Exercise 1 Show that $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = \begin{pmatrix} \sin(x) + y^2 + yz \\ x^2 + y^2 + 2z \\ y^3 - z^3 \end{pmatrix}$ is injective in a neighbourhood of $(0, 1, 1)$. Determine the Jacobian of its inverse function at $f(0, 1, 1)$.

Exercise 2 Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}^n$ be of class C^1 and $N(x)$ any norm on $\mathbb{R}^n$. Assume that $\det J_f(x) \neq 0$ on Ω. Show that $g(x) = N(f(x))$ cannot have a local maximum inside Ω. Hint: argue by contradiction. Suppose g has a local maximum at $x_0 \in \Omega$. Now use the local inversion theorem at x_0 applied to f to find a contradiction: carefully read the theorem and think of your topology lectures in $\mathbb{R}^n$!

Exercise 3 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function, $a \in \mathbb{R}^n$ and assume that $0 \neq v \in \ker(J_f(a))$. Show that f is not invertible in any neighbourhood of a.

Exercise 4 Let $f(x, y) = \arctan(x + y) - \sinh(y - x)$. Show that there exists some $\varepsilon > 0$ and a function $g : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that $g(0) = 0$ and $f(x, g(x)) = 0$ for all $|x| < \varepsilon$. Calculate $g'(0)$.

Exercise 5 Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x + z > 0\}$ and $f : \Omega \to \mathbb{R}$ given by $f(x, y, z) = y + z + 2\ln(x + z)$.

a) Show that there exists a function g defined in a neighbourhood U of $(0, -1) \in \mathbb{R}^2$ taking values in a neighbourhood V of $1 \in \mathbb{R}$ such that

$$f(x, y, g(x, y)) = 0 \qquad \forall (x, y) \in U$$

b) Show that $g_x + g_y = -1$ on U.

Exercise 6 Show that in a neighbourhood U of $(0, -1)$ exists a function $g : U \to \mathbb{R}$ such that

$$y^2 + \cos(x) + g(x, y)^2 \cosh(xg(x, y)) = 2$$

Provide the Taylor polynomial of degree 1 of this function at the development point $(0, -1)$.

Exercise 7 Show that the equation $y^2 \sinh(x)(3x + z^2)e^{y^2} - \cos(x)\cos(y)\cos(z) = 4\pi^2 - 1$ has a solution of the form $(g(y, z), y, z)$ in a whole neighbourhood of $(0, 0, 2\pi)$. Calculate the gradient of g.

Exercise 8 Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $F(x, y, z) = \begin{pmatrix} \sin(xz) + yz^2 - e^{xy} + 1 \\ x^2yz + y + \cos(yz) - 1 \end{pmatrix}$. Show that in some neighbourhood U of $-1 \in \mathbb{R}$ are defined two functions g, h of class C^1 such that

$$g(-1) = h(-1) = 0 \qquad \text{and} \qquad f(g(z), h(z), z) = 0 \qquad \forall z \in U$$

Calculate $g'(-1) + h'(-1)$.

Exercise 9 Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $F(x,y,u,v) = \begin{pmatrix} x^2 + y^2 + u\cos(x) + \arctan(v) \\ xy + \cos(xv) + e^{u^2} + (x-1)v - 1 \end{pmatrix}$. Show that two neighbourhoods U, V of $(0,0)$ in $\mathbb{R}^2$ exist and a function $g : U \to V$ such that, for all $(x,y) \in U$, one has $F(x,y,g(x,y)) = (0,0)$. Is g invertible in a neighbourhood of $(0,0)$?

Exercise 10 Consider the non-linear sysytem of equations $\{xu + yvu^2 = 2, \quad xu^3 + y^2v^4 = 2\}$. Show that close to $(1,1,1,1)$ one can resolve x, y as functions of u, v.

Exercise 11 Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = x^3 + 4y^2 + 8xz^2 - 3z^3y$ Show that in a neighbourhood of $(0,1,1)$, the equation $f(x,y,z) = 1$ admits a solution of the form $(x,y,g(x,y))$. Then show that in a neighbourhood of $(0,1)$, the equation $g(x,y) = 1$ has a solution of the form $(t, h(t))$. Calculate $h'(0)$.

Additional exercises (hors programme)

Exercise 12

a) Let $F(x,y) = x^2y^2$. Solve the differential equation $F_x(x,y(x)) + F_y(x,y(x))y'(x) = 0$.
b) Now solve $\cos(x)y(x) + \sin(x)\cos(x) + (1 + \sin(x))y'(x)$ (hind: find F first).
c) Consider $y^2 - 3xy - 2x^2 + (xy - x^2)y' = 0$. To solve it, first find an "integrating factor" $\mu(x)$ that allows to determine F.

Exercise 13 Evaluate the given scalar line integral.

a) $\int_\Gamma y\,ds$, where Γ is the curve parameterised by $\gamma(t) = (3\cos(t), 3\sin(t))$ for $0 \le t \le \pi/2$.
b) $\int_\Gamma xy\,ds$, where Γ is the line segment between the points $(3,2)$ and $(6,6)$.
c) $\int_\Gamma (x^2 + y^2)\,ds$, where Γ is the curve parameterised by $(e^\theta\cos(\theta), e^\theta\sin(\theta))$ for $0 \le \theta \le \pi$.

(answers: 9, 95, and $\frac{e^{2\pi}-1}{\sqrt{2}}$).

Exercise 14 Evaluate the given vector line integral.

a) $\int_\Gamma (y,1) \cdot ds$, where Γ is the curve $\gamma(t) = (t^3 - t, t^2)$ from the point $(0,0)$ to the point $(6,4)$.
b) $\int_\Gamma (y,-x) \cdot ds$, where Γ is the portion of the curve $y = \frac{1}{x}$ from the point $(1,1)$ to the point $(2, 1/2)$.

(answers: $\frac{308}{15}$ and $2\ln(2)$).

Exercise 15 Evaluate the given scalar surface integral. $\int_S 6xy\,dS$ where S is the part of the plane $x + y + z = 1$ where $0 \le x \le 1$ and $0 \le y \le 2$. (answer: $6\sqrt{3}$)

Exercise 16 The upper half-sphere S in $\mathbb{R}^3$ is parameterised by

$$\phi(s,t) = (\cos(s)\cos(t), \sin(s)\cos(t), \sin(t))$$

where $(s,t) \in [0,2\pi] \times [0,\pi/2]$. Convince yourself that the right surface integral in this case is $\int_S f dS = \int_0^{2\pi}\int_0^{\pi/2} f(\phi(s,t))\|\phi_s \wedge \phi_t\|_2\,dt\,ds$. Now calculate $\int_S z dS$.