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On Lojasiewicz-Type Inequalities for Exponential Polynomials

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Let $f(z)=\sum_{n=1}^{N} f_n(z)\, e^{i\alpha_n z}$ be an exponential polynomial, that is, the coefficients $f_n(z)$ are polynomials in one complex variable. We discuss the question of how close distinct roots of the equation $f(z)=0$ can get to each other.

INTRODUCTION

A number of questions in harmonic analysis, complex analysis and applied mathematics, require a precise knowledge of the lower bounds of a plurisubharmonic function F of the form

$$F(z)=\log\left(\sum_{j=1}^{m} |f_j(z)|^2\right),$$

where $f_1, ..., f_m$ are exponential polynomials of n variables. One such example is the work of Symes on materials testing [25]. The authors have given other examples in [6, 7, 10, 31].

The kind of bounds we have in mind are the following. Let $V=\{z\in \mathbf{C}^n: f_1(z)=\cdots=f_m(z)=0\}$ and let $p(z)$ be a weight function (see definition below), then we would like to show the existence of positive constants A, B, and N such that for all $z\in\mathbf{C}^n$

$$F(z)\geqslant N\log \operatorname{dist}(z, V)-Ap(z)-B. \tag{1}$$

Sometimes, such an inequality is only needed for $z\in\mathbf{R}^n$ (as in the work of

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Symes). When $f_1, \ldots, f_n$ are polynomials, $p(z) = \log(1+|z|)$, this inequality is known as the (global)) Lojasiewicz inequality [21].

The reader will find that the methods used here to study the existence of such bounds are inspired by the work of A. O. Gelfond, A. Baker, and others on transcendental numbers. This is not surprising as shown by the following simple example [22], which makes clear why diophantine approximation questions appear naturally when considering deconvolution problems:

Let $\alpha \in \mathbf{R}$, $p(z) = |\operatorname{Im} z| + \log(1+|z|)$, then the following three conditions are equivalent:

(i) α is a non-Liouville number (i.e., for some $\varepsilon > 0$, $N > 0$, we have $|r\alpha - s| \geqslant \varepsilon(1+|r|)^{-N}$ for every $r, s \in \mathbf{Z}^*$);

(ii) there are constants A, $B > 0$ such that

$$\left|\frac{\sin z}{z}\right| + \left|\frac{\sin \alpha z}{\alpha z}\right| \geqslant A \exp(-Bp(z)) \qquad \text{for all} \quad z \in \mathbf{C};$$

(iii) there exist distributions S, T of compact support (which can be written down explicitly) such that for every $\phi \in C_0^\infty(\mathbf{R})$, such that $\int_{-\infty}^{\infty} \phi(x)\, dx = 0$, one has

$$\phi(x) = (S * \psi)(x+1) - (S * \psi)(x-1) + (T * \psi)(x+\alpha) - (T * \psi)(x-\alpha),$$

where $\psi(x) = \int_{-\infty}^{x} \phi(t)\, dt$.

We note that (ii) is in effect an inequality of the form (1) for the functions $f_1(z) = \sin z$, $f_2(z) = \sin \alpha z$, since the $V = \{0\}$ in this example.

In general very little is known about (1) and the aim of this paper is to prove it in a number of cases in one variable and relate it to our work [9] on ideals generated by exponential polynomials in n variables.

1. Exponential Polynomials in One Variable

Definition 1. A continuous non-negative plurisubharmonic function p in $\mathbf{C}^n$ is called a weight if

(a) $\log(1+|z|) = O(p(z))$ and

(b) there exist positive constants k_1, k_2 such that for every pair z, $\zeta \in \mathbf{C}^n$ with $|z-\zeta| \leqslant 1$ we have $p(z) \leqslant k_1 p(\zeta) + k_2$.

To such a weight one associates the algebra of those entire functions f in $\mathbf{C}^n$ such that for some A, $B > 0$

$$|f(z)| \leqslant A \exp(Bp(z)) \qquad \forall z \in \mathbf{C}^n.$$

We are only interested in the following weights:

(i) $p(z) = |\mathrm{Im}\, z| + \log(1 + |z|)$, A_p = space of Fourier transforms of distributions of compact support in $\mathbf{R}^n$;

(ii) $p(z) = |z|$, A_p = space of Fourier transforms of analytic functionals in $\mathbf{C}^n$;

(iii) $p(z) = |z|^k$, $k \in [1, \infty)$.

DEFINITION 2. An exponential polynomial of n variables with frequencies in a finite set $\Lambda \subseteq \mathbf{C}^n$ is an entire function of the form

$$f(z) = \sum_{\lambda \in \Lambda} p_\lambda(z)\, e^{-i\lambda \cdot z},$$

where $\lambda \cdot z = \lambda_1 z_1 + \cdots + \lambda_n z_n$ and the p_λ are non-zero polynomials. If all the p_λ are constant we say f is an exponential sum.

It is clear that $f \in A_p$ for any of the weights mentioned above, with the understanding that $\Lambda \subseteq \mathbf{R}^n$ if the weight $p(z) = |\mathrm{Im}\, z| + \log(1 + |z|)$ is considered.

Let $\mathscr{K}$ be a subfield of $\mathbf{C}$ and Γ an additive subgroup of $\mathbf{C}^n$ satisfying $i\Gamma \subseteq \mathscr{K}^n$. We denote by $\mathscr{F}(\Gamma; \mathscr{K})$ respectively $\mathscr{G}(\Gamma; \mathscr{K}))$ the family of all exponential sums (respectively exponential polynomials) with frequencies $\Lambda \subseteq \Gamma$, and coefficients $p_\lambda \in \mathscr{K}$ (respectively $\mathscr{K}[z]$).

These two families (with the function zero added) are subrings of A_p closed under differentiation.

For the remainder of this section we will restrict ourselves to the case of functions of one variable.

We recall some well-known properties of exponential polynomials of one variable.

PROPOSITION 1. *Let f be an exponential polynomial with frequencies in Γ and denote by $h(z) = h(z; \Lambda) := \max\{\mathrm{Im}(\lambda \cdot z) : \lambda \in \Lambda\}$ and $V = \{z \in \mathbf{C} : f(z) = 0\}$.*

(i) *There are two non-negative integers M_0, N_0 and a constant $c_0 > 0$ such that*

$$\sum_{j=0}^{N_0} |f^{(j)}(z)| \geqslant c_0 (1 + |z|)^{-M_0} \exp(h(z)). \tag{2}$$

(ii) *There are positive constants c_1, M_1, N_1 such that*

$$|f(z)| \geqslant c_1 (1 + |z|)^{-M_1} (\mathrm{dist}(z, V))^{N_1} \exp(h(z)). \tag{3}$$

(iii) *For every* $\varepsilon > 0$, $C > 0$ *there exist constants* $\varepsilon_1 > 0$, $C_1 > 0$ *such that if* z, ζ *are two points in the same connected component of the set*

$$\{z \in \mathbf{C}: |f(z)| < \varepsilon_1 e^{-C_1 p(z)}\}$$

we have $|z - \zeta| < \varepsilon e^{-Cp(z)}$.

(iv) *If* $\Lambda \subseteq \mathbf{R}$, *then* V *is contained in a region of the form*

$$|\operatorname{Im} z| \leqslant C \log(1 + |z|).$$

Furthermore, if f *is an exponential sum we can eliminate the logarithm from this bound.*

The first-three statements can be found in [17]. The last one is a very particular case of the work of Polya on zeros of exponential polynomials [4]. A more precise knowledge of the asymptotic behavior of the zeros, due also to Polya and others, will be used later on. We remark that very uniform bounds on the number of zeros of an exponential polynomial are known [28, 30], for instance, let $\Omega = \max |\lambda|$, $m = \sum_\lambda (\deg p_\lambda + 1)$, then if $n(z, r)$ denotes number of zeros of f in the closed disk $\Delta(z; r)$ of center z and radius r (counted with multiplicities),

$$n(z; r) \leqslant 2(m-1) + 4r \frac{\Omega}{\pi}.$$

It is easy to see we can change Ω to one-half the diameter of Λ, but in fact if $\mathscr{C}$ denotes the convex hull of Λ, τ the number of vertices of $\mathscr{C}$, and γ the perimeter of $\mathscr{C}$,

$$n(z; r) \leqslant \frac{1}{2} \tau(m-1) + \frac{\gamma r}{2\pi}.$$

Definition 3. We say that entire functions $f_1, ..., f_m$ satisfy the Lojasiewicz inequalities for the weight p if for every $\varepsilon > 0$, $C > 0$ there exist $\varepsilon' > 0$, $C' > 0$ such that

$$\sum_{j=1}^{m} |f_j(z)| < \varepsilon' e^{-C' p(z)} \quad \text{implies} \quad \operatorname{dist}(z, V) < \varepsilon e^{-Cp(z)}, \tag{4}$$

where $V = \{z: f_1(z) = \cdots = f_m(z) = 0\}$.

Note that this definition also makes sense in $\mathbf{C}^n$. Part (ii) of Proposition 1 is a very precise form of the Lojasiewicz inequalities for a single exponential polynomial (also valid in $\mathbf{C}^n$).

PROPOSITION 2. *For a given pair Γ, $\mathcal{K}$ the following three properties are equivalent*:

(a) *Every family $f_1, ..., f_m \in \mathcal{F}(\Gamma, \mathcal{K})$ (respectively $\mathcal{G}(\Gamma; \mathcal{K})$) satisfies the Lojasiewicz inequalities for the weight p.*

(b) *For every $f \in \mathcal{F}(\Gamma; \mathcal{K})$ (respectively $\mathcal{G}(\Gamma; \mathcal{K})$) there exist positive constants l, L such that if*

$$f(z_1) = f(z_2) = 0, \qquad z_1 \neq z_2, \qquad \textit{then} \quad |z_1 - z_2| \geqslant l e^{-Lp(z_1)}.$$

(*We will then say the zeros of f are* well-separated for the weight *p*.)

(c) *For every $f \in \mathcal{F}(\Gamma; \mathcal{K})$ (respectively $\mathcal{G}(\Gamma; \mathcal{K})$) the variety $V = \{z \in \mathbf{C}: f(z) = 0\}$ (with multiplicities) is an interpolation variety for the algebra A_p.*

Proof. If we index the points in V as z_k, and denote their multiplicities by m_k, we have a map into a space of sequences

$$\rho: A_p \to A_p(V) := \{(b_{kl}): \exists A > 0 \ |b_{kl}| \leqslant A e^{Ap(z_k)}, 0 \leqslant l < m_k\}$$

given by

$$\rho(\varphi) = \left(\frac{\varphi^{(l)}}{l!}(z_k)\right).$$

V is said to be an interpolating variety if the map ρ is onto. The equivalence of (b) and (c) can be found in [6].

To prove that (b) implies (a) we recall that the definition of weight implies the following:

Given $\eta > 0$, $A > 0$ there exist η_1, $A_1 > 0$ such that

$$\eta_1 e^{-A_1 p(z)} < (\eta/2)\, e^{-Ap(\zeta)} \tag{5}$$

for every pair z, ζ such that $|z - \zeta| \leqslant 1$.

Now consider the function $f(z) = \prod_1^m f_j(z)$ and the corresponding constants l, L given by the hypothesis (b), and we can assume $l \leqslant 1$. By the preceding observation we can choose $C > 0$, $0 < \varepsilon \leqslant 1$ satisfying (5) with $\eta = l$, $A = L$, $\eta_1 = \varepsilon$, $A_1 = C$. By Proposition 1(iii), we can choose ε_1, $C_1 > 0$ such that for any j, $1 \leqslant j \leqslant m$, and any pair of points z, ζ in the same connected component of $\Omega_j = \{z: |f_j(z)| < \varepsilon_1 e^{-C_1 p(z)}\}$ we have $|z - \zeta| < \varepsilon e^{-Cp(z)}$. Let ε', C' be the constants associated by (5) to ε_1, C_1 and let z_0 be a point such that

$$\sum_{j=1}^{m} |f_j(z_0)| < \varepsilon' e^{-C' p(z_0)}.$$

Call $\mathscr{C}_j$ the component of Ω_j containing z_0. Let us define

$$\Omega_j' = \{z\colon |f_j(z)| < \varepsilon' e^{-C'p(z_0)}\}$$

and $\mathscr{C}_j'$ the connected component of Ω' that contains z_0. By the choice of ε', C' we have $\mathscr{C}_j' \subseteq \mathscr{C}_j$. On the boundary of $\mathscr{C}_j'$ we have $|f_j(z)| = \text{constant}$, hence f_j has a zero ζ_j inside $\mathscr{C}_j'$. This gives us zeros $\zeta_1, ..., \zeta_m$ of f which satisfy, by the choices we made of the constants,

$$|\zeta_j - \zeta_k| < 2\varepsilon e^{-Cp(z_0)} < l e^{-Lp(\zeta_j)}.$$

By our hypothesis (b) this shows $\zeta_j = \zeta_k$. Therefore we have found a point $\zeta = \zeta_j \in V$ such that

$$\operatorname{dist}(z_0, V) \leqslant |z_0 - \zeta| = \varepsilon e^{-Cp(z_0)}.$$

We prove that (a) implies (b) by induction. Let us call $\nu(z_0)$ the multiplicity of a zero z_0 of f. The inductive hypothesis (P_k) is the following:

(P_k) There exist positive constants l_k, L_k such that if $\zeta_1 \neq \zeta_2$, $f(\zeta_1) = f(\zeta_2) = 0$, and $\max\{\nu(\zeta_1), \nu(\zeta_2)\} \geqslant k$ then $|\zeta_1 - \zeta_2| \geqslant l_k e^{-L_k p(\zeta_1)}$.

It is clear that (P_1) is what we want to prove. On the other hand, Proposition 1 (i) implies (P_k) is valid for $k = N_0$. Namely, in that case we have a lower bound for the first non-vanishing derivative of f at ζ_1 (if $\nu(\zeta_1) = N_0$). Consider now $\varepsilon = \inf\{l_q\colon q \geqslant k+1\}$, $C = \sup\{L_q\colon q \geqslant k+1\}$, and the family $f_j = f^{(j)}$, $0 \leqslant j \leqslant k$. By the hypothesis (a) we have constants ε', C' for which (4) is valid. Let ζ_1, ζ_2 be two distinct zeros of f such that $\nu(\zeta_i) = k$. Then, if $|f^{(k)}(\zeta_1)| \geqslant \varepsilon' e^{-C'p(\zeta_1)}$ we conclude that $|\zeta_2 - \zeta_1| \geqslant l_k' e^{-L_k' p(\zeta_1)}$ and hence by using the defining properties of a weight, we would have

$$|\zeta_2 - \zeta_1| \geqslant l_k \exp(L_K \min\{-p(\zeta_1), -p(\zeta_2)\})$$

for convenient l_k, $L_k > 0$. Otherwise we have

$$\sum_{j=0}^{k} |f_j(\zeta_1)| < \varepsilon' e^{-C'p(\zeta_1)}$$

and hence there is a zero ζ_3, common to all f_j, with $|\zeta_1 - \zeta_3| < \varepsilon e^{-Cp(\zeta_1)}$. Since $\nu(\zeta_3) \geqslant k+1$, this would contradict the inductive hypothesis. ∎

Remark 1. In the case where Γ is an abelian group of rank 1, i.e., where there is a non-zero number ω such that $\Gamma = \omega\mathbf{Z}$, we have proved in [9, Proposition 7.7] that the zeros of any function in $\mathscr{G}(\Gamma; C)$ are well-separated. If rank $\Gamma \geqslant 2$ one needs to impose conditions both on the field $\mathscr{K}$ as well as on Γ if one wants this property to hold.

On the other hand, for single functions one can prove under additional conditions that the zeros are well separated. A typical example of this kind is the unpublished work [24] of C. Moreno, where exponential polynomials with three or four distinct frequencies were considered.

Remark 2. If any of the equivalent properties of Proposition 2 hold for $\mathscr{F}(\Gamma; \mathscr{K})$ (or $\mathscr{G}(\Gamma; \mathscr{K})$) we have the following version of Hilbert's Nullstellensatz: Let $f_1, ..., f_m$ in $\mathscr{F}(\Gamma; \mathscr{K})$ (or $\mathscr{G}(\Gamma; \mathscr{K})$) have no common zeros, then there are $g_1, ..., g_m \in A_p$ such that

$$f_1 g_1 + \cdots + f_m g_m = 1.$$

(In the engineering literature this is often called the Bezout equation.)

EXAMPLE 1. Let $\lambda = \sum_{n=1}^{\infty} 10^{-\varphi(n)}$, $\varphi(1) = 1$, $\varphi(n+1) = 10^{\varphi(n)}$, then $f(z) = \sin \lambda z \cdot \sin z$ cannot satisfy (b) of Proposition 2 for any weight $p(z) = |z|^k$. In this case rank $\Gamma = 2$, but the problem is that λ is too well approximated by rational numbers. This could not happen if λ were an algebraic number.

EXAMPLE 2. Let $f(z) = \sin(z - \alpha) - \sin \sqrt{2}(z - \alpha)$ then one has $f \in \mathscr{F}(\Gamma; \mathbf{C})$, $\Gamma = \mathbf{Z} \oplus \sqrt{2}\,\mathbf{Z}$, for any choice of α. On the other hand, its zeros are located at points of the form

$$z = \frac{2k\pi}{1 - \sqrt{2}} \quad \text{and} \quad z = \frac{2\alpha + (2k+1)\pi}{1 + \sqrt{2}}, \qquad k \in \mathbf{Z}.$$

As in the previous example one chooses α so that $2\alpha/\pi$ is extremely well approximated by numbers in $(2\mathbf{Z} + 1) \oplus (2\mathbf{Z} + 1)\sqrt{2}$. We note that in this case the coefficients of f are either $\sin \alpha$ or $\cos \alpha$ and by Baker's work on linear forms of logarithms of algebraic numbers [2] these coefficients are transcendental numbers (cf. Lemma 1 below).

Whence we are led to consider the following open problem (cf. [14, p. 322]. See [1] for a cognate conjecture).

Problem 1. Let $\mathscr{K} = \bar{\mathbf{Q}}$, the set of algebraic numbers, and rank $\Gamma \geqslant 2$: Are the conditions of Proposition 2 valid for $\mathscr{F}(\Gamma; \mathscr{K})$ or $\mathscr{G}(\Gamma; \mathscr{K})$? (Recall that we are assuming $i\Gamma \subseteq \mathscr{K}$, hence $\Gamma \subseteq \bar{\mathbf{Q}}$ is automatic in this case.)

Optimally one would like to them to be true for $p(z) = |z|$ or $p(z) = |\operatorname{Im} z| + \log(1 + |z|)$ when $\Gamma \subseteq \mathbf{R} \cap \bar{\mathbf{Q}}$. But it would be already interesting if the zeros were well-separated for $p(z) = |z|^k$, with k depending on the given function f. Note that all we are asking is that the number 1 be badly approximated by quotients of zeros of f (cf. [10], where the related problem of approximation by quotients of zeros of Bessel functions has an interesting application).

Remark 3. For the case of real frequencies and $p(z) = |\operatorname{Im} z| + \log(1+|z|)$, the property that the zeros of an exponential polynomial be well-separated is usually stated as the apparently stronger condition:

$$\text{If}\quad \zeta_1 \neq \zeta_2, f(\zeta_1) = f(\zeta_2) = 0 \qquad \text{then}\quad |\zeta_1 - \zeta_2| \geqslant \frac{c}{(1+|\zeta_1|)^N}$$

for some positive constants c, N. Due to Proposition 1 (iv), this condition is actually equivalent to our definition.

If we know that the zeros of a certain exponential polynomial f with real frequencies are well-separated for $p(z) = |\operatorname{Im} z| + \log(1+|z|)$ then we can draw several interesting conclusions about the C^∞ solutions φ of the associated difference-differential equation $\mu * \varphi = 0$, where $\hat{\mu} = f$. First, from [6] we know that φ can be written as a series

$$\varphi(x) = \sum \varphi_\alpha(x)\, e^{i\alpha x},$$

where the α run over the zeros of f and φ_α are polynomials of degree $< m_\alpha =$ multiplicity of α. The convergence is absolute and in the topology of $C^\infty(\mathbf{R})$ (i.e., uniformly over compact sets for φ and all derivatives of φ and formal derivatives of the series). Furtermore, we know the coefficients of φ_α tend rapidly to zero as $|\alpha| \to \infty$. If we know that φ is bounded, then by Kahane's remark [20, p. 293] we have that all the φ_α are constant and the only ones that could be different from zero will be those for which α is real. The rapid decrease of the coefficients then implies the uniform convergence of the series in the whole line. Therefore φ will be almost periodic. The same is true without any assumption on the boundedness of φ, if all the zeros of f are real and simple. (This was also remarked by Gramain [16]).

We are now going to tackle the case of rank $\Gamma = 2$ and we need an auxiliary result.

LEMMA 1. *Let* $\omega \in \bar{\mathbf{Q}} \setminus \mathbf{Q}$, $A(x) = x - \alpha$, $B(x) = x - \beta$, $\alpha, \beta \in \bar{\mathbf{Q}}$. *Then*:

(a) *The equations* $A(e^{-i\zeta}) = B(e^{-i\omega\zeta}) = 0$ *have at most the solution* $\zeta = 0$.

(b) *There are positive constants* (*depending on* ω, α, β) δ, Δ *such that for* $|\zeta| \geqslant 1$ *we have*

$$|A(e^{-i\zeta})| + |B(e^{-i\omega\zeta})| \geqslant \frac{\delta}{(1+|\zeta|)^{\Delta}}. \tag{6}$$

Proof. Part (a) is just the theorem of Gelfond [15, Theorem 2, p. 106], which shows α^{ω} is transcendental as soon as α is algebraic $\neq 0, 1$, and ω is algebraic.

Part (b) is a simple consequence of the work of Baker on linear forms on logarithms of algebraic numbers [2, Theorem 1, p. 5], which we recall:

Let $\alpha_1, ..., \alpha_n \in \bar{\mathbf{Q}} \setminus \{0\}$ of degree at most d, $A_j = \max\{\text{height } \alpha_j, 4\}$, and $\log \alpha_1, ..., \log \alpha_n$ be the principal value of their logarithms. Let $\lambda_1, ..., \lambda_n \in \bar{\mathbf{Q}}$ have degree at most d, $H = \max\{4, \text{height } \lambda_1, ..., \text{height } \lambda_n\}$, $\Omega = \prod_1^n A_j$, $\Omega' = \prod_1^{n-1} A_j$, $C = (16nd)^{200n}$. Then either

$$\lambda_1 \log \alpha_1 + \cdots + \lambda_n \log \alpha_n = 0$$

or

$$|\lambda_1 \log \alpha_1 + \cdots + \lambda_n \log \alpha_n| \geqslant (H\Omega)^{-C\Omega \log \Omega'}. \tag{7}$$

In order to apply this inequality to obtain (6) we observe first that the inequality (3) of Proposition 1 can be made more precise in this case. Namely, it is easy to show that for some readily computable constant $c > 0$ we have

$$|e^{-iz} - 1| \geqslant c|z|\, e^{|\operatorname{Im} z|} \qquad \text{if} \quad |\operatorname{Re} z| < \pi.$$

This inequality leads to

$$|e^{-i\zeta} - \alpha| \geqslant c\, e^{|\operatorname{Im} \zeta|} \min_k |\zeta + i \log \alpha + 2k\pi|$$

and

$$|e^{-i\omega\zeta} - \beta| \geqslant c\, e^{|\operatorname{Im} \omega\zeta|} \min_j |\omega\zeta + i \log \beta + 2j\pi|,$$

where the minima are taken over $k, j \in \mathbf{Z}$. We can assume that both minima are smaller than 1 and that $|\zeta| \gg 1$. Then for the k, j that attain the minima we have $|k| \approx |j/\omega| \approx |\zeta|$. It follows that for some constant $c' > 0$,

$$|A(e^{-i\zeta})| + |B(e^{-i\omega\zeta})| \geqslant \frac{c'}{1+|\omega|} |i\omega \log \alpha + 2k\pi\omega - i \log \beta - 2\pi j|.$$

The term in the right-hand side can be estimated using Baker's theorem. First we need to know that the expression

$$\omega(\log\alpha - 2k\pi i) - (\log\beta - 2\pi ij) \neq 0.$$

If this expression were zero we would have $\alpha^{\omega} = \beta$, which can only occur, by Gelfond's theorem already quoted in part (a), when $\alpha = \beta = 1$. In that case the expression reduces to $i(2k\pi\omega - 2\pi j) \neq 0$, unless $k = j = 0$ since $\omega \notin \mathbf{Q}$. The condition $|\zeta| \gg 1$ indicates that this case is not possible. Therefore we can use (7) with $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = -1$, $\lambda_1 = \omega$, $\lambda_2 = -1$, $\lambda_3 = -2k\omega + 2j$. The estimate of the height H is $O((|k| + |j|)^D)$, $D = \text{degree } \omega$. That is, $H = O(|\zeta|^D)$. Using that all the other constants are independent of ζ we obtain

$$|A(e^{-i\zeta})| + |B(e^{-i\omega\zeta})| \geqslant \delta(1 + |\zeta|)^{-\Delta}$$

for some positive constants δ, Δ. ∎

Remark. Using [29] one gets a better dependence of the constants on the degrees of α, β, ω.

We are now ready to study the case of rank $\Gamma = 2$. The result below was originally proved by F. Gramain [16] under the assumption that all the zeros of the exponential polynomial f are simple and real.

PROPOSITION 3. *The zeros of functions in $\mathscr{F}(\Gamma; \bar{\mathbf{Q}})$ are well-separated for $p(z) = |\mathrm{Im}\, z| + \log(1 + |z|)$ (respectively $p(z) = |z|$) when* rank $\Gamma = 2$ *and $\Gamma \subseteq \mathbf{R}$ (respectively $\Gamma \nsubseteq \mathbf{R}$).*

Proof. We can certainly assume $\Gamma = \mathbf{Z} \oplus \omega\mathbf{Z}$, ω an algebraic irrational number. Given $f \in \mathscr{F}(\Gamma; \bar{\mathbf{Q}})$, up to multiplication by an exponential term, it can be written in the form

$$f(z) = P_0(e^{-iz}, e^{-i\omega z}),$$

where $P_0 \in \bar{\mathbf{Q}}[X, Y]$. The successive derivatives $f^{(j)}$ can also be written in the form $P_j(e^{-iz}, e^{-i\omega z})$, P_j a polynomial in two variables with algebraic coefficients. We are interested in considering only $0 \leqslant j \leqslant N$, where N is the largest multiplicity of a zero of f. (By Proposition 1(i), $N \leqslant N_0 < \infty$.) Factorize P_0 into powers of irreducible factors in $\bar{\mathbf{Q}}[X, Y]$, $P_0 = \prod_k R_k$.

Let ζ be a zero of f of multiplicity ν, $|\zeta| \geqslant 1$. We have an index k such that $R_k(e^{-i\zeta}, e^{-i\omega\zeta}) = 0$ and also $P_\nu(e^{-i\zeta}, e^{-i\omega\zeta}) \neq 0$. Hence R_k is coprime with P_ν and the variety of common zeros in $\mathbf{C}^2$ of P_ν and R_k is finite. It follows from Hilbert's Nullstellensatz [27] that there are two non-zero polynomials $A \in \bar{\mathbf{Q}}[X]$, $B \in \bar{\mathbf{Q}}[Y]$ and polynomials $S_1, S_2, S_3, S_4 \in \bar{\mathbf{Q}}[X, Y]$ such that

$$A(X) = S_1(X, Y)\, R_k(X, Y) + S_2(X, Y)\, P_\nu(X, Y)$$

$$B(X) = S_3(X, Y)\, R_k(X, Y) + S_4(X, Y)\, P_\nu(X, Y).$$

If $P_\nu(e^{-i\zeta}, e^{-i\omega\zeta})$ were sufficiently large we would have the other zeros of f away from ζ, hence we can assume

$$|A(e^{-i\zeta})| + |B(e^{-i\omega\zeta})| < \varepsilon e^{-Cp(\zeta)} \tag{8}$$

for some $0 < \varepsilon < 1$, $C > 1$. Since A, B are polynomials in one variable we can assume $A(X) = X - \alpha$, $B(Y) = Y - \beta$, $\alpha, \beta \in \bar{\mathbf{Q}}$. Lemma 1 shows that (8) is impossible if $\varepsilon < \delta$, $C > \Delta$. Due to the finitely many choices available of R_k, P_ν and roots of corresponding A, B we can conclude that for every root ζ of f, if ν is the multiplicity of this root then

$$|f^{(\nu)}(\zeta)| \geqslant \varepsilon' e^{-C'p(\zeta)}$$

for a convenient choice of ε', $C' > 0$. This certainly implies the zeros of R are well-separated with respect to the weight p. ∎

In view of the possible applications to exponential polynomials in several variables that will be mentioned in the next section, one would really need the same one-variable result for the case $\mathscr{G}(\Gamma; \bar{\mathbf{Q}})$ with rank $\Gamma \geqslant 2$. We have not succeeded yet in proving this in general even for rank $\Gamma = 2$, on the other hand, there are simple cases where it is easy to show that the separation of the zeros holds. The following proposition, based on the work of Polya, Dickson, and others [4], is just one example of simple geometric conditions that ensure the zeros are well-separated.

Let us write $f(z) = \sum_{j=1}^{n} P_j(z)\, e^{-i\lambda_j z}$, $\lambda_j \in \mathbf{R}$, $m_j = $ degree P_j. Consider the Newton polygon defined as follows: plot the set S of points $(\lambda_1, m_1), \ldots, (\lambda_n, m_n)$ and find the concave polygonal curve L whose vertices lie on S and such that no points of S lie above it.

PROPOSITION 4. *Let f and L be as above. Suppose the only points of S which lie on L are vertices of L. Then the zeros of f are well-separated for the weight $p(z) = |\mathrm{Im}\, z| + \log(1 + |z|)$.*

Proof. We follow the notation from [4, Chap. 12]. Let $\mu_1 > \mu_2 > \cdots > \mu_r$ be the family of successive slopes of the sides of L. For each of them there is a number $\gamma_j > 0$ such that the zeros of f are asymptotically close to one of the curves

$$\Gamma_j\colon |e^{-iz} z^{\mu_j}| = \gamma_j.$$

If we had a pair of zeros, ζ_1, ζ_2 extremely close to each other (with $|\zeta_1| \gg 1$) it follows that they have to be close to the same curve Γ_j. That is, for $|\zeta_1| \gg 1$ we would be in the region

$$\Omega_j\colon |\operatorname{Im} z + \mu_j \log|z| - \log \gamma_j| < 1, \qquad |z| \gg 1,$$

which has exactly two (simply) connected components, both of which avoid the imaginary axis, therefore we can define a branch of $\log z$ by choosing $\arg z \in (-\pi/2, 3\pi/2)$. One considers the auxiliary function f_j,

$$f_j(z) = c_h + c_k\, e^{-i(\lambda_k - \lambda_h) z}\, z^{\mu_j(\lambda_k - \lambda_h)},$$

where (λ_h, m_h) and (λ_k, m_k) are the successive vertices defining the side of L of slope μ_j, $\mu_j = (m_k - m_h)/(\lambda_k - \lambda_h)$, $\lambda_k > \lambda_h$, and c_h, c_k are the leading coefficients of P_h and P_k. It can be shown [4, Chap. 12] that given $\delta > 0$ there is a $K > 0$ such that if $f(z) = 0$, $|z| \geq K$ there is a j and a z', $|z' - z| < \delta$, $f_j(z') = 0$. An examination of the definition of f_j shows that there are positive constants ε_1, δ_1 such that the zeros of f_j are simple, separated by a distance $\geq \delta_1$, and for each zero z' of f_j we have $|f_j(z)| \geq \varepsilon_1$ on $|z - z'| = \frac{1}{2}\delta_1$. Hence, if ζ_1, ζ_2 are the zeros of f we were considering, we can assume they satisfy $|\zeta_1 - \zeta_2| < \delta_1/3$, $|\zeta_1| \geq K$, $|\zeta_2| \geq K$. Therefore, they correspond to a single zero z' of f_j with $|\zeta_1 - z'| < \delta_1/3$, $|\zeta_2 - z'| < \delta_1/3$. Furthermore,

$$\left| \frac{f(z)}{e^{-i\lambda_h z}\, z^{m_h}} - f_j(z) \right| < \frac{1}{2}\varepsilon_1 \qquad \text{if} \quad z \in \Omega_j, \qquad |z| \gg 1.$$

An application of Rouche's theorem shows now that the existence of two zeros of f near a single zero of f_j is impossible. ▮

Remark 5. One can see that even if there are more than two points of S in some sides of L, the above proof still works if $\mu_j \neq 0$ and the function f_j formed using all the points of S on the jth side does not have multiple zeros. We note that by introducing the variable $s = ze^{-iz/\mu_j}$, the function f_j becomes (after multiplication by an integral power of s) a polynomial in s, hence the verification that the zeros of f_j are simple is, in principle, easy to do. On the other hand, exponential sums correspond exactly to the troublesome case $\mu_j = 0$.

PROPOSITION 5. *Let* $\Gamma = \mathbf{Z}\omega_1 \oplus \mathbf{Z}\omega_2 \oplus \mathbf{Z}\omega_3$, *where the generators* $\omega_1, \omega_2, \omega_3$ *are algebraic numbers satisfying the condition that the two vectors in* $\mathbf{R}^3$, $(\operatorname{Re}\omega_1, \operatorname{Re}\omega_2, \operatorname{Re}\omega_3)$ *and* $(\operatorname{Im}\omega_1, \operatorname{Im}\omega_2, \operatorname{Im}\omega_3)$, *are linearly independent over* $\mathbf{R}$ (*that is,* $\omega_1, \omega_2, \omega_3$ *do not lie in a straight line*). *For every* $f \in \mathscr{F}(\Gamma; \bar{\mathbf{Q}})$ *the zeros are well-separated for the weight* $p(z) = |z|$.

Proof. As in Proposition 3 we can assume $f(z) = P_0(e^{-i\omega_1 z}, e^{-i\omega_2 z}, e^{-i\omega_3 z})$ with $P_0 \in \bar{\mathbf{Q}}[X, Y, Z]$. We can assume without loss of generality that P_0 has no multiple factors. Let P_j be also defined by $f^{(j)}(z) = P_j(e^{-i\omega_1 z}, e^{-i\omega_2 z}, e^{-i\omega_3 z})$.

Let ζ be a zero of f of multiplicity ν, $|\zeta| \gg 1$. Let R be an irreducible factor of P_0 such that $R(e^{-i\omega_1\zeta}, e^{-i\omega_2\zeta}, e^{-i\omega_3\zeta}) = 0$. We know that R is then relatively prime to P_ν, hence the subvariety of $\mathbf{C}^3$ defined by $R = P_\nu = 0$ has dimension at most 1. Writing

$$R(X, Y, Z) = \sum_{k=0}^{m} Z^k R_k(X, Y)$$

$$P_\nu(X, Y, Z) = \sum_{k=0}^{M=0} Z^k P_{\nu,k}(X, Y)$$

we have that the resultant of R and P_ν as polynomials in Z is a non-zero polynomial $\Delta_0(X, Y)$, which is in the ideal generated by them [27]. We can obtain in the same way a non-zero polynomial $\Delta_1(X, Z)$ in the same ideal. As we have done in Proposition 3, we can assume $f^{(\nu)}(\zeta)$ is very small and we obtain

$$|\Delta_0(e^{-i\omega_1\zeta}, e^{-i\omega_2\zeta})| + |\Delta_1(e^{-i\omega_1\zeta}, e^{-i\omega_3\zeta})| < \varepsilon e^{-C|\zeta|}, \tag{9}$$

where ε, $C > 0$ will be chosen later.

Writing $\Delta_0(X, Y)$ as a polynomial in Y with coefficients in $\bar{\mathbf{Q}}[X]$ we see there are constants $T > 0$, $m > 0$, $C_1 > 0$ such that the distance between $e^{-i\omega_2\zeta}$ to a root of the algebraic equation

$$\Delta_0(e^{-i\omega_1\zeta}, Y) = 0$$

is bounded by $C_1 e^{C_1|\zeta|} |\Delta_0(e^{-i\omega_1\zeta}, e^{-i\omega_2\zeta})|^m$ if $|\operatorname{Im}(\omega_1 \zeta)| \geq T$. One can do the same for $e^{-i\omega_3\zeta}$ and Δ_1.

Consider now the equation

$$\Delta_0(e^{-i\omega_1\zeta}, Y) = 0$$

when $\operatorname{Im}(\omega_1 \zeta) \geq T_1 \geq T$. If $T_1 \gg 1$, we can use the Puiseux development in fractional powers of X [18] of the roots of the algebraic equation

$$\Delta_0(X, Y) = 0, \qquad |X| \ll 1.$$

The different roots are given by expression of the form

$$Y = X^{k_0/p}[a_0 + O(|X|^{1/p})],$$

where $k_0 \in \mathbf{Z}$, $p \in \mathbf{N}$, $a_0 \in \bar{\mathbf{Q}} \setminus \{0\}$. Hence, given ε', $C' > 0$, ε, C can be chosen in (a) so that if $\operatorname{Im}(\omega_1 \zeta) \geqslant T_1$,

$$|e^{-i\omega_2\zeta} - e^{-i\omega_1\zeta(k_0/p)}[a_0 + O(|e^{-i\omega_1\zeta}|^{1/p})]| < \varepsilon' e^{-C'|\zeta|}, \tag{10}$$

since k_0, a_0 are chosen among a finite number of values. Under the same hypothesis $\operatorname{Im}(\omega_1\zeta) \geqslant T_1$ we have

$$|e^{-i\omega_3\zeta} - e^{-i\omega_1\zeta(k_1/q)}[a_1 + O(|e^{-i\omega_1\zeta}|^{1/q})]| < \varepsilon' e^{-C'|\zeta|}, \tag{11}$$

with the obvious meaning for the k_1, q, a_1.

After dividing the equation $\Delta_0(e^{-i\omega_1\zeta}, Y) = 0$, $\Delta_1(e^{-i\omega_1\zeta}, Z) = 0$ by convenient powers of $e^{-i\omega_1\zeta}$ one can apply the same idea in the region $\operatorname{Im}(\omega_1\zeta) \leqslant -T_1$ and obtain

$$|e^{-i\omega_2\zeta} - e^{-i\omega_1\zeta(k_0'/p)}[a_0' + O(|e^{-i\omega_1\zeta}|^{1/p'})]| < \varepsilon' e^{-C'|\zeta|}, \tag{12}$$

$$|e^{-i\omega_3\zeta} - e^{-i\omega_1\zeta(k_1'/q')}[a_1' + O(|e^{-i\omega_1\zeta}|^{1/q'})]| < \varepsilon' e^{-C'|\zeta|}. \tag{13}$$

Let $\Omega_2 = \omega_2 - \omega_1 k_0/p$, $\Omega_3 = \omega_3 - \omega_1 k_1/q$, and the condition (10) can be rewritten as

$$i\Omega_2\zeta = \log a_0 + 2\pi i j + O(|e^{-i\omega_1\zeta}|^{1/p}), \tag{14}$$

for some $j \in \mathbf{Z}$, if we choose $\varepsilon' > 0$, $C' > 0$ such that

$$\varepsilon' e^{(-C' + |\omega_2| + |\omega_1 k_0/p|)|\zeta|} = O(|e^{-i\omega_1\zeta}|^{1/p}).$$

Similarly, we obtain from (11)

$$i\Omega_3\zeta = \log a_1 + 2\pi i l + O(|e^{-i\omega_1\zeta}|^{1/q}), \qquad l \in \mathbf{Z}. \tag{15}$$

Eliminating ζ from (14) and (15) we have for some $\delta > 0$

$$|\Omega_3 \log a_0 - \Omega_2 \log a_1 + i\pi(2j\Omega_3 - 2l\Omega_2)| = O(|e^{-i\omega_1\zeta}|^{\delta}). \tag{16}$$

We note that the expression on the left-hand side of (16) can only be zero for at most one pair of integers j, l since $\omega_1, \omega_2, \omega_3$ are linearly independent over the rationals. Since $|j| \approx |l| \approx |\zeta|$ we have that for $|\zeta| \geqslant C_0 > 0$ we can apply Baker's theorem (7) and obtain that the left-hand side of (16) is bounded below by $(1 + |\zeta|)^{-A}$ for some $A > 0$. Therefore (16) implies that for some $B > 0$,

$$\operatorname{Im}(\omega_1\zeta) \leqslant B \log(2 + |\zeta|) \qquad \text{if} \quad |\zeta| \geqslant C_0 \quad \text{and} \quad \operatorname{Im}(\omega_1\zeta) \geqslant T_1.$$

Using (12) and (13) we finally obtain, when $|\zeta| \geqslant C_0$,

$$|\operatorname{Im}(\omega_1\zeta)| \leqslant T_1 + B \log(2 + |\zeta|).$$

Repeating this argument with ω_2, ω_3 in place of ω_1 we end up with

$$\sum_{j=1}^{3} |\mathrm{Im}(\omega_j \zeta)| \leqslant T_1 + B \log(2 + |\zeta|)$$

at every zero ζ of f for which $|\zeta| \geqslant C_0$ and (9) holds. The hypothesis of linear independence of the two vectors $(\mathrm{Re}\,\omega_1, \mathrm{Re}\,\omega_2, \mathrm{Re}\,\omega_3)$ and $(\mathrm{Im}\,\omega_1, \mathrm{Im}\,\omega_2, \mathrm{Im}\,\omega_3)$ implies that

$$|\zeta| \leqslant \text{const.} \sum_{j=1}^{3} |\mathrm{Im}(\omega_j \zeta)|.$$

This shows that if $C_0 \gg 1$ the inequality (9) cannot hold and the zeros of f are well-separated as we wanted to show. ∎

COROLLARY 1. *Let* $\Gamma = \mathbf{Z} \oplus \omega_2 \mathbf{Z} \oplus \omega_3 \mathbf{Z}$, $\mathrm{Im}\,\omega_2 \neq 0$, *then the zeros of any function in* $\mathscr{F}(\Gamma; \bar{\mathbf{Q}})$ *are well-separated for* $p(z) = |z|$.

The same ideas used in Proposition 5 allow us to deal with one case of polynomial coefficients and rank two.

PROPOSITION 6. *Let* $\Gamma = \mathbf{Z} \oplus \omega \mathbf{Z}$, $\mathrm{Im}\,\omega \neq 0$. *Then the zeros of any function* $\mathscr{G}(\Gamma; \mathbf{C})$ *are well-separated for* $p(z) = |z|$.

Sketch of the Proof. In the same way as in Proposition 5 we can limit ourselves to consider two non-zero polynomials of two variables P, Q and assume

$$|P(\zeta, e^{-i\zeta})| + |Q(\zeta, e^{-i\omega\zeta})| < \varepsilon e^{-C|\zeta|}$$

at a point ζ, with $|\zeta| \gg 1$. The use of Puiseux developments will give us expansions of the form, for some $l_1, l_2 \in \mathbf{Q}$, $a_1, a_2 \in \mathbf{C} \setminus \{0\}$,

$$e^{-i\zeta} \simeq a_1 \zeta^{l_1} + \cdots, \qquad e^{-i\omega\zeta} \simeq a_2 \zeta^{l_2} + \cdots.$$

Hence for some integers k, j we will have

$$-i\zeta\left(1 - l_1 \frac{\log \zeta}{i\zeta}\right) \simeq \log a_1 + 2k\pi i$$

$$-i\omega\zeta\left(1 - l_2 \frac{\log \zeta}{i\zeta}\right) \simeq \log a_2 + 2k\pi i$$

and we conclude (by comparing real parts) that $|\mathrm{Im}\,\zeta|$, $|\mathrm{Im}\,\omega\zeta|$ are bounded. Hence, $|\zeta|$ is bounded and the proof ends the same way as Proposition 5 did. ∎

The following proposition is the only case of an exponential sum of rank 3 with *real* frequencies that we know how to deal with. The method of proof is borrowed from the original theorem of Gelfond that shows that the transcendence degree $\bar{\mathbf{Q}}(e^{-i\zeta}, e^{-i\omega\zeta}, e^{-i\omega^2\zeta}) \geqslant 2$ [15]. We follow the work of Brownawell, which gave a transcendence measure for this case [11]. The modifications we made were necessary to keep track of a different set of constants than the ones that are usually important in number theory. Nevertheless, we have only been able to prove that zeros are well-separated for the weight $p(z) = |z|^{4+\varepsilon}$ and not $p(z) = |\operatorname{Im} z| + \log(1+|z|)$ (or, what is the same, $\log(1+|z|)$ as we explained in Remark 3) as one would like.

PROPOSITION 7. *Let ω be a cubic irrational, $\Gamma = \mathbf{Z} \oplus \omega\mathbf{Z} \oplus \omega^2\mathbf{Z}$, and if $f \in \mathscr{F}(\Gamma; \bar{\mathbf{Q}})$ then for every $\varepsilon > 0$ the zeros of f are well-separated for the weight $p(z) = |z|^{4+\varepsilon}$.*

Proof. The case where $\omega \notin \mathbf{R}$ is a direct consequence of Corollary 1 above (with the better weight $p(z) = |z|$). Hence we assume $\omega \in \mathbf{R}$ and therefore the zeros of f are located in a strip of the form $|\operatorname{Im} z| \leqslant A_0$ by Proposition 1. We can also assume that ω is an algebraic integer since this can be achieved by a simple change in scale.

As we have done before we can assume $f(z) = P_0(e^{-iz}, e^{-i\omega z}, e^{-i\omega^2 z})$, $P_0 \in \bar{\mathbf{Q}}[X, Y, Z]$. There is a non-zero polynomial $R_0 \in \mathbf{Z}[X, Y, Z] \cap P_0\bar{\mathbf{Q}}[X, Y, Z]$, and it is enough to show that the function $F(z) = R_0(e^{-iz}, e^{-i\omega z}, e^{-i\omega^2 z})$ has well-separated zeros. Let ζ be a zero of F of multiplicity v. If R_v is associated to $F^{(v)}$ by the same procedure as above, then its coefficients are algebraic integers, hence there is an integer m and polynomials $C_{k,v} = C_k \in \mathbf{Z}[X, Y, Z]$, $1 \leqslant k \leqslant m$, such that

$$R_v^m + C_1 R_v^{m-1} + \cdots C_m = 0 \qquad \text{(cf. [27])}. \tag{17}$$

Let S_0 be an irreducible factor of R_0 in $\mathbf{Z}[X, Y, Z]$ such that $S_0(e^{-i\zeta}, e^{-i\omega\zeta}, e^{-i\omega^2\zeta}) = 0$. Since $F^{(v)}(\zeta) \neq 0$, S_0 cannot divide every coefficient C_k of (17). Let j be the largest index such that C_j is not divisible by S_0. We then have

$$(R_v(e^{-i\zeta}, e^{-i\omega\zeta}, e^{-i\omega^2\zeta}))^{m-j} + \cdots + C_j(e^{-i\zeta}, e^{-i\omega\zeta}, e^{-i\omega^2\zeta}) = 0,$$

and therefore to find a lower bound for $|F^{(v)}(\zeta)|$ it is enough to find one for $|C_j(e^{-i\zeta}, e^{-i\omega\zeta}, e^{-i\omega^2\zeta})|$.

Since the total number of positive pairs of relative prime polynomials appearing above is finite, we drop the indices and consider two distinct irreducible polynomials $R, S \in \mathbf{Z}[X, Y, Z]$. We need to find a lower bound, when $|\text{Im}\,\zeta| \leqslant A_0$, of the form

$$|R(e^{-i\zeta}, e^{-i\omega\zeta}, e^{-i\omega^2\zeta})| + |S(e^{-i\zeta}, e^{-i\omega\zeta}, e^{-i\omega^2\zeta})| \geqslant \varepsilon_1 e^{-C_1|\zeta|^k}$$

for some $k \geqslant 1$, $\varepsilon_1 > 0$, $C_1 > 0$. (Later on we will set $k = 4 + \varepsilon$, $\varepsilon > 0$ fixed.)

As we have done in the proof of Proposition 5, we can assume R does not depend on the variable Z and S does not depend on the variable Y. There is the possibility, which we want to eliminate, that there is a non-zero polynomial $Q \in \mathbf{Z}[X]$ in the ideal generated by R and S.

Either R and S are both in $\mathbf{Z}[X]$, in which case 1 is in the ideal they generate, and there is no problem in obtaining the lower bound (in fact, in terms of the weight $p(z) = |\text{Im}\, z| + \log(1 + |z|)$), or taking the resultant of Q and R or of Q and S we obtain also a non-zero polynomial in $\mathbf{Z}[Y]$ or $\mathbf{Z}[Z]$ in the ideal, in which case we can apply Proposition 3 and obtain a lower bound in terms of the weight $p(z) = |\text{Im}\, z| + \log(1 + |z|)$.

This argument shows more, namely, given any non-zero polynomial $Q \in \mathbf{Z}[X]$ (not necessarily in the ideal generated by R and S) we always have the estimate

$$|Q(e^{-i\zeta})| + |R(e^{-i\zeta}, e^{-i\omega\zeta})| + |S(e^{-i\zeta}, e^{-i\omega^2\zeta})| \geqslant \frac{\varepsilon e^{-C|\text{Im}\,\zeta|}}{(1 + |\zeta|)^N} \tag{18}$$

for some positive constants ε, C, N which depend on Q, R, S. In particular, we can apply this estimate to $Q(X) = A(X)\, B(X)$, where A, B are the leading terms of R, S when expressed as polynomials in Y and Z, respectively. It follows that for $|\zeta| \geqslant 1$, given any $K_1 > 0$ there is a constant $K_2 > 0$ (independent of $k \geqslant 1$) such that if

$$|R(e^{-i\zeta}, e^{-i\omega\zeta})| + |S(e^{-i\zeta}, e^{-i\omega^2\zeta})| < e^{-K_2|\zeta|^k}, \tag{19}$$

then there are solutions ξ_1, ξ_2 of the equations

$$R(e^{-i\zeta}, \xi_1) = 0, \qquad S(e^{-i\zeta}, \xi_2) = 0 \tag{20}$$

such that

$$|e^{-i\omega\zeta} - \xi_1| + |e^{-i\omega^2\zeta} - \xi_2| < e^{-K_1|\zeta|^k}. \tag{21}$$

Since the aim of the proof it is to show that the inequality (19) is impossible anywhere when $|\zeta| \geqslant 1$ and $|\text{Im}\,\zeta| \leqslant A_0$ for k, K_2 conveniently chosen, we can always assume without loss of generality that ζ is a point such that $e^{-i\zeta}$ is transcendental. (Therefore $0 < |\xi_1| \leqslant A_1$, $0 < |\xi_2| \leqslant A_1$.) For $0 < \varepsilon \ll 1$, $k_0 > 0$, $k_1 > 1$ to be chosen, define $k = (6 + \varepsilon)\, k_0$,

$K_1 = K_1(k_1, \varepsilon)$ to be chosen below in (36) and assume that (19), and hence (21), hold at a point ζ with $e^{-i\zeta}$ transcendental. Define two constants

$$N_0 = [k_1 |\zeta|^{k_0}], \qquad N_1 = [N_0^2 \log N_0]. \tag{22}$$

For N integer, $N_0 \leqslant N \leqslant N_1$, we set as in [15, 11]

$$L = [N^{1/2}(\log N)^{1/4}], \tag{23}$$

$$P = \left[\frac{1}{12d^2} N^{3/2}(\log N)^{-3/4}\right], \tag{24}$$

$$H = [N^{3/2}(\log N)^{1/4}], \tag{25}$$

where $d = \max(\deg_Y R, \deg_Z S)$. It is clear that there is an absolute constant c_0 such that

$$NL + P \log N \leqslant c_0 H. \tag{26}$$

The idea of Gelfond has been to introduce an auxiliary exponential sum F_N with frequencies in Γ and which is very small, together with all its derivatives of order $< P$, at all points in the finite portion of a lattice, namely $\zeta(l_1 + l_2\omega + l_3\omega^2)$, $l_j \in \mathbf{Z}$, $|l_j| < L$. To simplify the writing denote $\boldsymbol{l} := (l_1, l_2, l_3) \in \mathbf{Z}^3$, $|\boldsymbol{l}| = \max |l_j|$, $\boldsymbol{\omega} := (1, \omega, \omega^2)$, and $\boldsymbol{l} \cdot \boldsymbol{\omega} = l_1 + l_2\omega + l_3\omega^2$. The function F_N has the form

$$F_N(z) = \sum_{\mathbf{n}} \varphi_{\mathbf{n}} e^{-i(\mathbf{n} \cdot \boldsymbol{\omega})z}, \qquad \mathbf{n} \in \mathbf{Z}^3, \quad |\mathbf{n}| < N, \tag{27}$$

where $\varphi_{\mathbf{n}} \in \mathbf{Z}[e^{-i\zeta}]$. We want to consider the expression $\Phi_{p,\boldsymbol{l}}$ one obtains by differentiating F_N p times, $p < P$, substituting $z = (\boldsymbol{l} \cdot \boldsymbol{\omega})\zeta$, and after rewriting the exponents in terms of $\zeta, \omega\zeta, \omega^2\zeta$, replacing $e^{-i\omega\zeta}$ by ξ_1 and $e^{-i\omega^2\zeta}$ by ξ_2. It is in the third step that we use that ω is a cubic. Namely, given $\mathbf{n} \in \mathbf{Z}^3$, $|\mathbf{n}| < N$, $\boldsymbol{l} \in \mathbf{Z}^3$, $|\boldsymbol{l}| < L$ there is a unique triple $\mathbf{m} \in \mathbf{Z}^3$ such that

$$(\mathbf{n} \cdot \boldsymbol{\omega})(\boldsymbol{l} \cdot \boldsymbol{\omega}) = \mathbf{m} \cdot \boldsymbol{\omega}, \tag{28}$$

and one can easily see $|\mathbf{m}| \leqslant c_1 NL$ for some integral constant c_1 (which depends only on ω). Similarly, for $0 \leqslant p < P$, we can take c_1 sufficiently large so that

$$(\mathbf{n} \cdot \boldsymbol{\omega})^p = \mathbf{r} \cdot \boldsymbol{\omega}, \qquad \log |\mathbf{r}| \leqslant c_1 P \log N. \tag{29}$$

Therefore,

$$\Phi_{p,\boldsymbol{l}} = \sum_{\mathbf{n}} \varphi_{\mathbf{n}} (\mathbf{r} \cdot \boldsymbol{\omega})(e^{-i\zeta})^{m_1} \xi_1^{m_2} \xi_2^{m_3}. \tag{30}$$

The exponents m_j that appear could well be negative, but by multiplying $\Phi_{p,l}$ by $(e^{-i\zeta}\xi_1\xi_2)^{c_1NL}$ this can be corrected. Now we use the fact that, by (20), $A(e^{-i\zeta})\,\xi_1$, $B(e^{-i\zeta})\,\xi_2$ are algebraic integers over $\mathbf{Z}[e^{-i\zeta}]$, where, as we recall, $A(X)$, $B(X)$ are the leading coefficients of $R(X, Y)$ and $S(X, Y)$, respectively. After multiplying by $(A(e^{-i\zeta})\,B(e^{-i\zeta}))^{2c_1NL}$ and rewriting we get

$$(e^{-i\zeta}\xi_1\xi_2)^{c_1NL}\,(A(e^{-i\zeta})\,B(e^{-i\zeta}))^{2c_1NL}\,\Phi_{p,l}$$

$$=\sum_{\mathbf{n}}\varphi_{\mathbf{n}}(\mathbf{r}\cdot\boldsymbol{\omega})\sum_{j_1,j_2}\Pi_{j_1,j_2}\cdot(A(e^{-i\zeta})\,\xi_1)^{j_1}\,(B(e^{-i\zeta})\,\xi_2)^{j_2} \tag{31}$$

with $0\leqslant j_1<\deg_Y R$, $0\leqslant j_2<\deg_Z S$, $\Pi_{j_1,j_2}=\Pi_{j_1,j_2}(e^{-i\zeta})$, $\Pi_{j_1,j_2}(X)\in\mathbf{Z}[X]$ and

$$\deg \Pi_{j_1,j_2}(X)\leqslant c_2\,NL, \qquad \log\text{height}\,\Pi_{j_1,j_2}(X)\leqslant c_2\,NL. \tag{32}$$

One can rewrite the right-hand side of (31) as combinations of $\omega^{j_0}(A\xi_1)^{j_1}(B\xi_2)^{j_2}$, where $0\leqslant j_0\leqslant 2$. We would like to chose $\varphi_{\mathbf{n}}$ so that the coefficients of each of these powers vanish identically. To do this for all p, $0\leqslant p<P$, $\boldsymbol{l}$, $|\boldsymbol{l}|<L$, one has to apply the Dirichlet box principle, that is, count the number E of equations and U of unknowns. We have

$$E=3d^2P(2L+1)^3, \qquad U=(2N+1)^3.$$

By the definitions (23) and (24) of L and P we get

$$E\simeq\tfrac{1}{4}\,U$$

when $N_0\gg 1$, i.e., after we choose $k_1\gg 1$. By a lemma due to Siegel [15, Lemma II, p. 135] there is a solution $\{\varphi_{\mathbf{n}}\}\subseteq\mathbf{Z}[e^{-i\zeta}]$, not all identically zero, with

$$\deg\varphi_{\mathbf{n}}\leqslant c_3\,NL, \qquad \log\text{height}\,\varphi_{\mathbf{n}}\leqslant c_3'(NL+P\log N)\leqslant c_3\,H, \tag{33}$$

and one can even assume the $\varphi_{\mathbf{n}}$ to be relatively prime in $\mathbf{Z}[e^{-i\zeta}]$.

Therefore we have, for $\boldsymbol{l}\in\mathbf{Z}^3$, $|\boldsymbol{l}|<L$, $0\leqslant p<P$,

$$|(A(e^{-i\zeta})\,B(e^{-i\zeta}))^{2c_1NL}\,(e^{-i\zeta}\xi_1\xi_2)^{c_1L}\,F_N^{(p)}((\boldsymbol{l}\cdot\boldsymbol{\omega})\,\zeta)|$$

$$=|(A^2B^2e^{-i\zeta}\xi_1\xi_2)^{c_1NL}\,[F_N^{(p)}((\boldsymbol{l}\cdot\boldsymbol{\omega})\,\zeta)-\Phi_{p,l}]|$$

$$\leqslant(2N+1)^3\max_{\mathbf{n}}|\varphi_{\mathbf{n}}|\,e^{c_1P\log N}e^{c_3NL}(|\xi_1-e^{-i\omega\zeta}|+|\xi_2-e^{-i\omega\zeta}|), \tag{34}$$

where we have used the mean value theorem to obtain the last inequality, and the fact that $|\mathrm{Im}\,\zeta|\leqslant A_0$, hence $|\xi_1|$, $|\xi_2|$, $|e^{-i\zeta}|$, $|e^{-i\omega\zeta}|$, $|e^{-i\omega^2\zeta}|$ are uniformly bounded by a positive constant A_1, was used to determine c_3. For the same reason these five quantities are bounded below by A_1^{-1}. By (18) we have also a lower bound

$$|A(e^{-i\zeta})\,B(e^{-i\zeta})| \geqslant |\zeta|^{-A_2}, \qquad A_2>0. \tag{35}$$

From (34), (35), (33), and (21), we obtain the following upper bound for $F_N^{(p)}((\boldsymbol{l}\cdot\boldsymbol{\omega})\,\zeta)$:

$$\begin{aligned}\log|F_N^{(p)}((\boldsymbol{l}\cdot\boldsymbol{\omega})\,\zeta)| &\leqslant c_4 H + c_5(\log|\zeta|)\,NL - K_1|\zeta|^k\\ &\leqslant c_6|\zeta|^{3k_0}(\log|\zeta|)^{3/2} - K_1|\zeta|^k < -N_1^3(\log N_1)^2.\end{aligned} \tag{36}$$

The second line was obtained using the definitions (22)–(25), $k=(6+\varepsilon)\,k_0$, and K_1 was then chosen in terms of k_1, ε so that the last inequality holds.

We now want to use (36) to obtain a similar inequality at more points. For that purpose one uses Hermite's interpolation formula [15]. For $|z| \leqslant c_7 L|\zeta|$, $c_7 = 5|\boldsymbol{\omega}|$, we have, as long as $k_1 \gg 1$ so that $c_7 \leqslant 2L^{\varepsilon}$,

$$\begin{aligned}F_N(z) = &\frac{1}{2\pi i}\int_{|s|=L^{1+\varepsilon}|\zeta|}\frac{F_N(s)}{s-z}\prod_{\boldsymbol{l}}\left(\frac{z-(\boldsymbol{l}\cdot\boldsymbol{\omega})\,\zeta}{s-(\boldsymbol{l}\cdot\boldsymbol{\omega})\,s}\right)^P ds\\ &-\frac{1}{2\pi i}\sum_{\boldsymbol{l},p}\frac{F_N^{(p)}((\boldsymbol{l}\cdot\boldsymbol{\omega})\,\zeta)}{p!}\int_{|s-(\boldsymbol{l}\cdot\boldsymbol{\omega})\,\zeta|=\delta}\left(\frac{s-(\boldsymbol{l}\cdot\boldsymbol{\omega})\,\zeta}{s-z}\right)^P\\ &\times\prod_{\boldsymbol{l}'\neq\boldsymbol{l}}\left(\frac{z-(\boldsymbol{l}'\cdot\boldsymbol{\omega})\,\zeta}{s-(\boldsymbol{l}'\cdot\boldsymbol{\omega})\,\zeta}\right)^P ds,\end{aligned} \tag{37}$$

where $\delta=\frac{1}{2}|\zeta|\min|\boldsymbol{l}\cdot\boldsymbol{\omega}-\boldsymbol{l}'\cdot\boldsymbol{\omega}| \geqslant c_8|\zeta|\,L^{-3}$ (This inequality is a consequence of Schmidt's theorem [13].)

The first term in the formula (37) can be estimated by

$$N^3 e^{c_8(H+NL^{1+\varepsilon}|\zeta|)-(\varepsilon/2)\,PL^3\log N} < e^{-c_9 N^3\log N} \tag{38}$$

as long as we choose

$$k_0 > \frac{2}{3-\varepsilon}. \tag{39}$$

The constant c_9 depends on ε and, for $N \gg 1$ (i.e., $k_1 \gg 1$), it can be made $\approx \frac{1}{4}$, that is, there is no hope of it being large. The condition (39) is the one that leads to the final result $k=4+\varepsilon$ in the statement of the proposition. It is easy to see that the second term of (37) can be estimated by

$$\left(\sum_{p=0}^{P}\frac{|\zeta|^p}{p!}\right)e^{c_{10}N^3\log N - N_1^3(\log N_1)^2} \leqslant e^{-(1-\delta)\,N_1^3(\log N_1)^2} \tag{40}$$

for any δ, $0<\delta<1$, as long as $N \gg 1$. Hence the first term in (37) dominates and we get

$$\max_{|z| = c_7 L |\zeta|} \log |F_N(z)| < -c_9 N^3 \log N, \tag{41}$$

where by abuse of language we have incorporated the estimate from (40) into the same constant as (38). Further along we will use (41) and the Cauchy inequalities to estimate $F_N^{(p)}$ at points of the form $(\boldsymbol{l} \cdot \boldsymbol{\omega})\,\zeta$, $|\boldsymbol{l}| < L$, but with $0 \leqslant p < c_{11} P$, that is, the same points that led to (41) but with more derivatives involved. The value $c_{11} \geqslant 1$ is fixed by the next step of the proof.

We want now to show that there is a point of the form $(\boldsymbol{l} \cdot \boldsymbol{\omega})\,\zeta$, $|\boldsymbol{l}| < L$, and an integer p, $0 \leqslant p < c_{11} P$, such that $\log|F_N^{(p)}((\boldsymbol{l} \cdot \boldsymbol{\omega})\,\zeta) \geqslant -c_{12} N^3 \log N$ for some $c_{12} > 0$. The non-existence of such a point will contradict the fact that the $\varphi_{\mathbf{n}}$ have been chosen to be relatively prime in $\mathbf{Z}[e^{-i\zeta}]$. This reasoning depends on the following three lemmas.

LEMMA 2. [26, Theorem 3]. *Let $E(z) = \sum_{\nu=0}^{m-1} a_\nu e^{-i\alpha_\nu z}$ be an exponential sum with real frequencies. Let $\{\beta_\sigma\}$ be a collection of s distinct real numbers and t a positive integer. Set*

$$A = \max_\nu |a_\nu|, \qquad E = \max\{|E^{(j)}(\beta_\sigma)| : 0 \leqslant j < t, 0 \leqslant \sigma < s\},$$

$$\alpha = \max_\nu \{|\alpha_\nu|, 1\}, \qquad \beta = \max\{|\beta_\sigma|, 1\},$$

$$a_0 = \min\{|\alpha_\nu - \alpha_\mu|, 1 : \mu \neq \nu\}, \qquad b_0 = \min\{|\beta_\sigma - \beta_\rho|, 1 : \rho \neq \sigma\}.$$

Assume that

$$st \geqslant 2m + 13\alpha\beta. \tag{42}$$

Then

$$A \leqslant \frac{s\, e^{7\alpha\beta}}{(3a_0\beta)^{m-1}} \left(\frac{72\beta}{b_0 s}\right)^{st} E. \tag{43}$$

LEMMA 3. [15, Lemma VI, p. 147, and Lemma II, p. 135]. *Let $0 \neq T \in \mathbf{Z}[X]$, $\gamma \in \mathbf{C}$ such that $|T(\gamma)| < e^{-\lambda d(h+d)}$, where $\lambda \geqslant 3$, $d \geqslant \deg T$, $h \geqslant \log \text{height}\, T$. Then there is a factor T_1 of T in $\mathbf{Z}[X]$ such that T_1 is a power of an irreducible polynomial,* log height $T_1 \leqslant d + h$, *and* $|T_1(\gamma)| < e^{-(\lambda - 1)d(h+d)}$.

LEMMA 4. [15, Lemma V, p. 145]. *Let the heights of $f, g \in \mathbf{Z}[X]$ be $|f|$, $|g|$ and their degrees m, n, respectively. If for some $\gamma \in \mathbf{C}$ we have*

$$\max\{|f(\gamma)|, |g(\gamma)|\} \cdot |f|^n |g|^m (m+n)^{m+n} < 1$$

then f, g have a common irreducible factor.

In order to apply Lemma 2 to F_N and the values $F_N^{(p)}((\boldsymbol{l} \cdot \boldsymbol{\omega})\zeta)$, $|\boldsymbol{l}| < L$, $0 \leqslant p < c_{11} P$, we need to choose c_{11} so that the condition (42) is satisfied. This can be done since both sides of (42) are $O(N^3)$ in our case. The conclusion of the lemma now gives

$$\max_{\mathbf{n}} |\varphi_{\mathbf{n}}(e^{-i\zeta})| \leqslant e^{C_{13} N^3 \log N} \max\{|F_N^{(p)}((\boldsymbol{l} \cdot \boldsymbol{\omega})\zeta)| : |\boldsymbol{l}| < L, 0 \leqslant p < c_{11} P\}.$$

Assume $\log |F_N^{(p)}((\boldsymbol{l} \cdot \boldsymbol{\omega})\zeta)| < -c_{12} N^3 \log N$ in this range of indices. By choosing c_{12} extremely big we get

$$\max_{\mathbf{n}} |\varphi_{\mathbf{n}}(e^{-i\zeta})| < e^{-C_{14} N^3 \log N} \tag{44}$$

for any choice of c_{14} we want. Pick any $\varphi_{\mathbf{n}}$ different from zero. By (33) we have

$$(\deg \varphi_{\mathbf{n}})(\deg \varphi_{\mathbf{n}} + \log \text{height } \varphi_{\mathbf{n}}) = O(N^3 (\log N)^{1/2}).$$

Then Lemma 3 says that there is a polynomial T_1 which is a power of an irreducible polynomial $T_2 \in \mathbf{Z}[X]$ which satisfies at $e^{-i\zeta}$ an estimate similar to (44). Applying Lemma 4 to T_1 and any other $\varphi_{\mathbf{n}}$ one sees T_2 is a common factor of all the $\varphi_{\mathbf{n}}$. This is impossible, hence it follows that for a convenient constant c_{12} there is a point $\boldsymbol{l}_0 \in \mathbf{Z}^3$, $|\boldsymbol{l}_0| < L$, and an index p_0, $0 \leqslant p_0 < c_{11} P$, such that

$$\log |F_N^{(p_0)}((\boldsymbol{l}_0 \cdot \boldsymbol{\omega})\zeta)| \geqslant -c_{12} N^3 \log N. \tag{45}$$

Using the Cauchy estimates and (41) we obtain that there is $c_{15} > 0$ such that

$$\log |F_N^{(p_0)}((\boldsymbol{l}_0 \cdot \boldsymbol{\omega})\zeta)| \leqslant -c_{15} N^3 \log N. \tag{46}$$

We have not yet shown that (19) leads to a contradiction. This will be done using (45) and (46) for every value of N, $N_0 \leqslant N \leqslant N_1$. (It is clear that $\boldsymbol{l}_0$ and p_0 depend on N.) From this moment on there is no further difference between our arguments and those in [15, 11]. We continue giving the details of the proof for the sake of completeness.

The first thing to do is find an algebraic integer η_N over $\mathbf{Z}[e^{-i\zeta}]$ such that

$$-c_{17}N^3 \log N \leqslant \log |\eta_N| \leqslant -c_{16}N^3 \log N. \tag{47}$$

This is done by reversing the argument that led to (34), that is, replacing in $F_N^{(p_0)}((l_0 \cdot \omega)\,\zeta)$ the powers of $e^{-i\omega\zeta}, e^{-i\omega^2\zeta}$ by ξ_1, ξ_2 and multiplying by convenient powers of $A(e^{-i\zeta})\,B(e^{-i\zeta})$, $e^{-i\zeta}\xi_1\xi_2$. This η_N can be written

$$\eta_N = \sum \omega^{j_0}(A\xi_1)^{j_1}\,(B\xi_2)^{j_2}\,\Pi_{\mathbf{j}}\,(e^{-i\zeta}), \tag{48}$$

where the sum extends over $\mathbf{j} = (j_0, j_1, j_2)$, $0 \leqslant j_0 \leqslant 2$, $0 \leqslant j_1 < \deg_Y R$, $0 \leqslant j_2 < \deg_Z S$ and

$$\Pi_{\mathbf{j}}\,(X) \in \mathbf{Z}[X], \qquad \deg \Pi_{\mathbf{j}} \leqslant c_{18}NL, \qquad \log\text{height}\ \Pi_{\mathbf{j}} \leqslant c_{18}H. \tag{49}$$

We want to be able to apply once more Lemmas 3 and 4. These lemmas involve polynomials in $\mathbf{Z}[X]$, and to obtain them we use the fact that (48) shows that the degree n of η_N is at most $3d^2$, whereas before $d = \max(\deg_Y R, \deg_Z S)$. Hence η_N satisfies a monic irreducible equation

$$\eta_N^n + a_1\eta_N^{n-1} + \cdots + a_n = 0, \tag{50}$$

where $a_j \in \mathbf{Z}[e^{-i\zeta}]$. From (48) and (49) one can estimate the height and degree in $\mathbf{Z}[e^{-i\zeta}]$ of all the coefficients involved in writing all the conjugates of η_N, and hence one can estimate them for all symmetric powers of η_N and its conjugates. It follows [11] that

$$\deg a_j \leqslant c_{19}NL, \qquad \log\text{height}\ a_j \leqslant c_{19}H \tag{51}$$

as polynomials in $\mathbf{Z}[X]$.

At this point we can use the following result of Brownawell and Waldschmidt, which, as Professor Brownawell pointed out to us, is based on an idea of G. Chudnovsky.

LEMMA 5. [12]. *Let $\gamma \in \mathbf{C}$ be a transcendental number and $\eta \in \mathbf{C}$ satisfy the monic equation* (50) *of degree $\leqslant n$ over $\mathbf{Z}[\gamma]$, where the coefficients have degree $\leqslant D_1$ and height $\leqslant e^{D_2}$ as polynomials in $\mathbf{Z}[X]$. Suppose further that there exists two real numbers λ_1, λ_2 such that*

$$\lambda_1 > \lambda_2 > 6 + 2\log(n+1) + 2\log(|\gamma| + 1) \tag{52}$$

and

$$-\lambda_1 D_1(D_1 + D_2) \leqslant \log|\eta| \leqslant -\lambda_2 D_1(D_1 + D_2). \tag{53}$$

Then there is an irreducible polynomial $T \in \mathbf{Z}[X]$ *and an integer* $s \geq 1$ *such that* $T^s(X)$ *divides* $a_n(X)$ *in* $\mathbf{Z}[X]$ *and*

$$-3n\lambda_1 D_1(D_1 + D_2) \leqslant \log |T(\gamma)| \leqslant -\lambda_2 D_1(D_1 + D_2)/6s. \tag{54}$$

The unique difficulty in applying this lemma is to see that the estimate (47) can be put in the form (52), (53). Recall that the constant c_{16} is obtained ultimately from c_9 which could be very small, while λ_2 is required to be not too small in (52) when $\gamma = e^{-i\zeta}$. In fact the bound required by (52) lies between two constants which depend only on the original exponential sum f. On the other hand, in our case $D_1 = O(NL)$, $D_2 = O(H)$, hence

$$D_1(D_1 + D_2) = O(N^3(\log N)^{1/2}) = o(N^3 \log N),$$

so that in effect we have (52) and (53) satisfied. Furthermore, $s = s_N = \deg a_n = O(NL)$. We call T_N the polynomial obtained using Lemma 5; it satisfies

$$-c_{20}N^3 \log N \leqslant \log |T_N(e^{-i\zeta})| \leqslant -c_{21} \frac{N^3 \log N}{s_N} \tag{55}$$

$$\left.\begin{aligned} \deg T_N &\leqslant c_{22} NL/s_N \\ \log \text{height } T_N &\leqslant c_{22} H/s_N. \end{aligned}\right\} \tag{56}$$

The last two inequalities follow from [15, Lemma IV, p. 14]. By Lemma 4 all the polynomials T_N coincide, $N_0 \leqslant N \leqslant N_1$. This leads very quickly to a contradiction. Namely, the right-hand side of (55) implies

$$\log |T_{N_1}^{s_{N_1}}(e^{-i\zeta})| = s_{N_1} \log |T_{N_1}(e^{-i\zeta})| \leqslant -c_{21} N_1^3 \log N_1. \tag{57}$$

On the other hand, $T_{N_0} = T_{N_1}$, hence

$$\begin{aligned} \log |T_{N_1}^{s_{N_1}}(e^{-i\zeta})| &= s_{N_1} \log |T_{N_0}(e^{-i\zeta})| \\ &\geqslant -c_{20} s_{N_1} N_0^3 \log N_0 \\ &\geqslant -c_{22} N_1 L_{N_1} N_0^3 \log N_0. \end{aligned} \tag{58}$$

Using the definitions of N_0, N_1 we have

$$N_1^3 \log N_1 \approx N_0^6 (\log N_0)^4,$$

while

$$\begin{aligned} N_1 L_{N_1} N_0^3 \log N_0 &\approx (N_0^2 \log N_0)(N_0^2 \log N_0)^{1/2} (\log N_0)^{1/4} N_0^3 (\log N_0) \\ &= N_0^6 (\log N_0)^{11/4}, \end{aligned}$$

which shows that (57) and (58) cannot hold simultaneously if $N_0 \gg 1$ (this is achieved by having chosen $k_1 \gg 1$ to start with). This is the contradiction we were looking for. ∎

2. Exponential Polynomials in Several Variables

The aim of this section is to relate the questions considered in Section 1 to our ongoing work on ideals generated by exponential polynomials in the spaces $A_p(\mathbf{C}^n)$, $p(z) = |\operatorname{Im} z| + \log(1 + |z|)$ [9].

It is clear that Definition 3 makes sense for entire functions of several complex variables. The Lojasiewicz inequality for the weight $p(z) = |\operatorname{Im} z| + \log(1 + |z|)$ (respectively $p(z) = |z|$) for a family of Fourier transforms of distributions $\mu_1, ..., \mu_m$ of compact support in $\mathbf{R}^n$ (resp. analytic functionals) plays an important role in the study of the solutions φ of the system of convolution equations

$$\mu_1 * \varphi = \cdots = \mu_m * \varphi = 0.$$

Let us recall first what is known for a single exponential polynomial. The inequality (3) in Proposition 1 is still valid [17], i.e., if f is an exponential polynomial, Λ is set of frequencies in $\mathbf{C}^n$, $h(z) = \max\{\operatorname{Im}(\lambda \cdot z) : \lambda \in \Lambda\}$, and $V = \{z \in \mathbf{C}^n : f(z) = 0\}$ then

$$|f(z)| \geqslant c_1 (1 + |z|)^{-M_1} (\operatorname{dist}(z, V))^{N_1} e^{h(z)}$$

for some constants $c_1 > 0$, $M_1 \geqslant 0$, $N_1 \geqslant 0$. In fact one can prove more, and by following the method of proof from [7, Theorem 7.3] one sees that there is an algebraic variety $W \subseteq \mathbf{C}^n$ such that if $a \notin W$ there are positive constants c_2, M_2, N_2 such that for any complex line L through a, say $z = a + v\zeta$, $\zeta \in \mathbf{C}$, $v \in \mathbf{C}^n$, $|v| = 1$, we have

$$|f(a + v\zeta)| \geqslant c_2 (1 + |\zeta|)^{-M_2} (\operatorname{dist}(\zeta, V \cap L))^{N_1} e^{h(v\zeta)}.$$

The constants are independent of v; cf. [5].

One of the main questions that arises in $\mathbf{C}^n$ for finitely generated ideals is to find out whether they are slowly decreasing [7]. Let us recall that this means in case the variety $V = \{z \in \mathbf{C}^n : f_1(z) = \cdots = f_m(z) = 0\}$ is discrete, $f_1, ..., f_m \in A_p$.

Definition 4. $f_1, ..., f_m$ are slowly decreasing for the weight p if there exist positive constants ε, C, K_1, K_2 such that if z_1, z_2 are two points in the same connected component of the open set

$$S(\varepsilon, C) = \left\{ z \in \mathbf{C}^n \colon \sum_{j=1}^{m} |f_j(z)| < \varepsilon e^{-Cp(z)} \right\}$$

we have

$$p(z_1) \leqslant K_1 p(z_2) + K_2. \tag{59}$$

Note that the fact that (59) holds implies that the components of $S(\varepsilon, C)$ are bounded and hence the variety V must necessarily be discrete.

For exponential polynomials $f_1, ..., f_m$ we have shown the following equivalence:

PROPOSITION 8 [9, Theorem 3.1 and Remark 3.2]. *The family $f_1, ..., f_m$ of exponential polynomials is slowly decreasing if and only if for every $\varepsilon_0 > 0$, $C_0 > 0$ there are positive constants ε_1, C_1 such that if z_1, z_2 are in the same component of $S(\varepsilon_1, C_1)$ we have $|z_1 - z_2| < \varepsilon_0 e^{-C_0 p(z_1)}$.*

That is, the property of being slowly decreasing corresponds exactly to part (iii) of Proposition 1.

It is easy to see when rank $\Gamma \leqslant n$ (n = number of variables) that the Lojasiewicz inequality for the class $\mathscr{F}(\Gamma; \mathbf{C})$ is a consequence of the classical Lojasiewicz inequality for polynomials [21]. Namely, after a linear change of coordinates we can assume the generators of Γ are the canonical basis $e_1, ..., e_n$ of $\mathbf{C}^n$, and any family $f_1, ..., f_m \in \mathscr{F}(\Gamma; \mathbf{C})$ is (up to multiplication by an exponential factor) a family of polynomials in $e^{-iz_1}, ..., e^{-iz_n}$. This is exactly the way one proves $f_1, ..., f_m$ are slowly decreasing when V is discrete [7]. In the case of $n = 1$, as we pointed out in Remark 1, the Lojasiewicz inequality holds for $\mathscr{G}(Z; \mathbf{C})$ [9]. In [9] we considered, among other questions, the property of being slowly decreasing in $\mathscr{G}(\mathbf{Z}^n; \mathscr{K})$, $\mathscr{K}$ a subfield of $\mathbf{C}$. Namely, we discussed:

Problem 2. Let $f_1, ..., f_m \in \mathscr{G}(\mathbf{Z}^n; \mathscr{K})$. Does the fact that the variety V of common zeros is discrete imply $f_1, ..., f_m$ slowly decreasing for the weight $p(z) = |\mathrm{Im}\, z| + \log(1 + |z|)$?

For $n = 2$ we proved this result for $\mathscr{K} = \mathbf{C}$ [9] (cf. also [8]). We have also shown it is false without restrictions on $\mathscr{K}$ as soon as $n \geqslant 3$. It might hold for $\mathscr{K} = \bar{\mathbf{Q}}$ but we have been unable to prove this yet.

The link between Problem 2 and Problem 1 is the following. First, if the answer to Problem 2 were affirmative for a fixed field $\mathscr{K}$ and any dimension n then it would follow that Hilbert's Nullstellensatz is valid in $\mathscr{G}(\mathbf{Z}^n; \mathscr{K})$, i.e., given $f_1, ..., f_m \in \mathscr{G}(\mathbf{Z}^n; \mathscr{K})$ with no common zeros there are functions $g_1, ..., g_m \in A_p(\mathbf{C}^n)$ such that $f_1 g_1 + \cdots + f_m g_m = 1$. In fact, it is clear that $\mathscr{G}(\mathbf{Z}^n; \mathscr{K})$ can be considered a subset of $\mathscr{G}(\mathbf{Z}^\nu; \mathscr{K})$, for $\nu \geqslant n$ (namely the functions are independent of the last $(\nu - n)$ coordinates). The condition $V = \varnothing$ remains valid in $\mathbf{C}^\nu$ (if $V \neq \varnothing$ then it would not remain discrete when considered in $\mathbf{C}^\nu$, $\nu > n$). Therefore we can assume $\nu \geqslant m$.

Consider in $\mathbf{C}^\nu$ the function

$$u(z) = \log\left(\sum_{j=1}^{m} |f_j(z)|^2\right).$$

It is a continuous plurisubharmonic function which satisfies the homogeneous complex Monge—Ampère equation. (It is here where we use $\nu \geqslant m$.) By [3], u satisfies a minimum modulus principle. The condition that $f_1, ..., f_m$ are slowly decreasing in $\mathbf{C}^\nu$ implies that on the boundary of the components of the set $S(\varepsilon, C)$, $u \geqslant -C_1 p(z_1) - C_2$ for some $C_1 > 0$, $C_2 > 0$ and z_1 a point in the component. It follows that this holds also throughout the interior of the component and hence everywhere we have $u(z) \geqslant -D_1 p(z) - D_2$ for some $D_1, D_2 > 0$. Hörmander's work [19] then shows that the $g_1, ..., g_m$ exist.

Now, let $F_1, ..., F_r$ be exponential polynomials of one variable with frequencies in a subgroup Γ of $\mathbf{C}$ of rank n, $\Gamma = \omega_1 \mathbf{Z} \oplus \cdots \oplus \omega_n \mathbf{Z}$. Consider the r polynomials of $n+1$ variables which give

$$F_j(z) = P_j(z, e^{-i\omega_1 z}, ..., e^{-i\omega_n z})$$

(as before we have premultiplied F_j by an exponential factor). Let f_j, $j = 1, ..., \nu + n$, be defined in $\mathbf{C}^{n+1}$ by the equations

$$\begin{aligned} f_j(z_1, ..., z_{n+1}) &= P_j(z_1, e^{-iz_2}, ..., e^{-iz_n}), \qquad 1 \leqslant j \leqslant r \\ f_{r+1}(z) &= z_2 - \omega_1 z_1 \\ &\vdots \\ f_{r+m}(z) &= z_{n+1} - \omega_n z_1. \end{aligned}$$

It is clear that $F_1, ..., F_r$ have no common zeros in $\mathbf{C}$ if and only if $f_1, ..., f_{\nu+n}$ have no common zeros in $\mathbf{C}^{n+1}$. Hence a solution of the Bezout equation in $A_p(\mathbf{C}^{n+1})$ for $f_1, ..., f_{\nu+n}$ leads to a solution of the same equation in $A_p(\mathbf{C})$ for $F_1, ..., F_r$ (it could happen that some $\omega_j \notin \mathbf{R}$, then one has to consider $p(z) = |z|$ in $\mathbf{C}$, even if one had the result for $p(z) = |\text{Im}\, z| + \log(1 + |z|)$ in $\mathbf{C}^{n+1}$). As we pointed out in Remark 2, the solvability of the Bezout equation is tied to the equivalent properties of Proposition 2. This reasoning also shows why one needs to restrict the field $\mathscr{K}$ to $\bar{\mathbf{Q}}$ in Problem 2.

On the other hand, Theorems 8.1 and 8.2 from [9] show that to answer Problem 2 in the affirmative for $\mathscr{G}(\mathbf{Z}^2; \bar{\mathbf{Q}})$ one only needs to prove that the Nullstellensatz is valid in $\mathscr{G}(\mathbf{Z}^2; \bar{\mathbf{Q}})$. We will presently see that this question is closely related to the Lojasiewicz inequality for $\mathscr{G}(\mathbf{Z} \oplus \omega\mathbf{Z}; \bar{\mathbf{Q}})$, $\omega \in \bar{\mathbf{Q}}$.

Let $f_1, ..., f_m \in \mathscr{G}(\mathbf{Z}^2; \bar{\mathbf{Q}})$, with $V = \{z \in \mathbf{C}^2 : f_1(z) = \cdots = f_m(z) = 0\} = \varnothing$. We have as before polynomials $P_j \in \bar{\mathbf{Q}}[X, Y, Z, W]$ such that $f_j(z) = P_j(z_1, z_2, e^{-iz_1}, z^{-iz_2})$; they are relatively prime, and we can assume $P_j(X, Y, 0, W) \not\equiv 0$, $P_j(X, Y, Z, 0) \not\equiv 0$. We can further assume that the ideal generated by $P_1, ..., P_m$ is prime [9]. Let $\tilde{V}$ be the algebraic subvariety of $\mathbf{C}^4$ defined by them. Let S be the variety of singular points of $\tilde{V}$ and $Q_1, ..., Q_r$ the generators of the ideal of S (including $P_1, ..., P_m$ in the list). We have shown in [9, Theorem 8.2] that it is enough to show that the Nullstellensatz is valid for the family of exponential polynomials $g_j \in \mathscr{G}(\mathbf{Z}^2; \bar{\mathbf{Q}})$ defined by $g_j(z) = Q_j(z_1, z_2, e^{-iz_1}, e^{-iz_2})$. Now, since $\dim \tilde{V} \leqslant 2$ we have $\dim S \leqslant 1$. If $\dim S = 0$ one finds that the ideal generated by the Q_j contains non-zero polynomials $A \in \bar{\mathbf{Q}}[X]$, $B \in \bar{\mathbf{Q}}[Y]$. Hence one immediately has, for some $\varepsilon, C > 0$, $\sum_1^r |g_j(z)| \geqslant \varepsilon e^{-Cp(z)}$ outside a compact set, and since the $g_1, ..., g_r$ have no common zeros it is clear the Nullstellensatz is valid in this case.

Therefore the only case left to consider is that of a family $g_1, ..., g_r \in \mathscr{G}(\mathbf{Z}^2; \bar{\mathbf{Q}})$, $V = \{z \in \mathbf{C}^2 : g_1(z) = \cdots = g_r(z) = 0\} = \varnothing$, and $\dim S = 1$, where $S = \{z \in \mathbf{C}^4 : Q_1 = \cdots = Q_r = 0\}$. The same kind of reasoning we applied before in Section 1 using elimination theory shows that the only case that causes any difficulty is that where the ideal generated by $Q_1, ..., Q_r$ contains a nonzero polynomial R in $\bar{\mathbf{Q}}[X, Y]$ (cf. [9, Proposition 6.3]). Hence the Nullstellensatz we need is a particular case of the Lojasiewicz inequality for the *restrictions* of the exponential polynomials $g_1, ..., g_r$ to the subvariety of $\mathbf{C}^2$ given by $R(z_1, z_2) = 0$. We have already proved in [9] that the restriction of a single exponential polynomial to an algebraic variety of dimension one is either identically zero or it satisfies the condition (iii) of Proposition 1 on the variety. It follows from the proof of Proposition 2, if one replaces everywhere

$$\frac{d}{dz} \quad \text{by} \quad \frac{\partial R}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial R}{\partial z_1}\frac{\partial}{\partial z_2},$$

that to obtain the Lojasciewicz inequality all one has to prove is that the zeros of the restriction to $\{R = 0\}$ of a single exponential polynomial in $\mathscr{G}(\mathbf{Z}^2; \bar{\mathbf{Q}})$ are well separated.

We know how to prove this property in a number of cases. Namely, we can restrict ourselves, as in Section 1, to consider the situation

$$|P(z_1, z_2, e^{-iz_1})| + |Q(z_1, z_2, e^{-iz_2})| \quad \text{small} \tag{60}$$

and

$$R(z_1, z_2) = 0. \tag{61}$$

From (60) we obtain $|\operatorname{Im} z_1| + |\operatorname{Im} z_2| \leqslant C \log(2 + |z|)$. Consider the different asymptotic developments of the roots of (6.1). For those of the form

$$z_2 = a z_1^k (1 + o(1)), \qquad k \in \mathbf{Q} \setminus \{1\}, \text{ as } |z_1| \to \infty,$$

we have $|\operatorname{Re} z_1| = O(|\operatorname{Im} z_1| + |\operatorname{Im} z_2|)$, hence $|z_1|$, and therefore $|z_2|$, remain bounded. The same reasoning holds for

$$z_2 = \omega z_1 (1 + o(1)), \qquad \omega \notin \mathbf{R}.$$

The only case we have not yet been able to handle is where one of the branches of $R = 0$ has an asymptotic developement of the form

$$z_2 = \omega z_1 (1 + o(1)), \qquad \omega \in \bar{\mathbf{Q}} \cap \mathbf{R}, \tag{62}$$

which is almost the same situation as in Problem 1 for $\mathscr{G}(\mathbf{Z} \oplus \omega \mathbf{Z}; \bar{\mathbf{Q}})$. We note that this problem of resistrictions to algebraic varieties might be substantially harder due to the $o(1)$ in (62).

We hope that the above explanations have convinced the reader that Problems 1 and 2 are strongly tied to each other. Their complete solution seems to be very difficult at this moment.

Every case where one has been able to prove that a system of exponential polynomials in $\mathbf{C}^n$ is slowly decreasing has given interesting applications to the harmonic analysis of the solutions of the corresponding system of difference–differential equations. For instance, Meril and Struppa [23] have shown recently that the Hartogs continuation property for holomorphic functions (or more generally for solutions of overdetermined systems of partial differential equations) holds also for solutions of certain types of overdetermined systems of convolution equations. We will publish shortly examples of such systems as well as a study of systems of partial differential equations with time lag. The techniques involved in this work are a combination of those in the present paper as well as those in [9].

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