Exercise 1 Prove that $\|x\| := |x|$ is a norm on $\mathbb{R}$. Explicit the open and closed balls $B(2,3)$ and $B[2,3]$. Discuss if the following sets are open or closed (or not). Determine interior, boundary and closure in each case. Determine interior, boundary and closure in each case.

$A = \{0\}$	$B = \{0,1\}$	$C = (0,1)$	$D = [0,1]$
$E = (0,1]$	$F = \mathbb{Z}$	$G = \{1/n : n \geq 1\}$	$H = \{1/n : n \geq 1\} \cup \{0\}$

Exercise 2 Equip $\mathbb{R}^2$ with the Euclidean norm. Discuss if the following sets are open or closed (or not).

$A = \{(0,0)\}$	$B = \{0\} \times [0,1]$	$C = \{0\} \times (0,1)$
$D = (0,1) \times (0,1)$	$E = [0,1] \times (0,1)$	$F = \{(x,2x) : x \geq 0\}$
$G = \{(x,2x) : x > 0\}$	$H = \{(x,1/x) : x > 0\}$	$J = \{(x,1/x) : x \neq 0\} \cup \{(0,0)\}$

Exercise 3 Give proofs or counterexamples to the following assertions.

a) Arbitrary unions of open sets are open.
b) If $A \subset \mathbb{R}^d$ is not open then A must be closed.
c) Arbitrary intersections of closed sets are closed.
d) Arbitrary unions of closed sets are closed.
e) The set $\{x^2 + 3y^4 < 1\}$ is open / closed / bounded ?
f) The set $\{x^2 + 3y^4 \leq 1\}$ is open / closed / bounded ?

Exercise 4

Which of the following subsets of $\mathbb{R}^2$ are vector spaces?

a) $\{(x,y) : x = 0\}$
b) $\{(x,y) : x = 0, y = 1\}$
c) $\{(x,y) : x + y = 0\}$
d) $\{(x,y) : x + y = 1\}$
e) $\{(x,y) : x \cdot y = 0\}$
f) $\{(x,y) : x^2 + y^2 = 1\}$

Which of the following subsets of $E = C([0,1])$ are vector spaces?

a) E itself.
b) The polynomial functions: $\{f \in E : \exists p \in \mathbb{R}[X] : \forall_x : f(x) = p(x)\}$
c) $\{f \in E : \int_0^1 f(x)\,dx = 0\}$
d) $\{f \in E : \forall_x : f(x) \geq 0\}$
e) $\{f \in E : \int_0^1 f(x)^2\,dx \leq 1\}$.

Exercise 5 Let $n \geq 2$ be fixed. For any $n \times n$ matrix $A = (a_{ij})$ define $\|A\| = \sum_{i=1}^{n} \max\{|a_{ik}| : k = 1..n\}$ Show that this defines indeed a norm. Equip $\mathbb{R}^n$ with the $\|.\|_1$-norm. Show that $\|Ax\|_1 \leq \|A\|\|x\|_1$.

Exercise 6 Show that the closure of an open ball $B(x,a)$ in a normed vector space is the closed ball $B[x,r]$.

Exercise 7 Let $N(x, y) := \max(|x - 2y|, |2x - 3y|)$. Show that N is a norm on $\mathbb{R}^2$. Let U be the Euclidean unit ball in $\mathbb{R}^2$. Find $r > 0$ such that $B_N(0, r) \subset U$.

Exercise 8 Let E be a finite dimensional vector space with basis $\{b_1, \dots b_n\}$. For $x = \sum_j \xi_j b_j$, let $\|x\|_\infty := \max\{|\xi_1|, .., |\xi_n|\}$. Prove that this defines a norm on E. Now let $\|x\|_1 := \sum_j |\xi_j|$. Prove that this defines a norm on E. Can you find a sequence (x_n) that converges in $\|.\|_\infty$ but not in norm $\|.\|_1$? Or a sequence that converges in $\|.\|_1$ but not in norm $\|.\|_\infty$?

Exercise 9 Let $E = C([0, 1])$ and $\|f\|_\infty := \max\{|f(x)| : x \in [0, 1]\}$. Prove that this defines a norm on E. Now let $\|f\|_1 := \int_0^1 |f(x)|dx$. Prove that this defines a norm on E. Can you find a sequence (f_n) that converges in $\|.\|_\infty$ but not in norm $\|.\|_1$? Or a sequence that converges in $\|.\|_1$ but not in norm $\|.\|_\infty$?

Exercise 10 Let $E = C([0, 1])$. Prove that $\langle f, g\rangle := \int_0^1 f(t)\overline{g(t)}\, dt$ defines a scalar product on E.

Exercise 11 Let $\langle x, y\rangle_2$ be the euclidean scalar product on $\mathbb{R}^n$. Establish a criterion on a $n \times n$ matrix A to guarantee that $[x, y] := \langle Ax, Ay\rangle$ defines a scalar product on $\mathbb{R}^n$.

Exercise 12 Discuss continuity (or not) of the following functions on $\mathbb{R}^2$.

$$f_1(x,y) = \begin{cases} (x^2 + xy + y^2)\sin(\frac{1}{x^2+y^2}) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(x,y) = \begin{cases} \frac{x^2y+2x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

$$f_3(x,y) = \begin{cases} \frac{y\sin(x)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

$$f_4(x,y) = \begin{cases} \frac{\sin(f(x))-\sin(f(y))}{x-y} & \text{if } x \neq y \\ f'(x)\cos(f(y)) & \text{otherwise} \end{cases} \qquad \text{where } f \text{ is of class } C^1$$

$$f_5(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

$$f_6(x,y) = \begin{cases} \frac{\sin^2(x)(e^y-1)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

$$f_7(x,y) = \begin{cases} \frac{|xy|^\alpha}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } \alpha > 0$$

$$f_8(x,y) = \begin{cases} \frac{x^4y^2+x^3y^3}{x^6+2y^6} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

$$f_9(x,y) = \begin{cases} \frac{x^3-y^4}{\sqrt{x^2+y^4+1}-1} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$