A Random-Projection Based Procedure to Test if a Strictly Stationary Process is Gaussian

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V Santouval 2009
1 Tests of Gaussianity

2 Gaussianity Tests for Strictly Stationary Processes

3 The Random Projection Test (RP test)

4 Simulations

5 Conclusions
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we already know Gaussianity tests for this setting!

But, if the r.v.'s are dependent?
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**But, if the r.v.’s are dependent?**
What kind of dependence?

We’ll deal with **Strictly Stationary Processes**

- \( \{X_t\}_{t \in \mathbb{Z}}, \ldots, X_1, X_2, \ldots \)
- \( \{X_t\}_{t \in \mathbb{N}}, X_1, X_2, \ldots \)

\( \{X_t\}_{t \in \mathbb{Z}} \) is a strictly stationary process iff

\[(X_{t_1}, X_{t_2}, \ldots, X_{t_j}) \text{ and } (X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_j+k}) \text{ are i.d for all } k \in \mathbb{Z} \]

A strictly stationary process \( \{X_t\}_{t \in \mathbb{Z}} \) is Gaussian iff

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But, there are not already a bunch of tests for this?

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  Check if $\phi_{X_t}(\lambda_i) = \phi_{N(\mu, \sigma^2)}(\lambda_i)$ for $i = 1, ..., N$

- Lobato and Velasco’s test (2004)
  Check the skewness and kurtosis of $X_t$

However, they only test if the marginals of the process are Gaussian.

That is; they test if $X_t$ is Gaussian, not if $(X_1, ..., X_t)$ is Gaussian

So, is this O.K.?

No, because such tests do not reject non-Gaussian processes with Gaussian marginals

We need a new test that reject such kind of non-Gaussian processes!
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Given \( \{X_t\}_{t \in \mathbb{Z}} \) a strictly stationary process

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**Theorem** (Cuesta, Barrio, Fraiman and Matrán, 2007)

Let \( \eta \) be a dissipative distribution on a separable Hilbert space, \( H \). If \((..., X_t)\) is an \( H \)-valued random element and

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then \( X \) is Gaussian.

It follows,

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So, selecting \( h \) using a dissipative distribution we have,

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In Practice

- \( H = \{(x_n)_{n \in \mathbb{N}^*} : \sum_{n \in \mathbb{N}^*} x_n^2 a_n < \infty \} \),
- \( a_n = \min(1, n^{-2}) \) and \( \mathbb{N}^* = \mathbb{N} \cup \{0\} \)
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- **Dissipative distribution**
  - \( l_0 \) chosen with \( \beta(\alpha_1, \alpha_2) \)
  - \( l_n \) chosen with \( \beta(\alpha_1, \alpha_2)[0, 1 - \sum_{i=0}^{n-1} \eta_i] \), for \( n \geq 1 \)
  - \( h_n = (l_n/a_n) \) for \( n \geq 0 \)
  - \( (h_0, ...) \) has dissipative distribution

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Test Gaussianity of \( \{Y_t\}_{t \in \mathbb{Z}} \) with a procedure that check if the marginals of the process are Gaussian

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Simulations

Given an AR(1) process

\[ X_t = q \ast X_{t-1} + \epsilon_t \]

Compare results of

- E, Epps’ test (1987)

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Given an AR(1) process

\[ X_t = q \times X_{t-1} + \epsilon_t \]

Compare results of

- E, Epps’ test (1987)
- GE, combination of G and E using the Multiple Testing Procedure of Benjamini and Yekutieli (2001)
- RP, Random Projection Test using GE
Given an AR(1) process

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A Non-Gaussian Process with Gaussian Marginals

A family of this kind of processes, $f(p)$, is given in Cuesta and Matrán (1991), with $p$ a prime number.

Let $p = 5$,

$$
\ldots, X_{mp}, X_{mp+1}, X_{mp+2}, X_{mp+3}, X_{mp+4}, X_{(m+1)p}, \ldots, X_{(m+1)p+4}, \ldots
$$

- a strictly stationary process
- of pairwise independent variables
- with $X_t$ Gaussian for all $t \in \mathbb{Z}$
- without mutually independent variables

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Summarizing

1. Tests of Gaussianity
2. Gaussianity Tests for Strictly Stationary Processes
3. The Random Projection Test (RP test)
4. Simulations
5. Conclusions
Conclusions

Given a strictly stationary process \( \{X_t\}_{t \in \mathbb{Z}} \),

**the RP test check if \( \{X_t\}_{t \in \mathbb{Z}} \) is Gaussian**

**Procedure:**
- Take \( h \in H \) following \( \eta \)
- \( Y_t := \langle (\ldots, X_t), h \rangle \)
- Check if the marginals of the strictly stationary process \( \{Y_t\}_{t \in \mathbb{Z}} \) are Gaussian

**Advantage:**
- Reject non-Gaussian processes with Gaussian marginals
Thank you very much!