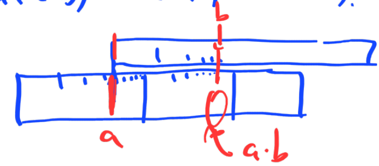
Stirling formula

Question: estimate $n!$.

similar question: estimate $\ln(n!)$: this leads to “Baby-Stirling”:

$$\ln(a \cdot b) = \ln(a) + \ln(b)$$



$$\ln(n!) = \ln(1) + \ln(2) + \dots + \ln(n)$$

Stirling formula

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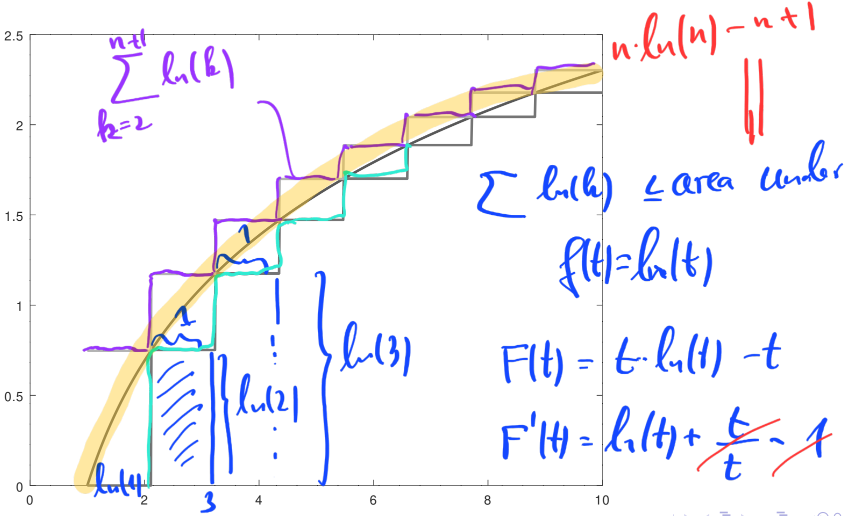

Stirling formula 2

$$\int_1^n \ln(t)dt \leq \ln(n!) \leq \int_1^n \ln(1+t)dt$$

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\int_1^n \ln(t)dt \le \ln(n!) \le \int_1^n \ln(1+t)dt$$

Knowing the anti-derivative $t\ln(t) - t$ we get

$$n\ln(n) - n \quad \le \ln(n!) \le \quad (n+1)\ln(n+1) - (n-1) - 2\ln(2)$$

or, writing $1 = \ln(e)$ to get only logarithms,

example: $n = n \cdot 1$
$= n \ln(e)$
$= \ln(e^n)$

$$\ln(n^n) - \ln(e^n)$$

$$= \ln\left(\left(\frac{n}{e}\right)^n\right)$$

$$\cancel{\ln}(n!) \approx \cancel{\ln}\left(\left(\frac{n}{e}\right)^n\right)$$

Stirling formula 2

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or, writing $1 = \ln(e)$ to get only logarithms,

$$\ln\left(\frac{n}{e}\right)^n \quad \leq \ln(n!) \leq \quad \ln\frac{1}{4}(n+1)^2\left(\frac{n+1}{e}\right)^{n-1}$$

Stirling formula 2

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$$\left(\frac{n}{e}\right)^n \quad \le n! \le \quad \frac{e}{4}(n+1)\left(\frac{n+1}{e}\right)^n$$

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or, writing $1 = \ln(e)$ to get only logarithms,

$$\left(\frac{n}{e}\right)^n \quad \leq n! \leq \quad \frac{e}{4}(n+1)\left(\frac{n+1}{e}\right)^n \approx \frac{e^2}{4}(n+1)\left(\frac{n}{e}\right)^n$$

$$\sum_{n=1}^{N} 0{,}1 = \frac{N}{10} \to +\infty. \quad \text{if } N \to \infty.$$

Stirling formula 2

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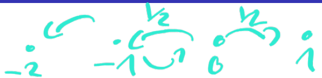
more precise is **Stirling's formula** (Student project!!) → Clara & Melanie

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$e^{i\pi} = -1$



What can be said about this formula? Using Stirling,

$$\mathbb{P}(S_{2n}=0) = \left(\tfrac{1}{2}\right)^{2n}\binom{2n}{n} = \left(\tfrac{1}{2}\right)^{2n}\frac{(2n)!}{n!.n!} \approx \frac{1}{\sqrt{\pi n}}$$

n out of 2n choices.

$$(+1, -1, +1, +1, +1, -1, -1, -1)$$

(2n) places, n times +1

n times −1

$$\mathbb{P}(S_{2n}=0) \approx \frac{1}{\sqrt{\pi \cdot n}}$$

Back to random walk

2dim. $\mathbb{P}(S_{4n}=0) \approx \frac{1}{\pi n}$

What can be said about this formula? Using Stirling,

3dim. $\mathbb{P}(S_{6n}=0) \approx \frac{1}{(\pi n)^{3/2}}$

$$\mathbb{P}(S_{2n}=0) = \left(\tfrac{1}{2}\right)^{2n} \binom{2n}{n} = \left(\tfrac{1}{2}\right)^{2n} \frac{(2n)!}{n!.n!} \approx \frac{1}{\sqrt{\pi n}}$$

Theorem We have with probablity 1 that $S_{2n}=0$ infinitely often if, and only if,

$$\sum_n \underbrace{\mathbb{P}(S_{2n}=0)}_{\approx \frac{1}{\sqrt{n}}} = +\infty$$

We will prove this theorem below.

$$\sum_{n=1}^{N} \frac{1}{\sqrt{n}} \approx \int_1^N \frac{dx}{\sqrt{x}} = \Big[2\sqrt{x}\Big]_{x=1}^{x=N} \approx 2\sqrt{N} \xrightarrow{N\to\infty} +\infty$$

$\frac{1}{n}$ $\frac{1}{n^{3/2}}$ $\frac{1}{x}$ $\frac{1}{x^{3/2}}$ $\ln(x)$ $\frac{1}{\sqrt{x}}$

Back to random walk

What can be said about this formula? Using Stirling,

$$\mathbb{P}(S_{2n}=0) = \left(\tfrac{1}{2}\right)^{2n}\binom{2n}{n} = \left(\tfrac{1}{2}\right)^{2n}\frac{(2n)!}{n!.n!} \approx \frac{1}{\sqrt{\pi n}}$$

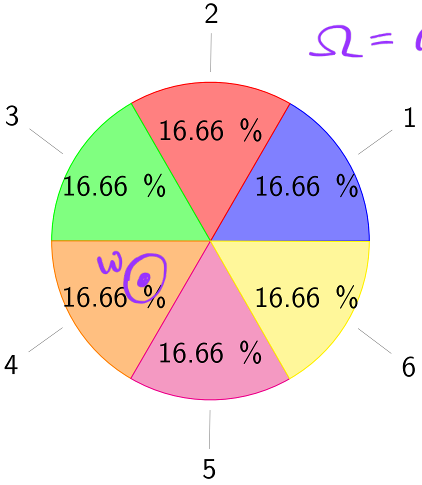
Theorem We have with probablity 1 that $S_{2n}=0$ infinitely often if, and only if,

$$\sum_n \mathbb{P}(S_{2n}=0) = +\infty$$

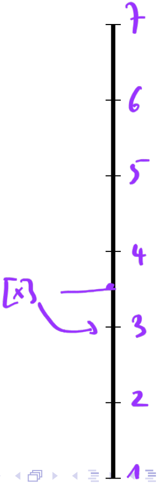
We will prove this theorem below.
Corollary: random walks in $\mathbb{Z}$ and $\mathbb{Z}^2$ return infinitely often at the origin, whereas in higher dimensions, this does not happen! (explaining this is a small student project !!)

An **event** is a set of outcomes of an observation of a random phenomenon. If you throw a die, $A = \{1\}$ is an event. $B = \{1, 2\}$ is another one. If the die is fair, A occurs with probability $1/6$, B with $2/6$.

$\Omega = $ disc.

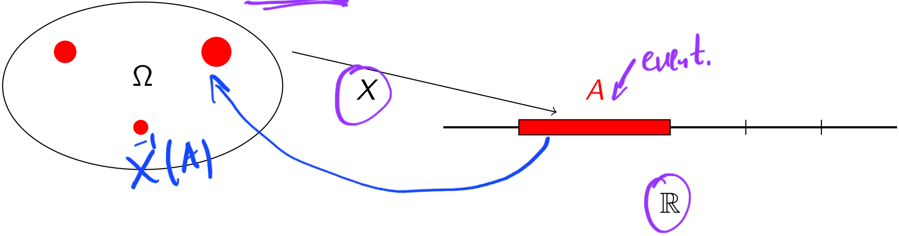


other "dice"



Probablity theory in a nutshell 2

What counts is the **observed** result, not the probability space !



A map $X : \Omega \to \mathbb{R}$ is called a **random variable**. Now for an event A,

$$\mathbb{P}(X \in A) := \nu(\underbrace{\{\omega \in \Omega : X(\omega) \in A\}}_{X^{-1}(A)})$$

The left is to be read "the probability that A is realised". It is calculated by the right, which is a *probability measure* of the pre-image of A under X.

Probablity theory in a nutshell 3

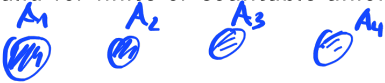
A “probability measure” is a measure for the “size” of a subset of Ω. But what does that mean? Formally, we have “measurable subsets” in Ω. The collection of these sets is called $\mathcal{M}$. For practical pruposes you can safely assume that all sets you ever encounter are measurable (you will need to play with the axion of choice to construct non-measurable sets).

- $\nu : \mathcal{M} \to \underline{[0,1]}$
- If $A_n \in \mathcal{M}$ for all n and if $A_n \cap A_k = \emptyset$ for $n \neq k$, then

no x is in A_k and A_n simultaneously

$$\nu\left(\bigcup_n A_n\right) = \sum_n \nu(A_n)$$

This is valid for finite or countable unions (indexed by natural numbers).



Example 1: Ω finite, $\mathcal{M} = \{$ all subsets of $\Omega\}$. $\nu(A) = \frac{\text{size of } A}{\text{size of } \Omega}$
Example 2: In $\Omega = \underbrace{[0,1]}$, we let $\nu([a,b]) = b - a$. This is the standard "uniform" measure.

Observe that $\nu(\{a\}) \leq \nu([a-\varepsilon, a+\varepsilon]) = 2\varepsilon$ for any $\varepsilon > 0$ so that $\nu(\{a\}) = 0$. *Probabilists will say, the "event" $\{a\}$ happens with probability 0.* Consequently,

$a \leq x \leq b$ $\quad$ $a < x \leq b$ $\quad$ $a < x < b$

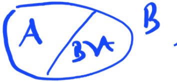
$$\nu([a,b]) = \nu([a,b)\,) = \nu(\,(a,b]) = \nu(\,(a,b)\,)$$

$a \leq x < b$

By the additivity property, $\nu([1/4, 1/2] \cup [2/3, 3/4]) = \frac{1}{2} - \frac{1}{4} + \frac{3}{4} - \frac{2}{3} = \frac{1}{3}$.

(back to general theory): Let $A \subset B$. Then $B = A \cup (B \backslash A)$, and so we have

$$\nu(B) = \nu(A) + \overbrace{\nu(B \backslash A)}^{\geq 0} \geq \nu(A)$$



which means that measures are *monotonic*: "smaller set" implies "smaller measure".

Example 3: Observe that in the previous example

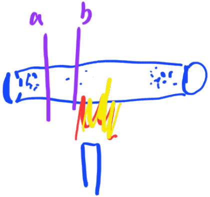
$$\nu([a,b]) = b - a = \int_a^b 1 dx$$

'density'

therefore, is a special case of the situation where we let

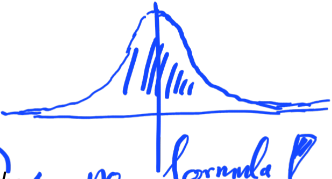
$$\nu([a,b]) = \int_a^b f \, dx$$

density.



for a suitable positive function f, a "density".
The most famous density is the Gaussian density on $\mathbb{R}$. It is given by

$$\gamma(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

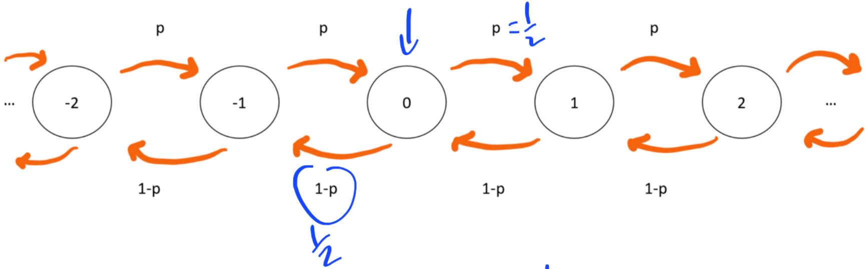


The Gaussian measure of an interval $[a,b] \subset \mathbb{R}$ is

$$\nu_{\text{Gauss}}([a,b]) := \int_a^b \gamma(x) \, dx$$

no formula!

Random walk - next step



Consider the event $S_{2k} = 0$. Its probability is $\nu(A_k)$, where $A_k = \{\omega \in \Omega : S_{2k}(\omega) = 0\}$. Then the probability that $S_{2k} = 0$ for at least one $k > n$ is

$$\nu\Big(\bigcup_{k>n} A_k\Big)$$

$\bigcup A_k$ union of the sets!

and so, the probability that $S_{2k} = 0$ "infinitely often" is cast by

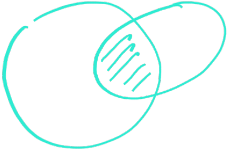
$$\nu\Big(\bigcap_{n=1}^{\infty} \bigcup_{k>n} A_k\Big)$$

for any time n

there is a time $k > n$ so that $S_{2k} = 0$.

Random walk - next step

$\bigcap$ = i$\bigcap$tersection



$S_{2k} = 0$ "infinitely often" is cast by

$$\nu\Big(\bigcap_{n=1}^{\infty} \bigcup_{k>n} A_k\Big) \leq \nu\Big(\bigcup_{k>N} A_k\Big)$$

for all N. This is just monoticity! Now assume $\sum_n \nu(A_n) < \infty$. Then

Sum over tail convergent series

$$\nu\Big(\bigcup_{k>N} A_k\Big) \leq \sum_{k>N} \nu(A_k) \quad \to \quad 0 \qquad (\text{when } N \to \infty)$$

Tail sum $\to 0$.

So "infinitely often" occurs with probability zero if $\sum_n \nu(A_n) < \infty$!!

$$\nu(S_{2n} = 0) \quad \sim \quad \frac{1}{\sqrt{\pi n}}$$

$\| \quad n^{-3/2}$

That does not tell anything in dimension 1,2 but it tells us that for dimesion > 2, random walks come back to the origin with probability 0.

Expectations. Let $X : \Omega \to \{x_1..x_N\}$ be a random variable and $p_n = \mathbb{P}(X = x_n)$. We define the expectation as

head : win 2€
tail : lose 1 €

$$\mathbb{E}(X) = \sum_{n=1}^{N} x_n\, p_n .$$

outcome x_i × prob. of outcome p_i

$\mathbb{E}(X) = +2€ \cdot \frac{1}{2} + (-1)€ \cdot \frac{1}{2} = 1€$

Think about coin flipping, dice throwing and similar experiments to understand that this is, what you actually *expect* when repeating independently the same random experiment.

← Countable

If the random variable $X : \Omega \to \mathbb{N}$ takes values in $\mathbb{N}$ we extend the above definition by

series = limit of partial sums.

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} x_n\, p_n$$