— DM 1 —

a) $f(x) = |x|$ est une fonction $L^1_{loc}(\mathbb{R})$ donc $T_f \in \mathcal{D}'(\mathbb{R})$.

$$\langle T_f', \varphi \rangle \overset{def}{=} -\langle T_f, \varphi' \rangle = -\int_{\mathbb{R}} f(x)\varphi'(x)\,dx$$

$$= \int_{-\infty}^{0} x\varphi'(x)\,dx - \int_0^{\infty} x\,\varphi'(x)\,dx$$

$$\overset{IPP}{=} \underbrace{[x\varphi]_{-\infty}^{0}}_{=0} - \int_{-\infty}^{0} \varphi(x) - \underbrace{[x\varphi]_0^{\infty}}_{=0} + \int_0^{\infty} \varphi(x)\,dx$$

$$= \langle T_g, \varphi \rangle \quad \text{où} \quad g(x) = \begin{cases} -1 &]-\infty, 0] \\ 1 & (0, \infty). \end{cases}$$

$g \in L^1_{loc}$ et C^∞ par morceaux.

$\overset{\text{Formule de sauts}}{\Longrightarrow}$ $T_g' = 2\cdot\delta_0$

Donc $T_f'' = 2\delta_0$.

$\Rightarrow T_f''' = (2\delta)' = 2\delta'$.

b) $g(x) = \log|x|$. g est continue donc L^1_{loc} sur $\mathbb{R}^*$.

Remarquons que $g \in L^1(0, A)$ $\forall A > 0$ (primitive $x\log x - x$)

donc $g \in L^1_{loc}(\mathbb{R}) \Rightarrow T_g \in \mathcal{D}'(\mathbb{R})$.

On a $\langle T_g, \varphi \rangle = \lim\limits_{\varepsilon \to 0+} \int\limits_{|x|>\varepsilon} g(x)\, \varphi(x)$.

Ainsi, $\langle T_g', \varphi \rangle = - \langle T_g, \varphi' \rangle$

$$= \lim_{\varepsilon \to 0+} - \int\limits_{|x|>\varepsilon} \log|x| \cdot \varphi'(x)$$

$$\overset{IPP}{=} \lim_{\varepsilon \to 0+} - \left[\log|x| \cdot \varphi(x) \right]_{-\infty}^{-\varepsilon} - \left[\log|x|\, \varphi(x) \right]_{\varepsilon}^{\infty} + \int\limits_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx$$

$$= \lim_{\varepsilon \to 0+} + \log|\varepsilon| \left(\varphi(\varepsilon) - \varphi(-\varepsilon) \right) + \int\limits_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx.$$

Or $\varphi(\varepsilon) - \varphi(-\varepsilon) = 2\varepsilon\, \varphi'(0) + O(\varepsilon^2)$, on a

$= \langle vp\left(\frac{1}{x}\right), \varphi \rangle$. Donc $T_g' = vp\left(\frac{1}{x}\right)$.

c) $u_n(x) = \frac{1}{n} \sin(nx)$. u_n continue $\Rightarrow L^1_{loc}(\mathbb{R})$.

$\Rightarrow T_n = T_{u_n} \in \mathcal{D}'(\mathbb{R})$

Soit $K \subseteq \mathbb{R}$ comp. et $\varphi \in \mathcal{D}_K(\mathbb{R})$.

On a $\left| \int\limits_{\mathbb{R}} \frac{1}{n} \sin(nx)\, \varphi(x)\, dx \right|$

$$\leq \frac{1}{n} \int\limits_{K} |\varphi(x)|\, dx \leq \frac{1}{n} \cdot |K|\, \|\varphi\|_\infty \xrightarrow{n \to \infty} 0$$

donc $\langle T_n, \varphi \rangle \xrightarrow{n \to \infty} \langle 0, \varphi \rangle \quad \forall \varphi$

$\Leftrightarrow T_n \longrightarrow 0$.

d) $f(x,y) = \begin{cases} 1 & x \geq y \\ 0 & \text{sinon} \end{cases}$

$f \in L^\infty(\mathbb{R}^2)$ donc $L^1_{loc}(\mathbb{R}^2) \Rightarrow T_f \in \mathcal{D}'(\mathbb{R}^2)$.

On a $\langle \frac{\partial T}{\partial x}, \varphi \rangle \overset{def}{=} (-1)^1 \langle T, \frac{\partial \varphi}{\partial x} \rangle$

$$= -\int_{\mathbb{R}} \int_y^\infty \frac{\partial \varphi}{\partial x}(x,y)\, dx \, dy$$

$$= \int_{\mathbb{R}} \varphi(y,y) \, dy.$$

e) $\int_{\mathbb{R}^2} f(x,y)\, \varphi(x,y)\, d(x,y) = \int_0^{2\pi} \int_0^\infty \frac{1}{r} \varphi(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

Si $K \subseteq \mathbb{R}^2$ comp., $\varphi \in \mathcal{D}_K(\mathbb{R}^2)$, $\|\cdot\|_2$ est une fonct. continue $\Rightarrow$ borné sur K donc $\exists M$: $\forall x \in K$: $\|x\|_2 \leq M$.

Ainsi, $\left| \int_{\mathbb{R}^2} f(x,y)\, \varphi(x,y)\, d(x,y) \right| = \left| \int_0^{2\pi} \int_0^M \varphi(r\cos\theta, r\sin\theta)\, dr\, d\theta \right|$

$$\leq \sup_{B(0,K)} |\varphi| \cdot 2\pi M$$

on a donc une distrib d'ordre 0.

On écrit $z = x + iy$.

$$\int_{\mathbb{R}^2} \frac{x - iy}{x^2 + y^2 + \varepsilon^2} \varphi(x,y) \, d(x,y)$$

$$= \int_0^{2\pi} \int_0^\infty \frac{r(\cos\varphi - i\sin\varphi)}{r^2 + \varepsilon^2} \varphi(r\cos\varphi, r\sin\varphi) \, r \, dr \, d\varphi$$

$$= \int_0^{2\pi} \int_0^\infty \frac{r^2}{r^2+\varepsilon^2} \underbrace{(\cos\varphi - i\sin\varphi)\varphi(r\cos\varphi, r\sin\varphi)}_{\substack{\text{borné à supp. compact,} \\ \text{donc majorant } \in L^1}} \, dr \, d\varphi$$

$$\Rightarrow \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \frac{x - iy}{x^2+y^2+\varepsilon^2} \varphi(x,y) \, dx dy$$

$$\overset{\text{Lebesgue}}{=} \int_0^{2\pi} \int_0^\infty \underbrace{\lim_{\varepsilon \to 0} \frac{r^2}{r^2+\varepsilon^2}}_{=1} (\cos\varphi - i\sin\varphi) \varphi(r\cos\varphi, r\sin\varphi) \, dr \, d\varphi$$

$$\overset{\cos^2+\sin^2=1}{=} \int_0^{2\pi} \int_0^\infty \frac{1}{\cos\varphi + i\sin\varphi} \varphi(r\cos\varphi, r\sin\varphi) \, dr \, d\varphi$$

$$= \int_0^{2\pi} \int_0^\infty \frac{1}{r(\cos\varphi + i\sin\varphi)} \varphi(r\cos\varphi, r\sin\varphi) \, r \, dr \, d\varphi$$

$$= \int_{\mathbb{C}} \frac{1}{z} \varphi(z) \, dz.$$

Puisque $\langle T_{f_\varepsilon}, \varphi \rangle \longrightarrow \langle T_{\frac{1}{z}}, \varphi \rangle \ \forall \varphi$, on a conv. au sens de distrib.

f) Soit $\tilde{\theta} \in \mathcal{D}(\mathbb{R}) \setminus \{0\}$.

$$\theta(x) = \frac{|\tilde{\theta}(x)|}{\int_{\mathbb{R}} |\tilde{\theta}(x)|\,dx}$$ satisfait $\int_{\mathbb{R}} \theta(x)\,dx = 1$.

$\varphi(x) = \psi(x) - c \cdot \theta(x)$ où $c = \int_{\mathbb{R}} \psi(t)\,dt$.

Puisque $\int_{\mathbb{R}} \varphi(x)\,dx = 0$, il ex. $\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,dt$

et $\Phi \in \mathcal{D}(\mathbb{R})$ (!) voir ex. 8, feuille 2.

Par hypothèse $\langle T', \Phi \rangle = 0$

$\Leftrightarrow \quad \langle T, \varphi \rangle = 0$

$\Leftrightarrow \quad \langle T, \psi \rangle = c \cdot \langle T, \theta \rangle$

$$= \int_{\mathbb{R}} \psi(x) \cdot \langle T, \theta \rangle \, dx$$

Ainsi, $T = T_f$ avec $f(x) \equiv \langle T, \theta \rangle$. $\blacksquare$