CONTROLLABILITY AND OBSERVABILITY FOR NON-AUTONOMOUS EVOLUTION EQUATIONS: THE AVERAGED HAUTUS TEST

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ABSTRACT. We consider the observability problem for non-autonomous evolution systems (i.e., the operators governing the system depend on time). We introduce an averaged Hautus condition and prove that for skew-adjoint operators it characterizes exact observability. Next, we extend this to more general class of operators under a growth condition on the associated evolution family. We give an application to the Schrödinger equation with time dependent potential and the damped wave equation with a time dependent damping coefficient.

1. INTRODUCTION

In this article we consider observation properties of first order non-autonomous evolution equations of the following form:

$$\textbf{(A,C)} \qquad \begin{cases} x'(t) + A(t)x(t) &= 0 \quad t \geq 0 \\ x(0) &= x_0 \\ y(t) &= C(t)x(t). \end{cases}$$

To be precise, let $\mathcal{T} > 0$ and assume that for $t \in [0, \mathcal{T}]$, the operator $A(t)$ generates a strongly continuous contraction semigroup $(e^{-sA(t)})_{s\geq 0}$ on the Hilbert space H. We suppose further that there exists a densely and continuously embedded subspace $\mathscr{D} \hookrightarrow H$ such that that for all $t \in [0, \mathcal{T}]$, $\mathscr{D}(A(t)) = \mathscr{D}$ and that $t \mapsto A(t)v$ is continuously differentiable in H for every $v \in \mathscr{D}$. These assumptions are sufficient to guarantee the existence of an evolution family $(U(t,s))_{0\leq s\leq t}$ such that the Cauchy problem $x'(t) = A(t)x(t)$, $x(s) = x_0$ (for $0 \leq s \leq t$) admits a unique solution given by $x(t) = U(t,s)x_0$ (we refer to section 2 below for details). Next, we consider for each $t \in [0, \mathcal{T}]$, a generally unbounded, closed operator $C(t) : H \to Y$, whose restriction to $\mathscr{D}$ is bounded (with respect to the stronger norm $\|.\|_{\mathscr{D}}$). Here Y is another Hilbert space. Since for initial data $x_0 \in \mathscr{D}$, the solution x to **(A,C)** will satisfy $x(t) \in \mathscr{D}$ for each $t \geq 0$, the observation $y(t)$ is well defined for all $t \geq 0$.

The question we discuss is that of exact or final-time observability. Observability consists of unique determination or recovery of the initial (or final) time state under the knowledge of the observed solution $y(\cdot)$. In an autonomous setting this question has attracted attention for several decades and there is rich literature on the subject. In the non-autonomous setting the probably best known result is a result of Silverman and Meadows [28], an extension of the well-known Kalman rank condition, that characterizes exact observability and controllability in the case where $A(t)$ and $C(t)$ are matrices. Their arguments have, in turn, been adapted to certain infinite dimensional settings, see for example [1, 2, 3]. Since this requires high regularity assumptions on the operator function $t \mapsto A(t)$ we shall not follow this way.

Instead, our starting point is the following reformulation of the Kalman test in the autonomous setting, going back to Hautus [13]:

$$\|Cx\|^2 + \|(\lambda I - A)x\|^2 \geq \kappa\|x\|^2. \tag{1.1}$$

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In Russell and Weiss [27] this condition was suggested as a replacement for the Kalman test for operators on infinite dimensional spaces, since its formulation is free from rank arguments. The authors actually conjectured that (1.1) would characterize exact observability in infinite-dimensional Hilbert spaces, showing the validity of their conjecture for bounded and invertible operators A. Jacob and Zwart [14] then proved the named "Hautus conjecture" for diagonal semigroup generator on a Riesz basis if the output space Y is finite dimensional. Even if the general conjecture was later proved to be wrong*, there exist other formulations of the Hautus (or spectral) condition which imply exact observability. The most prominent case is when A generates a unitary group. We refer to [20, 30] for early results with bounded observations, and [5, 22] for successive extensions. These have subsequently been generalized (see [16]) to groups with certain growth bounds (see also [29] for more information and references on this subject).

In this paper we will investigate possibilities to extend the Hautus condition (1.1) to a non-autonomous, infinite-dimensional setting. A major difficulty when dealing with non-autonomous equations is that the two-parameter evolution family $U(t,s)$ mentioned above has no simple representation formula in terms of the operators $A(t)$, or their resolvents or the associated (semi)groups.

Nevertheless, it is tempting to hope for a transfer of observability or controllability concepts for $(\mathbf{A},\mathbf{C})$ from properties of each system

$$(A_s, C_s) \qquad \begin{cases} x'(t) + A(s)x(t) &= 0 \quad t \in [0,\mathcal{T}] \\ x(0) &= x_0 \\ y(t) &= C(s)x(t). \end{cases}$$

where the parameter s runs through the interval $[0,\mathcal{T}]$. This is, however, hopeless in general: indeed, consider the simple case of 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad C(t)\begin{pmatrix} x \\ y \end{pmatrix} = \cos(t)x - \sin(t)y.$$

It is an easy exercise to see that $C(t)\exp(-tA)(0,1)^t = 0$ for all $t > 0$ and so all observability concepts fail miserably, despite the fact that for any fixed $s \geq 0$ the systems (A_s, C_s) is (exactly) observable, as the Kalman test shows. On the other hand, in [12, Theorem 2.4] an example of a non-autonomous wave equation with an admissible time-dependent observation is constructed where the non-autonomous system $(\mathbf{A},\mathbf{C})$ is exactly observable, but *none* of the equations (A_s, C_s) are†.

In view of these examples, it is remarkable that an "integrated" or "averaged" version of the Hautus condition (1.1) does allow to conclude observability in certain situations. We introduce in Section 3 the following averaged Hautus conditions

$$\|x\|^2 \leq m^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|C(s)e^{\lambda s}x\big\|^2\,\mathrm{d}s\Big) + M^2\Big(\tfrac{1}{\tau}\int_0^\tau e^{\mathrm{Re}\lambda.s}\big\|(\lambda + A(s))x\big\|\,\mathrm{d}s\Big)^2$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathscr{D}$, or

$$\|x\|^2 \leq m^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|C(s)x\big\|^2\,\mathrm{d}s\Big) + M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|(i\xi + A(s))x\big\|^2\,\mathrm{d}s\Big)$$

for all $\xi \in \mathbb{R}$ and $x \in \mathscr{D}$. These inequalities do coincide with the usual infinite-dimensional Hautus conditions if the operators A and C are independent of t. We prove that averaged Hautus conditions are necessary for exact observability and are sufficient when the operators $A(t)$ are skew-adjoint. This result is then refined in Section 4 to deal with invertible evolution families (not necessarily unitary) under certain growth constraint, thereby generalizing former work of Jacob and Zwart [16].

In a last section we apply our results to the Schrödinger equation with time dependent potential and to the damped wave-equation with time-dependent damping.

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*see [15], whose example is due to a "pathological" construction on conditional bases

†Since exact observation of autonomous equations is preserved by small bounded perturbations of the generator or of the observer, we see that one should not perceive the non-autonomous problem $(\mathbf{A},\mathbf{C})$ as perturbation of autonomous ones.

Finally, we mention that other approaches on observability (or controllability) of parabolic equations with time dependent coefficients exist (see [2, 8] and references therein). The results there are based on Carleman estimates and are very different from our approach.

2. Preliminary results

Recall that we suppose $A(t) : \mathscr{D} \to H$ to have a fixed domain, that $t \mapsto A(t)v$ is continuously differentiable in H for every $v \in \mathscr{D}$ and each semigroup $e^{-sA(t)}$ is a contraction on H. By [24, Sections 5.3 and 5.4] there exists a unique evolution family $(U(t,s))_{0\le s\le t}$ on H generated by $A(t)_{0\le t\le \mathcal{T}}$. This evolution family satisfies the following properties.

(1) $\|U(t,s)\| \le Me^{-\omega(t-s)}$ for some $\omega \in \mathbb{R}$
(2) For all $v \in \mathscr{D}$, $\frac{\partial^+}{\partial t}U(t,s)v|_{t=s} = -A(s)v, \quad \frac{\partial^+}{\partial t}U(t,s)v = -A(t)U(t,s)v$.
(3) For all $v \in \mathscr{D}$, $\frac{\partial}{\partial s}U(t,s)v = U(t,s)A(s)v$.
(4) $U(t,s)\mathscr{D} \subseteq \mathscr{D}$
(5) For all $v \in \mathscr{D}$, $(s,t) \mapsto U(t,s)v$ is continuous in $\mathscr{D}$ for $0 \le s \le t \le \mathcal{T}$.

For every $v \in \mathscr{D}$ and $0 \le s < \mathcal{T}$ the evolution equation

$$\left\{\begin{array}{lcl} \frac{d}{\mathrm{d}t}\eta(t) + A(t)\eta(t) & = & 0 \\ \eta(s) & = & v \end{array}\right. \tag{2.1}$$

has a unique solution. This solution is given by $\eta(t) = U(t,s)v$. Similarly, for $f \in L^1(0,\mathcal{T};H)$, the non-homogeneous problem

$$\left\{\begin{array}{lcl} \frac{d}{\mathrm{d}t}\eta(t) + A(t)\eta(t) & = & f(t) \\ \eta(s) & = & v \in H. \end{array}\right. \tag{2.2}$$

has then a mild solution given by

$$\eta(t) = U(t,s)v + \int_s^t U(t,r)f(r)\,\mathrm{d}r, \tag{2.3}$$

see e.g. [24, p.146]. If, in addition to the standing assumptions, $f \in C^1([s,\mathcal{T}];H)$ then (2.2) has a unique classical solution which coincides with the mild solution, see for example [24, Theorem 5.2, p.146]. We associate with $(\mathbf{A},\mathbf{C})$ the operator

$$(\Psi_{s,\mathcal{T}}x)(t) = \left\{\begin{array}{ll} C(t)U(t,s)x & t \in [s,\mathcal{T}] \\ 0 & t > \mathcal{T} \end{array}\right.$$

and define the following notions:

Definition 2.1 (admissible observations)**.** *Let $(C(t))_{t\in[0,\mathcal{T}]}$ be a family of bounded operators in $\mathscr{L}(\mathscr{D},Y)$, where Y is some Hilbert space. We say that $(C(t))_t$ are admissible observations for $(A(t))_{t\in[0,\mathcal{T}]}$ if there exists a constant $M_{\mathcal{T}} > 0$ such that*

$$\int_s^{\mathcal{T}} \left\|C(t)U(t,s)x\right\|^2 \mathrm{d}t \le M_{\mathcal{T}}^2\|x\|^2 \quad \forall x \in \mathscr{D}, s \in [0,\mathcal{T}].$$

(one can also consider a weaker admissibility notion by requiring the above inequality for $s = 0$, only). For a single operator $C(t_0)$ such that

$$\int_0^{\mathcal{T}} \left\|C(t_0)U(t,s)x\right\|^2 \mathrm{d}t \le M_{\mathcal{T}}\|x\|^2 \quad \forall x \in \mathscr{D}$$

we say that $C(t_0)$ is admissible for $(A(t))_{t\in[0,\mathcal{T}]}$.

For admissible observations, $\Psi_{s,\mathcal{T}}$ extends to a bounded operator from H to $L_2(s,\mathcal{T};Y)$ which we denote again by $\Psi_{s,\mathcal{T}}$.

In this definition the norm inside the integral is taken in Y and the norm of x is taken in H. We always use the same notation $\|\cdot\|$ for both, the difference will be clear from the context.

Definition 2.2. *Suppose that $(C(t))_t$ is an admissible observation for $(A(t))_t$. We say that the system $(\mathbf{A},\mathbf{C})$ is*

a) exactly averaged observable in time τ *if the map* $\Psi_{s,\tau}$ *is bounded from below in the sense that there exists a constant* $\kappa_\tau > 0$ *such that for all* $x \in \mathscr{D}$

$$\int_0^\tau \left\|C(t)U(t,0)x\right\|^2 \mathrm{d}t \geq \kappa_\tau \|x\|^2.$$

For a given $t_0 \in [0,\tau]$*, the system* $(\mathbf{A}, C(t_0))$ *is exactly observable at time* τ *if*

$$\int_0^\tau \left\|C(t_0)U(t,0)x\right\|^2 \mathrm{d}t \geq \kappa_\tau \|x\|^2.$$

b) final-time averaged observable in time τ *if there exists a constant* $\kappa_\tau > 0$ *such that*

$$\int_0^\tau \left\|C(t)U(t,0)x\right\|^2 \mathrm{d}t \geq \kappa_\tau \|U(\tau,0)x\|^2 \quad \forall x \in \mathscr{D}.$$

As above we define final observability for the simple operator $C(t_0)$ *for some* t_0 *as*

$$\int_0^\tau \left\|C(t_0)U(t,0)x\right\|^2 \mathrm{d}t \geq \kappa_\tau \|U(\tau,0)x\|^2.$$

c) approximately averaged observable in time τ *if* $\ker \Psi_{s,\tau} = \{0\}$ *for all* $0 \leq s < \tau$*. Again we define approximate observability for a single operator* $C(t_0)$ *if* $(A, C(t_0))$ *is approximate observable in average as above.*

In order to justify the use of the term "averaged" in the previous notions of observability, we note that it might be possible that $(A, C(t_0))$ is not exactly (or final or approximately) observable for some $C(t_0)$ or even for all $t_0 \in J$ for some subset J of $[0,\tau]$ but $(\mathbf{A},\mathbf{C})$ is exactly (or final or approximately) observable in average. In order to see this, we consider the autonomous case $A(t) = A$ and an observation operator C such that the autonomous system is exactly (or null or approximately) observable at time τ_0. Define

$$C(t) = \begin{cases} C, & t \in [0,\tau_0] \\ 0, & t \in (\tau_0, \tau]. \end{cases}$$

Then

$$\int_0^\tau \left\|C(t)e^{-tA}x\right\|^2 \mathrm{d}t \geq \int_0^{\tau_0} \left\|C(t)e^{-tA}x\right\|^2 \mathrm{d}t \geq \kappa_\tau \|x\|^2.$$

Hence the averaged observability property for $(A, C(t))$ at time τ holds but the system $(A, C(t_0))$ is not observable for $t_0 \in (\tau_0, \tau]$ at any time. The same observation is valid for null and approximate average observability.

To avoid confusion, we mention that in [21] the notion of "averaged control" is considered. This however refers to systems of the form

$$x'(t) + A(\omega)x(t) = B(t)u(t)$$

where the operator $A(\omega)$ depends on a random variable $\omega \in (\Omega, \mathbb{P})$. The authors discuss deterministic controls that will steer the ω-depending solution in $\mathbb{P}$-average to some target. Such approach is entirely unrelated to our work.

Along with $(\mathbf{A},\mathbf{C})$ we consider a controlled evolution equation. First, we recall the following: one can construct an extrapolation space H_{-1} and extrapolated operators $A_{-1}(t)$ such that the following diagram commutes

$$\begin{tikzcd} H \arrow[r, "A_{-1}(t)"] & H_{-1}(t) \\ \mathscr{D} \arrow[u, hook, "i"] \arrow[r, "A(t)"] & H \arrow[u, hook, "i"'] \end{tikzcd}$$

One way to realize $H_{-1}(t)$ is to take the completion of H with respect to a resolvent norm $\|(\lambda - A(t))^{-1}x\|_H$ or via its identification with $\mathscr{D}(A(t)^*)'$. For all this we refer to [9, Chapter II.5].

In order to keep the abstract setting simple we will suppose for the rest of this section that $\mathscr{D}(A(t)^*) =: \mathscr{D}^*$ is independent of time as well, and that the respective graph norms are equivalent with constants independent of t. Note that if for all $t \in [0,\mathcal{T}]$, $A(t) = A(0) + R_t$ with a bounded operator on H, then $A(t)^* = A(0)^* + R_t^*$ with domain $\mathscr{D}^* := \mathscr{D}(A(0)^*)$ independent of t.

Let U be another Hilbert space and let $B(t) : U \to H_{-1}$ is bounded for each $t \in [0,\mathcal{T}]$. We consider in H_{-1} the evolution equation

$$\text{(A,B)} \qquad \left\{ \begin{array}{lcll} x'(t) + A(t)x(t) & = & B(t)u(t) & t \in [0,\mathcal{T}] \\ x(s) & = & 0. & \end{array} \right.$$

Since the mild solution is of the form (2.3), we have for $0 \le \tau \le \mathcal{T}$ the naturally associated operator

$$\Phi_{s,\tau} u = \int_s^\tau U(\tau, r)B(r)u(r)\,\mathrm{d}r \qquad (\tau \le \tau) \tag{2.4}$$

to (A,B).

Definition 2.3 (admissible controls)**.** *Let $(B(t))_{t\in[0,\mathcal{T}]}$ be a family of bounded operators in $\mathscr{L}(U; H_{-1})$. We say that $(B(t))_t$ are admissible controls for $(A(t))_{t\in[0,\mathcal{T}]}$ if there exists a constant $M_\mathcal{T} > 0$ such that the solution x to (A,B) satisfies $x(t) \in H$ and for all $s \in [0,\mathcal{T})$*

$$\Big\| \int_s^\mathcal{T} U(\mathcal{T}, r)B(r)u(r)\,\mathrm{d}r \Big\|^2 \le M_\mathcal{T}^2 \|u\|^2_{L_2(s,\mathcal{T};U)}$$

for all $u \in \mathscr{D}(0,\mathcal{T}; U)$ (one can also consider a weaker admissibility notions by requiring the above inequality for $s = 0$, only).

Let us consider the retrograde final-value problem

$$\left\{ \begin{array}{lcl} z'(t) - A(t)^* z(t) & = & 0 \\ z(\mathcal{T}) & = & z_\mathcal{T}. \end{array} \right. \tag{2.5}$$

Observe that for $x \in \mathscr{D}$ and $x^* \in \mathscr{D}^*$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle x, U(\mathcal{T},t)^* x^*\rangle = \tfrac{\mathrm{d}}{\mathrm{d}t}\langle U(\mathcal{T},t)x, x^*\rangle = -\langle U(\mathcal{T},t)A(t)x, x^*\rangle = \langle x, -A(t)^* U(\mathcal{T},t)^* x^*\rangle$$

so that $z(t) = U(\mathcal{T},t)^* z_\mathcal{T}$ solves the retrograde equation (2.5) on $[s,\mathcal{T}]$ for all $0 \le s < \mathcal{T}$.

Lemma 2.4. *The family $(B(t))_{t\in[0,\mathcal{T}]}$ are admissible controls for $(A(t))_{t\in[0,\mathcal{T}]}$ if and only if the family $(B(t)^*)_{t\in[0,\mathcal{T}]}$ are admissible observations for the retrograde equation (2.5).*

Proof. The following calculation is standard.

$$\begin{aligned} \sup_{\|u\|_2 \le 1} \Big\| \int_s^\mathcal{T} U(\mathcal{T},r)B(r)u(r)\,\mathrm{d}r \Big\| &= \sup_{\|u\|_2\le 1} \sup_{\|x^*\|\le 1} \Big| \int_s^\mathcal{T} \langle U(\mathcal{T},r)B(r)u(r), x^*\rangle\,\mathrm{d}r \Big| \\ &= \sup_{\|x^*\|\le 1} \sup_{\|u\|_2\le 1} \Big| \int_s^\mathcal{T} \langle u(r), B(r)^* U(\mathcal{T},r)^* x^*\rangle\,\mathrm{d}r \Big| \\ &= \sup_{\|x^*\|\le 1} \Big(\int_s^\mathcal{T} \big\| B(r)^* U(\mathcal{T},r)^* x^* \big\|^2\,\mathrm{d}r \Big)^{1/2}. \end{aligned}$$ □

Definition 2.5. *Let $(B(t))_t$ be admissible controls for $(A(t))_{t\in[0,\mathcal{T}]}$. We say that (A,B) is*

a) Exactly averaged controllable in time τ *if for any $s \in [0,\tau)$ and $x_s, x_\tau \in H$, there exist $u \in L^2(s,\tau;U)$ such that the mild solution x satisfies $x(s) = x_s$ and $x(\tau) = x_\tau$.*
This definition coincides with the usual one in the autonomous case, that is, given two states $x_s, x_\tau \in H$ we find a control u such that the solution takes the value x_s at the initial time $t = s$ and the value x_τ at time $t = \tau$.
b) approximately averaged controllable in time τ *if for any $0 \le s < \tau$ and any $x_s, x_\tau \in H$ and $\varepsilon > 0$, there exist $u \in L^2(0,\tau;U)$ such that $x(s) = x_s$ and $\|x(\tau) - x_\tau\| < \varepsilon$.*
c) averaged null controllable in time τ *if for every $0 \le s < \tau$ and every $x_s \in H$, there exist $u \in L^2(s,\tau;U)$ such that the mild solution x satisfies $x(s) = x_s$ and $x(\tau) = 0$.*

Since the mild solution is given by

$$x(t) = U(t,s)x_s + \int_s^t U(t,r)B(r)u(r)\,\mathrm{d}r$$

it is clear that in order to obtain exact averaged controllability it suffices to consider the case where $x(s) = 0$.

Proposition 2.6. *Let $B(t) \in \mathscr{L}(U, H_{-1})$ be a family of admissible controls for $(A(t))_{t\in[0,\mathcal{T}]}$. Then*

a) *Exact averaged controllability for* (A,B) *in time τ is equivalent to exact averaged observability of the retrograde final-value problem (2.5) with the observation operators $C(t) = B(t)^*$.*
b) *Approximate averaged controllability for* (A,B) *in time τ is equivalent to approximate averaged observability of the retrograde final-value problem (2.5) with the observation operators $C(t) = B(t)^*$.*
c) *Averaged null controllability for* (A,B) *in time τ is equivalent to averaged observability of $z(s)$, $0 \le s < \tau$ where z is the solution of the retrograde final-value problem (2.5) with the observation operators $C(t) = B(t)^*$.*

Proof. First note that $(\Phi^*_{s,\tau} z_s)(t) = B(t)^* U^*(\tau,t) z_s$ for $t \in [s,\tau]$. For simplicity we extend this function by zero for other values of t. Exact averaged controllability for (A,B) at τ is equivalent to $\mathrm{range}(\Phi_{s,\tau}) = H$ for all s. Since these operators are bounded, the latter property is equivalent to the fact that their adjoints $\Phi^*_{s,\tau}$ is bounded from below on $L^2(s,\tau;H)$, i.e., there exists $\kappa_{s,\tau}$ such that

$$\int_s^\tau \|B(t)^* U(\tau,t)^* z_s\|^2\,\mathrm{d}t \ge \kappa_{s,\tau}\|z_s\|^2$$

for all $z_s \in \mathscr{D}^*$. Approximate averaged controllability is equivalent to $\mathrm{range}(\Phi_{s,\tau})$ being dense for all $s \in [0,\tau)$, or, equivalently, the respective adjoints being injective. Finally, averaged null controllability in time τ is equivalent to $\mathrm{range}(U(\tau,s)) \subset \mathrm{range}(\Phi_{s,\tau})$ for all $0 \le s < \tau$. Applying [29, Proposition 12.1.2], averaged null controllability is equivalent to

$$\|U(\tau,s)^* z_\tau\|^2 \le \delta^2 \|\Phi^*_{s,\tau} z_\tau\|^2 = \delta^2 \int_s^\tau \left\|B(t)^* U(\tau,t)^* z_\tau\right\|^2 \mathrm{d}t$$

for some constant $\delta > 0$. But $U(\tau,s)^* z_\tau = z(s)$ where $z(\cdot)$ is the solution of the retrograde equation (2.5). □

3. The averaged Hautus test: skew-adjoint operators

Throughout this section, the family of operators $(A(t))_{0\le t\le \mathcal{T}}$ is as before. Let $C(t)_{0\le t\le\mathcal{T}}$ be a family of bounded operators from $\mathscr{D}$ to a Hilbert space Y. In the autonomous case $A(t) = A$ and $C(t) = C$ for all t, it is well known that for admissible C the exact observability of the system (**A,C**) implies the so-called Hautus test (or spectral condition)

$$\|x\|^2 \le m^2\|Cx\|^2 + M^2\|(i\xi + A)x\|^2 \tag{3.1}$$

for some positive constants m and M and all $\xi \in \mathbb{R}$ and $x \in \mathscr{D}(A)$. There is also another condition with $\lambda \in \mathbb{C}$ in place of $i\xi$, see below. In the general non-autonomous situation we introduce an integrated (or averaged) version of this test. We also study, as in the autonomous case, when the averaged Hautus test is necessary and/or sufficient for averaged observability. We start with the "necessary" part.

Proposition 3.1. *Suppose that $(C(t))$ is admissible for $(A(t))$. If the system* (**A,C**) *is exactly averaged observable at time $\tau > 0$ then there exist positive constants m and M such that:*

$$\|x\|^2 \le m^2\Big(\tfrac{1}{\tau}\int_0^\tau \left\|C(s)e^{\lambda s}x\right\|^2 \mathrm{d}s\Big) + M^2\Big(\tfrac{1}{\tau}\int_0^\tau e^{Re\lambda\cdot s}\|(\lambda + A(s))x\|\,\mathrm{d}s\Big)^2 \tag{AH.1}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathscr{D}$,

$$\|x\|^2 \le m^2\Big(\tfrac{1}{\tau}\int_0^\tau \left\|C(s)x\right\|^2 \mathrm{d}s\Big) + M^2\Big(\tfrac{1}{\tau}\int_0^\tau \left\|(i\xi + A(s))x\right\|\mathrm{d}s\Big)^2 \tag{AH.2}$$

for all $\xi \in \mathbb{R}$ *and* $x \in \mathscr{D}$.

Remark 3.2. *If* $C(s) = C$ *for all* s *then* (AH.1) *can be rewritten as:*

$$\text{(AH.3)} \qquad \|x\|^2 \le \tfrac{e^{2\tau Re(\lambda)}-1}{2\tau Re(\lambda)} m^2 \|Cx\|^2 + M^2 \big(\tfrac{1}{\tau} \int_0^\tau e^{Re\lambda\cdot s} \big\|(\lambda{+}A(s))x\big\| \,\mathrm{d}s\big)^2$$

If, in addition, $A(s){=}A$ *then both assertions coincide with the classical Hautus (or spectral) conditions. We call the conditions* (AH.1) *and* (AH.2) *averaged Hautus tests.*

Proof. The proof is similar to the autonomous case. We start from $\frac{d}{\mathrm{d}s}\big(e^{\lambda s}C(t)U(t,s)x\big) = \lambda e^{\lambda s}C(t)U(t,s)x + e^{\lambda s}C(t)U(t,s)A(s)x$ for $x \in \mathscr{D}$. Integrating on $[0,\tau]$ yields

$$e^{\lambda t}C(t)x - C(t)U(t,0)x = \int_0^t C(t)U(t,s)(A(s)+\lambda)xe^{\lambda s}\,\mathrm{d}s.$$

Hence,

$$\int_0^\tau \big\|C(t)U(t,0)x\big\|^2 \,\mathrm{d}t \le 2\int_0^\tau \big\|C(t)xe^{\lambda t}\big\|^2 \,\mathrm{d}t + 2\int_0^\tau \Big\|\int_0^t C(t)U(t,s)(\lambda{+}A(s))xe^{\lambda s}\,\mathrm{d}s\Big\|^2 \,\mathrm{d}t.$$

Since (**A**,**C**) is exactly averaged observable on $[0,\tau]$, the left hand side is bounded below by $m_0\|x\|^2$ for some constant $m_0 > 0$. We estimate the second term on the right hand side

$$\begin{aligned}
I :=& \Big(\int_0^\tau \Big\|\int_0^t C(t)U(t,s)(\lambda{+}A(s))xe^{\lambda s}\,\mathrm{d}s\Big\|^2 \,\mathrm{d}t\Big)^{1/2} \\
=& \sup\Big\{\Big|\int_0^\tau \int_0^t \Big\langle C(t)U(t,s)(\lambda{+}A(s))xe^{\lambda s}, g(t)\Big\rangle_H \,\mathrm{d}s\,\mathrm{d}t\Big| \ : \ \|g\|_{L_2(0,\tau;H)} \le 1\Big\} \\
=& \sup_{\|g\|_{L_2}\le 1} \Big|\int_0^\tau \Big\langle (\lambda{+}A(s))x\,e^{\lambda s}, \int_s^\tau U(t,s)^*C(t)^*g(t)\,\mathrm{d}t\Big\rangle_H \,\mathrm{d}s\Big| \\
\le& \sup_{\|g\|_{L_2}\le 1} \Big(\int_0^\tau \big\|(\lambda{+}A(s))x\,e^{\lambda s}\big\|_H \,\Big\|\int_s^\tau U(t,s)^*C(t)^*g(t)\,\mathrm{d}t\Big\|_H \,\mathrm{d}s\Big).
\end{aligned}$$

By Lemma 2.4 and the admissibility assumption of $(C(t))$, there exists a constant $K_\tau > 0$ such that

$$I \le\ K_\tau \int_0^\tau \big\|(\lambda{+}A(s))xe^{\lambda s}\big\|\,\mathrm{d}s =\ K_\tau \int_0^\tau \big\|(\lambda{+}A(s))x\big\|e^{\mathrm{Re}\lambda.s}\,\mathrm{d}s.$$

and (AH.1) follows. The second assertion is obtained from the first one by taking $\lambda = i\xi$. ☐

Now we study the converse. In the autonomous case, i.e., $A(s) = A$ and $C(t) = C$, it is well known that condition (AH.2) implies the exact observability if the single operator A is skew-adjoint. We extend this result to our more general situation.

Theorem 3.3. *Suppose that* $A(t) \in \mathscr{L}(\mathscr{D}; H)$ *be a family of skew-adjoint operators generating an evolution family* $U(t,s)_{0\le s\le t\le\mathcal{T}}$. *Suppose that the* differences $D_s(t) = A(t) - A(s)$ *of the operators* $A(t)$ *are admissible observations for* $U(t,s)$ *with a constant, say* $N(s)$. *Assume that* $C(t) \in \mathscr{L}(\mathscr{D}; Y)$ *is a family of admissible observation operators and that the second averaged Hautus condition* (AH.2) *holds with positive constants* m *and* M. *Then, if for some* $r > 0$ *and* $\tau \in [0,\mathcal{T}]$

$$\text{(3.2)} \qquad \frac{\tau^2}{2} - \tfrac{(1+r)}{2}\pi^2 M^2 > (1{+}r^{-1})M^2 \int_0^\tau N(s)^2\,\mathrm{d}s$$

then the exact averaged observability estimate

$$\text{(3.3)} \qquad \tfrac{1}{\tau}\int_0^\tau \int_0^\tau \big\|C(s)U(t,0)x\big\|^2 \,\mathrm{d}t\,\mathrm{d}s \ge \tfrac{\kappa_\tau}{m^2}\|x\|^2$$

holds for all $x \in \mathscr{D}$. *In particular, if* $C(s) = C$ *is constant, then the system* (**A**,**C**) *is exactly observable in time* τ, *i.e, for all* $x \in \mathscr{D}$,

$$\int_0^\tau \big\|CU(t,0)x\big\|^2 \,\mathrm{d}t \ \ge\ \tfrac{\kappa_\tau}{m^2}\ \|x\|^2.$$

Remark 3.4. a) *The theorem states that the system is exactly observable at time* $\tau > \tau^*$ *where*

$$\tau^* = \inf\{\tau \in [0,\mathcal{T}] : \frac{\tau^2}{2} - \tfrac{(1+r)}{2}\pi^2 M^2 > (1+r^{-1})M^2 \int_0^\tau N(s)^2 \,\mathrm{d}s \text{ for some } r > 0\}.$$

The observation time $\tau > \pi M$ *is the best known one in the case that* $A(t) = A$ *generates a unitary group. Our condition (3.2) recovers this since in the autonomous case* $N(s) = 0$ *for all* $s > 0$*, so that we may chose* $r > 0$ *as small as we wish.*

b) *If* $A(t) = A + R(t)$ *where* $R(t) \in \mathscr{L}(H)$ *the admissibility of differences* $D_s(t) = R(t) - R(s)$ *is obvious. Hence for* $R \in L^2([0,\mathcal{T}];\mathscr{L}(H))$ *with sufficiently small norm will ensure the applicability of our result.*

c) *If we assume additionally that* $\|C(s) - C(t)\| \le L|t-s|^\alpha$ *for some positive constants* α *and* L *we obtain that for* L *small enough, the system* $(\mathbf{A},\mathbf{C})$ *is exactly averaged observable. Indeed, we have from* (3.3)

$$\begin{aligned}\kappa\|x\|^2 &\le 2\int_0^\tau\int_0^\tau \big\|(C(s)-C(t))U(t,0)x\big\|^2 \,\mathrm{d}s\,\mathrm{d}t + 2\int_0^\tau\int_0^\tau \big\|C(t)U(t,0)x\big\|^2\,\mathrm{d}s\,\mathrm{d}t\\ &\le 2L\int_0^\tau\int_0^\tau |t-s|^{2\alpha}\,\mathrm{d}s\,\mathrm{d}t\|x\|^2 + 2\tau\int_0^\tau \big\|C(t)U(t,0)x\big\|^2\,\mathrm{d}t\\ &= \frac{2L\tau^{2\alpha+2}}{(2\alpha+1)(\alpha+1)}\|x\|^2 + 2\tau\int_0^\tau\big\|C(t)U(t,0)x\big\|^2\,\mathrm{d}t.\end{aligned}$$

d) *We have assumed in the theorem that* $A(t)$ *are skew-adjoint operators in order to have* $U(t,s)$ *is a unitary operator on* H*. The proof of the previous theorem works under the weaker assumption that*

$$K_0\|x\| \le \|U(t,0)x\| \le K_1\|x\|,\ x \in H$$

for some positive constants K_0 *and* K_1*. The statement of the theorem still holds with different a different observation time* τ^* *(depending additionally on* K_0 *and* K_1*).*

Proof of Theorem 3.3. We proceed in a similar way as in the autonomous case. Let $\tau > 0$, $\varphi \in H_0^1(0,\tau)$ and $x \in \mathscr{D}$. For $t, s \in [0,\tau]$, let $h(t) := \varphi(t)U(t,0)x$ and $f(t,s) := h'(t) + A(s)h(t)$. Note that h and $f(.,s)$ can be extended continuously by zero outside $(0,\tau)$ since $\varphi \in H_0^1(0,\tau)$. We write $\widehat{f}(\xi,s)$ for the partial Fourier transform of f with respect to the first variable, and observe that

$$\widehat{f}(\xi,s) = \int_{\mathbb{R}} e^{-it\xi} f(t,s)\,\mathrm{d}t = \int_{\mathbb{R}} e^{-it\xi}h'(t)\,\mathrm{d}t + \int_{\mathbb{R}} e^{-it\xi}A(s)h(t)\,\mathrm{d}t \;=\; i\xi\widehat{h}(\xi) + A(s)\widehat{h}(\xi)$$

where we use the fact that each operator $A(s)$ is closed in order to have $\widehat{A(s)h}(\xi) = A(s)\widehat{h}(\xi)$. We apply (AH.2) with $z_0 = \widehat{h}(\xi)$ to obtain

$$\begin{aligned}\|\widehat{h}(\xi)\|^2 &\le \tfrac{m^2}{\tau}\int_0^\tau \big\|C(s)\widehat{h}(\xi)\big\|^2\,\mathrm{d}s + \tfrac{M^2}{\tau}\int_0^\tau \big\|(i\xi + A(s))\widehat{h}(\xi)\big\|^2\,\mathrm{d}s\\ &= \tfrac{m^2}{\tau}\int_0^\tau \big\|C(s)\widehat{h}(\xi)\big\|^2\,\mathrm{d}s + \tfrac{M^2}{\tau}\int_0^\tau \big\|\widehat{f}(\xi,s)\big\|^2\,\mathrm{d}s.\end{aligned}$$

We integrate over all $\xi \in \mathbb{R}$ and use Plancherel's theorem together with the fact that $C(s)\widehat{h}(\xi) = \widehat{C(s)h(\xi)}$ to deduce

$$\int_0^\tau \big\|h(t)\big\|^2\,\mathrm{d}t \le \tfrac{m^2}{\tau}\int_0^\tau\int_0^\tau \big\|C(s)h(t)\big\|^2\,\mathrm{d}t\,\mathrm{d}s + \tfrac{M^2}{\tau}\int_0^\tau\int_0^\tau \big\|f(t,s)\big\|^2\,\mathrm{d}t\,\mathrm{d}s. \tag{3.4}$$

We estimate the last term on the right hand side as follows

$$\begin{aligned}&\int_0^\tau\int_0^\tau \big\|f(t,s)\big\|^2\,\mathrm{d}t\,\mathrm{d}s\\ =\ &\int_0^\tau\int_0^\tau \big\|h'(t) + A(s)h(t)\big\|^2\,\mathrm{d}t\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\tau \int_0^\tau \big\|\varphi'(t)U(t,0)x - \varphi(t)A(t)U(t,0)x + \varphi(t)A(s)U(t,0)x\big\|^2 \,\mathrm{d}t\,\mathrm{d}s \\
&\le (1{+}r)\tau \int_0^\tau \big\|U(t,0)x\big\|^2 |\varphi'(t)|^2 \,\mathrm{d}t + (1{+}r^{-1}) \int_0^\tau \int_0^\tau \big\|(A(t)-A(s))U(t,0)x\big\|^2 |\varphi(t)|^2 \,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

for all $r > 0$. By skew-adjointness,

$$
\frac{d}{dt}\big\|U(t,s)x\big\|^2 = -2\mathrm{Re}\langle A(t)U(t,s)x, U(t,s)x\rangle = 0 \tag{3.5}
$$

for $x \in \mathscr{D}$ and so $U(t,s)$ is unitary for $0 \le s \le t \le \mathcal{T}$. Since $D_s(t) = A(t) - A(s)$ is admissible for $U(t,s)$ with constant $N(s)$

$$
\int_0^\tau \int_0^\tau \big\|f(t,s)\big\|^2 \,\mathrm{d}t\,\mathrm{d}s \le \big\|x\big\|^2 \left((1{+}r)\tau \int_0^\tau |\varphi'(t)|^2 \,\mathrm{d}t + (1{+}r^{-1})\|\varphi\|_\infty^2 \int_0^\tau N(s)^2 \,\mathrm{d}s \right).
$$

and so (3.4) implies

$$
\begin{aligned}
\|x\|^2 \int_0^\tau \big|\varphi(t)\big|^2 \,\mathrm{d}t \le \tfrac{m^2}{\tau} \int_0^\tau \int_0^\tau \big\|C(s)U(t,0)x\big\|^2 \varphi(t)^2 \,\mathrm{d}t\,\mathrm{d}s \\
+ \big\|x\big\|^2 M^2 \left((1+r) \int_0^\tau |\varphi'(t)|^2 \,\mathrm{d}t + \tfrac{1+r^{-1}}{\tau} \|\varphi\|_\infty^2 \int_0^\tau N(s)^2 \,\mathrm{d}s \right).
\end{aligned}
$$

Observe that this inequality is invariant under scalar multiplications of φ. We may therefore suppose without loss of generality that $\|\varphi\|_\infty = 1$. Then (3.3) holds with

$$
\kappa_\tau = \kappa_\tau(\varphi) = \Big(\int_0^\tau \big|\varphi(t)\big|^2 \,\mathrm{d}t - (1{+}r)M^2 \int_0^\tau \big|\varphi'(t)\big|^2 \,\mathrm{d}t - (1{+}r^{-1})\tfrac{M^2}{\tau} \int_0^\tau N(s)^2 \,\mathrm{d}s \Big).
$$

We want to chose φ such that the constant $\kappa_\tau(\varphi)$ is positive. Taking the normalized first eigenfunction of the Dirichlet Laplacian on $(0,\tau)$, i.e., $\varphi(t) := \sin\big(\frac{t\pi}{\tau}\big)$, we maximize $\kappa(\varphi)$ and obtain, as desired, (3.3) with

$$
\kappa_\tau = \Big(\frac{\tau}{2} - \tfrac{(1+r)\pi^2 M^2}{2\tau} - (1{+}r^{-1})\tfrac{M^2}{\tau} \int_0^\tau N(s)^2 \,\mathrm{d}s \Big).
$$

By our hypotheses (3.2) $\kappa_\tau > 0$ which ensures exact observation. □

4. The averaged Hautus test: a more general class of operators

In this section we extend Theorem 3.3 to a more general class of operators. More precisely, we consider operators $A(t)$ for which the corresponding evolution family $U(t,s)$ is not necessarily an isometry but satisfies an estimate of the form

$$
ke^{\alpha(t-s)}\|x\| \le \|U(t,s)x\| \le Ke^{\beta(t-s)}\|x\|,\ x \in H \tag{4.1}
$$

for some constants k, K, α and β. This question was considered in the autonomous case $A(t) = A$ and $C(t) = C$ by Jacob and Zwart [16]. Note however, even in this autonomous case, the result is very much less precise than in the case of unitary groups. In particular, the minimal time for observability obtained in [16] is $\frac{1}{\beta-\alpha}$. This value becomes large as α and β are close and this is not consistent with the result on unitary groups. The proof in [16] is based on optimal Hardy inequalities. In our general setting of non-autonomous equations, we give a very short proof and obtain a better minimal observability time.

Let us first make a basic remark on evolution families $U(t,s)_{0\le s\le t}$. Given $U(t,s)$ which is exponentially bounded, i.e., $\|U(t,s)x\| \le Ke^{\beta(t-s)}\|x\|$. If in addition each $U(t,s)$ is invertible then writing $V(t) := U(t,0)$ gives

$$
V(t) = U(t,0) = U(t,s)U(s,0) = U(t,s)V(s) \quad \Longleftrightarrow \quad U(t,s) = V(t)V(s)^{-1}.
$$

Then $I = V(t)V(t)^{-1}$ gives $\|x\| \le Ke^{\beta t}\|V(t)^{-1}x\|$ and so $\|V(t)^{-1}x\| \ge \frac{1}{K}e^{-\beta t}\|x\|$ so that

$$
ke^{\alpha(t-s)}\|x\| \le \|U(t,s)x\| \le Ke^{\beta(t-s)}\|x\|. \tag{4.2}
$$

holds for $\alpha = -\beta$ and $k = \frac{1}{K}$. If A is 'shifted', i.e., replaced by $A{+}\omega$, this symmetry $\alpha = -\beta$ will break, and we will therefore use only (4.2) for *some* constants $k, K > 0$ and $\alpha \le \beta$.

Theorem 4.1. *Let $A(t)_{0\le t\le\tau} \in \mathscr{L}(\mathscr{D};H)$ be a family of operators generating an evolution family $U(t,s)$ and let $0<k\le K$ and $\alpha<\beta$ be such that (4.2) holds. We denote by $\omega:=\beta-\alpha$. Suppose that the* differences *$D_s(t)=A(t)-A(s)$ of the operators $A(t)$ are admissible observations for $U(t,s)$ with a constant, say $N(s)$.*

Let $C\in\mathscr{L}(\mathscr{D};Y)$ satisfy the averaged Hautus condition (AH.3). *We let*

$$\tau^* := \sqrt{\tfrac{\pi^3}{1-e^{-2\pi}}}\,\tfrac{4MK}{k}. \tag{4.3}$$

Suppose that

$$\omega < \sqrt{\tfrac{1-e^{-2\pi}}{\pi}}\,\tfrac{k}{4MK} \qquad \text{as well as} \qquad \int_0^{\tau^*} N(s)^2\,\mathrm{d}s < \tfrac{\pi^2}{2}K^2. \tag{4.4}$$

Then the system (**A**,**C**) *is exactly observable in any time $\tau\ge\tau^*$, i.e., for $\tau\ge\tau^*$ we have*

$$\int_0^\tau \big\|CU(t,0)x\big\|^2\,\mathrm{d}t \ge \tfrac{\kappa_\tau}{m^2}\,\big\|x\big\|^2 \quad \forall x\in H$$

for some constant $\kappa_\tau>0$.

Proof. Observe that exact (averaged) observability is invariant under spectral shifts (replacing A by $A+\omega$), which in turn allows to assume $\beta=0$ and $\alpha=-\omega$ for $\omega=\beta-\alpha>0$.

We follow the lines of the proof of Theorem 3.3 until (3.5). Using (4.2) instead of unitary evolution family leads to consider a new function

$$\kappa_\tau(\varphi)=\Big(k^2\int_0^\tau \big|\varphi(t)\big|^2 e^{-2\omega t}\,\mathrm{d}t - 2M^2K^2\int_0^\tau\big|\varphi'(t)\big|^2\,\mathrm{d}t - \tfrac{2M^2}{\tau}\int_0^\tau N(s)^2\,\mathrm{d}s\Big).$$

where, again, we normalize $\|\varphi\|_\infty=1$, and chose $r=1$. Now, it is not clear what is the optimal function φ to be chosen, so we take $\varphi(t)=\sin(\pi/\tau t)$ as in the proof of Theorem 3.3. It follows that $\kappa_\tau(\varphi)>0$ is equivalent to

$$k^2\pi^2\tfrac{1-e^{-2\omega\tau}}{4\omega(\pi^2+\omega^2\tau^2)} - \tfrac{\pi^2}{\tau}M^2K^2 > \tfrac{2M^2}{\tau}\int_0^\tau N(s)^2\,\mathrm{d}s. \tag{4.5}$$

Obviously, the left hand side is bounded from below by $\frac{\pi^2}{\tau}M^2K^2$ if

$$k^2\pi^2\tfrac{1-e^{-2\omega\tau}}{4\omega(\pi^2+\omega^2\tau^2)} \ge 2\times\tfrac{\pi^2}{\tau}M^2K^2$$

or

$$\tau^2\ \tfrac{1-e^{-2\omega\tau}}{\omega\tau(\pi^2+\omega^2\tau^2)} \ge \tfrac{8M^2K^2}{k^2}.$$

By tedious elementary calculus, we see that $f(x):=\frac{1-e^{-2x}}{x(\pi^2+x^2)}$ is strictly decreasing on $\mathbb{R}_+$. Consequently, for any chosen $\lambda>0$, the condition $\omega\tau\le\lambda$ implies $f(\omega\tau)\ge f(\lambda)>0$. This amounts that for $\lambda>0$, we look for τ such that

$$\tfrac{8M^2K^2}{k^2}\le\tau^2 f(\lambda) \qquad\text{and}\qquad \tau<\tfrac{\lambda}{\omega}.$$

This double inequality for τ can only have a solution if

$$\tfrac{8M^2K^2}{k^2} < \tfrac{f(\lambda)\lambda^2}{\omega^2}.$$

Optimizing on $\lambda>0$ suggests the choice $\lambda=\pi$ that gives the spectral height condition

$$\tfrac{8M^2K^2}{k^2} < \tfrac{1-e^{-2\pi}}{2\pi\omega^2} \qquad\text{or}\qquad \omega<\sqrt{\tfrac{1-e^{-2\pi}}{\pi}}\,\tfrac{k}{4MK}$$

that we now decide to *impose* by (4.4). It guarantees that we can find some τ satisfying

$$\tfrac{16\pi^3M^2K^2}{(1-e^{-2\pi})k^2}\le\tau^2<\tfrac{\pi^2}{\omega^2},$$

notably $\tau=\tau^*=\sqrt{\pi^3/(1-e^{-2\pi})}\,\frac{4MK}{k}$, which is (4.3). Then a sufficient condition for $\kappa_\tau>0$ and hence exact observability can be read off (4.5):

$$\int_0^\tau N(s)^2\,\mathrm{d}s < \tfrac{\pi^2}{2}K^2$$

which is the second condition in (4.4). □

5. Applications to the wave and Schrödinger equations with time dependent potentials

In this section we give applications of our results to observability of the Schrödinger and wave equations both with time dependent potentials. We also consider the damped wave equation with time dependent damped term. Before considering these examples we explain the general idea. It is based on a perturbation argument. We perturb a given operator A_0 by time dependent operators $R(t)$. For each fixed t, the Hautus condition carries over from A_0 to $A_0 + R(t)$. Then, by integrating with respect to t we obtain an averaged Hautus test for time dependent family $(A_0 + R(t))$. We note however that the perturbation argument which allows to obtain the Hautus condition for $A_0 + R(t)$, for fixed t, is done at the expense of assuming rather restrictive conditions on the size of $R(t)$. For this reason we do not obtain precise results in our examples and additional investigations must be carried out. The sole advantage of our strategy in this section lies in its simplicity.

Let A_0 be the generator of unitary group on H. We assume that $C : \mathscr{D}(A_0) \to Y$ is an admissible operator and such that the system (A_0, C) is exactly observable at time τ_0. Therefore the Hautus test is satisfied by the operators A_0 and C. Now let $R(t)_{0\le t\le \mathcal{T}}$ be a family of uniformly bounded operators on H. By classical bounded perturbation result (see, e.g., [9, Theorem 9.19]) the operators given by $A(t) = A_0 + R(t)$, $t \in [0, \mathcal{T}]$, generate an evolution family $U(t,s)$ on H. Note that for every $x \in H$

$$e^{-\beta(t-s)}\|x\| \le \|U(t,s)x\| \le e^{\beta(t-s)}\|x\| \tag{5.1}$$

with $\beta = \sup_{t\in[0,\tau]} \|R(t)\|$. Indeed, one has for every $x \in \mathscr{D}(A_0)$, $\mathrm{Re}\langle (A_0 + R(t))x, x\rangle = \mathrm{Re}\langle R(t)x, x\rangle$ and hence

$$-\beta\|x\|^2 \le \mathrm{Re}\langle (A_0 + R(t))x, x\rangle \le \beta\|x\|^2.$$

We apply this with $U(t,s)x$ at the place of x and obtain

$$-\beta\|U(t,s)x\|^2 \le \tfrac{1}{2}\tfrac{\partial}{\partial t}\big\|U(t,s)x\big\|^2 \le \beta\|U(t,s)x\|^2.$$

We integrate and obtain (5.1). Note that if $\mathrm{Re}\langle R(t)x, x\rangle = 0$, then $U(t,s)$ is unitary.

Let now $x \in \mathscr{D}(A_0)$ and $\xi \in \mathbb{R}$. The Hautus test for (A_0, C) gives

$$\begin{aligned} \|x\|^2 &\le m^2\|Cx\|^2 + M^2\|(i\xi + A_0)x\|^2 \\ &\le m^2\|Cx\|^2 + 2M^2\|(i\xi + A_0 + R(s))x\|^2 + 2M^2\|R(s)\|^2\|x\|^2. \end{aligned}$$

Integrating on $[0,\tau]$ with respect to s gives

$$\|x\|^2 \le m^2\|Cx\|^2 + 2M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|(i\xi + A_0 + R(s))x\big\|^2\,\mathrm{d}s\Big) + 2M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|R(s)\big\|^2\,\mathrm{d}s\Big)\|x\|^2.$$

Suppose in addition that there exists $\tau_1 > 0$ and $\mu < 1$ such that for $\tau \ge \tau_1$

$$2M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|R(s)\big\|^2\,\mathrm{d}s\Big) \le \mu. \tag{5.2}$$

Then

$$(1-\mu)\|x\|^2 \le m^2\|Cx\|^2 + 2M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|(i\xi + A_0 + R(s))x\big\|^2\,\mathrm{d}s\Big). \tag{5.3}$$

Note that we could also replace $i\xi$ by $\lambda \in \mathbb{C}$ and obtain the Hautus test (AH.3). Next we assume that C is admissible for the unitary group e^{tA_0} generated by A_0. That is there exists a constant $K_\tau > 0$ such that

$$\int_0^\tau \big\|Ce^{tA_0}x\big\|^2\,\mathrm{d}t \le K_\tau\|x\|^2,\ x \in \mathscr{D}(A_0). \tag{5.4}$$

We prove that C is admissible for $(A_0+R(t))$. In order to do so, we start from Duhamel's formula[‡]

$$U(t,s)x - e^{(t-s)A_0}x = \int_s^t e^{(t-r)A_0}R(r)U(r,s)x\,\mathrm{d}r. \tag{5.5}$$

We use (5.4) so that

$$\begin{aligned}\int_0^\tau \left\|CU(t,s)x\right\|^2 \mathrm{d}t &\le 2\int_0^\tau \left\|Ce^{(t-s)A_0}x\right\|^2 \mathrm{d}t + 2\int_0^\tau \left\|\int_s^t Ce^{(t-r)A_0}R(r)U(r,s)x\,\mathrm{d}r\right\|^2 \mathrm{d}t\\ &\le 2K_\tau\|x\|^2 + 2\tau\int_s^\tau\int_r^\tau \left\|Ce^{(t-r)A_0}R(r)U(r,s)x\right\|^2 \mathrm{d}t\,\mathrm{d}r\\ &\le 2K_\tau\|x\|^2 + 2\tau K_\tau\int_s^\tau \|R(r)U(r,s)x\|^2\,\mathrm{d}r \le\ K'_\tau\|x\|^2,\end{aligned}$$

where we use the fact that the operators $R(r)$ are uniformly bounded and $U(t,s)$ is exponentially bounded.
We have admissibility of C and the averaged Hautus test (5.3). Now we conclude either by Theorem 3.3 or Theorem 4.1 that, as soon as μ in (5.2) is small enough, we have exact observability of the system $(A_0+R(.),C)$ at time $\tau>\tau^*$ for some $\tau^*>0$. Note that (5.2) always holds for τ large enough, if $R(t)=0$ for $t\ge t_0$ for some $t_0>0$.

In the sequel we give two examples that fit in the framework described here.

The Schrödinger equation. Let Ω be a bounded domain of $\mathbb{R}^d$ with a C^2-boundary Γ. Let Γ_0 be an open subset of Γ satisfying the geometric optics conditions in [4]. By [29, Theorems 6.7.2 and 7.5.1] the Schrödinger equation

$$\left\{\begin{array}{lcl} z'(t,x) &=& i\Delta z(t,x) \quad (t,x)\in[0,\tau]\times\Omega\\ z(0,.) &=& z_0\in H^2(\Omega)\cap H_0^1(\Omega)\\ z(t,x) &=& 0 \quad (t,x)\in[0,\tau]\times\Gamma\end{array}\right. \tag{5.6}$$

satisfies the double inequality

$$\kappa_\tau^2\|z_0\|^2_{H_0^1(\Omega)} \le \int_0^\tau\int_{\Gamma_0} |\tfrac{\partial z}{\partial\nu}(t,x)|^2 d\sigma\,\mathrm{d}t \le K_\tau^2\|z_0\|^2_{H_0^1(\Omega)} \tag{5.7}$$

for every $\tau>0$. Let C_0 be the normal derivative $\frac{\partial}{\partial\nu}$ on Γ_0, $Y=L^2(\Gamma_0,d\sigma)$, $H=H_0^1(\Omega)$ and $A_0=\Delta_D$ the Laplacian with Dirichlet boundary conditions. The previous inequality then reads as admissibility (with constant K_τ) and exactly observability (with constant κ_τ) in time τ of the system (A_0,C_0). It follows that (A_0,C_0) satisfies the Hautus condition with $m=\sqrt{2\tau\kappa_\tau}$ and $M=\tau\sqrt{\kappa_\tau M_\tau}$.
Let now $R(t)f=iV(t,.)f$ where $V(t,.)\in W^{1,\infty}(\Omega)$ is a real-valued potential. We assume that

$$2M^2\left(\frac{1}{\tau}\int_0^\tau \|V(t,.)\|^2_{W^{1,\infty}(\Omega)}\,\mathrm{d}t\right) < 1$$

for all τ larger than some τ_1. This means that (5.2) is satisfied and hence the averaged Hautus condition holds for $(i(\Delta_D+V(t)),C_0)$. We then conclude by Theorem 3.3 that the non-autonomous system $(i(\Delta_D+V(t)),C_0)$ is exactly observable in time $\tau>\tau^*$ for τ^* given in Theorem 3.3. In other words, (5.7) is satisfied for the solution of the Schrödinger equation with time dependent potential

$$\left\{\begin{array}{lcl} z'(t,x) &=& i\Delta z(t,x)+iV(t)z(t,x) \quad (t,x)\in[0,\tau]\times\Omega\\ z(0,.) &=& z_0\in H^2(\Omega)\cap H_0^1(\Omega)\\ z(t,x) &=& 0 \quad (t,x)\in[0,\tau]\times\Gamma.\end{array}\right. \tag{5.8}$$

This generalizes results on autonomous perturbations of Δ by $V(t,x)=V(x)$ which can be found in [29, Sections 7.3, 6.7].

[‡]in order to prove this formula one takes the derivative of $f(r):=e^{(t-r)A_0}U(r,s)x$ for $s\le r\le t$ and then integrate from s to t.

The wave equation. Let again Ω be a bounded smooth domain of $\mathbb{R}^d$. We consider the wave equation

$$\left\{\begin{array}{lcl} z''(t,x) & = & \Delta z(t,x) \in [0,\tau]\times\Omega \\ z(0,.) & = & z_0 \in H_0^1(\Omega),\ z'(0,.) = z_1 \in L^2(\Omega) \\ z(t,x) & = & 0 \quad (t,x)\in[0,\tau]\times\Gamma. \end{array}\right. \tag{5.9}$$

Let Γ_0 be an open subset of the boundary Γ. Observability for the wave equation with the Neumann observation operator $C = \frac{\partial}{\partial\nu}_{|\Gamma_0}$ holds under geometrical conditions on Γ_0. One of the well known condition is that for some $x_0 \in \mathbb{R}^d$,

$$\Gamma_0 \supseteq \{x \in \Gamma : (x - x_0)\,.\,\nu(x) > 0\}$$

where $\nu(.)$ is the outward normal vector on the boundary Γ. For $\tau > 2\sup_{x\in\Omega}|x - x_0|$, the wave equation is exactly observable in time τ. That is, there exists a positive constant κ_τ such that

$$\kappa_\tau\Big(\int_\Omega |z_1|^2 + \int_\Omega |\nabla z_0|^2\Big) \le \int_0^\tau \int_{\Gamma_0} |\tfrac{\partial z}{\partial\nu}|^2 d\sigma\, \mathrm{d}t. \tag{5.10}$$

We refer to [4, 19, 17], and [29, Section 7.2] and the references therein.

Let $A_0 = \begin{pmatrix} 0 & I \\ -\Delta_D & 0 \end{pmatrix}$ on $H := H_0^1(\Omega)\times L^2(\Omega)$. It is a standard fact that A_0 generates a unitary group $U(t)_{t\in\mathbb{R}}$ on H. Set $\widetilde{C}(f,g) := (\frac{\partial f}{\partial\nu}_{|\Gamma_0}, 0)$. Then the energy estimate (5.10) is precisely the observability inequality

$$\kappa_\tau\|(z_0,z_1)\|_H^2 \le \int_0^\tau \big\|\widetilde{C}U(t)(z_0,z_1)\big\|_{L^2(\Gamma_0)}^2\,\mathrm{d}t. \tag{5.11}$$

Now we consider the damped wave equation with a potential

$$\left\{\begin{array}{lcl} z''(t,x) & = & \Delta z(t,x) + b(t,x)z'(t,x) + V(t,x)z(t,x) \in [0,\tau]\times\Omega \\ z(0,.) & = & z_0 \in H_0^1(\Omega),\ z'(0,.) = z_1 \in L^2(\Omega) \\ z(t,x) & = & 0 \quad (t,x)\in[0,\tau]\times\Gamma. \end{array}\right. \tag{5.12}$$

Going to the first order system on H, the wave equation (5.12) can be rewritten as $Z' = A(t)Z$ with $A(t) = \begin{pmatrix} 0 & I \\ \Delta + V(t) & b(t) \end{pmatrix} = A_0 + R(t)$ where $R(t) = \begin{pmatrix} 0 & 0 \\ V(t) & b(t) \end{pmatrix}$. As in the case of the Schrödinger equation we can apply the previous discussion to see that the Hautus test for A_0 implies our averaged Hautus test for $(A(t))_{t\ge0}$. In order to do so we need to verify (5.2). Clearly,

$$\|R(t)(u,v)\|_H \le \sqrt{2}\max\{\|V(t,.)\|_\infty, \|b(t,.)\|_\infty\}\|(u,v)\|_H.$$

Since $\frac{1}{2}(a+b) \le \max(a,b) \le a+b$ for $a,b>0$, (5.2) holds if

$$\frac{1}{\tau}\int_0^\tau \big\|V(t,.)\big\|_\infty^2 + \big\|b(t,.)\big\|_\infty^2\,\mathrm{d}t < \mu$$

for some μ small enough. On the other hand, this condition also ensures the condition (4.4) of Theorem 4.1 that allows to obtain exact averaged observability for (5.12) at some time τ. That is, we obtain the energy estimate (5.10) for τ large enough for the solution z to (5.12). If V and b are independent of t then observability results are known (see [29, Section 7.3]). If $b(t) = 0$ and V depends on t, then a more precise result can be found in [25] for a special class of Γ_0. The proof in [25] is different from ours and it is based on Carleman estimates.

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