P is a polynomial in $\mathbb{C}[x, y]$ and $(\alpha, \beta) \in \mathbb{C}^2$. In the original example of Delsarte, whose complete proof was only given later [9, 41], the only case considered was when both ν_1 and ν_2 where measures.

Let us finish this introduction with an example that seems to us hard to obtain by methods different from those which we will use here, and which, it seems to us, might have further applications.

Let F_1, F_2 be two exponential-polynomials of three variables, with rational frequencies and no common zeroes, then the pair (F_1, F_2) satisfies an estimate of the form (0.1) (Proposition 8.7).

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We will be concerned with non-zero functions in the space $A_p = A_p(\mathbb{C}^n)$, where p is the weight

$$p(\zeta) := \sum |\operatorname{Im} \zeta_j| + \log(1 + \|\zeta\|) \tag{1.1}$$

and A_p is the space of holomorphic functions F in $\mathbb{C}^n$ such that

$$|F(\zeta)| \leqslant C e^{Cp(\zeta)}, \qquad \zeta \in \mathbb{C}^n$$

for some constant $C = C(F) \geqslant 0$. This space, with its natural topology of inductive limit, coincides with the space $\hat{\mathscr{E}}'(\mathbb{R}^n)$ of Fourier transforms of distributions of compact support in $\mathbb{R}^n$ [7, 20]. One could also consider other weights, for instance, the weight $p(\zeta) = \|\zeta\|$ corresponding to the space of analytic functionals.

Given a finitely generated ideal $\mathscr{I}$ in A_p, we say that $\mathscr{I}$ is *strongly* *slowly decreasing* (s.s.d.) if it has a system of generators $(F_1,\ldots,F_m)$ satisfying the following condition:

> $\forall \varepsilon>0, \forall C>0, \exists \delta>0, \exists D>0$ such that if ζ, ζ' are two points in the same connected component of the open set
>
> $$S(F_1,\ldots,F_m;\delta,D) := \{\zeta\in\mathbb{C}^n : \sum |F_j(\zeta)| < \delta e^{-Dp(\zeta)}\}$$
>
> we have $\|\zeta-\zeta'\| < \varepsilon e^{-Cp(\zeta)}$. (1.2)

We note that under this hypothesis the variety V of common zeroes of the ideal $\mathscr{T}$ is automatically discrete.

The condition (1.2) above is stronger than the condition "jointly slowly decreasing" introduced in [9, Definition 4.1], on the other hand no restriction is imposed to the number of generators of the ideal. It is also clear that (1.2) is a property of the ideal $\mathscr{T}$ and not only of the system of generators chosen.

Let us recall some properties attached to s.s.d. ideals. The first one relates to the Spectral Synthesis problem. Given a system of m convolution equations

$$\mu_1 * f \equiv \cdots \equiv \mu_m * f \equiv 0 \qquad (f\in C^\infty(\mathbb{R}^n)), \tag{1.3}$$

where $\mu_1,\ldots,\mu_m \in \mathscr{E}'(\mathbb{R}^n)$, the *spectrum* of the system (1.3) is the analytic variety V in $\mathbb{C}^n$

$$V := \{\zeta\in\mathbb{C}^n : F_1(\zeta) = \cdots = F_m(\zeta) = 0\}, \tag{1.4}$$

where $F_j(\zeta) = \hat{\mu}_j(\zeta) := \langle \mu_j(t), e^{it\cdot\zeta}\rangle = \int e^{i\Sigma t_k\zeta_k}\, d\mu_j(t)$. We say the system (1.3) is *non-redundant* if V is discrete (or empty). The spectral synthesis holds if all the solutions of (1.3) are the limits, in $C^\infty(\mathbb{R}^n)$, of linear combinations of solutions of the form

$$P(t)\, e^{i\lambda\cdot t}, \qquad \lambda\in V \text{ and } P\in\mathbb{C}[t_1,\ldots,t_n]. \tag{1.5}$$

The spectral synthesis always holds when $n=1$ [45], on the other hand it is in general false for certain systems of equations when $n\geqslant 2$ [27]. A theorem due to Gurevich [28] and Kelleher and Taylor [34] implies that if the ideal $\mathscr{T}$ generated by $(F_1,\ldots,F_m)$ is s.s.d., then the spectral synthesis holds (in fact, it is enough to know that the components of the set $SF_1,\ldots,F_m;\ \delta, D)$ are relatively compact for some choice of δ, D).

In the particular case $m=n$, one can go further than the spectral synthesis [9] and show that every s.s.d. ideal is also closed in A_p and, moreover, that every solution of the system (1.3) can be represented by a series of solutions of the form (1.5). This result has been extended to the case in which $\mu_1,..., \mu_m$ and f are vector-valued by Struppa [47]. In the case $m=n$ and $\mathscr{T}$ s.s.d. with spectrum V empty, one has also a kind of Nullstellensatz, that is, a decomposition of the function 1:

$$\exists G_j, \qquad 1=\sum_{j=1}^{m} F_j G_j, \qquad G_j \in A_p; \tag{1.6}$$

hence the condition (1.2) implies in this case the Hörmander Condition [31]

$$\exists \kappa>0 \qquad \Sigma |F_j(\zeta)| \geqslant \kappa e^{-\kappa p(\zeta)}, \qquad \forall \zeta \in \mathbb{C}^n. \tag{1.7}$$

The proof of (1.6) is just the fact that $1 \in \bar{\mathscr{T}}$ (by the spectral synthesis) and $\mathscr{T}$ is closed. It is well known that even when $m \neq n$ the existence of $(G_1,..., G_m)$ satisfying (1.6) is equivalent to (1.7) [31, 33]. Since we will be using this reasoning later, let us give here a direct proof that if $m=n$ and $V=\varnothing$ we have $(1.2) \Rightarrow (1.7)$. Namely, introduce the continuous plurisubharmonic function in $\mathbb{C}^n$:

$$u(\zeta) := \log\left(\sum_{j=1}^{n} |F_j(\zeta)|^2\right), \tag{1.8}$$

which is a solution of the Monge–Ampere equation

$$(dd^c u)^n = 0$$

(it is essential here that $m=n$). Hence one can apply to each connected component of $S(F_1,..., F_n; \delta, D)$ the minimum principle [4, Theorem A] and the fact that the weight p can be considered to be constant in such a component. It follows that the estimate (1.7) that holds on the boundary of the components holds in the interior and hence everywhere. (Note that this reasoning works if we only know that $F_1,..., F_n$ are jointly slowly decreasing in the sense of [9].) The advantage of the proof we have just given is that it also works if we replace $\mathbb{C}^n$ by a complex manifold of dimension n, this is simply a consequence of the fact that conditions satisfied by the function u defined by (1.8) are invariant under holomorphic changes of coordinates.

Explicit solutions G_j to (1.6) and even to (0.2) have been discussed elsewhere (e.g., [12]) and have interesting applications to engineering and optics problems. Similarly, the spectral synthesis for s.s.d. systems has applications to control theory, mathematical biology, etc., and we plan to return to these applications in the near future.

We would like also to show that the condition (1.2) for an n-tuplet of elements in A_p implies an estimate of the number of points of V. The counterexample of Cornalba and Shiffman [17] shows that in dimensions bigger than or equal to two, the order of growth of an equidimensional holomorphic mapping $\mathbf{F}$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ does not allow us to estimate the number of points in a ball of radius R of the variety $\{F_1 = \cdots F_n = 0\}$. We have, nevertheless, thanks to the fact that the Bezout estimates hold "on the average" [25, 46], that in the case we consider we can obtain true estimates.

PROPOSITION 1.1. *Let $(F_1, \ldots, F_n)$ be an n-tuplet of elements in $A_p(\mathbb{C}^n)$ s.s.d. with respect to the weight p defined by* (1.1) (*or with respect to the weight $p(\zeta) = \|\zeta\|$*). *Denote by $N(R)$ the number of points* (*counted with multiplicities*) *of the variety $V = \{F_1 = \cdots = F_n\} = 0$ which belong to the ball of center* 0 *and radius R. There exist two constants C_1, C_2 such that*

$$\forall R \in \mathbb{R}^+, \qquad N(R) \leqslant C_1 R^n + C_2. \tag{1.9}$$

Remark 1.1. When $F_1, \ldots, F_n$ are exponential-polynomials such an estimate always holds due to the work of Khovanskii on Liouville functions [35–37, 14].

Proof of Proposition 1.1. Let us fix $R > 0$ and denote by V_R the set of points of V in the open ball $\mathring{B}(0, R)$. Given $\zeta_0 \in V_R$, since $(F_1, \ldots, F_n)$ are s.s.d. we can construct a compact set $\Gamma(\zeta_0)$ of smooth boundary, $\zeta_0 \in \Gamma(\zeta_0) \subseteq B(\zeta_0, 1)$, and such that on the boundary of $\Gamma(\zeta_0)$ we have

$$\sum_{j=1}^{n} |F_j(\zeta)|^2 \geqslant \varepsilon_1 e^{-C_1 p(\zeta)} \tag{1.10}$$

for some positive constants ε_1, C_1 depending only on $(F_1, \ldots, F_n)$.

One can write V_R as a disjoint union of sets $\Gamma(\zeta_1), \ldots, \Gamma(\zeta_{m_R})$. This follows from the fact that given $\zeta_0 \in V_R$ and $\zeta_0' \in V_R \backslash \Gamma(\zeta_0)$ we have for some constants ε_2, C_2

$$d(\zeta_0', \partial\Gamma(\zeta_0)) \geqslant \varepsilon_2 e^{-C_2 p(\zeta_0')}$$

(this is just a consequence of the mean-value theorem). We can equally assume (see [9, Lemma 1.5]) that each $\Gamma(\zeta)$ will be a small deformation of a set of the form

$$|F_1| \leqslant \varepsilon_1' e^{-C_1' p(\zeta)}, \ldots, \qquad |F_n| \leqslant \varepsilon_1' e^{-C_1' p(\zeta)}$$

and hence that

$$\operatorname{meas}(\partial\Gamma(\zeta_i)) \leqslant A e^{Bp(\zeta_i)}, \qquad i = 1, \ldots, m_R,$$

for some constants A, B independent of R.

An application of Kronecker's formula [24, p. 369] shows that there are positive constants ε, c such that for every $\theta \in 2\pi(\mathbb{R}/\mathbb{Z})^n$, the analytic functions $G_j^{(\theta)}$, $j=1,\dots,n$, defined by

$$G_j^{(\theta)}(\zeta)=F_j(\zeta)-\frac{\varepsilon}{(1+R)^c}\,e^{i\theta_j} \tag{1.11}$$

have, in every $\Gamma(\zeta_i)$, the same number of common zeroes as the functions F_j. The constants ε, c depend only on ε_1, C_1, A, B and on the fact that on each $\partial\Gamma(\zeta_i)$ we have, for some convenient K_1, K_2,

$$p(\zeta)\leqslant K_1+K_2 R.$$

Let $N(R,\theta)$ be the number of common zeroes of the functions $G_1^{(\theta)},\dots,G_n^{(\theta)}$ in $\mathring{B}(0,R+1)$. By a result of Gruman [25, Theorem 2.9] or using the work of Stoll, one has for any $\gamma>1$:

$$\begin{aligned}\int_0^{2\pi}\cdots\int_0^{2\pi} & N(R,\theta)\,d\theta_1\cdots d\theta_n\\ &\leqslant\left(\frac{1}{(\gamma^2-1)(R+1)}\right)^n \operatorname{vol} B(0,\gamma^n(R+1))\\ &\quad\times\prod_{j=1}^{n}(\log^+(M_j(R+1))+\log^-\varepsilon+c\log(R+1)),\end{aligned} \tag{1.12}$$

where $M_j(r)=\max\{|F_j(\zeta)|:\zeta\in B(0,r)\}$.

The left-hand-side term in (1.12) is bigger than or equal to

$$(2\pi)^n\sum_{i=1}^{m_R}\operatorname{card}(\Gamma(\zeta_i)\cap V),$$

which is itself an upper bound for $(2\pi)^n N(R)$. The inequality (1.12) leads immediately to (1.9) since there are constants A_1, B_1 such that

$$\log^+ M_j(r)\leqslant A_1+B_1 r,\qquad \forall j. \quad ∎$$

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While our aim is to study the condition s.s.d., we will see in the next section that its verification is tied to the discreteness of certain varieties. We give here some useful criteria to check such discreteness.

PROPOSITION 2.1. *Let W be an analytic variety in $\mathbb{C}^n$ $(n>1)$, suppose W*

is of pure dimension d, $d \geqslant 1$. Assume further that there is a non-zero polynomial $P \in \mathbb{C}[X_1, ..., X_n]$ and two positive constants C, K such that

$$\zeta \in W \Rightarrow |P(\zeta)| \exp\left(\sum_{j=1}^{n} |\operatorname{Im} \zeta_j|\right) \leqslant C(1 + \|\zeta\|)^K. \tag{2.1}$$

Then the variety W is included in the algebraic hypersurface $\{P = 0\}$.

Proof. We are going to show that we can choose constants $\varepsilon > 0$, $M > 0$ such that

$$f_{\varepsilon,M}(\zeta) := (P(\zeta))^2 \left(\frac{\sin \varepsilon\zeta_1}{\zeta_1} \times \cdots \times \frac{\sin \varepsilon\zeta_n}{\zeta_n}\right)^M$$

tends to zero when $\|\zeta\| \to \infty$ along W.

Let us fix a constant L, $L > K$. Denote $\mathcal{U}_L$ the open set in $\mathbb{C}^n$ defined by

$$\mathcal{U}_L = \{\zeta \in \mathbb{C}^n : |P(\zeta)| < (1 + \|\zeta\|)^{-L}\}. \tag{2.2}$$

By (2.1), there is a constant $C_0 = C_0(L) > 0$ such that

$$\zeta \in W, \zeta \notin \mathcal{U}_L \Rightarrow \sum |\operatorname{Im} \zeta_j| \leqslant C_0(1 + \log(1 + \|\zeta\|)). \tag{2.3}$$

Let $\varepsilon > 0$, and M a positive integer so that

$$M > 2 \deg(P) + 1, \qquad \varepsilon M \leqslant 1, \varepsilon M C_0 \leqslant 1, \tag{2.4}$$

where deg (P) is the total degree of the polynomial P. We are going to estimate $f_{\varepsilon,M}(\zeta)$ when $\zeta \in W$ and $\|\zeta\|$ is large.

(*) Let us assume first that $\zeta \in W \cap \mathcal{U}_L$, then

$$|f_{\varepsilon,M}(\zeta)| \leqslant C_1 |P(\zeta)| |P(\zeta)| \exp\left(\varepsilon M \sum |\operatorname{Im} \zeta_j|\right)$$

for some constant $C_1 = C_1(\varepsilon, M)$. This follows from the inequalities

$$\left|\frac{\sin z}{z}\right| \leqslant e^{|\operatorname{Im} z|} \quad \text{and} \quad |\sin z| \leqslant e^{|\operatorname{Im} z|} \quad \forall z \in \mathbb{C}. \tag{2.5}$$

Since $\varepsilon M \leqslant 1$ (by (2.4)) we have

$$\zeta \in W \cap \mathcal{U}_L \Rightarrow |f_{\varepsilon,M}(\zeta)| \leqslant C_1 |P(\zeta)| |P(\zeta) e^{\sum |\operatorname{Im} \zeta_j|}|,$$

or, using (2.2) and (2.1),

$$\zeta \in W \cap \mathcal{U}_L \Rightarrow |f_{\varepsilon,M}(\zeta)| \leqslant CC_1(1 + \|\zeta\|)^{K-L}. \tag{2.6}$$

(∗) Let us suppose now that $\zeta \in W \setminus \mathscr{U}_L$. We can suppose that $\zeta \neq 0$ and, say, $|\zeta_1| \geqslant (1/\sqrt{n})\|\zeta\|$ at that point. Let us write

$$f_{\varepsilon,M}(\zeta) = g_{\varepsilon,M}(\zeta) \cdot \frac{1}{\zeta_1^M}. \tag{2.7}$$

It follows from (2.5) and (2.3) that for some $C_2 = C_2(\varepsilon, M, P) > 0$

$$\begin{aligned} |g_{\varepsilon,M}(\zeta)| &\leqslant C_2(1 + \|\zeta\|)^{2\deg(P) + C_0 \varepsilon M} \\ &\leqslant C_2(1 + \|\zeta\|)^{2\deg(P) + 1} \end{aligned} \tag{2.8}$$

(use (2.4) to obtain the last inequality). From (2.7) and (2.8) we obtain

$$|f_{\varepsilon,M}(\zeta)| \leqslant (2n)^M C_2(1 + \|\zeta\|)^{2\deg(P) + 1 - M}. \tag{2.9}$$

By our choice of M, it follows that

$$f_{\varepsilon,M}(\zeta) \to 0 \qquad \text{as} \quad \zeta \in W,\ \|\zeta\| \to \infty.$$

By the maximum principle for varieties [26, Theorem III B-16] it follows that $f_{\varepsilon,M} \equiv 0$ on W (it is here we use that W is of pure dimension d, $d \geqslant 1$).

Suppose that W has an irreducible component W' not included in the hypersurface $\{P = 0\}$. Once ε, M are chosen arbitrarily but satisfying (2.4) we have $j = j(\varepsilon) \in \{1, ..., n\}$ and $k = k(\varepsilon) \in \mathbb{Z}^*$ such that

$$W' \subseteq \left\{\zeta_j = \frac{k\pi}{\varepsilon}\right\} \tag{2.10}$$

because $f_{\varepsilon,M} \equiv 0$ but $P \not\equiv 0$ on W'. One can fix M, and pick a sequence $\{\varepsilon_m\}$, such that for all $m \in \mathbb{N}$, ε_m and M satisfy (2.4), and furthermore for every pair of distinct indices m, m' the quotient $\varepsilon_m/\varepsilon_{m'}$ is irrational. Since the sequence $j(\varepsilon_m)$ has a stationary subsequence one sees that (2.10) is impossible for all $\varepsilon = \varepsilon_m$, $m \in \mathbb{N}$. This leads to a contradiction. We conclude that every branch of W, and hence W itself, is included in $\{P = 0\}$. ∎

Let us point out a strengthening of Proposition 2.1.

PROPOSITION 2.2. *Let W be an analytic variety in $\mathbb{C}^n$, $n > 1$, of pure dimension d, $d \geqslant 1$. Assume that there is a non-zero polynomial $P \in \mathbb{C}[\zeta_1, ..., \zeta_n]$, an integer $k \in \{1, ..., n-1\}$, and two positive constants C, K such that*

$$\zeta \in W \Rightarrow \left| P(\zeta) \exp\left(\sum_{j=1}^{k} |\operatorname{Im} \zeta_j|\right)\right| \leqslant C(1 + \|\zeta\|)^K \tag{2.11}$$

and

$$\zeta \in W \Rightarrow \sum_{j=k+1}^{n} |\zeta_j| \leqslant C\left(1 + \sum_{j=1}^{k} |\zeta_j|\right)^K. \tag{2.12}$$

Then the variety W is contained in the hypersurface $\{P=0\}$.

Proof. It is very similar to the previous one. We can clearly assume $K \geqslant 1$. Let $L > K$ and $\mathscr{U}_L$ be the open set defined by (2.2). From (2.11) we know there is a constant $C_0 = C_0(L)$ such that

$$\zeta \in W \setminus \mathscr{U}_L \Rightarrow \sum_{j=1}^{k} |\operatorname{Im} \zeta_j| \leqslant C_0 \log(1 + \|\zeta\|). \tag{2.13}$$

We choose ε, M similarly as done before so that

$$M > (2\deg(P) + 1)\, K, \qquad \varepsilon M \leqslant 1, \varepsilon M C_0 \leqslant 1. \tag{2.14}$$

Consider the auxiliary entire function $h_{\varepsilon, M}$

$$h_{\varepsilon, M}(\zeta) = (P(\zeta))^2 \prod_{j=1}^{k} \left(\frac{\sin \varepsilon\zeta_j}{\zeta_j}\right)^M.$$

As in the previous proposition $h_{\varepsilon, M}$ satisfies the estimate (2.6) in $W \cap \mathscr{U}_L$. The assumption (2.12) implies the existence of constants $R_0 > 0$, $C_2 > 0$ (independent of ε, M) such that

$$\zeta \in W, \qquad \|\zeta\| \geqslant R_0 \Rightarrow \|\zeta\| \leqslant C_2 \sum_{1}^{k} |\zeta_j|^K. \tag{2.15}$$

For those points for which $|\zeta_1| = \max\{|\zeta_j|, j = 1, \ldots, k\}$ we then obtain

$$\|\zeta\| \leqslant C_2 k\, |\zeta_1|^K \tag{2.16}$$

and writing down

$$h_{\varepsilon, M}(\zeta) = \frac{1}{\zeta_1^M}\, g_{\varepsilon, M}(\zeta)$$

we obtain, for those points in $W \setminus \mathscr{U}_L$ where $|\zeta_1|$ dominates that for some $C_3 = C_3(\varepsilon, M) > 0$,

$$\begin{aligned} |h_{\varepsilon, M}(\zeta) &\leqslant C_3(1 + \|\zeta\|)^{(2\deg(P)+1)}\, |\zeta_1|^{-M} && \text{(by (2.13))} \\ &\leqslant C_4(1 + \|\zeta\|)^{2\deg(P)+1-M/K} && \text{(by (2.16)).} \end{aligned} \tag{2.17}$$

This last inequality holds in $W \setminus \mathscr{U}_L$ no matter which $|\zeta_j|$ dominates

$(j=1,\ldots,k)$. Since $2\deg(P)+1-M/K<0$ by (2.14), we obtain that $h_{\varepsilon,M}(\zeta)\to 0$ as $\|\zeta\|\to\infty$ in W. The rest of the proof is the same as in Proposition 2.1. ∎

COROLLARY 2.3. *Let W be an analytic variety in $\mathbb{C}^n$, $n>1$. Suppose that there are two positive constants C, K such that*

$$\zeta\in W\Rightarrow\sum_{j=1}^{n}|\operatorname{Im}\zeta_j|\leqslant C\log(1+\|\zeta\|), \tag{2.18}$$

then W is a discrete variety.

Proof. If W is not discrete there is an irreductible branch W' of dimension d, $d\geqslant 1$, W' being irreducible is pure dimensional. Proposition 2.1 with $P\equiv 1$ contradicts the existence of W'. ∎

Remark 2.1. If W were assumed to be algebraic then this corollary is a consequence of the Seidenberg–Tarski Theorem [22]. Note that there is a modification of this corollary corresponding to Proposition 2.2.

We use Proposition 2.1 in the verification that certain analytic varieties in $\mathbb{C}^n$ are discrete once we possess enough geometric information about them. We give here a very simple example of application of that proposition.

We consider the analytic variety V in $\mathbb{C}^n$ defined by the equations

$$e^{i\zeta_1}=P_1(\zeta),\ldots,\qquad e^{i\zeta_n}=P_n(\zeta), \tag{2.19}$$

where $P_1,\ldots,P_n$ are elements in $\mathbb{C}[\zeta_1,\ldots,\zeta_n]$. We show that V is discrete.

In fact, if $k_j\geqslant\deg(P_j)$, for some $C_1>0$ we have, just using the first equation,

$$\zeta\in V\Rightarrow|e^{i\zeta_1}|\leqslant C_1(1+\|\zeta\|)^{k_1}$$

which only tells that $-\operatorname{Im}\zeta_1$ is bounded above. But we also have

$$|P_1(\zeta)|\,|e^{-i\zeta_1}|=1$$

which allows us to bound $\operatorname{Im}\zeta_1$ above. Using all the equations we see that the hypotheses of Proposition 2.1 are satisfied. It follows that if V had an irreducible branch W of dimension $\geqslant 1$ we would have $W\subseteq\{P_1P_2\cdots P_n=0\}$, hence $W\subseteq\{P_j=0\}$ for some j, which is clearly impossible.

We take the opportunity to introduce a different method to show the discreteness of the V we have just considered. This method does not depend on the geometry of V but on arithmetical conditions satisfied by the equations of V. It consists in using the work of *Ax* [3] on the Schanuel conjecture for formal power series. (It is interesting also to compare with the work of Chabauty [15], Kolchin [37], and Coleman [16].)

PROPOSITION 2.4. [3, Corollary 2, p. 253]. *Let k be an integer bigger than or equal to one and let $y_1,..., y_n$ be n functions of the complex variables $(t_1,..., t_k)$, holomorphic in $\{\|t\|<r\}$, $r>0$. Let $\mathscr{P}$ be an ideal in $\mathbb{C}[\zeta_1,..., \zeta_n, X_1,..., X_n]$ such that*

$$\forall P\in\mathscr{P},\qquad P(y_1,..., y_n, e^{y_1},..., e^{y_n})\equiv 0\qquad \text{in } \{\|t\|<r\}. \tag{2.20}$$

If $\mathscr{X}$ denotes the algebraic variety in $\mathbb{C}^{2n}$ of common zeroes of the elements in $\mathscr{P}$ and if dim $\mathscr{X}\leqslant n$, *then there exist rationals $r_1,..., r_n$, not all zero, and a complex number α such that*

$$r_1 y_1(t)+\cdots+r_n y_n(t)\equiv\alpha,\qquad \|t\|<r. \tag{2.21}$$

In the example (2.19) the variety $\mathscr{X}$ is given by

$$X_1=P_1(\zeta),..., X_n=P_n(\zeta)$$

and it is exactly of dimension n in $\mathbb{C}^{2n}$. If the variety V has an irreducible branch W of dimension k, $k\geqslant 1$, one can parametrize that branch in a neighborhood of a regular point and apply Proposition 2.4, hence one concludes that the branch W is included in a hyperplane of the form

$$r_1\zeta_1+\cdots+r_n\zeta_n=\alpha.$$

One can assume that $r_1=-1$ and study in $\mathbb{C}^{n-1}$ the variety defined by

$$e^{i\zeta_2}=Q_2(\zeta_2,..., \zeta_n),\ldots,\qquad e^{i\zeta_n}=Q_n(\zeta_2,..., \zeta_n),$$

where $Q_j(\zeta_2,..., \zeta_n)=P_j(\alpha-r_2\zeta_2-\cdots-r_n\zeta_n, \zeta_2,..., \zeta_n)$. By induction one concludes that the existence of W is impossible.

Note that the major difference in the two approaches lies in the fact that using Proposition 2.1 one gets a fixed algebraic hypersurface which contains all possible irreducible branches of V of dimension $\geqslant 1$, while using Proposition 2.4 one gets a hyperplane, but this hyperplane depends on the branch we are considering. Let us finish this section with an example generalizing (2.19) but which Proposition 2.4 seems badly adapted to handle.

Recall that an *exponential-polynomial* of n variables (with frequencies in $\mathbb{R}^n$—or sometimes one says $i\mathbb{R}^n$) is a function of the form

$$\sum_{\lambda\in\Lambda} P_\lambda(\zeta)\, e^{i\lambda\cdot\zeta},$$

where the *set of frequencies* Λ is a finite subset of $\mathbb{R}^n$, $(P_\lambda)_{\lambda \in \Lambda}$ is a family of polynomials in $\mathbb{C}[\zeta_1, \ldots, \zeta_n]$ which we suppose nonzero, and $\lambda \cdot \zeta$ denotes always the bilinear form

$$\lambda \cdot \zeta = \sum_{j=1}^{n} \lambda_j \zeta_j.$$

PROPOSITION 2.5. *Let $F_1, \ldots, F_n$ be exponential-polynomials whose sets of frequencies are $\Lambda_1, \ldots, \Lambda_n$, respectively, and satisfy*

$$\forall \lambda_1, \lambda_1' \in \Lambda_1, \ldots, \forall \lambda_n, \lambda_n' \in \Lambda_n, \lambda_j \neq \lambda_j' \text{ for all } j, \tag{2.22}$$

we have

$$\det \| \lambda_1 - \lambda_1', \ldots, \lambda_n - \lambda_n' \| \neq 0.$$

Let $Z_j = \{\zeta \in \mathbb{C}^n : P_\lambda(\zeta) = 0 \ \forall \lambda \in \Lambda_j\}$.

If W is an irreducible branch of dimension strictly positive of the variety $V = \{F_1 = \cdots = F_n = 0\}$, there is $j \in \{1, \ldots, n\}$ such that $W \subseteq Z_j$.

COROLLARY 2.6. *If all the varieties Z_j are discrete, then V is also discrete.*

Remark 2.2. If one applies Proposition 2.4 to this situation one is bothered by the fact that the $\mathbb{Z}$-rank of the abelian group generated by $\Lambda_1 \cup \cdots \cup \Lambda_n$ could be very big, hence even under the conditions of Corollary 2.6 one obtains a rather bad bound for the dimension of W.

Remark 2.3. As the proof of Proposition 2.5 will show one can improve on the statement if one uses geometric properties of each Λ_j and not only the relative position of the different Λ_j.

Proof of Proposition 2.5. Let Q_j be the product of the polynomial coefficients of the exponential-polynomial F_j, denote $Q = Q_1 \cdots Q_n$, and let W be a branch of V of dimension strictly positive. We are going to show that Q is identically zero on W. The proof follows immediately out of this by a simple induction on the cardinality of the Λ_j.

Let $\zeta_0 \in W$, F_1 has the form

$$F_1(\zeta) = \sum_{\lambda \in \Lambda_1} P_\lambda(\zeta) e^{i\lambda \cdot \zeta},$$

either $P_\lambda(\zeta_0) = 0$ for all $\lambda \in \Lambda_1$ (in which case $Q_1(\zeta_0) = 0$ and hence $Q(\zeta_0) = 0$ and we are done) or there are two distinct indices $\lambda_1 = \lambda_1(\zeta_0)$, $\lambda_1' = \lambda_1'(\zeta_0)$ in Λ_1 such that $P_{\lambda_1}(\zeta_0) \neq 0$ and $P_{\lambda_1'}(\zeta_0) \neq 0$, furthermore, one can show that they can be chosen in such a way that the following inequalities hold:

$$\begin{gathered} P_{\lambda_1}(\zeta_0)\cdot P_{\lambda_1'}(\zeta_0)\neq 0, \\ \frac{1}{2}\leqslant\left|\frac{P_{\lambda_1}(\zeta_0)}{P_{\lambda_1'}(\zeta_0)}\exp[i(\lambda_1-\lambda_1')\cdot\zeta_0]\right|\leqslant 2. \end{gathered} \tag{2.23}$$

Let us assume (2.23) for the moment and continue with the proof. It follows from (2.23) that there are two constants C_1, K_1, independent of ζ_0, such that

$$\zeta_0\in W\Rightarrow |Q_1(\zeta_0)|\, e^{|(\lambda_1-\lambda_1')\cdot \operatorname{Im}\zeta_0|}\leqslant C_1(1+\|\zeta_0\|)^{K_1}. \tag{2.24}$$

One can repeat the reasoning for the same point ζ_0 with the other F_j. On the other hand, the hypothesis (2.22) implies the existence of two constants $\theta_1>0$, $\theta_2>0$ (independent of ζ_0) such that

$$\begin{aligned} \theta_1\sum_{j=1}^{n}|\operatorname{Im}\zeta_{0,j}| &\leqslant \sum_{j=1}^{n}|(\lambda_j-\lambda_j')\cdot\operatorname{Im}\zeta_0| \\ &\leqslant \theta_2\sum_{j=1}^{n}|\operatorname{Im}\zeta_{0,j}|, \end{aligned} \tag{2.25}$$

where $\zeta_0=(\zeta_{0,1},..., \zeta_{0,n})$. From (2.24) and (2.25) we obtain two constants C, K such that

$$\zeta\in W\Rightarrow |Q(\zeta)|\exp\left(\theta_1\sum_{j=1}^{n}|\operatorname{Im}\zeta_j|\right)\leqslant C(1+\|\zeta\|)^K.$$

Thanks to Proposition 2.1 we obtain that Q is identically zero in W.

Let us prove (2.23), the simplest proof consists in using a trick from [40, Lemma 2, p. 280]. Let $a_1,..., a_k$ be non-zero complex numbers such that $\sum a_j=0$, then there are two distinct indices j_1, j_2 such that

$$\tfrac{1}{2}<|a_{j_1}|/|a_{j_2}|<2.$$

If this were not true one could rearrange the a_j so that their absolute values were decreasing and conclude that

$$|a_k|\leqslant\tfrac{1}{2}|a_{k-1}|,\qquad |a_{k-1}|\leqslant\tfrac{1}{2}|a_{k-2}|,...,\qquad |a_2|\leqslant\tfrac{1}{2}|a_1|,$$

hence

$$\begin{aligned} |a_1+\cdots+a_k| &\geqslant |a_1|-(|a_2|+\cdots+|a_k|) \\ &\geqslant |a_1|\left(1-\frac{1}{2}-\frac{1}{4}-\cdots\frac{1}{2^{k-1}}\right)>0. \quad ∎ \end{aligned}$$

3

We propose to give here for an ideal $\mathcal{I}$ in A_p, finitely generated by exponential-polynomials, a necessary and sufficient condition for $\mathcal{I}$ to be s.s.d. That is, we consider m exponential-polynomials $F_1,\dots, F_m$ (with frequencies in $\mathbb{R}^n$) and try to find under which conditions does the property (1.2) hold. In what follows the sets of frequencies $\Lambda_1,\dots, \Lambda_n$ will be considered as a subset of subgroup Γ of $\mathbb{R}^n$, Γ of finite type. It is well known [48] that there are N elements $\alpha_1,\dots, \alpha_N$ in $\mathbb{R}^n$ such that

$$\Gamma = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_N. \tag{3.1}$$

In the case of rational frequencies we will always assume that $\Gamma \subseteq \mathbb{Z}^n$. We can associate to each function F_j a polynomial $p_j \in \mathbb{C}[\zeta_1,\dots, \zeta_n, X_1,\dots, X_N]$ such that, up to an exponential factor, we have

$$F_j(\zeta) = p_j(\zeta_1,\dots, \zeta_n, e^{i\alpha_1\cdot\zeta},\dots, e^{i\alpha_N\cdot\zeta}) \qquad \text{for all } \zeta \in \mathbb{C}^n. \tag{3.2}$$

Since the exponential factors are invertible in A_p we could really consider polynomials in $\mathbb{C}[\zeta_1,\dots, \zeta_n, X_1,\dots, X_N, 1/X_1,\dots, 1/X_N]$. We will use implicitly this remark in the future.

Associated to the polynomials $p_1,\dots, p_m$ we consider the algebraic variety Y in $\mathbb{C}^{n+N}$ defined by

$$Y = \{(\zeta, X): p_1(\zeta, X) = \cdots = p_m(\zeta, X) = 0\}. \tag{3.3}$$

Thanks to the above remark we can suppose that no irreducible component of Y is included in the variety $\{X_1 \cdots X_N = 0\}$.

There is a natural action of $(\mathbb{C}^*)^N$ on those algebraic subvarieties of $\mathbb{C}^{n+N}$ that have no components included in $\{X_1 \cdots X_N = 0\}$. It is the transformation which associates to the variety Y the variety

$$Y^{(\rho)} = \{(\zeta, X): p_j(\zeta, \rho_1 X_1,\dots, \rho_N X_N) = 0, j = 1,\dots, m\}, \tag{3.4}$$

where $\rho \in (\mathbb{C}^*)^N$. This transformation is an algebraic isomorphism between Y and $Y^{(\rho)}$.

Denote $F_j^{(\rho)}$ the exponential-polynomials defined by

$$F_j^{(\rho)}(\zeta) := p_j(\zeta, \rho_1 e^{i\alpha_1\cdot\zeta},\dots, \rho_N e^{i\alpha_N\cdot\zeta}) \qquad \forall \zeta \in \mathbb{C}^n. \tag{3.5}$$

We also denote $V^{(\rho)}$ the subvariety of $\mathbb{C}^n$ of common zeroes to the functions $F_j^{(\rho)}$, $j = 1,\dots, m$.

THEOREM 3.1. *Let $F_1,\ldots,F_m$ be exponential-polynomials as above, the m-tuplet $(F_1,\ldots,F_m)$ generates a s.s.d. ideal if and only if:*

$$\begin{aligned}&\exists\varepsilon_0>0,\ C_0>0,\ \delta_0>0,\ D_0>0 \text{ such that:}\\ &\forall\zeta_0\in\mathbb{C}^n\quad \forall\rho\in(\mathbb{C}^*)^N\\ &\sup_{1\leqslant j\leqslant N}|1-\rho_j|<\delta_0 e^{-D_0 p(\zeta_0)}\Rightarrow V^{(\rho)}\cap\{\|\zeta-\zeta_0\|<\varepsilon_0 e^{-C_0 p(\zeta_0)}\}\\ &\text{is discrete.}\end{aligned}\tag{3.6}$$

Proof of the sufficiency of the condition (3.6). This proof is extremely technical and that compels us to give all its details; on the other hand the reader can get a good idea of its basic principles by comparing with [11, pp. 127–129], where a particular case is dealt with.

We will denote $B(Z,r)$ the open ball of center $Z=(\zeta,X)\in\mathbb{C}^{n+N}$ and radius $r>0$. Recalling that p is the weight defined by (1.1) in $\mathbb{C}^n$, we introduce a weight P in $\mathbb{C}^{n+N}$ by

$$P(\zeta,X)=\log(1+\|\zeta\|+\|X\|)+\sum_{j=1}^{n}|\operatorname{Im}\zeta_j|.$$

We introduce further the exponential map

$$\varphi_j(\zeta,X):=X_j e^{-\alpha_j\cdot\zeta}-1,\qquad j=1,\ldots,N.\tag{3.7}$$

We are going to prove in fact that the $m+N$-tuplet $\mathbf{G}=(p_1,\ldots,p_m,\phi_1,\ldots,\phi_N)$ of elements in $A_P(\mathbb{C}^{n+N})$ is s.s.d.; i.e., for every pair (ε,C) of positive constants we can find another pair (ε_1,C_1) of positive constants such that if $\mathscr{C}$ denotes a connected component of the set

$$S(\mathbf{G};\varepsilon_1,C_1)=\left\{(\zeta,X)\in\mathbb{C}^{n+N}:\sum_{j=1}^{m}|p_j(\zeta,X)|+\sum_{j=1}^{N}|\phi_j(\zeta,X)|<\varepsilon_1 e^{-C_1P(\zeta,X)}\right\},\tag{3.8}$$

and if (ζ_0,X_0) and (ζ,X) are two points in $\mathscr{C}$, we have

$$\|\zeta-\zeta_0\|+\|X-X_0\|<\varepsilon e^{-CP(\zeta_0,X_0)}.\tag{3.9}$$

It will follow immediately that the m-tuplet $(F_1,\ldots,F_m)$ is s.s.d. in $\mathbb{C}^n$ with respect to the weight p.

Let us prove by induction on the integer $k\in\{0,\ldots,n+N-1\}$ the following result:

(3.10) *Let W be an algebraic subvariety of $\mathbb{C}^{n+N}$,* $\dim W = k$, $W \subseteq Y$, *assume that W is defined by the following algebraic equations:*

$$W = \{(\zeta, X) \in \mathbb{C}^{n+N} : q_1(\zeta, X) = \cdots = q_l(\zeta, X) = 0\}.$$

Let ε, C be two positive constants. There are two positive constants $\varepsilon_1 = \varepsilon_1(\varepsilon, C, W)$, $C_1 = C_1(\varepsilon, C, W)$ such that:

(a) *if $\mathscr{C}$ is a connected component of the set $S(W; \varepsilon_1, C_1)$,*

$$S(W; \varepsilon_1, C_1) := \left\{(\zeta, X) : \sum_{j=1}^{l} |q_j(\zeta, X)| + \sum_{j=1}^{N} |\varphi_j(\zeta, X)| < \varepsilon_1 e^{-C_1 P(\zeta, X)}\right\},$$

then $\mathscr{C}$ is a bounded subset of $\mathbb{C}^{n+N}$;

(b) *if (ζ_1, X_1), (ζ_2, X_2) are points in $\mathscr{C}$, $\mathscr{C}$ as in part* (a), *then*

$$\|\zeta_1 - \zeta_2\| + \|X_1 - X_2\| < \varepsilon e^{-CP(\zeta_1, X_1)}.$$

This result (3.10) is trivial when $k = 0$ or when W is empty. It is also clear that in (3.10) the constants ε_1, C_1 are not really dependent only on W but on the choice of generators $q_1, \ldots, q_l$ for W.

We will assume that $k \geqslant 1$ and that (3.10) is true up to dimension $k-1$. The first step is to show that we can reduce the problem to those varieties W which correspond to a prime ideal $\mathscr{P}$ in $\mathbb{C}[\zeta, X]$, and $q_1, \ldots, q_l$ are the generators of $\mathscr{P}$.

Consider hence the prime divisors $\mathscr{P}_1, \ldots, \mathscr{P}_t$ of the ideal $\mathscr{P}$ associated to W [48], for each $\mathscr{P}_s$ we have a corresponding algebraic variety W_s and polynomials $q_{s,1}, \ldots, q_{s,n_s}$ generating $\mathscr{P}_s$,

$$W_s = \{(\zeta, X) \in \mathbb{C}^{n+N} : q_{s,j}(\zeta, X) = 0, j = 1, \ldots, n_s\}.$$

To every pair (ε_2, C_2) of positive constants we can associate a pair (ζ_1, C_1) also of positive constants such that

$$\begin{aligned} &\sum_{j=1}^{l} |q_j(\zeta, X)| < \varepsilon_1 e^{-C_1 P(\zeta, X)} \Rightarrow \exists s \in \{1, \ldots, t\}, \\ &\sum_{j=1}^{n_s} |q_{s,j}(\zeta, X)| < \varepsilon_2 e^{-C_2 P(\zeta, X)} \end{aligned} \tag{3.11}$$

(it is clear that s depends on the point (ζ, X)). Suppose (3.11) does not hold, hence for each $s \in \{1, \ldots, t\}$ there is an index $j(s) \in \{1, \ldots, n_s\}$ such that

$$|q_{s,j(s)}(\zeta, X)| \geqslant \frac{\varepsilon_2}{n_s} e^{C_2 P(\zeta, X)}.$$

On the other hand, the polynomial $\prod_s q_{s,j(s)}$ is in the radical of the ideal $\mathscr{P}$ generated by $q_1,\dots, q_l$, and hence it satisfies an inequality of the form

$$\left|\prod_s q_{s,j(s)}(\zeta, X)\right|^r \leqslant A e^{BP(\zeta, X)} \sum_{j=1}^{l} |q_j(\zeta, X)|,$$

for some positive integer r, and positive constants A, B. This shows that one can in fact find ε_1, C_1 so that (3.11) holds.

We will suppose hence that (3.10) holds whenever W is a variety of dimension $k'<k$ (which is the case for the varieties $W_s \cap W_{s'}$, $s \neq s'$, which are also included in Y) and also that (3.10) is true when W is one of the varieties W_s, $s=1,\dots, t$.

Fix ε, C and let ε'_1, C'_1 denote positive constants to be fixed later satisfying the inequalities $\varepsilon'_1 < \frac{1}{2}\varepsilon_1(\varepsilon, C, W_{s_0} \cap W_{s'_0})$, $C'_1 > C_1(\varepsilon, C, W_{s_0} \cap W_{s'_0})$ for a pair of distinct indices s_0, s'_0 in $\{1,\dots, t\}$. Let $\mathscr{C}$ be a connected component of $S(W; \varepsilon'_1, C'_1)$. By the inductive hypothesis either $\mathscr{C}$ satisfies (3.9) or there is a point $Z_{\mathscr{C}} = (\zeta_{\mathscr{C}}, X_{\mathscr{C}}) \in \mathscr{C}$ such that

$$\sum_{j=1}^{n_{s_0}} |q_{s_{0,j}}(Z_{\mathscr{C}})| + \sum_{j=1}^{n_{s'_0}} |q_{s'_{0,j}}(Z_{\mathscr{C}})| \geqslant \tfrac{1}{2}\varepsilon_1 e^{-C_1 P(Z_{\mathscr{C}})}, \tag{3.12}$$

where $\varepsilon_1 = \varepsilon_1(\varepsilon, C, W_{s_0} \cap W_{s'_0})$, $C_1 = C_1(\varepsilon, C, W_{s_0} \cap W_{s'_0})$. This inequality remains valid with $\frac{1}{2}$ replaced by $\frac{1}{4}$ in a ball $B(Z_{\mathscr{C}}, \alpha e^{-AP(Z_{\mathscr{C}})})$ for some α, $A>0$, which we can choose $\alpha<\varepsilon$, $A>C$. Given a second pair s_1, s'_1 if we had taken

$$\varepsilon'_1 < \inf\{\tfrac{1}{2}\varepsilon_1(\varepsilon, C, W_{s_0} \cap W_{s'_0}), \tfrac{1}{2}\varepsilon_1(\alpha, A, W_{s_1} \cap W_{s'_1})\}$$

$$C'_1 > \sup\{C_1(\varepsilon, C, W_{s_0} \cap W_{s'_0}), C_1(\alpha, A, W_{s_1} \cap W_{s'_1})\},$$

then either $\mathscr{C}$ is a subset of $B(Z_{\mathscr{C}}, \alpha e^{-AP(Z_{\mathscr{C}})})$ or it would exist a point $Z'_{\mathscr{C}}$ in $\mathscr{C}$ where one has two inequalities:

$$\sum_{j=1}^{n_{s_0}} |q_{s_{0,j}}(Z'_{\mathscr{C}})| + \sum_{j=1}^{n_{s'_0}} |q_{s'_{0,j}}(Z'_{\mathscr{C}})| \geqslant \tfrac{1}{4}\varepsilon_1 e^{-C_1 P(Z'_{\mathscr{C}})} \tag{3.13}$$

with the same ε_1, C_1 as in (3.12), and

$$\sum_{j=1}^{n_{s_1}} |q_{s_{1,j}}(Z'_{\mathscr{C}})| + \sum_{j=1}^{n_{s_1}} |q_{s'_{1,j}}(Z'_{\mathscr{C}})| \geqslant \tfrac{1}{2}\tilde{\varepsilon}_1 e^{-\tilde{C}_1 P(Z'_{\mathscr{C}})}, \tag{3.14}$$

where $\tilde{\varepsilon}_1 = \varepsilon_1(\alpha, A, W_{s_1} \cap W_{s'_1})$, $\tilde{C}_1 = C_1(\alpha, A, W_{s_1} \cap W_{s'_1})$. By iteration we obtain two constants λ, μ such that if ε'_1, C'_1 are correctly chosen only two kinds of situations can occur for the connected components $\mathscr{C}$ of $S(W; \varepsilon_1'', C_1'')$, where ε_1'', C_1'' are any pair, $0<\varepsilon_1''<\varepsilon_1'$, $C_1''>C_1'$:

(*) either $\mathscr{C}$ satisfies (a) and (b) from (3.10) for ε, C, or

(*) there is $Z\in\mathscr{C}$ such that

$$\forall s, s'\in\{1,..., t\}, s\neq s' \Rightarrow \sum_{j=1}^{n_s} |q_{s,j}(Z_{\mathscr{C}})| + \sum_{j=1}^{n_{s'}} |q_{s',j}(Z_{\mathscr{C}})| \geqslant \lambda e^{-\mu P(Z_{\mathscr{C}})}. \tag{3.15}$$

Remark 3.1. We have just proved a version of Proposition 2.2 from [11], valid for any number of variables. That is, given a finite family $\{V_1, V_2,...\}$ of algebraic subvarieties of Y of dimension strictly smaller than k, V_j defined by $V_j=\{R_{i,j}=0\}$, and a pair (ε, C) of positive constants, there are, by the induction hypothesis, two pairs (ε_1, C_1), (η, K) such that if $\varepsilon_1'<\varepsilon_1$, $C_1'>C_1$ only two situations can occur for any connected component $\mathscr{C}$ of $S(Y; \varepsilon_1', C_1')$:

(*) either $\mathscr{C}$ satisfies the conditions (a) and (b) from (3.10 , or

(*) there is a point $Z\in\mathscr{C}$ such that

$$\forall j \qquad \sum_i |R_{ij}(Z)| + \sum_{h=1}^{N} |\varphi_h(Z)| > \eta e^{-KP(Z)}.$$

Let us return to the proof of the theorem and suppose (3.15) holds while we suppose (3.10) has been proved for all the varieties W_s, $s\in\{1,..., t\}$.

Consider two positive numbers σ, τ such that

$$\forall Z\in B(Z_{\mathscr{C}}, \sigma e^{-\tau P(Z_{\mathscr{C}})}), \forall s, s'\in\{1,..., t\}, s\neq s',$$
$$\sum_{j=1}^{n_s} |q_{s,j}(Z)| + \sum_{j=1}^{n_{s'}} |q_{s',j}(Z)| \geqslant \frac{\lambda}{2} e^{-\mu P(Z)}. \tag{3.16}$$

We suppose now that ε_1'', C_1'' have been further restricted by the condition that (3.11) holds with (ε_1'', C_1'') in the role of (ε_1, C_1) and $\varepsilon_2=\inf\{\lambda/8, \frac{1}{2}\varepsilon_1(\sigma, \tau, W_s)\,\forall s\}$. $C_2=\sup\{\mu, C_1(\sigma, \tau, W_s)\,\forall s\}$. In this case, for every $Z\in B(Z_{\mathscr{C}}, \sigma e^{-\tau P(Z_{\mathscr{C}})})$ there is an $s=s(Z)\in\{1,..., t\}$ such that

$$\sum_{j=1}^{n_s} |q_{s,j}(Z)| < \varepsilon_2 e^{-C_2 P(Z)}.$$

Because of the choice of ε_2, C_2, if we take into account (3.16) we see that the index s is independent of Z, we could as well take it as $s=s(Z_{\mathscr{C}})$. If we also impose on ε_1'', C_1'' the further condition that $\varepsilon_1''\leqslant\frac{1}{2}\varepsilon_1(\sigma, \tau, W_s)$, $C_1''\geqslant$

$C_1(\sigma, \tau, W_s)$ $\forall s$, then we would have that for all points $Z \in \mathscr{C} \cap B(Z_{\mathscr{C}}, \sigma e^{-\tau P(Z_{\mathscr{C}})})$, where as always $s = s(Z_{\mathscr{C}})$,

$$\sum_{j=1}^{n_s} |q_{s,j}(Z)| + \sum_{j=1}^{N} |\phi_j(Z)| < \varepsilon_1(\sigma, \tau, W_s)\, e^{-C_1(\sigma,\tau,W_s)P(Z)}.$$

Since we have assumed (3.10) to be valid for $W = W_s$, if we had taken the care of choosing $\sigma < \varepsilon$, $\tau > \mathscr{C}$ we see that $\mathscr{C}$ also satisfies (a), (b) from (3.10).

The above lengthy argument shows that we can limit ourselves to prove (3.10) when $W = W_s$ for $s \in \{1, ..., t\}$. Let us suppose that $\varepsilon < \varepsilon_0$, $C > C_0$, where ε_0, C_0 are the constants appearing in (3.6).

We are supposing from now on that W corresponds to a prime ideal $\mathscr{P}$ generated by the polynomials q_j, $j = 1, ..., l$, and we assume that (3.10) has been proved for $k' < k$. Let $\{\Delta_j\}_{j=1,...,L}$ be the family of all $(n+N-k)$-minors of the matrix $\|\partial q_j / \partial \zeta_i\|$, where for simplicity we have denoted by $\zeta_{n+1}, ..., \zeta_{n+N}$ the variables $X_1, ..., X_n$. By Theorem 5.3 from [30, Chap. 1, Sect. V], we have $l \geqslant n + N - k$ and, moreover,

$$\dim\{Z \in W; \Delta_j(Z) = 0, j = 1, ..., L\} < k. \tag{3.17}$$

The algebraic variety W' appearing in (3.17) could be empty.

Let us choose ε_1, C_1 arbitrarily for the moment, but satisfying $0 < \varepsilon_1 < \delta_0/2$, $C_1 > 2D_0$, where δ_0, D_0 are the constants appearing in (3.6). Let $\mathscr{C}$ be a component of the set $S(W; \varepsilon_1, C_1)$. By the induction hypothesis applied to W', there is a pair of constants $\varepsilon_1(\varepsilon, C, W')$, $C_1(\varepsilon, C, W')$ associated to (ε, C) by the condition (3.10). We can also assume that

$$\varepsilon_1 < \eta = \tfrac{1}{2}\varepsilon_1(\varepsilon, C, W'), \qquad C_1 > K = C_1(\varepsilon, C, W').$$

If for every $Z \in \mathscr{C}$ we have

$$\sum_{j=1}^{L} |\Delta_j(Z)| < \eta e^{-KP(Z)}$$

then the component $\mathscr{C}$ satisfies the conditions (a), (b) in (3.10). Hence we can assume that no matter which is the choice of (ε_1, C_1) there is a point $Z_0 \in \mathscr{C}$ such that

$$\sum_{j=1}^{L} |\Delta_j(Z_0)| \geqslant \eta e^{-KP(Z_0)}. \tag{3.18}$$

There are two constants η_1, K_1 such that

(i) $\eta_1 < (\varepsilon/2)\, e^{-C}$, $K_1 > C$,

(ii) if Z_0 is a point satisfying (3.18) then

$$\|Z-Z_0\|<\eta_1 e^{-K_1P(Z_0)} \Rightarrow \sum_{j=1}^{L} |\Delta_j(Z)| \geqslant \frac{\eta}{2} e^{-KP(Z)}.$$

After reordering the variables and the polynomials q_j we can suppose

$$\|Z-Z_0\|<\eta_1 e^{-K_1P(Z_0)} \Rightarrow |\Delta_1(Z)| \geqslant \frac{\eta}{2L} e^{-KP(Z)}, \tag{3.19}$$

where Δ_1 is the Jacobian determinant of the first $n+N-k$ polynomials q_j with respect to the first $n+N-k$ variables, let us call these variables $z=(z_1,..., z_{n+N-k})$ and denote $z'=(z_{n+N-k+1},..., z_{n+N})$ the last k variables, i.e., $Z=(z, z')$ in the reordered variables, $Z=(\zeta, X)$ in the original variables. Therefore, except for modifying conveniently η_1, K_1, we can assume that $(q_1,..., q_{n+N-k}, z_{n+N-k+1},..., z_{n+N})$ is a system of local coordinates in the ball of center Z and radius $\eta_1 e^{-K_1P(Z)}$ for all possible Z in (3.19). (All one has to obtain is the injectivity of the map $(z, z') \to (q_1,..., q_{n+N-k}, z_{n+N-k+1},..., z_{n+N})$ which follows from (3.19) and Taylor's formula, so the choice of η_1, K_1 is dictated by the polynomials q_j as well as η, K—compare with the argument in (3.21) below.) The choice of η_1, K_1 is not modified anymore.

Let us pick two further constants $\eta_0<\eta_1$, $K_0>K_1$. Since we have assumed $\varepsilon_1<\frac{1}{2}\delta_0$, $C_1>2D_0$, and $Z_0\in\mathscr{C}$, we can choose η_0, K_0 so that

$$\|Z-Z_0\|<\eta_0 e^{-K_0P(Z_0)} \Rightarrow \sup_{1\leqslant j\leqslant N} |\varphi_j(Z)|<\delta_0 e^{-D_0P(Z)}. \tag{3.20}$$

We want to show, and this is the critical point of the proof, that the algebraic variety W_0 given by

$$W_0=\{Z: q_1(Z)=\cdots=q_{n+N-k}(Z)=0\}$$

intersects the set defined by the left-hand side of (3.20) if ε_1, C_1 are chosen correctly. Recall that $Z_0=(z_0, z_0')$. Writing the Taylor expansion of $q_j(z, z_0')-q_j(z_0, z')$ and using Cramer's rule to solve a system of $n+N-k$ equations, one sees that

$$\begin{aligned}\|Z-Z_0\|<\eta_0 e^{-K_0P(Z_0)} \Rightarrow &\sum_{1}^{n+N-k} |q_j(z, z_0')-q_j(z_0, z_0')|\\ &\geqslant \|z-z_0\| \,\tilde{\eta} e^{-\tilde{K}P(Z_0)},\end{aligned} \tag{3.21}$$

where $\tilde{\eta}$ and $\tilde{K}$ have been obtained using (3.19) and the bounds we have on the q_j and their partial derivatives. We concludes that if in the $\mathbb{C}^{n+N-k}$ of equation $z'=z_0'$, we consider the ball B of center z_0 and radius $\eta_0 e^{-K_0 P(Z_0)}$, we have on the boundary ∂B simultaneous lower bounds for the functions $q_j(z, z_0') - q_j(z_0, z_0')$. These functions vanish simultaneously, of course, for $z=z_0$, and Kronecker's formula tells us that if $\sum |q_j(Z_0)|$ is sufficiently small, then the functions $q_j(z, z_0')$ have exactly one common zero in B; hence if ε_1, C_1 are chosen conveniently we will have a point $Z_1=(z_1, z_1') \in W \cap B(Z_0, \eta_0 e^{-KP(Z_0)})$ with $z_1'=z_0'$. The conditions we have imposed on the minor Δ_1 and what we have just shown tell us that only one branch of W_0 (and hence at most one branch of W) intersects the ball of center Z_1 and radius $\eta_1 e^{-K_1 P(Z_1)}$, that this branch (which we will call W_0') has dimension k, and that the coordinates z' form a system of local coordinates on W_0'. One can also see that W_0' is contained in W, if not the dimension of $W \cap W_0'$ would be strictly smaller than k and applying the Remark 3.1 we could have used a point $Z_0 \in \mathscr{C}$ such that the existence of Z_1 would be impossible. By considerations of dimension we have then

$$W_0' \cap B(Z_1, \eta_1 e^{-K_1 P(Z_1)}) = W \cap B(Z_1, \eta_1 e^{K_1 P(Z_1)}).$$

Let B' be the ball in $\mathbb{C}^k$ of center z_1' and radius $n_1 e^{-K_1 P(Z_1)}$. In this ball we consider the holomorphic functions $f_1, \ldots, f_N$ obtained by restriction of the functions $\varphi_1, \ldots, \varphi_N$ to the variety W parametrized by the coordinates z'.

Thanks to the hypothesis (3.6) and the fact that $W \subseteq Y$ and one of the minors Φ of maximal rank of the matrix $\| \partial f_j / \partial z_i \|$ is not identically zero in B', the same hypothesis allows us to conclude that $k \leqslant N$. In fact, if that were not true, by [26, Theorem 10, p. 160], the subvariety of B' defined by the equations

$$f_j(z') = f_j(z_i'), \qquad j = 1, \ldots, N \tag{3.22}$$

would not be discrete in B'; on the other hand, for every $j \in \{1, \ldots, N\}$ we have

$$f_j(z') = f_j(z_1') \Leftrightarrow \exp(-i\alpha_j \cdot \zeta)\, X_j = \varphi_j(Z_1) + 1.$$

It now follows that if the variety (3.22) is non-discrete in B', then the variety $V^{(\rho)}$ will have non-discrete intersection with $B(\zeta_1, \varepsilon_0 e^{-C_0 p(\zeta_1)})$ (ζ_1 is the ζ-coordinate of Z_1), where $\rho_j = 1 + \varphi_j(Z_1)$. By (3.20) this would contradict (3.6).

Now we can compute the minor Φ by the chain rule and using the fact that the f_j are the restrictions of the ϕ_j to the algebraic variety W, and we see that there are indices $s_1, \ldots, s_k \in \{1, \ldots, N\}$ and a polynomial R non-identically zero on W such that, in B', we have

$$\Delta_1^k(Z)\, \Phi(z') = R(Z) \exp\left(-i \sum_{j=1}^{k} \alpha_{s_j} \cdot \zeta\right), \tag{3.23}$$

where, as always, $Z=(\zeta, X)$ corresponds to the point of W associated to $z' \in B'$.

Since the variety $W \cap \{R=0\}$ has dimension strictly smaller than k we can assume, by the Remark 3.1, that we have

$$|R(Z_0)| \geqslant \eta e^{-KP(Z_0)}.$$

One can assume also that η_0, K_0 are such that

$$\|Z-Z_0\| < \eta_0 e^{-K_0 P(Z_0)} \Rightarrow |R(Z)| \geqslant \frac{\eta}{2} e^{-KP(Z)}.$$

We have then

$$|R(Z_1)| \geqslant \frac{\eta}{2} e^{-KP(Z_1)}. \tag{3.24}$$

The inequality (3.24) and the identity (3.23) give us a lower bound for $\Phi(z_1')$. Using now the reasoning based on the Taylor expansion of the functions $f_j(z')-f_j(z_1')$, as we did to prove (3.21), we construct a ball $B'' \subseteq B'$ of center z_1' and radius $\eta_2 e^{-K_2 P(Z_1)}$ such that on $\partial B''$ we have

$$\sum_{j=1}^{N} |f_j(z')-f_j(z_1')| \geqslant \eta_3 e^{-K_3 P(Z_1)}, \tag{3.25}$$

where η_2, η_3, K_2, K_3 can be explicitly determined in terms of the η, K, η_1, K_1, the coefficients of the polynomials q_j, R, and the size of the α_j.

Choosing carefully η_0, K_0 (i.e., imposing extra conditions on ε_1, C_1) we can assume

$$\sum_{1}^{N} |f_j(z_1')| < \frac{\eta_3}{2} e^{-K_3 P(Z_1)}. \tag{3.26}$$

This essentially ends the proof, but let us just finish up the last details.

Recall that $(q_1,..., q_{n+N-k}, z_{n+N-k+1},..., z_{n+N})$ form a system of local coordinates in the ball B, $B=B(Z_1, \eta_1 e^{-K_1 P(Z_1)})$. We can always go back to the original coordinates, and the quantitative part of the reasoning can always be taken care of by just using the estimate (3.19) for Δ_1 in B. Let us define a "box"

$$\left\{ \sum_{j=1}^{n+N-k} |q_j| < a \right\} \times B''. \tag{3.27}$$

Once a has been chosen, one can choose ε_1, C_1 in such a way that Z_0

belongs to the box (3.27) (i.e., $\sum_{j=1}^{n+N-k} |q_j(Z_0)| < a$, since $z'_0 = z'_1$ is the center of B''). We need to assume that in $\{\sum |q_j| < a\} \times \partial B''$ we have

$$\sum_1^N |f_j(z')| \geq \frac{\eta_3}{4} e^{-K_3 P(Z_1)}.$$

The choice of a can be done effectively since on $W = \{q_1 = \cdots = q_{n+N-k} = 0\}$ we have

$$\sum_1^N |f_j(z')| \geq \frac{\eta_3}{2} e^{-K_3 P(Z_1)} \qquad \forall z' \in \partial B''$$

by (3.25) and (3.26). In fact, a is of the form $\eta_4 e^{-K_4 P(Z_1)}$. On the remaining portion of the boundary of the box we will have then

$$\sum_{j=1}^{n+N-k} |q_j| = \eta_4 e^{-K_4 P(Z_1)}.$$

After making certain that $\varepsilon_1 < \inf\{\eta_3/8, \eta_4/8\}$, $C_1 > \sup\{K_3, K_4\}$, we see that the component $\mathscr{C}$ remains necessarily in the interior of the box, hence is a subset of the ball of center Z_0 and radius $\varepsilon e^{-CP(Z_0)}$, since we had already chosen η_1, K_1, η_0, K_0 so that this was precisely the case.

Proof of the necessity of the condition (3.6). Since the m-tuplet $F_1, ..., F_m$ is s.s.d. there is a pair (ε_1, C_1) such that every connected component of $S(F_1, ..., F_m; \varepsilon_1, C_1)$ has diameter less than one. There is a pair (δ_1, D_1) such that

$$\sup_j |1 - \rho_j| < \delta_1 e^{-D_1 p(\zeta)}$$

and

$$F_j^{(\rho)}(\zeta) = 0 \quad \forall j \Rightarrow \sum_1^m |F_j(\zeta)| < \frac{\varepsilon_1}{2} e^{-C_1 p(\zeta)}.$$

By the properties of the weight p, there is a pair (δ_0, D_0) such that

$$\forall \zeta_0 \in \mathbb{C}^n \quad \forall \zeta \in B(\zeta_0, 2), \qquad \delta_0 e^{-D_0 p(\zeta_0)} < \delta_1 e^{-D_1 p(\zeta)}.$$

Hence, if $V^{(\rho)}$ (with $\sup |1 - \rho_j| < \delta_0 e^{-D_0 p(\zeta_0)}$) intersects the ball $B(\zeta_0, 1)$, every point of $V^{(\rho)} \cap B(\zeta_0, 2)$ will be in $S(F_1, ..., F_m; \varepsilon_1, C_1)$ and $V^{(\rho)}$ should have a connected component which escapes from $B(\zeta_0, 2)$ if $V^{(\rho)} \cap B(\zeta_0, 2)$ is not discrete. Hence $S(F_1, ..., F_m; \varepsilon_1, C_1)$ will have a component of diameter bigger than one, which is impossible. ∎

Remark 3.2. Recall that in the sense of Berenstein and Taylor [9], an

m-tuplet F of elements of A_p is "jointly slowly decreasing" if there are constants $\varepsilon_1, C_1, K_1, K_2$ such that the connected components of $S(\mathbf{F}; \varepsilon_1, C_1)$ are bounded and, if ζ, ζ' are points in the same component,

$$p(\zeta') \leqslant K_1 \, p(\zeta) + K_2.$$

If the entries $F_1, ..., F_m$ of $\mathbf{F}$ are exponential-polynomials, then the proof of the necessity of (3.6) shows that given $r > 0$ there are δ_0, D_0 such that

$$\| 1 - \rho \| < \delta_0 e^{-D_0 p(\zeta_0)} \Rightarrow V^{(\rho)} \cap B(\zeta_0, r) \text{ is discrete.}$$

That shows that condition (3.6) is satisfied and hence for such m-tuplets, it is equivalent to be s.s.d. and to be jointly s.d.

Let us point out that in the proof of Theorem 3.1 one finds also a proof of the following:

PROPOSITION 3.2. *Let $\mathscr{J}$ be the algebraic ideal in $C[\zeta, X]$ asociated to the exponential-polynomials $F_1, ..., F_m$ (via the polynomials $p_1, ..., p_m$). Let $\mathscr{Q}_1, ..., \mathscr{Q}_r$ be the radicals of the different primary components of $\mathscr{J}$ and let $\mathscr{T}_1, ..., \mathscr{T}_r$ be the ideals in A_p generated by the corresponding exponential-polynomials. Then, $\mathscr{T}$ is s.s.d. if and only if $\mathscr{T}_1, ..., \mathscr{T}_r$ are s.s.d.*

Remark 3.3. Following the proof of Theorem 3.1 one can see that under the condition (3.6), one can make explicit the relation between the pairs (ε, C) and (δ, D) that appear in Definition 1.2. Namely, there are three constants $\beta > 0$, $B > 0$, $k \in \mathbb{N}^*$ such that δ, D can be chosen as follows:

$$\begin{aligned} \delta &= \beta \varepsilon^k \\ D &= kC + B \end{aligned} \tag{3.28}$$

(we assume $\varepsilon \leqslant 1$).

In Section 8, we will need to study exponential-polynomials depending on k parameters in an algebraic way; more precisely, these are formal expressions which can be considered as functions from $\mathbb{C}$ to $\overline{\mathbb{C}(u_1, ..., u_k)}$ (where $\overline{\mathbb{C}(u)}$ denotes an algebraic closure of $\mathbb{C}(u)$) of the form

$$\zeta \to F(\zeta) = \sum_{\lambda \in \Gamma} \left(\sum_{l} A_{\lambda, l} \zeta^l \right) e^{i\lambda \cdot \zeta};$$

here $\{A_{\lambda, l}\}$ denotes a family of elements in $\overline{\mathbb{C}(u)}$ and Γ is a fixed subgroup of $\mathbb{R}^n$.

Let us consider m such exponential-polynomials $F_1, \dots, F_m$, with frequencies in Γ, each F_j being of the form

$$F_j(\zeta) = \sum_{\lambda \in \Gamma} \left(\sum_l A_{\lambda,l}^{(j)} \zeta^l \right) e^{i\lambda \cdot \zeta}.$$

We may associate through the procedure (3.2) to such F_j polynomials p_j in $\overline{\mathbb{C}(u)}[\zeta, X]$.

Let us use here the terminology of [48, II, Sect. 32, p. 49]. As soon as u is, in $\mathbb{C}^k$, an allowable system of argument values of the elements $(A_{\lambda,l}^{(j)})$ (which is equivalent for u to be outside an algebraic hypersurface V of $\mathbb{C}^k$ depending on the $A_{\lambda,l}^{(j)}$), one may introduce the exponential-polynomials of m variables

$$F_j(u)(\zeta) = \sum_{\lambda \in \Gamma} \left(\sum_l A_{\lambda,l}(u) \zeta^l \right) e^{i\lambda \cdot \zeta}, \qquad j = 1, \dots, m,$$

where the complex numbers $A_{\lambda,l}^{(j)}(u)$ are function values for the $A_{\lambda,l}^{(j)}$ belonging to the allowable arguments $(u_1, \dots, u_k)$ [48]. Of course, this definition is not quite unique, for one has many possible choices for the numbers $A_{\lambda,l}^{(j)}(u)$ when u is fixed in $\mathbb{C}^k$. Anyway, when the $A_{\lambda,l}^{(j)}(u)$ have been chosen, one can also define as elements in $\mathbb{C}[\zeta, x]$ the polynomials $p_j(u, \zeta, X)$ associated through (3.2) to the exponential-polynomials $F_j(u)$.

In all the following, "u generic" will mean "u outside a countable union of algebraic hypersurfaces W_l of $\mathbb{C}^k$," with $V \subset W_l$.

We can now state the following proposition:

PROPOSITION 3.3. *Let $F_1, \dots, F_m$ be exponential-polynomials depending on parameters $(u_1, \dots, u_k)$ such that the condition (3.6) holds for u generic for the exponential-polynomials $F_j(u)(\zeta)$ with constants ε_0, C_0, δ_0, D_0 independent of u and of the determinations of the numbers $A_{\lambda,l}^{(j)}(u)$ among the function values for the $A_{\lambda,l}^{(j)}$ belonging to the allowable argument u. Denoting by φ_j the functions defined by (3.7) and P the weight in $\mathbb{C}^{n+N}$, $P(\zeta, X) = \operatorname{Log}(1 + \|\zeta\| + \|x\|) + \sum_1^n |\operatorname{Im} \zeta_j|$, there is a non-zero polynomial $R \in \mathbb{C}[u]$ and three positive constants β, B, κ such that, given $0 < \varepsilon \leqslant 1$, $c > 0$, if u is generic and*

$$\delta = \beta \frac{|R(u)|}{(1 + \|u\|)^\kappa} \varepsilon^\kappa, \qquad D = \kappa C + B$$

then every connected component $\mathscr{C}$ of the open subset of $\mathbb{C}^{n+N}$ defined by

$$S'_u(p_1, \dots, p_m, \varphi_1, \dots, \varphi_N; \delta, D)$$
$$= \left\{ (\zeta, X) : \sum_1^m |P_j(u, \zeta, X)| + \sum_1^N |\varphi_j(\zeta, X)| < \zeta e^{-DP(\zeta, X)} \right\}$$

satisfies

$$\forall(\zeta, X) \in \mathscr{C}, \forall(\zeta', X') \in \mathscr{C}, \qquad \|\zeta - \zeta'\| + \|x - x'\| < \varepsilon e^{-CP(\zeta, X)}.$$

Proof. The method of proof of Proposition 3.3 is exactly the same as that of Theorem 3.1. The ring $\mathbb{C}[\zeta, X]$ being replaced by $\overline{\mathbb{C}(u)}[\zeta, X]$, the induction is always on the dimension (in $\overline{\mathbb{C}(u)}[\zeta, X]$) of the ideals containing the original ideal generated by $p_1,..., p_m$. The only extra thing we need to study is what happens on the initial point of the induction, that is, when the ideal generated by the $q_j(\zeta, X)$ is of dimension 0. But in this case, thanks to Noether's normalization lemma [48], there are in the ideal $n + N$ irreducible polynomials in $\overline{\mathbb{C}(u)}[\zeta, X]$ of the form

$$\sum_0^{\nu_j} A_{k,j} \zeta_j^k, \qquad j = 1,..., n$$

and

$$\sum_0^{\nu_{j+n}} A_{k,j+n} X_j^k, \qquad j = 1,..., N.$$

Let us choose allowable arguments for all the elements of $\overline{\mathbb{C}(u)}$ written above and function values belonging to these arguments.

By a well-known lemma due to Polya [7, 20], if we denote by ν the maximum of the degrees of the above polynomials ($\nu = \max\{\nu_j\}$), there is a positive constant α which depends only on ν such that if $0 < \varepsilon < 1$ is given, there is for any $z \in \mathbb{C}$ a real number $0 < r(z) \leqslant \varepsilon$ such that

$$\min_{|w - z| = r(z)} \left| \sum A_{k,j}(u)\, w^k \right| \geqslant \alpha \varepsilon^\nu \,|A_{\nu_j,j}(u)|.$$

To arrive at the final estimation one needs only to observe that if we consider the product r of all the $A_{\nu_{j,j}}$ we obtain an element R of $\mathbb{C}[u]$ and a constant $L > 0$ such that

$$|R(u)| \leqslant L \min_j \{|A_{\nu_{j,j}}(u)|\}(1 + \|u\|)^L. \quad \blacksquare$$

Before going any further we give here an immediate application of Theorem 3.1 and Proposition 2.5.

PROPOSITION 3.4. *Let $\mu_1,..., \mu_n$ be n distributions in $\mathbb{R}^n$ with finite supports $A_1,..., A_n$. Assume the sets $A_1,..., A_n$ satisfy the condition* (2.22) *and write down the $\mu_1,..., \mu_n$ in the form*

$$\mu_j = \sum_{\lambda \in A_j} p_\lambda\left(i \frac{\partial}{\partial x}\right) \delta_\lambda.$$

Suppose that the algebraic subvarieties of $\mathbb{C}^n$ *given by*

$$Z_j = \{p_\lambda = 0\ \forall \lambda \in \Lambda_j\}$$

are all of dimension $\leqslant 0$.

Then the ideal generated by the $\hat{\mu}_1, ..., \hat{\mu}_n$ *is s.s.d.*

We end this section with the remark that if one is interested in the analytic functionals corresponding to exponential-polynomials with complex frequencies, every result we have stated holds after replacing the weight p by the radial weight $\|\zeta\|$. The auxiliary weight P that appears in the proof of Theorem 3.1 would this time be the weight $P(\zeta, X) = \log(1 + \|X\|) + \|\zeta\|$.

4

Theorem 3.1, which we have just proved, reduces us to studying the set of elements $(\rho_1, ..., \rho_N) \in (\mathbb{C}^*)^N$, which we will call *exceptional*, for which the variety $V^{(\rho)}$ defined by (3.5) has dimension bigger than or equal to one. Since the dimension of the algebraic varieties $Y^{(\rho)}$ in $\mathbb{C}^{n+N}$ remains constant, one could ask whether there are simple conditions about the algebraic variety Y which imply that the corresponding analytic variety V is discrete.

Let us recall that the relation between the varieties V and Y seems very simple:

$$V = \{\zeta \in \mathbb{C}^n : \exists X \in \mathbb{C}^N \text{ such that } (\zeta, X) \in Y \cap \mathrm{Exp}\}, \tag{4.1}$$

where Exp is the n-dimensional submanifold of $\mathbb{C}^{n+N}$ defined by the equations

$$X_1 = e^{i\alpha_1 \cdot \zeta}, ..., X_N = e^{i\alpha_N \cdot \zeta}.$$

When V is discrete but not empty, in the set of varieties Y such that (4.1) is satisfied one can find an algebraic variety Y_0, not necessarily unique, of minimal dimension, and one has then

$$\dim Y_0 \leqslant N. \tag{4.2}$$

In fact, since Y_0 is of minimal dimension and V is non-empty, one can always assume there is an irreducible branch Y_0' of Y_0, with $\dim Y_0' = \dim Y_0$, and such that Y_0' intersects Exp in a non-singular point of Y_0'. At that point one can use the formula [30, Proposition 7.1] which gives a lower bound of the intersection at a point of two analytic varieties in $\mathbb{C}^{n+N}$:

$$\dim(Y_0' \cap \mathrm{Exp}) \geqslant \dim Y_0' + \dim(\mathrm{Exp}) - (n+N).$$

By Remmert's theorem the dimension of $Y_0' \cap \mathrm{Exp}$ at the point we are considering is the same as the dimension of its projection V in $\mathbb{C}^n$, hence we have

$$0 \geqslant \dim Y_0' - N$$

which is what we wanted to show.

The problem about the variety Y_0 we have just introduced is that it is not at all related to the generators of the original idea $\mathscr{T}$. We can, on the other hand, consider first the irreducible components of Y, then the singular varieties of those components; by repeating this process, one finds a variety Y_1 which satisfies both (4.1) with Y_1 instead of Y and (4.2) with Y_1 instead of Y_0.

On the other hand, the condition $\dim Y \leqslant N$ does not ensure that V is discrete. Let us give the following example ($n = N = 2$):

$$F_1(\zeta) = \zeta_1 - \zeta_2$$
$$F_2(\zeta) = e^{i\zeta_1} - e^{i\zeta_2},$$

in this case V is of dimension 1 in $\mathbb{C}^2$ and Y is the algebraic variety of $\mathbb{C}^4$ defined by

$$\zeta_1 = \zeta_2, \qquad X_1 = X_2$$

which is an irreducible variety (also smooth) of dimension 2.

Modifying this example we can see that in general there are exceptional values of ρ even though (4.2) is satisfied for Y. For instance, let us consider in $\mathbb{C}^2$

$$F_1(\zeta) = \zeta_1 - \zeta_2$$
$$F_2(\zeta) = e^{i\zeta_1} - 2e^{i\zeta_2},$$

then $\{\rho \in (\mathbb{C}^*)^2 : \rho_1 = 2\rho_2\}$ is the exceptional set.

Even under restrictive conditions on Y the problem of the discreteness of V as well as the nature of the exceptional set appears to be tied to arithmetical conditions on the coefficients of the exponential-polynomials, even when the frequencies are rational. To give a more precise idea let us recall here Schanuel's conjecture [3]:

Given n numbers $y_1, ..., y_n$ $\mathbb{Q}$-linearly independent, the trancendency degree over $\bar{\mathbb{Q}}$ of the extension $\bar{\mathbb{Q}}(y_1, ..., y_n, e^{y_1}, ..., e^{y_n})$ is at least n.

Let us admit Schanuel's conjecture for $n=2$. (For $n=1$ it is true. It is just the Gelfond–Schneider theorem.) Assume $F_1, ..., F_m$ are exponential-polynomials of two variables, not all zero at the origin, with integral frequencies and such that the variety Y is irreducible, dim $Y=1$, and Y is defined over $\bar{\mathbb{Q}}$. Furthermore, let us assume, so that the problem is one really of two variables, that Y is not included in any hyperplane of $\mathbb{C}^4$ of the form

$$r_1\zeta_1 + r_2\zeta_2 = 0, \qquad (r_1, r_2) \in \mathbb{Q}^2 \backslash (0).$$

Under all these conditions we can conclude that V is empty. If not, let $(y_1, y_2) \in V$, and the transcendency degree of $\bar{\mathbb{Q}}(y_1, y_2, e^{iy_1}, e^{iy_2})$ is at most 1 since the point $(y_1, y_2, e^{iy_1}, e^{iy_2}) \in Y$. By Schanuel's conjecture there are rationals r_1, r_2 not both zero such that $r_1 y_1 + r_2 y_2 = 0$. Now, the algebraic variety $Y \cap \{r_1\zeta_1 + r_2\zeta_2 = 0\}$ is a variety of dimension 0 defined over $\bar{\mathbb{Q}}$. Hence the pairs (y_1, e^{iy_1}) and (y_2, e^{iy_2}) belong to $\bar{\mathbb{Q}}^2$ which contradicts the theorem of Gelfond and Schneider since $(y_1, y_2) \neq (0, 0)$.

This group of very simple examples leads us to pose the following problem.

PROBLEM 1. Given m exponential-polynomials of n variables with frequencies in $\mathbb{Q}^n$ and algebraic coefficients which define a variety V discrete (or empty), is the ideal generated by $F_1, ..., F_m$ s.s.d.?

When $n=1$, the ideal generated by a single exponential-polynomial with real frequencies (and a posteriori one generated by any finite number of exponential-polynomials) is s.s.d. [20, 9]. Hence the answer to Problem 1 in this case is positive without restrictions on the frequencies or on the coefficients.

When $n=2$, we have given a positive answer to this problem when m is also equal to 2 and without any restriction on the coefficients [11]. In the next section we will show that the condition $m=2$ is not necessary. On the other hand, the conditions on the frequencies are necessary, even when $n=2$. For instance, the pair $\cos\zeta_1$, $\cos\lambda\zeta_1$, $\lambda \notin \mathbb{Q}$, considered in $A_p(\mathbb{C}^2)$ is s.s.d. if and only if λ is not a Liouville number. In fact, since λ is irrational the spectrum V is empty; if this pair was s.s.d., the remarks after (1.7) show that we would have a Nullstellensatz, i.e., a pair of elements in $A_p(\mathbb{C}^2)$ such that

$$1 = G_1(\zeta)\cos\zeta_1 + G_2(\zeta)\cos\lambda\zeta_1. \tag{4.3}$$

Taking $\zeta_2 = 0$, one sees that one would have a Nullstellensatz in dimension 1 for the functions $\cos\zeta$, $\cos\lambda\zeta$, which is equivalent, in this case, to λ being non-Liouville (cf. the example given in the Introduction after (0.1)).

Modifying the same example, we see the necessity to impose conditions on the coefficients in Problem 1, even when the frequencies are rational. In fact, consider in $\mathbb{C}^3$ the following three exponential-polynomials, with integral frequencies and empty variety V:

$$F_1(\zeta)=\cos\zeta_1, \qquad F_2(\zeta)=\cos\zeta_2, \qquad F_3(\zeta)=\zeta_2-\lambda\zeta_1 \qquad (\lambda\in\mathbb{R}\setminus\mathbb{Q}).$$

The ideal generated by F_1, F_2, F_3 being s.s.d. is equivalent to the Nullstellensatz for this triplet, which by restriction to the line $\zeta_3=0$, $\zeta_2-\lambda\zeta_1=0$ implies a Nullstellensatz in dimension 1 for the pair $\cos\zeta$, $\cos\lambda\zeta$, hence one has to impose conditions on λ. If λ is algebraic it will be non-Liouville and we can solve (4.3) with $G_1(\zeta_1)$, $G_2(\zeta_1)\in A_p(\mathbb{C})$, so we will have

$$\begin{aligned}1&=G_1(\zeta_1)\cos\zeta_1+G_2(\zeta_1)\cos\lambda\zeta_1\\ &=G_1(\zeta_1)\cos\zeta_1+(G_2(\zeta_1)\cos(\lambda\zeta_1-\zeta_2))\cos\zeta_2\\ &\quad+\left(G_2(\zeta_1)\sin\zeta_2\frac{\sin(\lambda\zeta_1-\zeta_2)}{\lambda\zeta_1-\zeta_2}\right)(\zeta_2-\lambda\zeta_1)\end{aligned}$$

which is the Nullstellensatz for the above triplet.

These examples show that to obtain a positive answer to Problem 1 it is necessary to solve the following:

Problem 2. Let $F_1,\dots,F_m$ be m exponential-polynomials in $\mathbb{C}^n$ with frequencies in $\mathbb{Q}^n$, algebraic coefficients, and empty spectrum V, are there m elements $G_1,\dots,G_m$ in $A_p(\mathbb{C}^n)$ such that

$$1=F_1G_1+\cdots+F_mG_m?$$

It is easy to see that Problem 2 has a positive answer for $n=1$ (see Section 7), but we do not know the answer for any other value of n.

Let us add to this list a problem similar to the above and which has been posed by Ehrenpreis.

Problem 3. Given an exponential-polynomial F of a single variable with algebraic coefficients and real algebraic frequencies, do the distinct zeroes stay away from each other? More precisely, are there positive constants c, N such that

$$F(\zeta)=F(\zeta')=0, \qquad \zeta\neq\zeta'\Rightarrow|\zeta-\zeta'|>ce^{-Np(\zeta)}? \tag{4.4}$$

Using the work of Polya on zeros of exponential-polynomials in the last inequality the factor $e^{-Np(\zeta)}$ can be replaced by $(1+|\zeta|)^{-M}$ for some

$M>0$. A positive solution to this problem will have several applications in Harmonic Analysis. The first one is that the variety $V=\{F=0\}$ will be an interpolating variety in the sense of [10]. The second one is that, under the additional assumption that the spectrum is real and simple, every continuous solution ϕ of the equation $\mu * \phi = 0$ (where $\hat{\mu} = F$) must be almost-periodic [23].

When Γ denotes, as above, the subgroup of $\mathbb{R}$ generated by the frequencies of F and rank $\Gamma = 1$, then a positive answer to Problem 3 is equivalent to a positive answer in dimension 1 to Problem 1. When rank $\Gamma = 2$, under the additional hypothesis that V is real and simple, it was shown by F. Gramain [23] that the answer to Problem 3 is also positive.

5

We give here some applications, in the case of two variables, of the results in Sections 2 and 3.

Let us give first a generalization of the main theorem in [11].

THEOREM 5.1. *Let $\mathscr{I}$ be a finitely generated ideal in $A_p(\mathbb{C}^2)$, generated by exponential-polynomials with rational frequencies, assume also that the spectrum V of $\mathscr{I}$ is discrete or empty. Then the ideal $\mathscr{I}$ is s.s.d.*

Proof. We can assume that the generators of $\mathscr{I}$ are m exponential-polynomials $F_1, \ldots, F_m$ with frequencies in $\mathbb{N}^2$. Consider the algebraic variety Y in $\mathbb{C}^4$ associated to $F_1, \ldots, F_m$ via (3.3), and defined by polynomials $p_1, \ldots, p_m$. We can suppose that none of these polynomials are divisible by X_1 or X_2, in fact, by Proposition 3.2 we can assume that Y is irreducible and the p_j generate a prime ideal $\mathscr{I}$.

We are going to show that there is a constant δ, $0<\delta<1$, such that

$$\sup_{j=1,2} |1-\rho_j| < \delta \Rightarrow \dim V^{(\rho)} \leqslant 0. \tag{5.1}$$

The theorem will then follow from Theorem 3.1.

If the algebraic variety Y has dimension bigger than or equal to 3, there is a polynomial $q \in \mathbb{C}[\zeta, X]$ dividing all the polynomials p_j, which is impossible (it would even contradict the discreteness of V). Hence $\dim Y \leqslant 2$.

Suppose $\rho \in (\mathbb{C}^*)^2$ is an exceptional value. Let us consider an irreducible branch $W^{(\rho)}$ of dimension 1 of $V^{(\rho)}$ (the case of $\dim V^{(\rho)} = 2$ being impossible since the $F_j^{(\rho)}$ are non-zero). By the Proposition 2.4 there are $(r_1, r_2) \in \mathbb{Q}^2 \backslash (0)$ and $\gamma \in \mathbb{C}$ such that

$$W^{(\rho)} \subseteq \{\zeta \in \mathbb{C}^2 : r_1 \zeta_1 + r_2 \zeta_2 = \gamma\}. \tag{5.2}$$

By dimensionality considerations we obtain that $W^{(\rho)}$ is the line

$$r_1\zeta_1 + r_2\zeta_2 = \gamma. \tag{5.3}$$

It follows from [11, Remark 6, p. 122], since the variety V is discrete, that the pair (r_1, r_2) can be taken to lie in finite subset $\mathscr{F}$ of $\mathbb{Q}^2$ which is related only to the frequencies of the exponential-polynomials $F_1,..., F_m$. We will prove this in a more general context in Proposition 7.1.

Let us assume first that the polynomial $r_1\zeta_1 + r_2\zeta_2 - \gamma$ does not belong to the algebraic ideal $\mathscr{I}$ generated by $p_1,..., p_m$; hence the subvariety of $\mathbb{C}^4$ $\{p_1 = \cdots = p_m = r_1\zeta_1 + r_2\zeta_2 - \gamma = 0\}$ has dimension at most 1 and one can use another time Proposition 2.4 and obtain a second line $s_1\zeta_1 + s_2\zeta_2 - \gamma' = 0$ containing also $W^{(\rho)}$ and such that $r_1s_2 - r_2s_1 \neq 0$, and this leads to a contradiction. Hence, in order that $W^{(\rho)}$ could exist, one must have $r_1\zeta_1 + r_2\zeta_2 - \gamma \in \mathscr{I}$. This is only possible for fixed (r_1, r_2) and for a single value of γ, otherwise $1 \in \mathscr{I}$, $Y = \emptyset$, and, a fortiori, $W^{(\rho)} = \emptyset$.

We conclude that $W^{(\rho)}$ could only be a line belonging to a certain finite set of lines, independent of ρ. If there were a sequence of exceptional values $\rho_k, \rho_k \to (1, 1)$, one could extract a subsequence such that $W^{(\rho_k)}$ is stationary, namely, a line W_0, with equation of the form (5.3). But on this line W_0 all the $F_j^{(\rho_k)} \equiv 0$, hence by letting $k \to \infty$ we have

$$F_1 \equiv \cdots F_m \equiv 0 \qquad \text{on } W_0$$

contradicting the discreteness of V. ▮

Remark 5.1. One can in fact prove, decomposing the variety $Y = \{p_1 = \cdots = p_m = 0\}$ in irreducible components and taking the smallest δ corresponding to the different components, that (5.1) is always valid under the assumptions of Theorem 5.1. We can in fact go further and see that there are no exceptional ρ unless there is an equation of the form (5.3), considered in $\mathbb{C}^4$, which is satisfied on a whole irreducible component of Y. Looking at this in more detail, one arrives at the conclusion that there is a finite number of pairs (r_1, r_2) in $\mathbb{Z}^2\backslash(0)$ and non-zero complex numbers $\gamma \neq 1$ such that every ρ exceptional satisfies one of the equations

$$\rho_1^{r_1}\rho_2^{r_2} = \gamma.$$

Let us now give an application of Theorem 3.1 to a type of system of convolution equations proposed by J. Delsarte [18]. We need a certain amount of extra notation. Q denotes a convex compact polygon with non-empty interior in $\mathbb{R}^2$. Its vertices ordered counterclockwise are $A_1,..., A_n$.

We adopt the convention $A_{n+1}=A_1$. We fix two n-tuplets $(a_1,..., a_n)$ and $(b_1,..., b_n)$ in $\mathbb{C}^n$ satisfying the condition

$$\prod_{j=1}^{n} (a_j b_{j+1} - a_{j+1} b_j) \neq 0 \tag{5.4}$$

(with the convention $a_{n+1}=a_1$, $b_{n+1}=b_1$).

One considers two distributions with support in Q of the form

$$\begin{aligned} \mu &= \sum_{j=1}^{n} a_j \delta_{A_j} + \sigma + \varphi \\ \nu &= \sum_{j=1}^{n} b_j \delta_{A_j} + \tau + \psi, \end{aligned} \tag{5.5}$$

where δ_{A_j} denotes the Dirac mass at the point A_j, σ, τ are two distributions with finite support in the interior $\mathring{Q}$ of the polygon Q, and φ, ψ are two C^∞ functions with compact support contained in Q. The case not considered in [9, 18, 41] is the case where the order of at least one of the distributions σ, τ is strictly positive, and this case seems to escape the previously known methods. We prove the following theorem.

THEOREM 5.2. *Let μ, ν be two distributions of the form* (5.5), *then the ideal $\mathscr{T}$ generated by $\hat{\mu}$, $\hat{\nu}$ is s.s.d. in $A_p(\mathbb{C}^2)$.*

Proof. We first assume that $\varphi = \psi = 0$, hence we are in the case that $\hat{\mu}$, $\hat{\nu}$ are exponential-polynomials with frequencies in $\mathbb{R}^2$. We are going to show that in this case there are no exceptional values ρ, hence the theorem will follow immediately from Theorem 3.1. Fix $\rho \in (\mathbb{C}^*)^N$, where N denotes the rank of the group Γ associated to $\hat{\mu}$ and $\hat{\nu}$ by (3.1).

Let us consider a unit vector $\mathbf{u}$ in $\mathbb{R}^2$, and after a rotation we can assume that its direction is that of the x-axis. Thanks to the condition (5.4) one can find a linear combination of the measures $\sum a_j \delta_{Aj}$ and $\sum b_j \delta_{Aj}$ such that the support of this new measure contains the point $(\alpha, 0)$ as the only point in the support with maximal abscissa. Using exactly the same linear combination one finds a function in the ideal $\mathscr{T}^{(\rho)}$ of the form

$$e^{i\alpha\zeta_1}\left(1 + \sum P_k(\zeta)\, e^{i(\alpha_k \zeta_1 + \beta_k \zeta_2)}\right), \tag{5.6}$$

where all the $\alpha_k < 0$. Hence there are three positive constants c, C, T such that if

$$|\zeta_2| \leqslant c\,|\zeta_1| \qquad \text{and} \qquad \operatorname{Im} \zeta_1 \leqslant -T$$

the set of zeros of the function (5.6) is contained in a logarithmic strip of the form

$$|\operatorname{Im} \zeta_1| \leqslant C \log(1 + |\zeta_1|).$$

Using a compactness argument (whith respect to the unit sphere of directions $\mathbf{u}$) we see that there is a constant C_0 such that the variety $V^{(p)}$ is contained in a set of the form

$$\|\operatorname{Im} \zeta\| \leqslant C_0(1 + \log(1 + \|\zeta\|)). \tag{5.7}$$

We can then appeal to Corollary 2.3 and conclude that $V^{(p)}$ is discrete. That shows that the pair $(\hat{\mu}, \hat{\nu})$ is s.s.d. in the case $\varphi \equiv \psi \equiv 0$.

In order to finish the proof let us denote $\check{Q}$ the polygon with vertices $-A_1, ..., -A_n$ and $H_{\check{Q}}$ the indicator function of the set $\check{Q}$ (i.e., $H_{\check{Q}}(x) = \max\{x \cdot y : y \in \check{Q}\}$ for $x \in \mathbb{R}^2$). What we have shown by (5.6) is that there are two positive constants K and k such that

$$\begin{aligned} \|\operatorname{Im} \zeta\| &\geqslant K \log(1 + \log(1 + \|\zeta\|)) \\ &\Rightarrow |\hat{\mu}(\zeta)| + |\hat{\nu}(\zeta)| \geqslant k e^{H_{\check{Q}}(\operatorname{Im} \zeta)}, \end{aligned} \tag{5.8}$$

always under the condition $\varphi \equiv \psi \equiv 0$. Let us return to the general case. Since the two functions $\varphi, \psi \in C_0^\infty(Q)$ we have, outside a compact subset of $\mathbb{C}^2$,

$$|\hat{\varphi}(\zeta)| + |\hat{\psi}(\zeta)| \leqslant \tfrac{1}{2} k e^{H_{\check{Q}}(\operatorname{Im} \zeta)}.$$

One can conclude that, for a convenient choice of (ε_1, C_1), all the components of the set $S(\hat{\mu}, \hat{\nu}; \varepsilon_1, C_1)$ are contained in a set of the form (5.7). Using now the fact that we have proved that the pair of exponential-polynomials appearing in $\hat{\mu}$, $\hat{\nu}$ are s.s.d., that φ, ψ are C_0^∞, and all the components of $S(\hat{\mu}, \hat{\nu}; \varepsilon_2, C_2)$ satisfy (5.7) for any $\varepsilon_2 \leqslant \varepsilon_1$, $C_2 \geqslant C_1$, we see without any difficulty that the pair $\hat{\mu}$, $\hat{\nu}$ generates an ideal s.s.d. in $A_p(\mathbb{C}^2)$. ∎

Remark 5.2. If we replace $\sigma + \varphi$ and $\tau + \psi$ in (5.5) by distributions of compact support contained in $\mathring{Q}$, we see, thanks to (5.8), that V is discrete. One can then ask whether the spectral synthesis still holds for the system $\mu * f = \nu * f = 0$.

We give here a third example where the geometry of the support of the distributions associated to the exponential-polynomials plays a role.

THEOREM 5.3. *Let $\mu_1, ..., \mu_m$ be distributions with finite support in $\mathbb{R}^2$ whose Fourier transforms define a discrete variety of $\mathbb{C}^2$. We assume that the supports A_1 and A_2 of the two distributions μ_1, μ_2 satisfy the condition (2.22). Thus the ideal generated by $(\hat{\mu}_1, ..., \hat{\mu}_m)$ is s.s.d. in $A_p(\mathbb{C}^2)$.*

Proof. Thanks to Proposition 2.5 we see that every sequence $W^{(\rho_k)}$ of irreducible branches of dimension 1 of the $V^{(\rho_k)}$ has a subsequence which is stationary when the $\rho_k \to (1,1)$. Thanks to the reasoning already used in the proof of Theorem 5.1 we see that this is incompatible with V being discrete. ∎

The typical example of application of the previous theorem is when μ_1 is the characteristic function of a convex polygon and μ_2 is obtained from μ_1 by a convenient rotation.

Remark 5.3. We can also show–and we will in Section 7–that if μ_1, μ_2 are two distributions with finite support in $\mathbb{R}^2$ and their supports satisfy (2.22), then the spectral synthesis always holds (even if the spectrum is not discrete) for the system

$$\mu_1 * f = \mu_2 * f = 0, \qquad f \in C^\infty(\mathbb{R}^2).$$

6

From what we said in Section 4 it is clear that, at least for the moment, we can only give partial answers, when $n \geqslant 3$, to the problems raised there. While the methods developed in Section 3 apply to a system of exponential-polynomials with real frequencies, we will restrict ourselves in this section to the case of rational frequencies.

Let us return to the triplet $(\cos \zeta_1, \cos \zeta_2, \zeta_2 - \lambda \zeta_1)$ considered in $(A_p(\mathbb{C}^3))^3$. We have seen that the ideal they generate cannot be s.s.d. On the other hand, from the results of Section 5 (or by an easy direct verification) one obtains that the system of convolution equations they define does not admit any non-zero solution and hence the spectral synthesis still holds since the spectrum V is empty in this case. We could then add the following to the list of problems in Section 4.

PROBLEM 4. If $\mu_1, \ldots, \mu_m$ are distributions with finite support in $\mathbb{R}^n$, does the system

$$\mu_1 * f = \cdots = \mu_m * f = 0$$

have the spectral synthesis property?

When $n = 2$ a positive answer to Problem 4 was given by Gurevich in [29] when the supports of the μ_j lie in $\mathbb{Q}^2$ reducing the case of spectrum non-discrete to that of empty spectrum by a method that was used also in [8, 34]. On the other hand, it is also known that there are convolution systems in $\mathbb{R}^2$ for which the spectral synthesis does not hold [27], but the known examples do not correspond to distributions of finite support.

The proof of Theorem 3.1 gives, at least theoretically, another sufficient condition to ascertain that an ideal $\mathscr{T}$ generated by exponential-polynomials with integral frequencies is s.s.d. when dim $Y \leqslant n$. Let us try to describe this mechanism: we begin by decomposing the variety $Y = Y_0$ in irreducible components $Y_0^{(1)}$, $Y_0^{(2)}$,..., and we denote $Z_0^{(1)}$, $Z_0^{(2)}$,..., the singular locus of $Y_0^{(1)}$, $Y_0^{(2)}$,.... Given one of the components $Y_0^{(j)}$, say, $Y_0^{(1)}$, consider the polynomials $g_1,..., g_l$ generating the prime ideal corresponding to $Y_0^{(1)}$ and regard the matrix

$$\left\| \frac{\partial G_k}{\partial \zeta_h} \right\|_{\substack{k=1,...,l \\ h=1,...,n}},$$

where $G_k(\zeta) = g_k(\zeta, e^{i\zeta})$. Since the minors of rank n of this matrix are of the form $h(\zeta, e^{i\zeta})$, h a polynomial, one must verify, so that the method works, that at least one of these polynomials h is not identically zero on $Y_0^{(1)}$. One adds the equation $\{h = 0\}$ to those of $Y_0^{(1)}$ and obtains an algebraic variety $W_0^{(1)}$ of smaller dimension. One makes the same verification for the other components $Y_0^{(2)}$,.... Consider now the algebraic variety

$$Y_1 = Z_0^{(1)} \cup W_0^{(1)} \cup Z_0^{(2)} \cup W_0^{(2)} \cup \cdots,$$

whose dimension is strictly smaller than that of Y_0. Repeat the procedure starting with Y_1. If we never find any trouble with the above steps we get to an algebraic variety of dim $\leqslant 0$, and that implies that the original ideal is s.s.d.

We have hence a method, completely algebraic, to ensure that a system of exponential-polynomials with integral frequencies defines an s.s.d. ideal. It is only a sufficient condition and usually very hard to verify, and for such a reason we will provide later other sufficient conditions.

We give here an extremely simple example to which we apply the above decision method. We consider in $\mathbb{C}^3$

$$\begin{aligned} F_1(\zeta) &= \zeta_3 e^{i\zeta_1} - 1 \\ F_2(\zeta) &= \zeta_1 - \zeta_2 \\ F_3(\zeta) &= e^{i\zeta_1} - 2e^{i\zeta_3}. \end{aligned} \tag{6.1}$$

Y_0 is then the irreducible smooth variety of dimension 3 given by

$$Y_0 = \{\zeta_3 X_1 - 1 = \zeta_1 - \zeta_2 = X_1 - 2X_3 = 0\}.$$

The only thing to compute here is the Jacobian

$$\det \left\| \frac{\partial F_j}{\partial \zeta_k} \right\| = -2\zeta_3 e^{i(\zeta_1+\zeta_3)} + ie^{i2\zeta_1}. \tag{6.2}$$

We can take as polynomial h the polynomial $-2\zeta_3 X_3 + iX_1$, since the factor $e^{i\zeta_1}$ in (6.2), being invertible in $A_p(\mathbb{C}^3)$, does not play any role. The variety Y_1 is also irreducible and smooth and dim $Y_1 = 2$, and it is defined by

$$Y_1 = \left\{\zeta_1 - \zeta_2 = \zeta_3 - i = X_3 + \frac{i}{2} = X_1 + i = 0\right\}.$$

Repeating this procedure we obtain that the variety Y_2 is empty, hence the ideal generated by (F_1, F_2, F_3) is s.s.d. We note that in this very simple example we have just considered, the study of the varieties $V^{(\rho)}$ is immediate; one sees that the problem can be reduced to one in dimension two and applying Proposition 2.2 one obtains that the possible irreducible branches of dimension $\geqslant 1$ of $V^{(\rho)}$ must be contained in the hyperplane $\{\zeta_3 = 0\}$, which is impossible by the first equation in (6.1).

We try here to give other types of conditions which allow a direct application of Theorem 3.1. Let us recall that we suppose that we are given m exponential-polynomials $F_1, \ldots, F_m$ with integral frequencies defining a variety V which is discrete or, possibly, empty. We want to find conditions on the algebraic variety Y which will allow us to pinpoint, if they exist, the irreducible branches of strictly positive dimension of the analytic varieties $V^{(\rho)}$ for $\rho \in (\mathbb{C}^*)^n$ exceptional. In what follows all the algebraic varieties are subvarieties of $\mathbb{C}^{2n}$, the variables are denoted $\zeta_1, \ldots, \zeta_n, X_1, \ldots, X_n$, and Ω is the open subset of $\mathbb{C}^n$:

$$\Omega = \{(\zeta, X) : X_1 \cdots X_n \neq 0\}. \tag{6.3}$$

Definition 6.1. A coherent change of coordinates on Ω is a bijection T of Ω into Ω such that there is a matrix $A = \|a_{kl}\| \in GL(n, \mathbb{Z}^+)$ such that

$$\forall (\zeta, X) \in \Omega,\ (\zeta', X') = T(\zeta, X) \leftrightarrow \zeta = A\zeta' \text{ and } \forall l \in \{1, \ldots, n\},$$

$$X_l = \prod_{k=1}^{n} (X'_k)^{a_{k,l}}.$$

It is clear that a coherent change of coordinates is a proper mapping of Ω into itself, and if W is an analytic subvariety of pure dimension p of a domain $U \subseteq \Omega$, then $T(W)$ is an analytic subvariety of dimension p in $T(U)$. This is a consequence of Remmert's theorem [26].

We will use the following two preliminary lemmas.

LEMMA 6.1. *Let q be a polynomial in $\mathbb{C}[\zeta_1, ..., \zeta_n]$, then there is a matrix A in $GL(n, \mathbb{Z}^+)$ such that*

$$\forall \zeta \in \mathbb{C}^n, \qquad q(A\zeta) = a\zeta_1^k + \sum_{j=0}^{k-1} u_j(\zeta_2, ..., \zeta_n)\, \zeta_1^j, \tag{6.4}$$

where $a \in \mathbb{C}^*$, $u_j \in \mathbb{C}[\zeta_2, ..., \zeta_n]$, and k is the total degree of the polynomial q.

Proof. The existence of a matrix $A \in GL(n, \mathbb{C})$ such that (6.4) holds is very well known (see, e.g., [7, Chap. 6]), and the only thing to observe is that in the proof of that lemma we can impose the extra condition that the coefficients of A are in $\mathbb{Z}^+$. ∎

LEMMA 6.2. *Let F be an exponential-polynomial of n variables with frequencies in $\mathbb{Z}^n$. There is an element $u \in \mathbb{Z}^n$, a matrix $A \in GL(n, \mathbb{Z}^+)$, a strictly positive integer N, a non-zero polynomial $P_0 \in \mathbb{C}[\zeta_1, ..., \zeta_n]$, and a family $\{G_q\}_{q=1}^N$ of exponential-polynomials of n variables with frequencies in $\{0\} \times \mathbb{Z}^{n-1}$, such that*

$$\forall \zeta \in \mathbb{C}^n, \qquad e^{iu \cdot \zeta} F(A\zeta) = P_0(\zeta)\, e^{iN\zeta_1} + \sum_{q=1}^{N} G_q(\zeta)\, e^{i(N-q)\zeta_1}. \tag{6.5}$$

Proof. After multiplication by a convenient exponential we can assume that the frequencies $\alpha_1, ..., \alpha_L$ of F are all elements in $(\mathbb{N}^*)^n$. Let $a \in (\mathbb{N}^*)^n$ such that

$$\forall l \in \{1, ..., L\}, \qquad l \neq 1 \Rightarrow a \cdot \alpha_1 > a \cdot \alpha_l.$$

After an eventual rearrangement of the α_l such an element always exists. We choose a family $\{a_j\}_{j=2,...,n}$ of elements in $(\mathbb{N}^*)^n$ such that $(a, a_2, ..., a_n)$ form a basis for $\mathbb{R}^n$, and we denote by A the matrix whose columns are the vectors $a, a_2, ..., a_n$. Let $\{C_l\}_{l=1,...,L}$ be the non-zero polynomials in $\mathbb{C}[\zeta_1, ..., \zeta_n]$ such that

$$F(A \cdot \zeta) = \sum_{l=1}^{L} C_l(\zeta) \exp\left[i\left(\alpha_l \cdot a\zeta_1 + \sum_{j=2}^{n} \alpha_l \cdot a_j \zeta_j\right)\right].$$

Denote

$$u = (0, -\alpha_1 \cdot a_2, ..., -\alpha_1 \cdot a_n) \in \mathbb{Z}^n.$$

Hence

$$e^{iu\cdot\zeta}F(A\cdot\zeta)=C_1(\zeta)\,e^{i\alpha_1\cdot a\zeta_1}$$
$$+\sum_{l=2}^{L}C_l(\zeta)\exp\left[i\left(\alpha_l\cdot a\zeta_1+\sum_{j=2}^{n}(\alpha_l-\alpha_1)\cdot a_j\zeta_j\right)\right].$$

If we set $N=\alpha_1\cdot a$ we have immediately the expression (6.5). ∎

Remark 6.1. If we denote by T the coherent change of variables corresponding to the matrix A appearing in Lemma 6.2 we see that there is an integer $N>0$, a non-zero polynomial $P_0\in\mathbb{C}[\zeta_1,...,\zeta_n]$, and a family $\{P_q\}_{q=1}^{N}$ of elements in $\mathbb{C}[\zeta_1,...,\zeta_n, X_2,..., X_n, 1/X_2,..., 1/X_n]$ such that if g is the polynomial associated to F by (3.2)

$$\forall(\zeta,X)\in\Omega,\qquad g(T^{-1}(\zeta,X))=0$$
$$\leftrightarrow P_0(\zeta)\,X_1^N+\sum_{q=1}^{N}P_q(\zeta,\tilde{X})\,X_1^{N-q}=0,$$

where $\tilde{X}=(X_2,..., X_n)$.

We are now ready to prove the following proposition.

PROPOSITION 6.3. *Let $F_1,..., F_m$ be m exponential-polynomials of n variables with frequencies in $\mathbb{N}^n$, and we assume that the polynomials $P_1,..., P_m\in\mathbb{C}[\zeta,X]$ associated via (3.2) are in fact in $\mathbb{C}[\zeta_1,...,\zeta_k, X_1,..., X_n]$, $1\leqslant k\leqslant n$, and that the dimension (in $\mathbb{C}^{2n}$) of the algebraic variety Y defined by (3.3) is smaller or equal to n. Then, there is a non-zero polynomial P in $\mathbb{C}[\zeta_1,...,\zeta_k]$ such that all the irreducible branches of dimension bigger or equal to 1 (in $\mathbb{C}^n$) of the analytic variety $V=\{\zeta\in\mathbb{C}^n: F_1(\zeta)=\cdots=F_m(\zeta)=0\}$, if they exist, are included in the algebraic hypersurface*

$$\{\zeta\in\mathbb{C}^n: P(\zeta_1,...,\zeta_k)=0\}.$$

Remark 6.2. Given an n-tuplet $\rho\in(\mathbb{C}^*)^n$, it will be immediate, from the proof of Proposition 6.3, to see that every irreducible branch of positive dimension of $V^{(\rho)}$ is also included in the same hypersurface $\{P(\zeta_1,...,\zeta_k)=0\}$ given in Proposition 6.3.

The theorem of Ax, i.e., Proposition 2.4, allows us to say that for a given irreducible branch W of V there is a polynomial P_W, affine with rational coefficients (except for the independent term), such that $W\subseteq\{P_W=0\}$. What is surprising about Proposition 6.3 is that the polynomial P is independent of the branch W.

Proof of the Proposition 6.3. Let us suppose W is an irreducible branch of positive dimension of V and denote by Z the subvariety of $\mathbb{C}^{2n}$:

$$Z = \{(\zeta, X) \in \mathbb{C}^{2n} : \zeta \in W,\ X_1 = e^{i\zeta_1}, \ldots, X_n = e^{i\zeta_n}\}.$$

Since W is irreducible, Z is contained in one of the irreducible components of the variety Y, hence we are not making any restriction if we suppose that $P_1, \ldots, P_m$ are the generators of a prime ideal $\mathscr{P}$ in $\mathbb{C}[\zeta, X]$ associated to one of the irreducible components of Y, and it is clear that the hypotheses of Proposition 6.3 still hold for these new polynomials $p_1, \ldots, p_m$.

If a non-zero element of $\mathbb{C}[\zeta_1, \ldots, \zeta_n]$ is in $\mathscr{P}$, then there is a non-zero polynomial in $\mathbb{C}[\zeta_1, \ldots, \zeta_k]$ which belongs to $\mathscr{P}$ and the conclusion of Proposition 6.3 is immediate in this case, and we will assume hence that

$$\mathbb{C}[\zeta_1, \ldots, \zeta_n] \cap \mathscr{P} = \{0\}. \tag{6.6}$$

The polynomial P_1 does not depend only on the variables ζ. Consider P_1 as a polynomial in $\mathbb{C}[\zeta_1, \ldots, \zeta_k][X_1, \ldots, X_n]$. Let $(\zeta_0, X_0) \in Z$, and either one of the coefficients of P_1 vanishes at ζ_0 or none of them vanishes at ζ_0. Let us suppose we are in the latter situation.

The point (ζ_0, X_0) belongs to the open set Ω, and thanks to Lemma 6.2 and Remark 6.1 there is a coherent change of coordinates T_1 such that, in a neighborhood of (ζ_0, X_0), the variety Y is defined in the new coordinates by the following equations:

$$\begin{aligned} P_0(\zeta')(X_1')^{N_1} + \sum_{l=1}^{N} P_l(\zeta', X_2', \ldots, X_N')(X_1')^{N-l} &= G_1(\zeta', X') = 0 \\ G_2(\zeta', X') &= 0 \\ \vdots \qquad & \\ G_m(\zeta', X') &= 0, \end{aligned} \tag{6.7}$$

where $0 \neq P_0 \in \mathbb{C}[\zeta']$, $P_l \in \mathbb{C}[\zeta'][X_2', \ldots, X_n', 1/X_2', \ldots, 1/X_n']$ for $l = 1, \ldots, N_1$, $G_j \in C[\zeta'][X_1', \ldots, X_n', 1/X_1', \ldots, 1/X_n']$ for $j = 2, \ldots, m$. Moreover, if Q denotes the polynomial P_0 or any of the coefficients (in $\mathbb{C}[\zeta']$) of the P_l or G_j, and if A_1 is the matrix associated to the change of variable T_1, the function

$$\zeta \to Q(A_1^{-1}\zeta)$$

is a polynomial function depending only on the variables $\zeta_1, \ldots, \zeta_k$.

Let $(\zeta_0', X_0') = T_1(\zeta_0, X_0)$, and we have, thanks to the hypothesis we have made above,

$$P_0(\zeta_0') = P_0(A_1^{-1}\zeta_0) \neq 0.$$

Denote $P_0'(\zeta) = P_0(A_1^{-1}\zeta)$.

By elimination theory [48], there is a family $R_2,\dots, R_L$ of elements in $\mathbb{C}[\zeta'][X'_2,\dots, X'_n, 1/X'_2,\dots, 1/X'_n]$ such that, in the variables (ζ', X'), the variety Y is defined in a neighborhood of (ζ'_0, X'_0) by the equations

$$\begin{aligned} G_1(\zeta', X') &= 0 \\ R_2(\zeta', X') &= 0 \\ &\vdots \\ R_L(\zeta', X') &= 0. \end{aligned} \tag{6.8}$$

The analytic subvariety of $\mathbb{C}^{2n-1}$ (the variables being $\zeta'_1,\dots, \zeta'_n, X'_2,\dots, X'_n$) defined in a neighborhood of $(\zeta'_0, X'_{0,2},\dots, X'_{0,n})$ by the equations

$$R_2(\zeta', X'_2,\dots, X'_n) = \cdots = R_L(\zeta', X'_2,\dots, X'_n) = 0$$

is, by the hypothesis, Remmert's theorem and Remark 6.1, of dimension smaller or equal to n in $\mathbb{C}^{2n-1}$ if $n \geqslant 2$, something we have implicitly assumed. Hence, at least one of the functions R_q, for instance, R_2, is not identically zero. Moreover, if Q denotes any one of the polynomial coefficients (i.e., in $C[\zeta']$) of R_2, the function $\zeta \to Q(A_1^{-1}\zeta)$ is still in $\mathbb{C}[\zeta_1,\dots, \zeta_k]$.

If $n = 2$, we stop the procedure, and we will return to it at the end of the proof. If $n \geqslant 3$, we continue as follows. Again we have two cases, either at least one of the coefficients of R_2 vanishes at ζ'_0 or none of them vanishes at this point. Again we suppose we are in the latter situation.

As before we can find a coherent change of coordinates T_2, this time leaving untouched the variables ζ'_1, X'_1, and such that in a neighborhood of (ζ'_0, X'_0) the variety Y is defined, in the new coordinates (ζ'', X''), by the following equations:

$$\begin{aligned} Q_{0,1}(\zeta'')(X''_1)^{N_1} + \sum_{l=1}^{N_1} Q_{l,1}(\zeta'', X''_2,\dots, X''_n)(X''_1)^{N-l} &= H_1(\zeta'', X'') = 0 \\ Q_{0,2}(\zeta'')(X''_2)^{N_2} + \sum_{l=1}^{N_2} Q_{l,2}(\zeta'', X''_3,\dots, X''_n)^{N-l} &= H_2(\zeta'', X'') = 0 \\ H_3(\zeta'', X'') &= 0 \\ &\vdots \\ H_L(\zeta'', X'') &= 0, \end{aligned} \tag{6.9}$$

where $Q_{0,1}, Q_{0,2} \in \mathbb{C}[\zeta'']$, $Q_{l,1} \in \mathbb{C}[\zeta'', X''_2,\dots, X''_n, 1/X''_2,\dots, 1/X''_n]$ for $l \neq 1$, $Q_{l,2} \in \mathbb{C}[\zeta'', X''_3,\dots, X''_n, 1/X''_3,\dots, 1/X''_n]$ for $l \neq 1$, and $H_j \in \mathbb{C}[\zeta'', X''_1,\dots, X''_n, 1/X''_1,\dots, 1/X''_n]$ for $j = 3,\dots, L$. Moreover, if Q denotes any of the polynomial coefficients (i.e., in $\mathbb{C}[\zeta'']$) of the H_j and if A_2 is the matrix associated to

T_2, then the function $\zeta \to Q(A_2^{-1}A_1^{-1}\zeta)$ is a polynomial in $\mathbb{C}[\zeta_1,..., \zeta_k]$. Denote $(\zeta_0'', X_0'') = T_2(\zeta_0', X_0')$, then, in the case we are considering,

$$Q_{0,2}(\zeta_0'') = Q_{0,2}(A_2^{-1}A_1^{-1}\zeta_0) \neq 0.$$

We denote $P_1'(\zeta) = Q_{0,2}(A_2^{-1}A_1^{-1}\zeta)\, P_0'(\zeta)$.

Again by elimination theory we obtain a family $S_3,..., S_M$ in $\mathbb{C}[\zeta'', X_3'',..., X_n'', 1/X_3'',..., 1/X_n'']$ such that the variety Y is defined in a neighborhood of (ζ_0'', X_0'') by the system

$$\begin{aligned} &H_1(\zeta'', X'') = 0\\ &H_2(\zeta'', X'') = 0\\ &S_3(\zeta'', X'') = 0\\ &\vdots \cdots\cdots\\ &S_M(\zeta'', X'') = 0. \end{aligned} \tag{6.10}$$

The analytic variety in $\mathbb{C}^{2n-2}$ (variables $\zeta_1'',..., \zeta_n''$, $X_3'',...,X_n''$) defined near $(\zeta_0'',..., X_{0,3}'',..., X_{0,n}'')$ by the $M-3$ last equations of (6.10) is again of dimension less equal to n; and, since we have assumed $n \geqslant 3$, at least one of the S_m, say, S_3, is not identically zero. Moreover, if Q is any one of the coefficients (in $\mathbb{C}[\zeta'']$) of S_3 considered as an element of $\mathbb{C}[\zeta''][X_3'',..., X_n'', 1/X_3'',..., 1/X_n'']$, the function $\zeta \to Q(A_2^{-1}A_1^{-1}\zeta)$ is in $\mathbb{C}[\zeta_1,..., \zeta_k]$.

When $n=3$ we stop here, if not we continue in the same way. Therefore, after having gone through this procedure $n-1$ times, we construct a polynomial $P' \in \mathbb{C}[\zeta_1,..., \zeta_k]$, independent of (ζ_0, X_0), and a coherent change of coordinates T, with associated matrix A, such that :

—either $P'(\zeta_0) = 0$

—or the variety Y is defined in a neighborhood of (ζ_0, X_0) in new coordinates (w, Z) by equations of the form

$$\begin{aligned} &\lambda_{0,1}(w)\, Z_1^{N_1} + \sum_{l=1}^{N_1} \lambda_{l,1}(w, Z_2,..., Z_n)\, Z_1^{N_1-l} = 0\\ &\lambda_{0,2}(w)\, Z_2^{N_2} + \sum_{l=1}^{N_2} \lambda_{l,2}(w, Z_3,..., Z_n)\, Z_2^{N_2-l} = 0\\ &\vdots\\ &\lambda_{0,n}(w)\, Z_n^{N_n} + \sum_{l=1}^{N_n} \lambda_{l,n}(w)\, Z_n^{N_n-l} = 0\\ &K_{n+1}(w, Z_n) = 0\\ &\vdots\\ &K_r(w, Z_n) = 0, \end{aligned} \tag{6.11}$$

where none of the polynomial coefficients (i.e., in $\mathbb{C}[w]$) of the $\lambda_{l,j}$ vanishes at the point w_0; moreover, if Q is one of those polynomial coefficients, the function $\zeta \to Q(A^{-1}\zeta)$ is a polynomial function in the variables $\zeta_1,..., \zeta_k$ alone.

In the case where Y is defined near (ζ_0, X_0) by equations of the form (6.11) in the coordinates (w, Z), the classical method of elimination theory of computing successive Sylvester determinants shows us that there is, for every $j \in \{1,..., n\}$, a family $\{\mu_{j,l}\}_{l=0}^{M_j}$ of elements in $\mathbb{C}[w_1,..., w_n]$ independent of the point (ζ_0, X_0) such that

(a) $\forall j,\ \mu_{j,0} \not\equiv 0$,

(b) $\forall j \in \{1,..., n\},\ \forall l \in \{0,..., M_j\},\ \mu_{j,l}(A^{-1}(\cdot)) \in \mathbb{C}[\zeta_1,..., \zeta_k]$,

(c) if $(w, Z) \in Y$ is near (w_0, Z_0) then

$$\sum_{l=0}^{M_j} \mu_{j,l}(w)\, Z_j^{M_j-l} = 0 \qquad \forall j. \tag{6.12}$$

After dividing (6.12) by convenient powers of the Z_j we can assume that

$$\forall j \qquad \mu_{j,M_j} \not\equiv 0.$$

Let $P'' \in C[\zeta_1,..., \zeta_k]$ be defined by

$$P''(\zeta) = \prod_{j=1}^{n} \mu_{j,M_j}(A^{-1}\zeta)\, \mu_{j,0}(A^{-1}\zeta).$$

From (6.12) we conclude there are two constants C', K' such that for every point ζ in W close to ζ_0, we have

$$|P''(\zeta_1,..., \zeta_k)| \prod_{j=1}^{n} \left(|\psi_j(\zeta)| + \frac{1}{|\psi_j(\zeta)|} \right) \leqslant C'(1 + \|\zeta\|)^{K'}, \tag{6.13}$$

where

$$\psi_j(\zeta) = \exp\left(\sum_{l=1}^{n} \alpha_{l,j} \operatorname{Im} \zeta_l \right), \qquad A^{-1} = \|\alpha_{l,j}\|.$$

It follows from (6.13), since $\det A \neq 0$, that there are two constants C, K such that for every $\zeta \in W$ near ζ_0 we obtain

$$|P''(\zeta_1,..., \zeta_k)| \exp\left(\sum_{1}^{n} |\operatorname{Im} \zeta_j| \right) \leqslant C(1 + \|\zeta\|)^{K}. \tag{6.14}$$

Now, every constant that has appeared, as well as all the polynomials we have introduced, are independent of ζ_0 and the branch W. Hence, setting

$P = P'P''$, we see that every point of a branch of positive dimension is contained in the set

$$\left\{\zeta\colon |P(\zeta)| \exp\left(\sum_1^n |\mathrm{Im}\,\zeta_j|\right) \leqslant C(1 + \|\zeta\|)^K\right\}.$$

We can finish the proof using Proposition 2.1. ∎

Remark 6.3. The main difficulty in the above proof is the lack of a theorem of the type of Noether's normalization theorem [30, 48], which we cannot prove since are limited to coherent changes of variables and not arbitrary linear changes of coordinates in the $2n$ variables as would be necessary for the proof of such a normalization theorem. It is precisely to get around this difficulty that we have given the above proof with practically all details.

Remark 6.4. If the coefficients of the exponential-polynomials $F_1,\dots, F_m$ are in a subfield of $\mathbb{C}$ or in a field of the type $\mathbb{C}(u)$, the polynomial P of Proposition 6.3 has coefficients in the same field.

We can also prove the following proposition.

PROPOSITION 6.4. *Let $P_1,\dots, P_m \in \mathbb{C}[\zeta, X]$, Y the corresponding algebraic variety. Denote by W_0 the subset of $\mathbb{C}^n$ defined by*

$$\zeta \notin W_0 \Rightarrow \dim\, Y \cap (\{\zeta = \zeta_0\} \times \mathbb{C}^n) \leqslant 0. \tag{6.15}$$

Every irreducible branch of positive dimension of the analytic subvariety V of $\mathbb{C}^n$ defined by the equations

$$P_1(\zeta, e^{i\zeta_1},\dots, e^{i\zeta_n}) = \cdots = P_m(\zeta, e^{i\zeta_1},\dots, e^{i\zeta_n}) = 0 \tag{6.16}$$

is contained in the closure $\bar{W}_0$ of W_0 in $\mathbb{C}^n$.

We need two further lemmas:

LEMMA 6.5. *Under the hypotheses of the preceding proposition, every irreducible component of Y not included in $W_0 \times \mathbb{C}^n$ has dimension at most n.*

Proof of Lemma 6.5. Let Y_1 be a component of Y not included in $W_0 \times \mathbb{C}^n$, $(\zeta_0, X_0) \in Y_1$ with $\zeta_0 \notin W_0$. By (6.15), if Z is an irreducible component of $Y_1 \cap (\{\zeta = \zeta_0\} \times \mathbb{C}^n)$ containing (ζ_0, X_0) we have

$$\dim Z = 0.$$

As pointed out before, by [30, Proposition 7.1], one has

$$\dim Z \geqslant \dim\, Y_1 + \dim(\{\zeta = \zeta_0\} \times \mathbb{C}^n) - 2n.$$

It follows that

$$0 \geqslant \dim Y_1 + n - 2n = \dim Y_1 - n. \quad \blacksquare$$

LEMMA 6.6. *Under the hypotheses of the preceding proposition, we have, for every $j \in \{0, ..., n\}$, for every algebraic subvariety V_j of $\mathbb{C}^n$ of dimension less or equal to j (in $\mathbb{C}^n$)*

> $(\zeta_0, X_0) \in Y \cap (V_j \times \mathbb{C}^n)$ *and* $\zeta_0 \notin W_0 \Rightarrow$ *every irreducible component of* $Y \cap (V_j \times \mathbb{C}^n)$ *containing* (ζ_0, X_0) *has dimension less or equal to j (in $\mathbb{C}^{2n}$).* (6.17)

Proof of Lemma 6.6. The conclusion of the lemma is correct when $j = 0$, and this follows immediately from the hypotheses. It is also clear one can assume Y is irreducible in $\mathbb{C}^{2n}$ and V_j in $\mathbb{C}^n$ (j is now fixed in the range $1 \leqslant j \leqslant n$), hence $V_j \times \mathbb{C}^n$ is also irreducible in $\mathbb{C}^{2n}$. We will assume the lemma valid for $j-1$ and $\dim V_j = j$.

Given $(\zeta_0, X_0) \in Y \cap (V_j \times \mathbb{C}^n)$, let Z be an irreducible component of this variety containing the point (ζ_0, X_0). Since the conclusion is valid for $j-1$ we can assume ζ_0 is a regular point of V_j. If no point of Z has a regular ζ-coordinate we would have $Z \subseteq Y \cap (V_j' \times \mathbb{C}^n)$, $V_j' =$ singular variety of V_j, and since $\dim V_j' < \dim V_j \leqslant j$ we could apply the inductive hypothesis.

Since ζ_0 is a regular point of V_j we can construct a linear variety L in $\mathbb{C}^n$, $\dim L = n - j$, $\zeta_0 \in L$, and ζ_0 is an isolated point of $V_j \cap L$. By the case $j = 0$ of the lemma we have

$$\dim Y \cap (V_j \times \mathbb{C}^n \cap L \times \mathbb{C}^n) = 0 \tag{6.18}$$

On the other hand, let Z' be an irreducible component of $Z \cap (L \times \mathbb{C}^n)$ containing (ζ_0, X_0), and by the now familar argument

$$\dim Z' \geqslant \dim Z + \dim(L \times \mathbb{C}^n) - 2n = \dim Z - j.$$

But, by (6.18)

$$\dim Z' \leqslant \dim((Y \cap V_j \times \mathbb{C}^n) \cap L \times \mathbb{C}^n) = 0.$$

Hence $\dim Z \leqslant j$, which proves the lemma. $\blacksquare$

Proof of Proposition 6.4. Let W be an irreducible branch of positive dimension of the subvariety V of $\mathbb{C}^n$. Suppose W is not included in $\bar{W}_0$. Let

$$Z = \{(\zeta, X) \in \mathbb{C}^{2n} : \zeta \in W, X_1 = e^{i\zeta_1}, ..., X_n = e^{i\zeta_n}\}.$$

Since Z is connected it is contained in an irreducible branch Y_1 of Y. We assume $\dim Y_1 = n - q$, $0 \leqslant q \leqslant n$, and this is justified by Lemma 6.5 since

Y_1 is not contained in $W_0 \times \mathbb{C}^n$. Consider a point $(\zeta_0, X_0) \in Z$, $\zeta_0 \notin W_0$. By the Proposition 2.4 there is an $r \in \mathbb{Q}^n \setminus (0)$ such that

$$W \subseteq \{\zeta \in \mathbb{C}^n : r_1 \zeta_1 + \cdots + r_n \zeta_n = r_1 \zeta_{0,1} + \cdots + r_n \zeta_{0,n}\}. \tag{6.19}$$

We denote $\gamma_1 = r_1 \zeta_{0,1} + \cdots + r_n \zeta_{0,n}$. After a coherent change of coordinates we can assume that the hyperplane in (6.19) has the form

$$\zeta_1 = \gamma_1.$$

The analytic variety Z is now contained in an irreducible component Y_2 of the algebraic variety $Y_1 \cap \{\zeta_1 = \gamma_1\}$. By Lemma 6.6, dim $Y_2 \leqslant n-1$. Replacing ζ_1 by γ_1 in the exponential-polynomials corresponding to the generators of Y_2, we obtain exponential-polynomials of $n-1$ variables whose variety of zeroes contains a copy of W. We can apply again Proposition 2.4 in $\mathbb{C}^{n-1}$, make a new coherent change of coordinates that does not touch ζ_1 or X_1, and see that the copy of W is contained in the set $\zeta_2 = \gamma_2$. Now, every component of $Y \cap \{\zeta_1 = \gamma_1\} \cap \{\zeta_2 = \gamma_2\}$ containing the point (ζ_0, X_0) has dimension $n-2$ by Lemma 6.6. Iterating this procedure one sees that W cannot have positive dimension. ∎

COROLLARY 6.7. *Let $P_1, ..., P_m \in \mathbb{C}[\zeta, X]$, with $m \geqslant n$. Assume there is a closed set $W_0 \subseteq \mathbb{C}^n$ such that*

$$\zeta_0 \notin W_0 \Rightarrow \text{rank} \left\| \frac{\partial P_j}{\partial X_k}(\zeta_0, X) \right\|_{\substack{j=1,...,m \\ k=1,...,n}} = n \qquad \forall X. \tag{6.20}$$

Then, every irreducible branch of positive dimension of the variety $\{\zeta \in \mathbb{C}^n : P_1(\zeta, e^{i\zeta}) = \cdots = P_m(\zeta, e^{i\zeta}) = 0\}$ is contained in W_0.

Proof. This follows from the fact that the hypothesis (6.20) implies (6.15). In fact, if $\zeta_0 \notin W_0$, for X_0 fixed, a minor of rank n of the matrix $\|(\partial P_j / \partial X_k)(\zeta_0, X_0)\|$ is not zero, hence the variety

$$\{X \in \mathbb{C}^n : P_1(\zeta_0, X) = \cdots = P_m(\zeta_0, X) = 0\}$$

has dimension at most zero at X_0, which is precisely (6.15). ∎

Application. If the exceptional set W_0 of (6.20) is either a variety of dimension $\leqslant 0$ or a finite union of irreducible varieties of dimension 1 (or a union of both of those things), then the ideal generated by the exponential-polynomials $P_1(\zeta, e^{i\zeta}), ..., P_m(\zeta, e^{i\zeta})$ is s.s.d., under the additional assumption, in the second case, that the variety V be empty or discrete.

7

In this section we will study in detail a particular kind of system of exponential-polynomials which will allow us to study Problem 2 posed in Section 4 in the case $n=1$ and to improve upon the Ritt theorem proved in [6, 43].

Let us begin with the following proposition.

PROPOSITION 7.1. *Let F be an exponential-polynomial of n variables with the set of frequencies $\Lambda \subseteq \mathbb{C}^n$, which we write as*

$$F(\zeta) = \sum_{\lambda \in \Lambda} A_\lambda(\zeta)\, e^{i\lambda\cdot\zeta}, \tag{7.1}$$

where none of polynomials A_λ is identically zero.

Let V_0 be an irreducible algebraic subvariety of $\mathbb{C}^n$ of dimension 1 such that F vanishes identically on V_0. Then only two situations can occur:

(a) *either all the polynomials A_λ, $\lambda \in \Lambda$, vanish identically on V_0,*

(b) *or there are λ, $\lambda' \in \Lambda$, $\lambda \neq \lambda'$, and $\gamma \in \mathbb{C}$ such that*

$$V_0 \subseteq \{\zeta \in \mathbb{C}^n : (\lambda - \lambda')\cdot\zeta = \gamma\}.$$

To prove this proposition we need several lemmas.

LEMMA 7.2. *Let $\mathscr{P}$ be a prime ideal in $\mathbb{C}[\zeta_1,..., \zeta_n]$ $(n \geqslant 2)$, such that the zero locus of $\mathscr{P}$ is an algebraic variety of dimension 1, denoted $V(\mathscr{P})$; we assume $V(\mathscr{P})$ is not contained in any hyperplane in $\mathbb{C}^n$ of the form $\zeta_1 = \gamma$. Then, there is a constant C such that*

$$\zeta = (\zeta_1,..., \zeta_n) \in V(\mathscr{P}) \text{ and } |\zeta_1| > C$$

imply

$$\sum_{j=2}^{n} |\zeta_j| \leqslant C(1 + |\zeta_1|)^C.$$

Proof of Lemma 7.2. We will prove this lemma by induction on n. When $n=2$ the lemma is trivial since $V(\mathscr{P})$ is defined by a single equation $P(\zeta_1, \zeta_2) = 0$, P irreducible and of degree $\geqslant 1$ in the variable ζ_2. We will assume that the result is correct for dimensions $n' < n$. Let $P_1,..., P_m$ be the generators of $\mathscr{P}$, then

$$V(\mathscr{P}) = \{\zeta \in \mathbb{C}^n : P_1(\zeta) = \cdots = P_m(\zeta) = 0\}.$$

Due to the hypothesis that $V(\mathscr{P})$ is not contained in a hyperplane $\zeta_1 = \gamma$

we can use Lemma 6.1 on the variables $\zeta_2, \ldots, \zeta_n$ and find new coordinates $(\zeta_1, \zeta_2', \ldots, \zeta_n')$ such that, in the new coordinates,

$$P_1(\zeta_1, \zeta_2', \ldots, \zeta_n') = R_1(\zeta_1)(\zeta_2')^{\nu} + \varphi_1(\zeta_1, \zeta_2', \ldots, \zeta_m'), \tag{7.2}$$

where φ_1 is a polynomial of degree $< \nu$ in ζ_2' and R_1 is a polynomial in the single variable ζ_1 which cannot be identically zero on $V(\mathscr{P})$.

We also conclude without difficulty that $|\zeta_1|$ is not bounded on $V(\mathscr{P})$, otherwise ζ_1 will be a bounded holomorphic function on an algebraic variety and hence constant (see, e.g. [19]). Choose a constant K such that $|\zeta_1| \geqslant K$ implies $|R_1(\zeta_1)| \geqslant 1$ and consider a regular point $\zeta_0 \in V(\mathscr{P})$ such that $|\zeta_{0,1}| \geqslant K$. By elimination theory, the projection W of $V(\mathscr{P})$ into $\mathbb{C}^{n-1}$ defines an irreducible algebraic variety of dimension one. This variety cannot be contained in any hyperplane $\zeta_1 = \gamma$ otherwise $V(\mathscr{P})$ would also be contained in such a hyperplane. By the induction hypothesis, there is a constant C' such that

$$|\zeta_1| > C', \qquad (\zeta_1, \zeta_3', \ldots, \zeta_n') \in W \Rightarrow \sum_{j=3}^{n} |\zeta_j'| < C'(1 + |\zeta_1|)^{C'}.$$

Using now (7.2) and the fact that

$$|\zeta_1| > K \Rightarrow |R_1(\zeta_1)| \geqslant 1$$

one arrives to the desired conclusion. ∎

LEMMA 7.3. *Let V be an algebraic variety of pure dimension equal to 1 in $\mathbb{C}^n$, $n \geqslant 2$. Assume that no irreducible component of V is contained in a hyperplane of the form $\zeta_1 = \gamma$. Let $P, Q \in \mathbb{C}[\zeta_1, \ldots, \zeta_n]$ such that the varieties $V \cap \{P = 0\}$ and $V \cap \{Q = 0\}$ are of dimension zero or empty. Then the analytic variety W defined by*

$$W = \{\zeta \in V \colon P(\zeta)\, e^{i\zeta_1} + Q(\zeta) = 0\}$$

is discrete.

Proof of Lemma 7.3. Assume W contains an irreducible analytic variety of dimension one. It is then, by dimensionality considerations, an irreducible branch Z of V, hence algebraic. The algebraic varieties $Z \cap \{P = 0\}$ and $Z \cap \{Q = 0\}$ have non-positive dimensions, hence they are finite sets. Therefore there is a constant C such that

$$\zeta \in Z \Rightarrow |\operatorname{Im} \zeta_1| \leqslant C(1 + \log(1 + \|\zeta\|)).$$

We can then find a positive constant K such that the function

$$(\zeta_1, \ldots, \zeta_n) \to \frac{1}{\zeta_1 + Ki}$$

is holomorphic and bounded on Z, hence constant. This contradicts the hypothesis. ∎

Remark 7.1. Note that we could have assumed that for each irreducible component of V one of the two polynomials P or Q does not vanish identically and obtained the same conclusion.

Proof of Proposition 7.1. We will prove this proposition by induction on the number of frequencies in F. Let $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$ and write A_j instead of A_{λ_j}. The proposition is clear if $N = 1$, and it is also true when $N = 2$ by Lemma 7.3. Assume hence $N > 2$. We can also assume $V_0 = V(\mathscr{P})$, where $\mathscr{P}$ is a prime ideal in $\mathbb{C}[\zeta]$ and none of the polynomials $A_1, \ldots, A_N$ belongs to $\mathscr{P}$.

Let $\zeta_0 \in V_0$ be a regular point. If $P_1, \ldots, P_m$ are the generators of $\mathscr{P}$ we assume that

$$\det \left\| \frac{\partial(P_1, \ldots, P_{n-1})}{\partial(\zeta_1, \ldots, \zeta_{n-1})} (\zeta_0) \right\| \neq 0. \tag{7.3}$$

An easy computation shows that

$$J(\zeta) = \text{Jacobian of } (P_1, \ldots, P_{n-1}, F)(\zeta) = \sum_{j=1}^{N} B_j(\zeta)\, e^{i\lambda_j \cdot \zeta},$$

where $B_j \in \mathbb{C}[\zeta]$. It is clear that the exponential-polynomial G defined by

$$G(\zeta) = A_N(\zeta)\, J(\zeta) - B_N(\zeta)\, F(\zeta) \tag{7.4}$$

vanishes identically on V_0, since F and also J vanish on V_0 in a neighborhood of ζ_0 and V_0 is irreducible.

A computation similar to that performed for the case $n = 2$ in [11, Proof of Theorem 2], which, for the sake of completeness, we will give below, shows that if one of the polynomial coefficients of G is identically zero on V_0, then there is an index $j \in \{1, \ldots, N-1\}$ and $\gamma \in \mathbb{C}^*$ such that

$$V_0 \cap \{\zeta \in \mathbb{C}^n : A_j(\zeta) + \gamma A_N(\zeta)\, e^{i(\lambda_N - \lambda_j)\cdot \zeta} = 0\} \tag{7.5}$$

is not a discrete variety. By Lemma 7.3 it follows that V_0 is included in a hyperplane of the form

$$(\lambda_N - \lambda_j) \cdot \zeta = \gamma',$$

and we will be finished. When none of the polynomial coefficients of G vanishes on V_0 we reach the same conclusion by the induction hypothesis since the number of frequencies in G is $N-1$. This ends the proof of Proposition 7.1, modulo the above-mentioned computation.

To show that we can reduce everything to (7.5) if one of the coefficients of G vanishes identically on V_0 we need to make explicit the computation of J and the coefficients $A_N B_j - B_N A_j$ in (7.4), say, $j=1$. We have

$$J(\zeta) = \begin{vmatrix} \dfrac{\partial F}{\partial \zeta_1} \cdots \dfrac{\partial F}{\partial \zeta_n} \\ \dfrac{\partial P_1}{\partial \zeta_1} \cdots \dfrac{\partial P_1}{\partial \zeta_n} \\ \cdots\cdots\cdots\cdots \\ \dfrac{\partial P_{n-1}}{\partial \zeta_n} \cdots \dfrac{\partial P_{n-1}}{\partial \zeta_n} \end{vmatrix}$$

$$= \sum_{k=1}^{n} \Delta_k(\zeta) \frac{\partial F}{\partial \zeta_k}$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{n} \Delta_k(\zeta) \left(\frac{\partial A_j}{\partial \zeta_k} + i\lambda_{j,k} A_j \right) e^{i\lambda_j \cdot \zeta},$$

hence

$$A_N B_1 - B_N A_1 = \sum_{k=1}^{N} \Delta_k(\zeta) \left[A_N \frac{\partial A_1}{\partial \zeta_k} - A_1 \frac{\partial A_N}{\partial \zeta_k} + i(\lambda_{1,k} - \lambda_{N,k}) A_1 A_N \right].$$

Suppose this polynomial is identically zero on V_0, and we will have then the identity, on V_0,

$$\sum_{k=1}^{n} \Delta_k \left(A_N \frac{\partial A_1}{\partial \zeta_k} - A_1 \frac{\partial A_N}{\partial \zeta_k} \right) = \left[\sum_{k=1}^{N} i(\lambda_{N,k} - \lambda_{1,k}) \Delta_k \right] A_N A_1. \quad (7.6)$$

In a neighborhood of the point ζ_0 we can assume $A_1 A_N$ does not vanish and that, by (7.3), the algebraic variety V_0 is parametrized by

$$\zeta_1 = \varphi_1(\zeta_n), \ldots, \qquad \zeta_{n-1} = \varphi_{n-1}(\zeta_n),$$

and their derivatives are $\varphi'_k = d\varphi_k/d\zeta_n = \Delta_k/\Delta_n$. Introduce the holomorphic function $\psi(\zeta_n)$ defined in a neighborhood of $\zeta_{0,n}$ by

$$\psi(\zeta_n) = \frac{A_1}{A_N} (\varphi_1(\zeta_n), \ldots, \varphi_{n-1}(\zeta_n), \zeta_n),$$

ψ does not vanish in that neighborhood. We note that the polynomial identity (7.6) on V_0 is equivalent to the fact that ψ satisfies the differential equation

$$\frac{d\psi}{d\zeta_n} = i\psi(\zeta_n)\left[\sum_{k=1}^{n-1} (\lambda_{N,k} - \lambda_{1,k})\, \varphi_k'(\zeta_n) + (\lambda_{N,n} - \lambda_{1,n})\right].$$

This equation can be integrated immediately and yields

$$\psi(\zeta_n) = -\gamma \exp\left[i \sum_{k=1}^{n-1} ((\lambda_{N,k} - \lambda_{1,k})\, \varphi_k(\zeta_n) + (\lambda_{N,n} - \lambda_{1,n})\, \zeta_n)\right],$$

where $\gamma \in \mathbb{C}^*$. Hence we have locally on V_0 the identity

$$\frac{A_1}{A_N} = -\gamma e^{i(\lambda_N - \lambda_1)\cdot\zeta},$$

hence V_0 is contained in the variety

$$A_1 + \gamma A_N e^{i(\lambda_N - \lambda_1)\cdot\zeta} = 0$$

which is what we wanted to prove. ∎

Remark 7.2. We note that given an exponential-polynomial F with real frequencies and $P_1, ..., P_m$ a family of polynomials defining an algebraic variety of dimension smaller or equal to one, if the variety $V = \{F = P_1 = \cdots = P_m = 0\}$ is discrete then the ideal generated by $F, P_1, ..., P_m$ is s.s.d. by the same reasoning as that of Theorem 5.3. The Proposition 7.1 gives us a way of deciding whether V is discrete or not.

We can equally prove, following the ideas from [11, Proofs of Theorem 2 and of Lemma 1.2], the following proposition, which can be considered an improvement on the Ritt theorem.

PROPOSITION 7.4. *Let F be an exponential-polynomial of n variables of the form*

$$F(\zeta) = \sum_{k=1}^{N} A_k(\zeta)\, e^{i\lambda_k\cdot\zeta},$$

where the A_k, $k = 1, ..., N$, are non-zero polynomials and the λ_k are distinct elements of $\mathbb{C}^n$. If P is an irreducible polynomial dividing F (i.e., F/P is an entire function) and not dividing all the polynomials A_k, then there are two complex numbers γ, γ' and two distinct indices k, k' in $\{1, ..., N\}$ such that

$$P(\zeta) = \gamma(\lambda_k - \lambda_{k'})\cdot\zeta + \gamma'.$$

Proof of Proposition 7.4. We will prove this proposition by induction on the number N of exponentials appearing in F. The result is trivial for $N=1$, and let us assume it correct for $N=2$. The inductive step, for $N \geqslant 3$, is to assume it valid when there are at most $N-1$ exponentials. Hence we can assume that none of the A_k vanishes identically on $\{P=0\}$.

Let us consider a regular point ζ_0 of the variety $\{P=0\}$, where we can assume $(\partial P/\partial \zeta_n)(\zeta_0) \neq 0$. We have $n-1$ independent vector fields tangential to $\{P=0\}$ near ζ_0, namely,

$$\frac{\partial P}{\partial \zeta_n}\frac{\partial}{\partial \zeta_j} - \frac{\partial P}{\partial \zeta_j}\frac{\partial}{\partial \zeta_n}, \qquad j=1,..., n-1.$$

We apply them to the exponential-polynomial F and obtain $n-1$ exponential-polynomials $J_1,..., J_{n-1}$ also identically zero on $\{P=0\}$:

$$\begin{aligned} J_j(\zeta) &= \sum_{k=1}^{N} \left[\frac{\partial P}{\partial \zeta_n}\frac{\partial A_k}{\partial \zeta_j} - \frac{\partial P}{\partial \zeta_j}\frac{\partial A_k}{\partial \zeta_n} + iA_k \left(\lambda_{k,j}\frac{\partial P}{\partial \zeta_n} - \lambda_{k,n}\frac{\partial P}{\partial \zeta_j}\right)\right] e^{i\lambda_k \cdot \zeta} \\ &= \sum_{k=1}^{N} B_{k,j} e^{i\lambda_k \cdot \zeta}. \end{aligned}$$

We want to see what happens if all the exponential-polynomials $A_N J_j - B_{N,j}F$ have all their polynomial coefficients identically zero on the variety $\{P=0\}$. One will have then, for instance,

$$A_N B_{1,j} - B_{N,j} A_1 = 0 \qquad \text{on} \quad \{P=0\}. \tag{7.7}$$

This identity means that on $\{P=0\}$ we have

$$\begin{aligned} &\frac{\partial P}{\partial \zeta_n}\left(A_N \frac{\partial A_1}{\partial \zeta_j} - A_1 \frac{\partial A_N}{\partial \zeta_j}\right) - \frac{\partial P}{\partial \zeta_j}\left(A_N \frac{\partial A_1}{\partial \zeta_n} - A_1 \frac{\partial A_N}{\partial \zeta_n}\right) \\ &\quad = iA_1 A_N \left((\lambda_{1,n} - \lambda_{N,n}) \frac{\partial P}{\partial \zeta_j} - (\lambda_{1,j} - \lambda_{N,j}) \frac{\partial P}{\partial \zeta_n}\right). \end{aligned} \tag{7.8}$$

In a neighborhood of the point ζ_0 we can assume that $A_1 A_N$ does not vanish on $\{P=0\}$ and that this algebraic variety is parametrized by

$$\zeta_n = \varphi(\zeta_1,..., \zeta_{n-1}).$$

As in the proof of the previous proposition we introduce A_1/A_N as an auxiliary function on $\{P=0\}$ in a neighborhood of ζ_0, namely,

$$\psi(\zeta_1,..., \zeta_{n-1}) = \frac{A_1}{A_N}(\zeta_1,..., \zeta_{n-1}, \varphi(\zeta_1,..., \zeta_{n-1})).$$

The identity (7.8) reflects the fact that ψ satisfies a system of linear partial differential equations of first order which can be integrated explicitly to yield

$$\psi(\zeta_1,..., \zeta_{n-1}) = ce^{i(\lambda_N - \lambda_1)\cdot\zeta}$$

for some non-zero constant c. Hence we have on $\{P=0\}$, in a neighborhood of ζ_0, the identity

$$A_1 - cA_n e^{i(\lambda_N - \lambda_1)\cdot\zeta} = 0. \tag{7.9}$$

Since the variety $\{P=0\}$ is irreducible (or using the same reasoning in a neighborhood of each regular point) we see that (7.9) holds throughout and P divides the exponential-polynomial $A_1 - cA_n e^{i(\lambda_N-\lambda_1)\cdot\zeta}$. By the assumption about the case $N=2$ we see that P must be of the form $\gamma(\lambda_N - \lambda_1)\cdot\zeta + \gamma'$ as we wanted. On the other hand, if one of the exponential-polynomials $A_N J_j - B_{N,j} F$ does not have all its coefficients divisible by P, then we can apply the inductive hypothesis and also reach the desired conclusion about P.

It remains to consider the case $N=2$. It is clear then that after a linear change of coordinates we can assume F has the form

$$F(z) = A_0(z)\, e^{iz_1} + A_1(z), \tag{7.10}$$

where A_0, A_1 are relatively prime polynomials and not divisible by P. Let us assume P is not of the form $\gamma z_1 + \gamma'$. Write P as

$$P(z) = \sum_{j=0}^{r} u_j(z_1, z_3,..., z_n)\, z_2^{r-j}, \tag{7.11}$$

with $r \geq 1$, and let us consider a point z^0 such that

$$P(z^0) = 0, \qquad u_0(z_1^0, z_3^0,..., z_n^0) \neq 0.$$

The algebraic variety $\{A_0 = A_1 = P = 0\}$ has dimension less or equal to $n-2$ and, again appealing to elimination theory, we can find a polynomial $R(z_1, z_3,..., z_n)$, not identically zero, such that for all $(z_1, z_3,..., z_n)$ with $u_0(z_1, z_3,..., z_n) \neq 0$,

$$\exists z_2 \in \mathbb{C} \text{ such that}$$

$$P(z_1, z_2, z_3,..., z_n) = A_0(z_1, z_2, z_3,..., z_n) = A_1(z_1, z_2, z_3,..., z_n) = 0$$
$$\Leftrightarrow R(z_1, z_3,..., z_n) = 0. \tag{7.12}$$

Hence, we can choose the point z^0 so that

$$u_0(z_1^0, z_3^0,..., z_n^0) \neq 0, \qquad R(z_1^0, z_3^0,..., z_n^0) \neq 0, \qquad \text{and} \qquad P(z^0) = 0. \tag{7.13}$$

Consider the Sylvester resultant S in $\mathbb{C}[z_1,..., z_n, T]$ of the two polynomials

$$A_0(z_1,..., z_n)\, T + A_1(z_1,..., z_n) \qquad \text{and} \qquad P(z_1,..., z_n) \tag{7.14}$$

considered as polynomials in z_2. Since for $z_1 = z_1^0$, $z_3 = z_3^0,..., z_n = z_n^0$ the three polynomials in z_2, $P(z_1^0, z_2, z_3^0,..., z_n^0)$, $A_0(z_1^0, z_2, z_3^0,..., z_n^0)$, $A_1(z_1^0, z_2, z_3^0,... z_n^0)$, have no common zeroes the resultant S cannot be identically zero. We write it in the form

$$S(z_1, z_3,..., z_n, T) = \sum_{k=0}^{L} S_k(z_1, z_3,..., z_n)\, T^k,$$

where $S_L(z_1, z_3,..., z_n) \not\equiv 0$.

We note that if $u_0(z_1, z_3,..., z_n) \neq 0$ then we can always find z_2 such that $P(z_1, z_2,..., z_n) = 0$ by solving (7.11). Hence we can change slightly the point z^0 and suppose that (7.13) holds together with the condition $S_1(z_1^0, z_3^0,..., z_n^0) \neq 0$. By the same reasoning for each z_1 near z_1^0 we can find z_2 near z_2^0 so that

$$P(z_1, z_2, z_3^0,..., z_n^0) = 0 \qquad \text{and} \qquad u_0(z_1, z_3^0,..., z_n^0) \neq 0.$$

Since P divides F, we also have

$$e^{iz_1} A_0(z_1, z_2, z_3^0,..., z_n^0) + A_1(z_1, z_2, z_3^0,..., z_n^0) = 0,$$

which implies that the resultant S of the system (7.14) satisfies

$$0 = S(z_1, z_3^0,..., z_n^0, e^{iz_1}) = \sum_{k=0}^{L} S_k(z_1, z_3^0,..., z_n^0)\, e^{ikz_1} = 0. \tag{7.15}$$

Since the function appearing in (7.15) is entire holomorphic in the variable z_1, and it vanishes for z_1 near z_1^0, it will vanish identically. Hence it follows that $S_1(z_1, z_3^0,..., z_n^0) \equiv 0$, which is false when $z_1 = z_1^0$. This contradiction shows that P must be of the form $\gamma z_1 + \gamma'$. ∎

Ritt proved in [43] that, for $n = 1$, the quotient of exponential-sums cannot be an entire function unless this quotient is already an exponential-sum. The example $(\sin \zeta)/\zeta$ shows the difficulty in extending this result to exponential-polynomials. The best result known to date is the following [6]: if F, G are two exponential-polynomials of n variables (without any restriction on their frequencies), such that their quotient F/G is an entire function, then this quotient has the form H/P, where H is an exponential-polynomial and P a polynomial that does not divide any of the coefficients of H; moreover, one can show that if G is an exponential-sum, then P is equal to 1. One can in fact show that the example $(\sin \zeta)/\zeta$ is really typical, since by Proposition 7.4, every irreducible factor of the polynomial P is affine and its direction determined by the frequencies in H, therefore by those of F and G. We state this result in the form of a corollary.

COROLLARY 7.5. *If F, G are exponential-polynomials (with frequencies in $\mathbb{C}^n$) such that F/G is an entire function, then there is an exponential-polynomial H and a polynomial P, factorizable in affine factors, such that*

$$\forall \zeta \in \mathbb{C}^n \qquad \frac{F(\zeta)}{G(\zeta)} = \frac{H(\zeta)}{P(\zeta)}.$$

We are going to show that the techniques we have developed allow us to answer the question raised in Remark 5.3, meanwhile we need the following result.

LEMMA 7.6. *Let F be a non-zero exponential-polynomial of two variables with real frequencies; let P be an affine polynomial of the form $\alpha\zeta_1 + \beta\zeta_2 + \gamma$, with $(\alpha, \beta) \in \mathbb{R}^2 \backslash (0)$. If P divides F exactly q times, then the pair $(P, F/P^q)$ defines an ideal s.s.d. in $A_p(\mathbb{C}^2)$.*

Proof. Let us recall that thanks to a theorem due to Ehrenpreis and Martineau [7, 20], we know that for any $F \in A_p$, P polynomial if P^q divides F then F/P^q lies automatically in A_p.

After a linear change of coordinates of the form $\zeta \to A\zeta + \gamma'$, $A \in SO(2)$, $\gamma' \in \mathbb{C}$, we can assume P is a constant multiple of ζ_1. We see that if $G(\zeta) = F(\zeta)\,\zeta_1^{-q}$, we have

$$G(\zeta) = G(0, \zeta_2) + \zeta_1 H(\zeta_1, \zeta_2), \qquad H \in A_p(\mathbb{C}^2).$$

The ideal generated by G and ζ_1 coincides with the ideal generated by $G(0, \zeta_2)$ and ζ_1. On the other hand,

$$G(0, \zeta_2) = \frac{d^q}{d\zeta_1^q} F(\zeta)\,|_{\zeta_1 = 0},$$

hence $G(0, \zeta_2)$ is a non-zero exponential-polynomial of a single variable, hence it generates an ideal which is s.s.d. in $A_p(\mathbb{C})$.

It is easy to see that if both $f_1(\zeta_1)$ and $f_2(\zeta_2)$ generate principal ideals which are s.s.d. in $A_p(\mathbb{C})$, then the pair $(f_1(\zeta_1), f_2(\zeta_2))$ generates an ideal in $A_p(\mathbb{C}^2)$ which is also s.s.d.; this follows from the construction of "boxes" as already done in the proof of Theorem 3.1. ▮

We can now prove the following (compare with Theorem 5.3):

Proposition 7.7. *Let μ_1, μ_2 be two distribution with finite support in $\mathbb{R}^2$ whose supports Λ_1, Λ_2 satisfy the condition (2.22). Then the system of equations*

$$\mu_1 * f = \mu_2 * f = 0$$

has the spectral synthesis property.

Proof. Proving the spectral synthesis property is equivalent, via the Hahn–Banach theorem, to showing that every $H \in A_p(\mathbb{C}^2)$ which locally admits, near every point $\zeta \in \mathbb{C}^2$, a decomposition of the form

$$H = u_1 \hat{\mu}_1 + u_2 \hat{\mu}_2, \tag{7.16}$$

where u_1, u_2 are functions holomorphic near ζ, is in the closure of the ideal generated by $\hat{\mu}_1, \hat{\mu}_2$ in $A_p(\mathbb{C}^2)$.

Let Δ_1, Δ_2 be the greatest common divisors of the polynomial coefficients of $\hat{\mu}_1$ and $\hat{\mu}_2$, respectively. We can assume, by the Ehrenpreis–Martineau theorem already mentioned above, that Δ_1 and Δ_2 are relatively prime. If not, let Δ be their greatest common divisor, then one finds that H/Δ is an element of $A_p(\mathbb{C}^2)$ which belongs to the local ideal generated by the exponential-polynomials $\hat{\mu}_1/\Delta_1$ and $\hat{\mu}_2/\Delta_2$ and we are in the relatively prime situation.

Set $F_1 = \hat{\mu}_1/\Delta_1$ and $F_2 = \hat{\mu}_2/\Delta_2$, and note that they are still exponential-polynomials. Let $\mathscr{A}_1 = \{L_1, ..., L_a\}$ be the finite family of distinct affine polynomials which divides both Δ_1 and F_2 (affine polynomials which differ by a constant non-zero factor are considered the same). Similarly, $\mathscr{A}_2 = \{M_1, ..., M_a\}$ is defined with respect to Δ_2 and F_1.

We can write for convenient positive integers $r_j, s_j, \rho_j, \sigma_j$

$$\Delta_1 = \Delta_1' \prod_{j=1}^{a} L_j^{r_j}, \qquad F_2 = F_2' \prod_{j=1}^{a} L_j^{s_j}$$

$$\Delta_2 = \Delta_2' \prod_{j=1}^{b} M_j^{\rho_j}, \qquad F_1 = F_1' \prod_{j=1}^{b} M_j^{\sigma_j},$$

and denote $t_j = \min\{r_j, s_j\}$, $\tau_j = \min\{\rho_j, \sigma_j\}$.

A new application of the Ehrenpreis–Martineau theorem reduces us to studying the pair of elements in $A_p(\mathbb{C}^2)$ given by

$$\left(\Delta_1' F_1' \prod_{j=1}^{a} L_j^{r_j - t_j} \prod_{j=1}^{b} M_j^{\sigma_j - \tau_j}, \Delta_2' F_2' \prod_{j=1}^{a} L_j^{s_j - t_j} \prod_{j=1}^{b} M_j^{\zeta_j - \tau_j}\right) \tag{7.17}$$

which defines a discrete variety. We would finish the proof of the proposition if we show they define an ideal s.s.d. in $A_p(\mathbb{C}^2)$, since, as pointed out in Section 1, this ideal will be closed and coincides with its local ideal. In fact, this will show that the ideal generated by $\hat{\mu}_1$, $\hat{\mu}_2$ is also closed and coincides with its local ideal.

The proof of Theorem 1 [11, pp. 138–139] shows that to show the pair (7.17) is s.s.d. it is enough to show that all the pairs obtained by taking a factor from the first product and a factor from the second also generate s.s.d. ideals.

The pair $(\Delta'_1, \hat{\mu}_2)$ is s.s.d. since Remark 7.2 applies to this case. The pair (F'_1, F'_2) generates an ideal that contains F_1, F_2 hence it is s.s.d. by Theorem 5.3. We are left to consider the pairs of the type (F'_1, L_j) when $t_j < s_j$ and (F', M_j) when $\tau_j < \rho_j$, since all the remaining cases are similar. By Proposition 7.1, the definition of $\mathscr{A}_1$, and (2.22) we conclude that L_j does not divide F_1; this exponential-polynomial is in the ideal generated by F'_1 and L_j, and it follows hence, as pointed out in Remark 7.2, that this ideal is s.s.d. Let us consider finally the pair F'_1, M_j; they generate an ideal containing M_j and F_1/M^{σ_j}, hence we can apply Lemma 7.6 and conclude that it is s.s.d. ▮

We give now the solution to Problem 2 of Section 4 when $n = 1$.

PROPOSITION 7.8. *Let $F_1, \ldots, F_m$ be m exponential-polynomials of one variable with rational frequencies and without any common zeroes, there exist m functions $G_1, \ldots, G_m$ in $A_p(\mathbb{C})$ such that*

$$F_1 G_1 + \cdots + F_m G_m = 1.$$

Proof. By the remarks from Section 1 it is enough to show

$$\exists k > 0, \qquad \sum_1^m |F_j(z)| \geqslant k e^{-kp(z)}, \qquad z \in \mathbb{C}. \tag{7.18}$$

We will assume, as always, that the frequencies are all integral and non-negative so we can associate to $F_1, \ldots, F_m$, polynomials $P_1, \ldots, P_m$ of two variables, $P_j(z, e^{iz}) = F_j(z)$. Consider, in $\mathbb{C}^2$, the system

$$\zeta_2 - e^{i\zeta_1}, \qquad P_1(\zeta_1, \zeta_2), \ldots, P_m(\zeta_1, \zeta_2).$$

The algebraic variety defined by the polynomials $P_1, \ldots, P_m$ has dimension 0, hence there are constants C, k_0, $K_0 > 0$ such that

$$\|\zeta\| > C \Rightarrow \sum_1^m |P_j(\zeta)| \geqslant \frac{k_0}{(1 + \|\zeta\|)^{K_0}}.$$

On the other hand, for $\|\zeta\| \leqslant C$ the functions $\zeta_2 - e^{i\zeta_1}$, $P_1, \ldots, P_m$ have no common zeroes, hence there is a global estimate

$$\sum_{j=1}^{m} |P_j(\zeta)| + |\zeta_2 - e^{i\zeta_1}| \geqslant k_1 \frac{1}{(1+\|\zeta\|)^{K_0}}$$

from which (7.18) follows immediately. ∎

PROPOSITION 7.9. *Let F be a non-zero exponential-polynomial of one variable with rational frequencies, then its zeroes satisfy* (4.4).

Proof. We can assume as above that F has entire non-negative frequencies, and hence for some polynomial $f \in \mathbb{C}[\zeta_1, \zeta_2]$ we have $f(z, e^{iz}) = F(z)$. We can also assume f has no multiple factors in its decomposition in irreducible factors in $\mathbb{C}[\zeta_1, \zeta_2]$.

If f_1, f_2 are two distinct irreducible factors of f then there are constants C, k, K such that

$$\|\zeta\| > C \Rightarrow |f_1(\zeta)| + |f_2(\zeta)| \geqslant k \frac{1}{(1+\|\zeta\|)^K}$$

which implies

$$|z| > C \Rightarrow |f_1(z, e^{iz})| + |f_2(z, e^{iz})| \geqslant \delta e^{-Dp(z)}$$

for some positive constants δ, D. Hence

$$|z| > C, \qquad f_1(z, e^{iz}) = 0 \Rightarrow |f_2(z, e^{iz})| \geqslant \delta e^{-Dp(z)}$$

which says that outside a compact set the zeroes of f_1 stay away from the zeroes of f_2. Therefore it is enough to prove the proposition when f is an irreducible polynomial.

Consider the polynomial $g \in \mathbb{C}[\zeta_1, \zeta_2]$:

$$g(\zeta) = \frac{\partial f}{\partial \zeta_1} + i\zeta_2 \frac{\partial f}{\partial \zeta_2}. \tag{7.19}$$

We have

$$g(z, e^{iz}) = F'(z).$$

We want to show that the variety $\{f = g = 0\}$ is discrete (hence finite). If not, f would divide g, and by degree considerations there is $\lambda \in \mathbb{C}$ such that

$$g = \lambda f. \tag{7.20}$$

If $\lambda \neq 0$, we have

$$\frac{\partial f}{\partial \zeta_1}(\zeta_1, 0) = \lambda f(\zeta_1, 0)$$

which implies, since f is a polynomial,

$$f(\zeta_1, 0) \equiv 0.$$

In this case, since f is irreducible, f is the polynomial ζ_2 and F has no zeroes. Suppose now that $\lambda = 0$. We can write

$$f(\zeta) = B(\zeta_1)\,\zeta_2^m + \cdots \qquad \text{with } B \not\equiv 0.$$

By (7.19) and (7.20), with $\lambda = 0$, we have

$$B'(\zeta_1) + im\, B(\zeta_1) = 0. \tag{7.21}$$

Since B is a polynomial this implies that $m = 0$, and hence (7.21) says B is constant. Therefore f is also constant in this case. Hence we can suppose f and g have only a finite number of common zeroes, therefore we have c, k, $K > 0$ such that

$$\|\zeta\| > c \Rightarrow |F(\zeta)| + |g(\zeta)| \geqslant \frac{k}{(1 + \|\zeta\|)^K},$$

which in turn implies

$$|z| > c \Rightarrow |F(z)| + |F'(z)| \geqslant \delta e^{-Dp(z)}.$$

This last inequality implies also that the distinct zeroes stay away from each other (and the only multiple zeroes occur in $|z| \leqslant c$). ∎

8

We propose to study here systems of exponential-polynomials in $\mathbb{C}^3$ with frequencies in $\mathbb{N}^3$, always under the assumption that they define a variety V discrete or empty. We know, by the example of Section 4, that in general they do not generate an ideal s.s.d. in $A_p(\mathbb{C}^3)$. We will also try to show why this example $(\cos \zeta_1, \cos \zeta_2, \zeta_2 - \lambda \zeta_1)$ is essentially the only type of example where the property of being s.s.d. fails. This study will allow us to introduce a new method, based on the concept of geometric duality, which looks promising for use in a more general context (in particular, studying analogous systems in more dimensions).

We have seen that in the above example the difficulty lies in the fact that the three functions do not depend on the variable ζ_3, and this leads to the following definition.

DEFINITION 8.1. *Let Y be an irreducible algebraic variety of dimension* 3 *in* $\mathbb{C}^6$ (*where the coordinates are always denoted* $(\zeta_1, \zeta_2, \zeta_3, X_1, X_2, X_3)$ *and* Y *is not contained in* $\{X_1 X_2 X_3 = 0\}$), *we say* Y *is* incomplete *if there is a coherent change of coordinates such that in the new coordinates* Y *can be defined by a system of equations where the variables* ζ_3, X_3 *do not appear explicitly.*

This definition means that if Y is incomplete there is an irreducible curve Y' in a 4-dimensional space (i.e., the space given by $\zeta_3 = X_3 = 0$ in the new coordinates) such that Y is fibered by linear varieties of dimension 2 over Y'.

A way to decide whether Y is incomplete is using a family $\mathscr{F}$ of pairs of differential operations, $\mathscr{F}$ invariant under coherent changes of coordinates, namely, the $\mathbb{Q}$-vector space generated by the three pairs $(\partial/\partial\zeta_j, X_j\partial/\partial X_j)$, $j = 1, 2, 3$. If Y is incomplete then there is an element of $\mathscr{F}$ leaving invariant the ideal $I(Y)$ of Y. For every element of $\mathscr{F}$ leaving invariant $I(Y)$ we can check whether its kernel contains a system of generators of Y, and all we need to compute is the dimension of the algebraic variety defined by the elements in this kernel; if it is 3 then Y is an incomplete variety.

We can state the following theorem:

THEOREM 8.1. *Let* $F_1, \ldots, F_m$ *be exponential-polynomials of three variables with frequencies on* $\mathbb{N}^3$ *defining a variety* V *of dimension* $\leqslant 0$. *Assume that the variety* Y *associated to them via* (3.3) *has dimension at most* 3 *and no irreducible component of* Y *is incomplete. Then there is a constant* $\delta > 0$ *such that*

$$\forall \rho \in (\mathbb{C}^*)^3, \qquad \|1 - \rho\| < \delta \Rightarrow \dim V^{(\rho)} \leqslant 0, \tag{8.1}$$

and hence the ideal generated by $F_1, \ldots, F_m$ *is s.s.d.*

Remark 8.1. No condition is imposed on the irreducible components of Y of dimension less than or equal to 2 if Y itself has dimension $\leqslant 2$, then the conclusion of the theorem holds.

Proof of Theorem 8.1. By Proposition 3.2 to prove that the system is s.s.d. one can assume Y is irreducible, but in fact one can assume the same thing to prove (8.1). Namely, if W is an irreducible branch of dimension $\geqslant 1$ of $V^{(\rho)}$ for some exceptional ρ, then the variety $Z = \{(\zeta, X) \in \mathbb{C}^6 : \zeta \in W, X_j = \rho_j e^{i\zeta_j}, j = 1, 2, 3\}$ is contained in one of the irreducible components of Y

We will therefore assume that the $p_1, \ldots, p_m$ defined by (3.2) generate a prime ideal $\mathscr{P}$ in $\mathbb{C}[\zeta, X]$. Assume first that no affine polynomial of the form $r \cdot \zeta - \gamma$, $r \in \mathbb{Q}^3 \backslash (0)$, $\gamma \in \mathbb{C}$, belongs to $\mathscr{P}$.

Consider an exceptional value ρ and a branch W of $V^{(\rho)}$, $\dim W \geqslant 1$. We will show that W is a line and furthermore there is only a finite number of possible lines to choose from. This will prove (8.1) as done in Theorem 5.3.

By Proposition 2.4 there exists $r \in \mathbb{Q}^3 \backslash (0)$, $\gamma \in \mathbb{C}$ such that

$$W \subseteq \{r \cdot \zeta - \gamma = 0\}. \tag{8.2}$$

We make a coherent change of coordinates in $\mathbb{C}^6$ so that the equation of the plane containing W becomes $\zeta'_1 = \gamma'$. The algebraic subvariety of $\mathbb{C}^6$, $Y \cap \{\zeta'_1 = \gamma'\}$, has dimension at most 2 by the hypothesis made on the ideal $\mathscr{P}$. Let $q_1 = 0, \ldots, q_m = 0$ be the equations of Y in the new coordinates (ζ', X'). There is an element $\rho' \in (\mathbb{C}^*)^3$ related to ρ as X' is to X such that, in the new coordinates,

$$W \subseteq \{G_j^{(\rho)}(\zeta'_2, \zeta'_3) := q_j(\gamma', \zeta'_2, \zeta'_3, \rho'_1 e^{i\gamma'}, \rho'_2 e^{i\zeta'_2}, \rho'_3 e^{i\zeta'_3}) = 0, \forall j\}. \tag{8.3}$$

The algebraic subvariety of $\mathbb{C}^4$

$$\{q_j(\gamma', \zeta'_2, \zeta'_3, \rho'_1 e^{i\gamma'}, X'_2, X'_3) = 0,\ j = 1, \ldots, m\} \tag{8.4}$$

has dimension $\leqslant 2$, hence a new application of Proposition 2.4 gives us $(s_2, s_3) \in \mathbb{Q}^2 \backslash (0)$, $\gamma'' \in \mathbb{C}$ such that

$$W \subseteq \{s_2 \zeta'_2 + s_3 \zeta'_3 = \gamma''\} \cap \{\zeta'_1 = \gamma'\}. \tag{8.5}$$

We see that W is indeed a line.

We need now to show that the number of possible directions of W is finite. Since the variety V is discrete it is impossible that all the polynomial coefficients of all the exponential-polynomials $F_1, \ldots, F_m$ vanish identically on W (recall that the polynomial coefficients of the $F_j^{(\rho)}$ differ from those of F_j only by non-zero multiplicative constants). By Proposition 7.1

$$W \subseteq \{(\lambda - \lambda') \cdot \zeta = \alpha\},$$

where λ and λ' are two distinct frequencies of one of the F_j and $\alpha \in \mathbb{C}$. We can now redo the above proof starting at (8.2) with r replaced by $\lambda - \lambda'$ and γ by α. We arrive at the situation (8.3) where we see that the frequencies of the exponential-polynomials $G_j^{(\rho)}(\zeta'_2, \zeta'_3)$ depend only on those of $F_1, \ldots, F_m$.

By Proposition 7.1 only two things could take place, either W is contained in a plane of the form $(\mu_2 - \mu'_2)\zeta'_2 + (\mu_3 - \mu'_3)\zeta'_3 = \beta$, where (μ_2, μ_3) and (μ'_2, μ'_3) are two distinct frequencies of one of the $G_j^{(\rho)}$, and this fixe completely the direction of W among a finite number of directions, or every polynomial coefficient of all the $G_j^{(\rho)}$ vanishes at W, and this implies that the variety defined by (8.4) has dimension 3 in $\mathbb{C}^4$, which is impossible.

Now fix the direction of W (it is after all one among finitely many possible ones), and we want to see that the parameters α, β that appeared above are also in a finite set. We make a new coherent change of coordinates so that in the new coordinates (ζ', X'), W is given by the equations

$$W: \zeta'_1 = \alpha', \qquad \zeta'_2 = \beta'.$$

Let $q_1,..., q_m$ be the equations of Y in the new coordinates. There is a $\rho' \in (\mathbb{C}^*)^3$ related to ρ as X' is to X such that

$$q_j(\zeta', \rho' e^{i\zeta'}) = 0 \qquad \text{in } W \text{ for } j = 1,..., m. \tag{8.6}$$

Let us write the equations from (8.6) in the form

$$q_j = \sum_{k,l} (\zeta'_3)^k (\rho'_3 e^{i\zeta'_3})^l A_{j,k,l}(\zeta'_1, \zeta'_2, \rho'_1 e^{i\zeta'_1}, \rho'_2 e^{i\zeta'_2}) = 0.$$

The identities (8.6) are equivalent to

$$\forall j, k, l \qquad A_{j,k,l} = 0 \qquad \text{if} \quad \zeta'_1 = \alpha', \zeta'_2 = \beta'. \tag{8.7}$$

Consider the algebraic variety Z in $\mathbb{C}^4$ defined by

$$Z = \{A_{j,k,l}(\zeta'_1, \zeta'_2, X'_1, X'_2) = 0 \qquad \forall j \, \forall k \, \forall l\}.$$

This variety has necessarily dimension $\leqslant 1$ since dim $Y \leqslant 3$. If dim $Z = 1$, the Y coincides with an algebraic variety fibered over Z and it is an incomplete variety. This case has been excluded by hypothesis, hence dim $Z = 0$, hence Z is a finite subset of $\mathbb{C}^4$ and this says that α', β' take values in a finite set. Let us remark that we have also provided in this case a description of the exceptional set, namely, as a finite union of subsets of $(\mathbb{C}^*)^3$ of the form $\{\rho^{v_1} = c_1, \rho^{v_2} = c_2\}$, where $v_1, v_2 \in \mathbb{N}^3 \setminus (0)$ are $\mathbb{Q}$-linearly independent and c_1, c_2 are two complex numbers distinct from 0 and 1. Since $(1, 1, 1)$ does not belong to the algebraic set we have just described this proves (8.1) in this case (i.e., without appealing to the limit argument of Theorem 5.3).

It remains to consider the case where the ideal $\mathscr{P}$ contains an affine polynomial of the form $r \cdot \zeta - \gamma$ with $r \in \mathbb{Q}^3 \setminus (0)$, $\gamma \in \mathbb{C}$. In this case, for every such $r \in \mathbb{Q}^3 \setminus (0)$ there is a single γ, unless the ideal $\mathscr{P}$ contains the function 1 (in which case there are no exceptional values ρ). If there are in $\mathscr{P}$ two affine polynomials $r_1 \cdot \zeta - \gamma_1$, $r_2 \cdot \zeta_2 - \gamma_2$ with r_1, r_2 $\mathbb{Q}$-linearly independent, then every possible branch W of dimension $\geqslant 1$ of $V^{(\rho)}$ would always be the same line and we see directly, since V is discrete, that (8.1) is satisfied. In any case, let us suppose that $r \cdot \zeta - \gamma \in \mathscr{P}$ and make a coherent change of coordinates so that in the new coordinates we can take as generators of the ideal $\mathscr{P}$

$$\zeta_1' - \gamma', \qquad q_j(\zeta_2', \zeta_3', X_1', X_2', X_3'), \qquad j = 1, \ldots, M. \tag{8.8}$$

Consider the exponential-polynomials

$$q_j(\zeta_2', \zeta_3', \rho_1' e^{i\gamma'}, \rho_2' e^{i\zeta_2'}, \rho_3' e^{i\zeta_3'}), \qquad j = 1, \ldots, M. \tag{8.9}$$

They form a family of exponential-polynomials of two variables with integral frequencies and defining a discrete variety in $\mathbb{C}^2$, by Theorem 5.1, they generate an ideal s.s.d. in $A_p(\mathbb{C}^2)$, hence the same holds for the ideal generated in $A_p(\mathbb{C}^3)$ starting from the polynomials in (8.8). Therefore our original ideal is s.s.d. But this reasoning does not prove the more refined statement (8.1); this is what we are going to do now, supposing, to simplify the notation, that in the original coordinates Y was generated by the polynomials (8.8).

Let ρ be an exceptional value in $(\mathbb{C}^*)^3$ and W an irreducible branch of $V^{(\rho)}$, $\dim W \geqslant 1$. Consider the algebraic variety $Z^{(\rho_1)}$ in $\mathbb{C}^4$:

$$\{q_j(\zeta_2, \zeta_3, \rho_1 e^{i\zeta_1}, X_2, X_3) = 0, j = 1, \ldots, M\}. \tag{8.10}$$

The dimension of this variety is at most 3, on the other hand there is at least one value of ρ_1 ($\rho_1 = 1$) such that this dimension is at most 2 since V is discrete.

Using elimination theory one sees that the condition $\dim Z^{(\rho_1)} = 3$ is a non-trivial algebraic condition on ρ_1, hence this condition can be satisfied by at most a finite number of values of ρ_1, all different from 1, and there is hence $\delta' > 0$ such that

$$\|1 - \rho\| < \delta' \Rightarrow \dim Z^{(\rho_1)} \leqslant 2.$$

(If we write everything in terms of the original ρ, we see that the set of exceptional values is contained in the union of a finite number of algebraic varieties in $(\mathbb{C}^3)^*$ which do not pass through $(1, 1, 1)$ and have for equations $\rho^\nu = c$, $\nu \in \mathbb{N}^3 \backslash (0)$.)

Assume now that $\dim Z^{(\rho_1)} \leqslant 2$, then Proposition 2.4 applies and it follows that W is a line. It is impossible that all the polynomial coefficients of all the exponential-polynomials $q_j(\zeta_2, \zeta_3, \rho_1 e^{i\zeta_1}, \rho_2 e^{i\zeta_2}, \rho_3 e^{i\zeta_3})$ are identically zero on W, for if they were $\dim Z^{(\rho_1)} = 3$. We find ourselves in the same situation as before since it is clear now, by Proposition 7.1, that the number of possible directions of W is finite. In the same way we arrive at the fact that W belongs to a finite family of lines. This ends the proof of Theorem 8.1. ▌

We still have to consider the case where the algebraic variety Y, which we assume to be irreducible, has dimension bigger than 3. Since we assume V is discrete it follows in this case that $\dim Y=4$. The variety Z of singular points of Y then has dimension less than or equal to 3, and let us consider a system of generators $g_1,..., g_l$ of the ideal $I(Z)$ and the corresponding exponential-polynomials $G_1,..., G_l$. It is clear that the analytic variety $\{G_1 = \cdots = G_l = 0\}$ is contained in V, hence it is discrete; on the other hand, if the $G_1,..., G_l$ generate an ideal $\mathscr{S}$ s.s.d. in $A_p(\mathbb{C}^3)$ this property of $\mathscr{S}$ is independent of the choice of generators of $I(Z)$.

THEOREM 8.2. *Let* $F_1,..., F_m$ *be exponential-polynomials of three variables with frequencies in* $\mathbb{N}^3$, *and assume that the variety* V *is discrete (or empty) and that the algebraic variety* Y *is irreducible and* $\dim Y=4$. *If the ideal* $\mathscr{S}$ *is s.s.d., then the ideal* $\mathscr{T}$ *generated by* $F_1,..., F_m$ *is also s.s.d.*

Remark 8.2. It is clear that if $\mathscr{T}$ is s.s.d. then $\mathscr{S}$ is also s.s.d.

Proof of Theorem 8.2. The idea of the proof is to add to the exponential-polynomials $F_1,..., F_m$ a new one, $u\cdot\zeta = u_1\zeta_1 + u_2\zeta_2 + u_3\zeta_3$, with u generic, show that one can arrive at estimates of the type mentioned in Remark 3.3, and, finally, using a method of geometric duality eliminate the parameters u.

Consider in $\mathbb{C}(u)[\zeta, X]$ the ideal $\mathscr{J}$ generated by $p_1,..., p_m, u\cdot\zeta$, where $p_1,..., p_m$ are the polynomials associated with the $F_1,..., F_m$ by (3.2). We decompose, in $\overline{\mathbb{C}(u)}[\zeta, X]$, the ideal $\mathscr{J}$ in primary components $\mathscr{J}_1,..., \mathscr{J}_r$ and denote $(q_{j,1},..., q_{j,n_j})$ a family of generators in $\overline{\mathbb{C}(u)}[\zeta, X]$ of the radical $\sqrt{\mathscr{J}_j}$. Off an algebraic variety in u, the algebraic subvarieties of $\mathbb{C}^6$ defined by

$$Y_j^{(u)} = \{(\zeta, X)\in\mathbb{C}^6 \colon q_{j,1}(u)(\zeta, X) = \cdots = q_{j,n_j}(u)(\zeta, X) = 0\} \tag{8.11}$$

are well defined and of dimension $\leqslant 3$, the numbers $q_{j,l}(u)$ being chosen as function values of $q_{j,l}$ belonging to the allowable arguments [48]. Denote $\Delta_{j,1},..., \Delta_{j,m_j}$ all the 3×3 minors of the matrix of partial derivatives of the polynomials $q_{j,k}(\zeta, X)$ with respect to the variables ζ, X; at least one of these minors does not belong to the ideal $\sqrt{\mathscr{J}_j}$. Hence, outside an algebraic variety in u, the algebraic varieties defined by

$$\begin{aligned}\{(\zeta, X)\in\mathbb{C}^6 \colon q_{j,1}(u)(\zeta, X) = \cdots &= q_{j,n_j}(u)(\zeta, X)\\ = \Delta_{j,1}(u)(\zeta, X) = \cdots &= \Delta_{j,m_j}(u)(\zeta, X) = 0\}\end{aligned} \tag{8.12}$$

are well defined and of dimension less than or equal to 2. The varieties we have introduced in (8.11) play the rôle of the varieties $Y_0^{(1)}, \ldots$ considered at the beginning of Section 6 (on the other hand, here the defining equations depend on parameters), while those introduced in (8.12) play the rôle of $Z_0^{(1)}, \ldots$.

Let us consider now the exponential-polynomials $Q_{j,k}(u)(\zeta)$, $j=1,\ldots,r$, $k=1,\ldots,n_j$, defined for generic values of u by

$$Q_{j,k}(u)(\zeta) = q_{j,k}(u)(\zeta, e^{i\zeta}). \tag{8.13}$$

All the minors of rank 3 of the matrix $\|\partial Q_{j,k}/\partial \zeta_i\|_{i,k}$ can be written in the form $h_{j,t}(u)(\zeta, e^{i\zeta})$, where $h_{j,t}(u)(\zeta, X) \in \overline{\mathbb{C}(u)}[\zeta, X]$.

We can at this moment classify the ideals $\sqrt{\mathscr{I}_j}$ into two classes:

(a) those for which the polynomials $h_{j,t} \in \sqrt{\mathscr{I}_j}$ for all t;

(b) those for which at least one of the polynomials $h_{j,t}$ does not belong to $\sqrt{\mathscr{I}_j}$.

In the class (b), for u generic, the algebraic subvariety of $\mathbb{C}^6$,

$$\{q_{j,k}(u)(\zeta, X) = 0 \ \forall k, \ j_{j,t}(u)(\zeta, X) = 0 \ \forall t\}, \tag{8.14}$$

has dimension $\leqslant 2$.

In order to study the exponential-polynomials depending on parameters associated with the polynomials in $\mathbb{C}(u)[\zeta, X]$ appearing in (8.12) or in (8.14) (class (b)), we need the following lemma.

LEMMA 8.3. *Let $S_1(\zeta), \ldots, S_M(\zeta)$ be exponential-polynomials with frequencies in $\mathbb{N}^3$ and coefficients in $\overline{\mathbb{C}(u)}[\zeta]$. Assume that the ideal generated in $\overline{\mathbb{C}(u)}[\zeta, X]$ by the polynomials $S_1(\zeta, X), \ldots, S_M(\zeta, X)$ contains a power of $u \cdot \zeta$ and that the algebraic subvariety of $[\overline{\mathbb{C}(u)}]^6$ that these polynomials define has dimension $\leqslant 2$. Then for u generic*

$$\forall \rho \in (\mathbb{C}^*)^3, \tag{8.15}$$

$$\dim\{\zeta \in \mathbb{C}^3 \colon S_1(u)(\zeta, \rho e^{i\zeta}) = \cdots = S_M(u)(\zeta, \rho e^{i\zeta}) = 0\} \leqslant 0.$$

Proof of Lemma 8.3. Let us consider a value u such that the algebraic variety defined by the $S_j(u)(\zeta, X)$ has dimension $\leqslant 2$ and that u_1, u_2, u_3 are $\mathbb{Q}$-linearly independent.

Let $\rho \in (\mathbb{C}^*)^3$ and W be a possible irreducible branch of dimension $\geqslant 1$ of the variety

$$\Sigma_u^{(\rho)} = \{\zeta \colon S_1(u)(\zeta, \rho e^{i\zeta}) = \cdots = S_M(u)(\zeta, \rho e^{i\zeta})\} = 0.$$

By Proposition 2.4 we know there are $\lambda \in \mathbb{Q}^3 \backslash (0)$ and $\alpha \in \mathbb{C}$ such that $W \subseteq \{\lambda \cdot \zeta - \alpha = 0\}$. Let us make a coherent change of coordinates so that this hyperplane becomes $\zeta_1' = \alpha'$. Consider the exponential-polynomials obtained by replacing ζ_1' by α' and $e^{i\zeta_1'}$ by $e^{i\alpha'}$ in the equations defining $\Sigma_u^{(\rho)}$ in the new coordinates. Since the associated algebraic variety remains of dimension less than or equal to 2 in $\mathbb{C}^4$ we can apply once more Proposition 2.4, and there is then a line with rational direction numbers in the plane of the variables ζ_2', ζ_3' which contains the projection of W into this plane. Returning to the original coordinates we have two elements $\lambda, \mu \in \mathbb{Q}^3 \setminus (0)$, $\mathbb{Q}$-linearly independent, and two complex numbers α, β such that

$$W \subseteq \{\lambda \cdot \zeta - \alpha = \mu \cdot \zeta - \beta = 0\}.$$

But on the other hand, due to the hypothesis of the lemma, we have also

$$W \subseteq \{u \cdot \zeta = 0\}.$$

The $\mathbb{Q}$-linear independence of u_1, u_2, u_3 implies that this branch W must be of dimension 0, which shows there cannot be any exceptional values ρ. ∎

We are hence exactly under the conditions needed to apply Proposition 3.3 to the ideals $\sqrt{\mathscr{I}_j}$ of class (b) or to the ideals associated to the equations in (8.12).

To study the exponential-polynomials corresponding to the generators of an ideal $\sqrt{\mathscr{I}_j}$ in the class (a) we need several lemmas.

LEMMA 8.4. *Let $p_1, \ldots, p_m \in \mathbb{C}[\zeta, X]$ generate a prime ideal of dimension 4 in $\mathbb{C}^6$, and assume that the variety V associated to the corresponding exponential-polynomials $F_1, \ldots, F_m$ is discrete. There exists a finite family $\mathscr{F}$ of non-zero elements of $\mathbb{Q}^3$ such that for u generic, $\rho \in (\mathbb{C}^*)^3$, and W an irreducible branch of positive dimension of $V^{(\rho)} \cap \{u \cdot \zeta = 0\}$, there are $\lambda_0 \in \mathscr{F}$, $\gamma = \gamma(\rho, \lambda_0, u) \in \mathbb{C}$ such that W is contained in the hyperplane $\lambda_0 \cdot \zeta - \gamma = 0$, where γ satisfies the conditions*

$$\begin{aligned} &\sum \gamma^k A_{k,\lambda_0}(u) = 0 \\ &\sum (e^{i\gamma} \rho^{\lambda_0})^k B_{k,\lambda_0}(u) = 0. \end{aligned} \tag{8.16}$$

The coefficients A_{k,λ_0}, B_{k,λ_0} belong to two finite families of polynomials (depending on the index $\lambda_0 \in \mathscr{F}$).

Remark 8.3. We see that the lemma implies the existence of polynomial Q and of a finite family $\mathscr{H}$ of holomorphic functions such that if $Q(u) \neq 0$ and u_1, u_2, u_3 are $\mathbb{Q}$-linearly independent, then the exceptional ρ corresponding to such a u satisfy an equation of the type

$$\rho^{\lambda_0} = h(u)$$

for some $\lambda_0 \in \mathscr{F}$ and some $h \in \mathscr{H}$.

Remark 8.4. When u_1, u_2, u_3 are fixed and $\mathbb{Q}$-linearly independent, the variety defined by $p_1 = \cdots = p_m = u \cdot \zeta = 0$ does not have incomplete irreducible components. It is then natural to find a description of the exceptional set as we have done in the proof of Theorem 8.1.

Proof of Lemma 8.4. When u has been chosen generically, the dimension of the algebraic variety $Y \cap \{u \cdot \zeta\} = 0$ is $\leqslant 3$. Let W be a branch of $V^{(\rho)} \cap \{u \cdot \zeta = 0\}$, dim $W \geqslant 1$. By Proposition 2.4 there are $\lambda \in \mathbb{Q}^3 \setminus (0)$, $\gamma \in \mathbb{C}$ so that W is the line of equations

$$W = \{\lambda \cdot \zeta - \gamma = u \cdot \zeta = 0\} \tag{8.17}$$

(recall that u_1, u_2, u_3 are $\mathbb{Q}$-linearly independent). Since W is a line we can use Proposition 7.1 and conclude, thanks to the discreteness of V, that λ can be replaced by an element $\lambda_0 \in \mathscr{F}$, $\mathscr{F}$ a finite subset of $\mathbb{Q}^3 \setminus (0)$.

We can assume, for instance, that

$$\Delta = u_1 \lambda_{0,2} - u_2 \lambda_{0,1} \neq 0$$

and parametrize the line W using the variable ζ_3:

$$\zeta_1 = \frac{\zeta_3(u_2 \lambda_{0,3} - u_3 \lambda_{0,2}) - \gamma u_2}{\Delta} = \alpha_1 \zeta_3 + \alpha_2$$

$$\zeta_2 = \frac{\zeta_3(u_3 \lambda_{0,1} - u_1 \lambda_{0,3}) + \gamma u_1}{\Delta} = \beta_1 \zeta_3 + \beta_2.$$

Let us make explicit the fact that $F_j^{(\rho)}$ vanish identically on W:

$$\sum_{k \in \mathbb{N}^3} A_{k,j}(\alpha_1 \zeta_3 + \alpha_2, \beta_1 \zeta_3 + \beta_2, \zeta_3)\, \rho^k e^{i\gamma((k_2 u_1 - k_1 u_2)/\Delta)} \times e^{i\zeta_3(\alpha_1 k_1 + \alpha_2 k_2 + k_3)} \equiv 0. \tag{8.18}$$

In order to group the terms in (8.18) following the frequencies of the different exponentials in ζ_3, one must find those $k \in \mathbb{N}^3$ such that

$$\alpha_1 k_1 + \alpha_2 k_2 + k_3 = 0.$$

This condition is equivalent to

$$\begin{vmatrix} u_1 & \lambda_{0,1} & k_1 \\ u_2 & \lambda_{0,2} & k_2 \\ u_3 & \lambda_{0,3} & k_3 \end{vmatrix} = 0. \tag{8.19}$$

The $\mathbb{Q}$-linear independence of the u_i shows that (8.19) is satisfied if and only if

$$k = r\lambda_0, \qquad r \in \mathbb{Q}. \tag{8.20}$$

The expression (8.20) defines an equivalence relation on the family of indices k appearing in (8.18). This leads to a finite number of equations (obtained after grouping together the terms in the same equivalence class):

$$\sum_r B_{j,\lambda_0,s,r}(u, \gamma)(\rho^{\lambda_0 + i\gamma})^r = 0. \tag{8.21}$$

The index s in (8.21) corresponds to the equivalence class, and the sum takes place along a finite collection of rationals r. The $B_{j,\lambda_0,s,r} \in \mathbb{C}(u)[\gamma]$. After chasing the denominators of the r and of the $B_{j,\lambda_0,s,r}$ we find polynomials $B_{j,\lambda_0,s} \in \mathbb{C}[u][z, T]$. Since $\rho^{\lambda_0} e^{i\gamma} \neq 0$ we can assume these polynomials are not divisible by T. Fix $\lambda_0 \in \mathscr{F}$, if there is a polynomial $B \in \mathbb{C}(u)[z, T]$ which divides $B_{j,\lambda_0,s}$ for all j and s, this polynomial cannot be a polynomial multiple of T, and this implies that for a generic value of u, $B(u)(\sigma, e^{i\gamma})$ has a zero γ_0, hence the line $W = \{\lambda_0 \cdot \zeta - \gamma_0 = u \cdot \zeta = 0\}$ is included in $V \cap \{u \cdot \zeta = 0\}$ and a fortiori in V, which is impossible. Hence we can use elimination theory and find, for the given $\lambda_0 \in \mathscr{F}$, two finite families of polynomials $A_{k,\lambda_0}, B_{k,\lambda_0} \in \mathbb{C}[u]$ such that the equations

$$B_{j,\lambda_0,s}(u)[z, T] = 0 \qquad \forall j \, \forall s$$

imply, for generic u,

$$\sum_k A_{k,\lambda_0}(u)\, z^k = \sum_k B_{k,\lambda_0}(u)\, T^k = 0.$$

This ends the proof of Lemma 8.4. ∎

LEMMA 8.5. *Let $\mathscr{K}$ be one of the ideals $\sqrt{\mathscr{I}_j}$ introduced above having the following properties:*

(i) *$\mathscr{K}$ has dimension 3 in $\overline{\mathbb{C}(u)}[\zeta, X]$ (this corresponds to the fact that for generic values of u the variety defined by (8.11) has dimension 3 in $\mathbb{C}^6$);*

(ii) *$\mathscr{K}$ contains an irreducible polynomial in $\overline{\mathbb{C}(u)}[\zeta_1]$ of the form $\pi_1 = A_n \zeta_1^n + \cdots + A_0$;*

(iii) *$\mathscr{K}$ contains an irreducible polynomial in $\overline{\mathbb{C}(u)}[X_1]$ of the form*

$$\pi_2 = B_N X_1^N + \cdots + B_0.$$

Then we have $n = 1$ and there is $a \in \mathbb{C}^*$ such that $A_0 = aA_1$.

Proof of Lemma 8.5. The ideal $\mathscr{K}$ contains the polynomial $u \cdot \zeta$ since this polynomial belongs to $\mathscr{J}$, hence $\mathscr{K}$ contains also the polynomial

$$A_n(-u_2\zeta_2 - u_3\zeta_3)^n + \cdots + A_0 u_1^n.$$

Since $\mathscr{K}$ is a prime ideal, it contains also an irreducible factor in $\overline{\mathbb{C}(u)}[\zeta_2, \zeta_3]$ of the above one, i.e., of the form

$$\pi_3 = \sum A_{s,t} \zeta_2^s \zeta_3^t.$$

The ideal generated by π_1, π_2, π_3 is prime and also has dimension 3, hence it coincides with $\mathscr{K}$. Since $u \cdot \zeta \in \mathscr{K}$, we can find polynomials $\chi_1, \chi_2, \chi_3 \in \overline{\mathbb{C}(u)}[\zeta, X]$ such that

$$u \cdot \zeta = \chi_1 \pi_1 + \chi_2 \pi_2 + \chi_3 \pi_3; \tag{8.22}$$

the parameter u being generic we can choose X_1, ζ_2, ζ_3 such that $\pi_2(X_1) = \pi_3(\zeta_2, \zeta_3) = 0$. The identity (8.22) is then an identity between polynomials in ζ_1 (coefficients in $\overline{\mathbb{C}(u)}$) and hence the degree n of π_1 is exactly 1.

Using this new information we obtain

$$\pi_3(\zeta_2, \zeta_3) = A_1(-u_2\zeta_2 - u_3\zeta_3) + A_0 u_1.$$

Recall that the original polynomials $p_1, ..., p_m$ have coefficients in $\mathbb{C}$ and define an irreducible variety of dimension 4 in $\mathbb{C}^6$. Using elimination we obtain a non-zero polynomial $f \in \mathbb{C}[\zeta, X_2, X_3]$ in the ideal generated by $p_1, ..., p_m$, hence $f \in \mathscr{K}$ also. There are $\psi_1, \psi_2, \psi_3 \in \overline{\mathbb{C}(u)}[\zeta, X]$ such that

$$f = \sum f_{s,t}(\zeta)\, X_2^s X_3^t = \pi_1\psi_1 + \pi_2\psi_2 + \pi_3\psi_3.$$

This identity implies that the polynomials $f_{s,t}$ are in the ideal generated by π_1, π_3 in $\overline{\mathbb{C}(u)}[\zeta]$. Fix one of these polynomials $f_{s,t}$ and set

$$\zeta_1 = \frac{A_0}{A_1}, \qquad \zeta_3 = -\frac{u_2}{u_3}\zeta_2 - \frac{u_1}{u_3}\zeta_1, \qquad \zeta_2 \in \mathbb{C}.$$

We will have

$$f_{s,t}(\zeta_1, \zeta_2, \zeta_3) \equiv 0 \qquad \text{as a function of } \zeta_2. \tag{8.23}$$

This identity tells us that the coefficient ξ of highest degree in ζ_2 in (8.23) and the coefficient η of lowest degree must be identically zero (for u generic). Now, $\xi = \xi(\zeta_1, u_2/u_3)$, where $\xi(\zeta_1, T)$ is a non-zero polynomial with constant coefficients. Similarly for $\eta = \eta(\zeta_1, u_1/u_3)$. Compute the Sylvester resultant of ξ, η with respect to the first variable. There are two possibilities:

—either this resultant is trivial, meaning that at least one of the two polynomials ξ, η does not depend on the second variable (in this case ζ_1 is one of the zeros of this polynomial and hence it is in $\mathbb{C}$, which is the desired conclusion),

—or both polynomials depend on the second variable and hence the Sylvester resultant gives a non-trivial relation between u_1, u_2, u_3 which must be identically zero for u generic, and this is clearly impossible. ∎

Let us go back to the proof of Theorem 8.2.

If $\sqrt{\mathscr{I}_j}$ is an ideal of the class (a) and dimension 2 in $\mathbb{C}(u)[\zeta, X]$, then we are in the situation of Lemma 8.3 and hence Proposition 3.3 can be applied to the system of exponential-polynomials with parameters corresponding to the generators of $\sqrt{\mathscr{I}_j}$.

To show that the same conclusion applies in the case $\sqrt{\mathscr{I}_j}$ is of class (a) and dimension 3 we will show that after a coherent change of coordinates the hypothesis of Lemma 8.5 holds. Assume this claim for the moment, the ideal contains the polynomial $u \cdot \zeta$, a polynomial of the form $\zeta_1 + a$, and a polynomial with coefficients in $\overline{\mathbb{C}(u)}$ of the form $B_N X_1^N + \cdots + B_0$ which are then its generators.

Let us consider the element of $\overline{\mathbb{C}(u)}$

$$r = B_N e^{-iNa} + \cdots + B_0.$$

This element cannot be zero, otherwise for u generic, the subvariety of $\mathbb{C}^3$ defined by the exponential-polynomials associated to $\sqrt{\mathscr{I}_j}$ would have as equations

$$\zeta_1 + a = 0$$
$$u \cdot \zeta = 0.$$

We have here a line contained in $V \cap \{u \cdot \zeta = 0\}$, hence in V, contradicting the fact that V is discrete.

The ideal, in $A_p(\mathbb{C}^3)$, generated by the exponential-polynomials $\mathbb{Q}_{j,k}(u)(\zeta)$ which corresponds to the original generators of $\sqrt{\mathscr{I}_j}$ contains then the element $r(u)$, hence we have the estimation

$$\sum_{k=1}^{n_j} |Q_{j,k}(u)(\zeta)| \geqslant \delta \frac{|R(u)|}{(1 + \|u\|)^k} e^{-Dp(\zeta)} \qquad \forall \zeta \in \mathbb{C}^3$$

for a convenient choice of positive constants δ, D, k, where $R \in \mathbb{C}[u]$.

Therefore it remains to show that in the case we are considering the hypothesis of Lemma 8.5 holds. The parameter u being generic, consider a regular point (ζ_0, X_0) of the variety (8.11), assume $(\zeta_0, X_0) \in \Omega$, and set

$$\rho_k = X_{0,k} e^{-i\zeta_{0,k}}, \qquad k = 1, 2, 3.$$

Consider the intersection of the algebraic variety $Y_j^{(u)}$ with the analytic variety $\{X_k = \rho_k e^{i\zeta_k}, k = 1, 2, 3\}$ in a neighborhood of the point (ζ_0, X_0). By the definition of ρ this intersection is not empty. We want to see that the analytic subvariety of $\mathbb{C}^3$ defined by $\{Q_{j,k}^{(\rho)}(u)(\zeta) = 0 \ \forall k\}$ has a branch of dimension $\geqslant 1$ passing through the point ζ_0. Since the point (ζ_0, X_0) is a regular point of the 3-dimensional algebraic variety $Y_j^{(u)}$, the analytic variety can be locally defined by exactly three equations, say,

$$Q_{j,1}^{(\rho)}(u)(\zeta) = \cdots = Q_{j,3}^{(\rho)}(u)(\zeta) = 0.$$

If this variety were discrete at the point ζ_0, the Jacobian det $\|\partial Q_{j,k}^{(\rho)}/\partial \zeta_i\|_{k,i}$ cannot belong, in the ring of germs of holomorphic functions at ζ_0, to the ideal generated by the $Q_{j,k}^{(\rho)}$ (this follows from the residue theorem, see, e.g., [9, 21]). But, this Jacobian can be written in the form $h(\zeta, \rho e^{i\zeta})$ where the polynomial $h(\zeta, X)$ belongs to the $\sqrt{\mathcal{J}_j}$; hence the variety $V^{(\rho)} \cap \{u \cdot \zeta = 0\}$ is not discrete and the point (ζ_0, X_0) is a point in an exceptional variety corresponding to the exceptional value $\rho = X_0 e^{-i\zeta_0}$. If we apply now Lemma 8.4 we see there is $\lambda_0 \in \mathcal{F}$ such that

$$\sum_k (\lambda_0 \cdot \zeta_0)^k A_{k,\lambda_0}(u) = 0 \tag{8.24a}$$

$$\sum_k (X_0^{\lambda_0})^k B_{k,\lambda_0}(u) = 0. \tag{8.24b}$$

Let us consider the polynomial in $\mathbb{C}(u)[\zeta]$

$$\prod_{\lambda_0} \left(\sum_k (\lambda_0 \cdot \zeta)^k A_{k,\lambda_0}(u) \right). \tag{8.25}$$

For u generic, this polynomial vanishes on the variety $Y_j^{(u)}$, it belongs hence to the ideal $\sqrt{\mathcal{J}_j}$; this ideal being prime, one of its irreducible factors already belongs to $\sqrt{\mathcal{J}_j}$, and we can assume this factor is given by (8.24a). On the other hand, one can assume without any problem that all the entries of λ_0 are integral. By Lemma 8.4, the second equation (8.24b) is also satisfied at every point of $Y_j^{(u)} \cap \Omega$. After a coherent change of coordinates we are in conditions to apply Lemma 8.5, which it what we wanted to show.

What we have just done was to follow the method described in Section 6 which allows us to verify whether Theorem 3.1 is applicable, and we conclude that the properties stated in Proposition 3.3 hold for the exponential-polynomials with parameters $F_1, \ldots, F_m$, $u \cdot \zeta$; this means, let us recall, that if P denotes the weight in $\mathbb{C}^6$, $P(\zeta, X) = \log(1 + \|\zeta\| + \|X\|) + \|\text{Im}\,\zeta\|$, then there is a non-zero polynomial $R \in \mathbb{C}[u]$ and three positive constants β, B, K such that, for $0 < \varepsilon \leqslant 1$, $C > 0$, if we define δ and D by

$$\delta = \delta(u) = \beta \frac{|R(u)|}{(1 + \|u\|)^K} \varepsilon^K, \qquad D = KC + B,$$

every connected component $\mathscr{C}$ of the open subset of $\mathbb{C}^6$

$$\tilde{S}_u(F_1, ..., F_m; \delta, D)$$
$$:= \left\{ (\zeta, X) : \sum_1^m |p_j| + \sum_1^3 |X_j e^{-i\zeta_j} - 1| + |u \cdot \zeta| < \delta e^{-DP(\zeta, X)} \right\}$$

satisfies the property

$$\forall (\zeta, X) \in \mathscr{C} \quad \forall (\zeta', X') \in \mathscr{C}:$$
$$\|\zeta - \zeta'\| + \|X - X'\| < \varepsilon e^{-CP(\zeta, X)}. \tag{8.26}$$

We proceed to eliminate the dependence on u from all the estimates. First we want to show that R in (8.26) can be chosen to be a homogeneous polynomial. Set, for $\lambda \in \mathbb{C}$,

$$R(\lambda u) = R_0(u) \lambda^n + \cdots + R_k(u) \lambda^{n+k}$$
$$= R_0(u) \lambda^n \left(1 + \frac{R_1(u)}{R_0(u)} \lambda + \cdots + \frac{R_k(u)}{R_0(u)} \lambda^k \right).$$

Define

$$\lambda(u) = \frac{R_0(u)}{A(1 + \|u\|)^L}$$

for some A, L sufficiently large so that $|\lambda(u)| \leqslant 1$ and

$$|R(\lambda(u) u)| \geqslant \frac{1}{2A^n(1 + \|u\|)^{Ln}} |R_0(u)|^{n+1}. \tag{8.27}$$

If we take now, for u fixed,

$$\delta_1 = \beta \frac{|R_0(u)|^{n+1}}{2A^n(1 + \|u\|)^{K+Ln}} \varepsilon^K,$$

then every connected component of $\tilde{S}_{\lambda(u)u}(F_1, ..., F_m; \delta_1, D)$ is contained in a component of $\tilde{S}_{\lambda(u)u}(F_1, ..., F_m; \delta(\lambda(u) u), D)$. On the other hand, because

$|\lambda(u)| \leqslant 1$, every connected component of $\tilde{S}_u(F_1,\ldots, F_m; \delta_1, D)$ is contained in a component of $\tilde{S}_{\lambda(u)u}(F_1,\ldots, F_m; \delta_1, D)$. This shows that after modifying the constant K in (8.26) we can assume R is homogeneous.

Our idea of the geometric duality is based on the following lemma.

LEMMA 8.6. *Let R be a non-zero homogeneous polynomial of n variables. There are a finite number of points $\alpha_1,\ldots, \alpha_N$ of the unit sphere of $\mathbb{C}^n$, generating distinct (complex) lines in $\mathbb{C}^n$, a constant $\eta > 0$, and a sequence of positive integers $\nu_1,\ldots, \nu_N$, such that for every ζ_0, $\|\zeta_0\| = 1$,*

$$\max_{\substack{u\cdot\zeta_0=0\\ \|u\|=1}} |R(u)| \geqslant \eta(d(\zeta_0, \alpha_1))^{\nu_1} \cdots (d(\zeta_0, \alpha_N))^{\nu_N}. \tag{8.28}$$

Here the distance d in the unit sphere is defined by

$$d(\zeta, \zeta') = (1 - |\zeta \cdot \bar{\zeta}'|^2)^{1/2}$$

(it is really the distance between ζ and the circle $\{\zeta' e^{i\theta}: \theta \in \mathbb{R}\}$).

Proof of Lemma 8.6. We factorize R in $\mathbb{C}[u]$, there are a finite number of distinct linear factors $u\cdot\alpha_1,\ldots, u\cdot\alpha_N$, $\|\alpha_j\| = 1$, and we can write

$$R(u) = (u\cdot\alpha_1)^{k_1} \cdots (u\cdot\alpha_N)^{k_N} Q(u),$$

where Q does not admit any linear factor.

We now prove the lemma by induction on the number N of linear factors.

We consider first the case $N=0$. The function on the unit sphere S^{2n-1}

$$\zeta_0 \to \max\{|R(u)|: u\cdot\zeta_0 = 0, \|u\| = 1\}$$

is lower semicontinuous, it is also strictly positive since given ζ_0, the polynomial $u\cdot\zeta_0$ does not divide $R(u)$. It follows that it has a positive lower bound in S^{2n-1}.

Suppose the lemma is correct when R has at most $N-1$ distinct linear factors. There is a constant $\bar{\eta} > 0$ and positive integers $\bar{\nu}_2,\ldots, \bar{\nu}_N$ such that for every $\zeta_0 \in S^{2n-1}$

$$\max_{\substack{u\cdot\zeta_0=0\\ \|u\|=1}} |u\cdot\alpha_2|^{k_2} \cdots |u\cdot\alpha_N|^{k_N} |Q(u)| \geqslant \bar{\eta} \prod_2^N d(\zeta_0, \alpha_j)^{\bar{\nu}_j}.$$

There is an element u_0 achieving the above maximum. By the mean-valu theorem there is a constant $0<\sigma<\frac{1}{2}$ which depends only on the polynomial R such that

$$\begin{aligned}\forall u\in\mathbb{C}^n,\qquad &\|u-u_0\|<\sigma\prod_{j=2}^{N} d(\zeta_0,\alpha_j)^{\bar{\nu}_j}\\ &\Rightarrow |u\cdot\zeta_2|^{k_2}\cdots|u\cdot\alpha_N|^{k_N}\,|Q(u)|\geqslant\frac{\bar{\eta}}{2}\prod_{j=2}^{N} d(\zeta_0,\alpha_j)^{\bar{\nu}_j}.\end{aligned}\tag{8.29}$$

We distinguish two cases:

$$\text{(i)}\qquad |u_0\cdot\alpha_1|\geqslant\frac{\sigma}{4}\,d(\zeta_0,\alpha_1)^2\prod_{j=2}^{N} d(\zeta_0,\alpha_j)^{\bar{\nu}_j}.$$

In this case one can immediately estimate $|R(u_0)|$, with $\nu_1=2k_1$, $\nu_j=(k_1+1)\,\bar{\nu}_j$ for $j=2,\dots,N$.

$$\text{(ii)}\qquad |u\cdot\alpha_1|<\frac{\sigma}{4}\,d(\zeta_0,\alpha_1)^2\prod_{j=2}^{N} d(\zeta_0,\alpha_j)^{\bar{\nu}_j}.$$

Consider $v=u_0+t(\bar{\alpha}_1-(\bar{\alpha}_1\cdot\zeta_0)\,\bar{\zeta}_0)$. Take $t=(\sigma/3)\prod_{j=2}^{N} d(\zeta_0,\alpha_j)^{\bar{\nu}_j}$. One verifies that $v\cdot\zeta_0=0$ since $u_0\cdot\zeta_0=0$ and $\|\zeta_0\|=1$. Moreover

$$\begin{aligned}|v\cdot\alpha_1|&=|u_0\cdot\alpha_1+(1-|\alpha_1\cdot\zeta_0|^2)|\\ &\geqslant\left(\frac{\sigma}{3}-\frac{\sigma}{4}\right)d(\zeta_0,\alpha_1)^2\prod_{2}^{N} d(\zeta_0,\alpha_j)^{\bar{\nu}_j}.\end{aligned}$$

Hence, thanks to (8.29), one finds

$$|R(v)|\geqslant\text{const}\prod_{j=1}^{N} d(\zeta_0,\alpha_j)^{\nu_j},$$

with $\nu_1=2k_1$, $\nu_j=(k_1+1)\,\bar{\nu}_j$ for $j=2,\dots,N$. Using that $\|v\|<2$ and R is homogeneous we arrive at (8.28). ∎

Given an element $\alpha\in S^{2n-1}$, let $\mathbb{C}\alpha$ denote the complex line through the origin and α and $\operatorname{dist}(\zeta,\mathbb{C}\alpha)$ the Euclidean distance from a point $\zeta\in\mathbb{C}^n$ to that line. If $\zeta\neq 0$ we have

$$\operatorname{dist}(\zeta,\mathbb{C}\alpha)=\|\zeta\|\,d\left(\frac{\zeta}{\|\zeta\|},\alpha\right).$$

Thanks to Lemma 8.6 we can construct in $\mathbb{C}^3$ a family of distinct lines $\mathbb{C}\alpha_1,\dots,\mathbb{C}\alpha_N$, $\alpha_j\in S^5$, associated with the homogeneous polynomial R from (8.26). Choose a point $\omega_0\in\mathbb{C}^3$ which does not belong to any of these lines.

Reasoning in the same way as we have done above but adding this time to the exponential-polynomials $F_1, ..., F_m$ the linear polynomial $u \cdot (\zeta - \omega_0)$, we obtain (8.26) with a homogeneous polynomial $T(u)$ in place of $R(u)$ and $u \cdot (\zeta - \omega_0)$ in place of $u \cdot \zeta$. Again by Lemma 8.6 we find a family $\beta_1, ..., \beta_M \in S^5$ associated with T. The two lines $\omega_0 + \mathbb{C}\beta_j$ and $\mathbb{C}\alpha_k$ either intersect at a single point or they are at a strictly positive distance in $\mathbb{C}^3$. Therefore there are two constants $\sigma > 0$, $E > 0$ such that

$$\operatorname{dist}(\mathbb{C}\alpha_k \cap \{\|\zeta\| \geq E\}, \omega_0 + \mathbb{C}\beta_j \cap \{\|\zeta\| \geq E\}) \geq \sigma \qquad \forall j, \forall k.$$

Hence, if we denote by $\nu_1, ..., \nu_N$ the integers corresponding to R and $\mu_1, ..., \mu_M$ those corresponding to T by Lemma 8.6, there are three constants $\sigma, E, L > 0$ such that

$$\|\zeta\| \geq E \Rightarrow \prod_{j=1}^{N} d\left(\frac{\zeta}{\|\zeta\|}, \alpha_j\right)^{\nu_j} + \prod_{j=1}^{M} d\left(\frac{\zeta - \omega_0}{\|\zeta - \omega_0\|}, \beta_j\right)^{\mu_j} \geq \frac{\sigma}{(1 + \|\zeta\|)^L}. \tag{8.30}$$

We can now conclude the proof of Theorem 8.2. Given $C > 0$, $0 < \varepsilon < 1$, consider a connected component $\mathscr{C}$ of the set $S(F_1, ..., F_m; \delta_1, D_1)$ for a pair (δ_1, D_1) which will be chosen later on. Since, by the hypothesis of Theorem 8.2, the ideal $\mathscr{S}$ is s.s.d. there are constants δ_2, D_2 tied to ε, C, and $\mathscr{S}$ such that if $\delta_1 < \frac{1}{2}\delta_2$, $D_1 > D_2$, and $\mathscr{C}$ does not satisfy (1.2), then there is a point $\zeta_0 \in \mathscr{C}$ verifying the condition

$$\|\zeta_0\| \geq E + 1 \qquad (E \text{ is the constant in (8.30)})$$

$$\sum |\Delta_k(\zeta_0, e^{i\zeta_0})| \geq \frac{1}{2} \delta_2 e^{-D_2 p(\zeta_0)},$$

where Δ_k are the 2×2 minors of the matrix $\|\partial p_j / \partial(\zeta, X)\|$.

Consider the point $(\zeta_0, e^{i\zeta_0}) \in \mathbb{C}^6$. As shown when going from (3.18) tc (3.19) we can see there are two constants η_1, K_1 tied to δ_2, D_2 and th(polynomials p_j such that

$$\operatorname{dist}((\zeta, X), (\zeta_0, e^{i\zeta_0})) < \eta_1 e^{-K_1 p(\zeta_0)}$$

$$\Rightarrow \sum |\Delta_k(\zeta, X)| \geq \frac{1}{4} \delta_2 e^{-D_2 p(\zeta_0)}.$$

Let us choose, for the moment, arbitrary $\eta_0 < \frac{1}{2}\eta_1$, $K_0 > K_1$. Except fo possible improvements in the choice of δ_1, D_1 (depending in η_0, K_0) on sees that one can find a point (ζ^1, X^1) of the algebraic variety Y such tha

$$\operatorname{dist}((\zeta_0, e^{i\zeta_0}), (\zeta^1, X^1)) < \eta_0 e^{-K_0 P(\zeta_0, X_0)},$$

the same weight P as in Theorem 3.1. (This follows also from the theorem of Lojasiewicz [38] in the algebraic case.) One chooses η_2, K_2 (this choice depends only on η_1, K_1) such that if

$$\operatorname{dist}((\zeta^1, X^1), (\zeta, X)) < \eta_2 e^{-K_2 P(\zeta^1, X^1)} \tag{8.31}$$

then

$$\operatorname{dist}((\zeta, X), (\zeta_0, e^{i\zeta_0})) < \eta_1 e^{-K_1 p(\zeta_0)}.$$

This is possible since $\eta_0 < \frac{1}{2}\eta_1$, $K_0 > K_1$. It follows that every point of Y satisfying (8.31) is a regular point of Y.

Since the variety Y has dimension 4 and the spectrum V is discrete, the functions $p_1, ..., p_m, \varphi_1, \varphi_2, \varphi_3$ (defined by (3.7)) do not have any common zero in the ball (8.31).

Let $u, v \in S^5$, consider the two families of exponential-polynomials $(p_1, ..., p_m, \varphi_1, \varphi_2, \varphi_3, u \cdot \zeta)$ and $(p_1, ..., p_m, \varphi_1, \varphi_2, \varphi_3, v \cdot (\zeta - \omega_0))$, and apply (8.26) with $\varepsilon = \eta_2$, $C = K_2$.

For the first family we have two possible cases:

(i) either $\sum_1^3 |\varphi_j(\zeta^1, X^1)| + |u \cdot \zeta^1| \geq \beta\, |R(u)|\, \eta_2^K e^{-(KK_2 + B)P(\zeta^1, X^1)}$ for a convenient β;

(ii) or the strict opposite inequality takes place at (ζ^1, X^1), hence the connected component containing (ζ^1, X^1) of the points in Y where the same inequality takes place is contained in the ball (8.31). In this case we have four holomorphic functions without common zeros on a manifold of dimension 4 and the application of the minimum principle to the solutions of the Monge–Ampère equation (see Section 1) tells us that the minimum of the quantity $\sum |\varphi_j|^2 + |u \cdot \zeta|^2$ is taken on the boundary of this component. We have then, except for a modification of the constant β, that the same inequality as in (i) takes place at (ζ^1, X^1).

The same reasoning applied to the second family shows that

$$\sum_1^3 |\varphi_j(\zeta^1, X^1)| + |u \cdot \zeta^1| + |v \cdot (\zeta^1 - \omega_0)|$$
$$\geq \frac{\beta}{2} (|R(u)| + |T(v)|)\, \eta_2^K e^{-(KK_2 + B)P(\zeta^1, X^1)}. \tag{8.32}$$

Now, we can estimate from below the quantity

$$\max_{\substack{(u,v) \in S^5 \times S^5 \\ u \cdot \zeta^1 = 0 \\ v \cdot (\zeta^1 - \omega_0) = 0}} (|R(u)| + |T(v)|) \geq \frac{\sigma}{(1 + \|\zeta^1\|)^L}$$

if $\|\zeta^1\| \geqslant E$, which is precisely the case here. Therefore,

$$\sum_{j=1}^{3} |\varphi_j(\zeta^1, X^1)| \geqslant \eta_3 e^{-K_3 P(\zeta^1, X^1)}, \tag{8.33}$$

where the η_3, K_3 have been obtained by replacing $|R(u)| + |T(v)|$ in (8.32) with $\sigma/(1 + \|\zeta^1\|)^L$.

On the other hand, $\varphi_j(\zeta_0, e^{i\zeta_0}) = 0$, hence choosing η_0, K_0 conveniently we have also

$$\sum_{j=1}^{3} |\varphi(\zeta^1, X^1)| < \frac{1}{2} \eta_3 e^{-K_3 P(\zeta^1, X^1)}. \tag{8.34}$$

This choice of η_0, K_0 fixes the choice of the original δ_1, D_1, and the existence of the original point ζ_0 leads to the contradiction between (8.33) and (8.34). This finishes the proof of Theorem 8.2. ∎

We can also obtain out of the proof of Theorem 8.2 the following special case of the Nullstellensatz:

PROPOSITION 8.7. *Let F_1, F_2 be two exponential-polynomials of three variables and rational frequencies. Suppose they have no common zeros in $\mathbb{C}^3$, then there are two elements G_1, $G_2 \in A_p(\mathbb{C}^3)$ such that*

$$1 = F_1 G_1 + F_2 G_2.$$

Proof. In fact, thanks to (8.26) and the minimum principle this time applied in $\mathbb{C}^3$ to the triplets

$$(F_1, F_2, u \cdot \zeta) \qquad \text{and} \qquad (F_1, F_2, v \cdot (\zeta - \omega_0)),$$

$u, v \in S^5$, one obtains (after repeating verbatim the end of the proof of the above theorem)

$$\exists \delta > 0 \; \exists D > 0, \qquad |F_1(\zeta)| + |F_2(\zeta)| \geqslant \delta e^{-Dp(\zeta)}. \;\blacksquare$$

It seems to us that this duality method will be useful in studying the systems of partial differential equations with delays that appear in control theory and in mathematical models in biology, where the delays appear only on the time variable. From the point of view of exponential-polynomials, this is a very degenerate situation, completely opposite to the conditions we gave in Section 6 (see, for instance, Corollary 6.7) to show that a non-redundant system is in fact s.s.d. It would also be interesting to study for this type of system the validity of the spectral synthesis or the asymptotic stability of the solutions [5]. We plan to return to these questions in the future.

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