

EXACT OBSERVABILITY, SQUARE FUNCTIONS AND SPECTRAL THEORY

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ABSTRACT. In the first part of this article we introduce the notion of a backward-forward conditioning (BFC) system that generalises the notion of zero-class admissibility introduced in [21]. We can show that unless the spectrum contains a halfplane, the BFC property occurs only in situations where the underlying semigroup extends to a group.

In a second part we present a sufficient condition for exact observability in Banach spaces that is designed for infinite-dimensional output spaces and general strongly continuous semigroups. To obtain this we make use of certain weighted square function estimates. Specialising to the Hilbert space situation we obtain a result for contraction semigroups without an analyticity condition on the semigroup.

1. INTRODUCTION

In this article we study exact observability of linear systems (A, C) on Banach spaces of the form

$$\begin{cases} x'(t) + Ax(t) & = 0 \\ x(0) & = x_0 \\ y(0) & = Cx(t) \end{cases}$$

We suppose throughout this article that $-A$ is the generator of a strongly continuous semigroup $T(t)_{t \geq 0}$ on a Banach space X . For details on semigroup theory used frequently in this article we refer to e.g. to the textbooks [5, 7, 16]. Since we deal with unbounded operators in general, we will note $\mathcal{D}(A)$ the domain of A and $\mathcal{R}(A)$ its range.

Let Y be another Banach space and suppose that the observation operator $C : \mathcal{D}(A) \rightarrow Y$ is bounded and linear when $\mathcal{D}(A)$ is endowed with the graph norm $\|x\|_{\mathcal{D}(A)} = \|x\| + \|Ax\|$. Here we denote by $\|\cdot\|$ the norm of X . Since the observation operator C is generally unbounded, the concept of admissibility is introduced. It means that the output y of the system (usually measured in L^2 norm) depends continuously on the initial value x_0 .

Definition 1.1. We say that C is L^2 -admissible in time $\tau > 0$ (for A or for $T(t)_{t \geq 0}$) if there exists a constant $M(\tau) > 0$ such that

$$\sup_{x \in \mathcal{D}(A), \|x\|=1} \int_0^\tau \|CT(t)x\|_Y^2 dt =: M(\tau)^2 < \infty.$$

Date: February 16, 2011.

1991 Mathematics Subject Classification. 93B07, 43A45, 47A60.

Key words and phrases. Exact observability, admissibility, spectral theory of semigroups and groups, H^∞ functional calculus, square function estimates.

The first author was partially supported by the ANR project ANR-09-BLAN-0058-01.

Definition 1.2. We say that C is exactly L^2 -observable for A (or for $T(t)$) in time $\eta > 0$ if there exists a constant $m(\eta) > 0$ such that

$$\inf_{x \in \mathcal{D}(A), \|x\|=1} \int_0^\eta \|CT(t)x\|_Y^2 dt =: m(\eta)^2 > 0.$$

For more information the notion of admissible observation (or control) operators we refer the reader to the overview article [8] or, both for admissibility and observability issues to the recent book [20] and references therein. We summarise some well-known facts and notations:

When there is no risk of confusion, 'admissible' means L^2 admissible in *some* finite time $\tau > 0$ and 'exact observable' means exactly L^2 -observable for A in *some* finite time $\eta > 0$. We say that C is infinite-time admissible if $M(\infty) < \infty$ and exactly observable in infinite time if $m(\infty) > 0$.

Finite-time admissibility does not depend on the choice of $\tau > 0$. Nevertheless it turns out to be useful to study the (clearly non-decreasing) functions $t \mapsto m(t)$ and $t \mapsto M(t)$. In the 'dual' situation of a control operator B the quantity $m(\eta)^{-1}$ is often referred to as *control cost* of a system. We refer e.g. to [13, 17, 18] and references therein for more details.

Independence of the time $\tau > 0$ of the notion of admissibility means that a lack of admissibility expresses either by $M(\tau) = \infty$ for all $\tau > 0$ or by $M(\tau) < \infty$ for finite τ while $M(\infty) = \infty$. On the other hand, a lack of exact observability expresses by $m(\eta) = 0$ for $0 < \eta < \eta_0$ for $\eta_0 \in (0, \infty]$. We remark that Example 2.3 below satisfies $m(\eta) = 0$ for $0 < \eta < 2$, $m(2) = 1$ while $m(\eta) \rightarrow +\infty$ for $\eta \rightarrow +\infty$.

Since most parabolic equations like for example the heat equation are not exactly observable unless very special observations are chosen whereas exact observability appears frequently for hyperbolic systems such as the wave equation, it appears natural to study necessary spectral conditions of the generator $-A$ that make exact observability possible or impossible. In this direction we extend and complete former results of [21]. We introduce the notion of backward-forward conditioning BFC-systems. These are admissible and exactly observable systems for which $M(\eta) < m(\tau)$ for some $\eta < \tau$. We analyse spectral properties of the generator $-A$ of the semigroup of such systems. In particular, we prove that the approximate point spectrum of A is contained in a vertical strip. Therefore, the boundary of the spectrum is also contained in a strip. We prove in addition that if (A, C) is an admissible BFC-system such that the spectrum of A does not contain a half-plane then the semigroup actually extends to a group. Note that every bounded group with an admissible operator C is a BFC-system. Since (BFC) is a frequent property that typically is more likely to hold the more 'regular' the operators A and C are, this shows that exact observability is considerably rare outside the group context.

A second part of this paper is devoted to a new sufficient criterion for exact observability. Under an assumption of square function type estimate we prove that a condition like

$$\|CA^{-\alpha}x\|_Y \geq \delta\|x\|,$$

implies exact observability. Here $\alpha \in (0, 1)$ and δ is a positive constant.

Without any further assumption, we show that if $-A$ is the generator of a contraction semigroup on a Hilbert space and $C : \mathcal{D}(A) \rightarrow Y$ is such that $\|CA^{-1/2}x\|_Y \geq \delta\|x\|$, then (A, C) is exactly observable.

In order to state and prove our criterion we make a heavy use of square function estimate of type

$$\|x\|^2 \leq K^2 \int_0^\infty \|(tA)^{-\beta}(T(2t^{2\beta}) - T(t^{2\beta}))x\|^2 \frac{dt}{t},$$

where K is a positive constant, $\beta \in (0, 1)$ and $T(t)$ denotes the semigroup generated by $-A$. In the case where $\beta = 1/2$, this corresponds to a lower square function estimate

$$\|x\|^2 \leq K^2 \int_0^\infty \|\varphi(tA)x\|^2 \frac{dt}{t},$$

where $\varphi(z) := z^{-1/2}(e^{-2z} - e^{-z})$. On Hilbert spaces, it is well known that such estimate is related to the holomorphic functional calculus of the operator A . The needed results on this functional calculus and associated square function estimates for sectorial operators will be sketched in the last two sections.

As we will explain later our criterion applies for bounded analytic semigroups on Hilbert spaces whose generator admits a bounded H^∞ -calculus – but the first part of this paper reveals this to be impossible for a large class of systems unless A is bounded. One important aspect of the criterion that might also help in other situations is therefore how to avoid making use of analyticity assumption of the semigroup. We discuss at the end of this paper two examples.

2. BFC-SYSTEMS

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_Y$, respectively. Throughout this section, $(T(t))_{t \geq 0}$ is a strongly continuous semigroup on X whose generator is denoted by $-A$.

Definition 2.1. An admissible observation operator C for A is called *zero-class admissible*, if $\lim_{\tau \rightarrow 0^+} M(\tau) = 0$.

Note that if the semigroup $T(t)$ is bounded analytic with generator $-A$ then for all $\alpha \in [0, 1/2)$, A^α is zero-class admissible. This follows from the inequality $\|A^\alpha T(t)\| \leq Mt^{-\alpha}$. Consequently, if C is bounded on such a fractional domain space $\mathcal{D}(A^\alpha)$ and thus

$$\|CT(t)x\|_Y \leq M [\|A^\alpha T(t)x\| + \|T(t)x\|],$$

C is zero-class admissible.

It is also an obvious fact that every bounded operator $C : X \rightarrow Y$ is zero-class admissible. Here $(T(t))_{t \geq 0}$ is merely a strongly continuous semigroup on X .

Consider now the linear operator $\tilde{\Psi}_\tau : X \rightarrow L^2(0, \tau; Y)$ defined by $\tilde{\Psi}_\tau x = CT(\cdot)x$. Then admissibility (i.e., $M(\tau) < \infty$) means that $\tilde{\Psi}_\tau$ is a bounded operator. If in addition $m(\tau) > 0$, then $\tilde{\Psi}_\tau$ is injective and has closed range. Therefore, we may consider the operator $\Psi_\tau : X \rightarrow \mathcal{R}(\tilde{\Psi}_\tau)$, $\Psi_\tau = \tilde{\Psi}_\tau$. We have

$$(2.1) \quad \frac{1}{m(\tau)} = \sup_{x \in \mathcal{D}(A), \|x\|=1} \frac{\|x\|}{\|\Psi_\tau x\|} = \sup_{x \in \mathcal{D}(A), \|x\| \neq 0} \frac{\|x\|}{\|\Psi_\tau x\|} = \|\Psi_\tau^{-1}\|.$$

We introduce the following definition.

Definition 2.2. We say that the system (A, C) has the backward-forward conditioning property or shortly that (A, C) is a BFC-system if there exists some $0 < \eta < \tau$ such that C is admissible and exactly observable in time τ and if

$$(BFC) \quad \|\Psi_\tau^{-1}\| \|\Psi_\eta\| < 1.$$

The condition (BFC) is clearly a conditioning property for the output operator with different times η and τ which correspond to a backward and forward evolution of the system. It also follows from (2.1) that (BFC) is equivalent to

$$(2.2) \quad M(\eta) < m(\tau) \quad \text{for some } \eta < \tau.$$

Therefore, if C is exactly observable in some time τ and of zero-class, then (2.2) holds trivially by letting η sufficiently small. Hence, the system is BFC. If C is admissible at any $\tau > 0$ and if $m(t) \rightarrow +\infty$ for $t \rightarrow +\infty$, then (2.2) holds and again the system is BFC.

Zero-class admissible operators are introduced and studied in [21]. See also [9] from which we borrow a concrete example leading to an BFC-system in which C is not zero-class.

Example 2.3. *The following example is taken from Jacob, Partington and Pott [9, Example 3.9]. We shall use Ingham inequalities to prove that our system is BFC. Similar ideas could be used in a more general class of examples. Consider an undamped wave equation on $[0, 1]$ with Dirichlet boundary conditions and Neumann type observation of the form*

$$\begin{cases} \frac{\partial^2}{\partial t^2} z(x, t) = \frac{\partial^2}{\partial x^2} z(x, t) & \text{for } x \in (0, 1), t \geq 0 \\ z(0, t) = z(1, t) = 0 & \text{for } t \geq 0 \\ z(x, 0) = z_0(x) \quad \text{and} \quad \frac{\partial}{\partial t} z(x, 0) = z_1(x) & \text{for } x \in (0, 1) \\ y(t) = \frac{\partial}{\partial x} z(0, t) \end{cases}$$

We rewrite the system as a first order Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} U = -AU(t), & t \geq 0 \\ U(0) = (z_0, z_1) \\ CU(x, t) = \frac{\partial}{\partial x} f(0) \end{cases}$$

where $A = \begin{pmatrix} 0 & -I \\ -\frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$ and $U = (f, g)$. The latter Cauchy problem is considered on the Hilbert space $H = H_0^1(0, 1) \times L^2(0, 1)$ endowed with the norm $\|(f, g)\| = \sqrt{\int_0^1 |f'|^2 dx + \int_0^1 |g|^2 dx}$. Note that by the Poincaré inequality, $\sqrt{\int_0^1 |f'|^2 dx}$ defines a norm on $H_0^1(0, 1)$ which is equivalent to the usual one.

It is a standard fact that $-A$ generates a strongly continuous semigroup $T(t)$ on H . It is easy to see that A has compact resolvent and the eigenvalues are $\lambda_n = -in\pi$, $n \in \mathbb{Z} \setminus \{0\}$ with normalised eigenfunctions $U_n(x) = \left(\frac{\sin(n\pi x)}{in\pi}, \sin(n\pi x) \right)$ which form an orthonormal basis of H . Fix $(f, g) \in H$ and denote by $\alpha_n = \langle (f, g), U_n \rangle_H$ (the scalar product in H). Then

$$\|(f, g)\|^2 = \sum_{n \in \mathbb{Z}, n \neq 0} |\alpha_n|^2$$

and

$$CT(t)(f, g) = \sum_n \alpha_n e^{in\pi t} CU_n = -i \sum_n \alpha_n e^{in\pi t}.$$

Using the well known Ingham inequalities (see e.g. [22, p. 162] or [10, Theorem 4.3]) we obtain the following estimates for all $\tau > 2$,

$$m(\tau)^2 \sum_n |\alpha_n|^2 \leq \int_0^\tau \left| \sum_n \alpha_n e^{in\pi t} \right|^2 dt \leq M(\tau)^2 \sum_n |\alpha_n|^2$$

with

$$m(\tau)^2 \geq \frac{2\tau}{\pi} \left(1 - \frac{4}{\tau^2} \right), \quad M(\tau)^2 \leq \frac{8\tau}{\pi} \left(1 + \frac{4}{\tau^2} \right).$$

This shows that C is admissible at any time $\tau > 0$ and exactly observable in time $\tau > 2$ with constant $m(\tau) \rightarrow +\infty$ as $\tau \rightarrow +\infty$. This shows (2.2) and hence the system (A, C) is backward-forward conditioning.

In order to see that C is not zero-class, we consider small $\tau > 0$ and $f \in H_0^1(0, 1)$ with Fourier coefficients α_n and note that

$$\int_0^\tau \left| \sum_n \alpha_n e^{in\pi t} \right|^2 dt = \|\chi_{[0, \tau]} f\|_2^2,$$

where $\chi_{[0, \tau]}$ denote the indicator function of $[0, \tau]$. From this equality it is clear that

$$1 = \sup_{\|f\|_2=1} \|\chi_{[0, \tau]} f\|_2^2 = \sup_{\|f\|_2=1} \int_0^\tau \left| \sum_n \alpha_n e^{in\pi t} \right|^2 dt = M(\tau)^2.$$

Therefore, the right hand side does not converge to 0 as $\tau \rightarrow 0$.

Remark 2.4. The above example is a special case of the following situation: let C be admissible in some arbitrary time $\tau > 0$ and exactly observable in some time $\eta > 0$ for a group $U(t)_{t \in \mathbb{R}}$. Observe that $\|x\| = \|U(-t)U(t)x\| \leq \|U(-t)\| \|U(t)x\|$ whence $\|U(t)x\| \geq \|U(-t)\|^{-1} \|x\|$. From

$$\begin{aligned} \int_0^{n\eta} \|CU(t)x\|^2 dt &= \sum_{j=0}^{n-1} \int_0^\eta \|CU(t)U(j\eta)x\|^2 dt \\ &\geq m(\eta)^2 \left(\sum_{j=0}^{n-1} \|U(-j\eta)\|^{-2} \right) \|x\|^2, \end{aligned}$$

we then infer $m(n\eta) \rightarrow +\infty$ for $n \rightarrow +\infty$ whenever the sum in the last expression diverges. This is in particular the case for bounded groups $U(t)_{t \in \mathbb{R}}$. By the admissibility of the system (A, C) one then obtains (BFC) by letting n sufficiently large.

We thank Hans Zwart for pointing out this remark to us.

3. SPECTRAL PROPERTIES OF BFC-SYSTEMS

We consider the same notation $X, Y, A, (T(t))_{t \geq 0}$ and $C : D(A) \rightarrow Y$ as in the previous section. Our aim here is to study spectral properties of BFC-systems. We will extend some results which have been proved in [21] in the context of zero-class operators. We note also that related ideas and results were obtained previously by Nikolski [14] in the particular case of bounded observation operators C on X .

Let us introduce the classical function $\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}^+$ defined by

$$\varepsilon(t) := \inf_{\|x\|=1} \|T(t)x\|.$$

It is clear that $\varepsilon(t)$ is strictly positive for all $t > 0$ if this holds for a single $t_0 > 0$. Indeed, from

$$\|T(s)\| \|T(t)x\| \geq \|T(t+s)x\| \geq \varepsilon(t) \|T(s)x\| \geq \varepsilon(t)\varepsilon(s) \|x\|$$

one infers that

$$\varepsilon(t) \|T(s)\| \geq \varepsilon(t+s) \geq \varepsilon(t)\varepsilon(s),$$

for all $t, s \geq 0$. For this reason we distinguish the cases that $\varepsilon(t)$ is strictly positive for all $t > 0$ or that it vanishes for all $t > 0$ and we note this by $\varepsilon(t) > 0$ or $\varepsilon(t) = 0$ respectively.

The following lemma is essentially contained in [14] and [21].

Lemma 3.1. *If (A, C) is an admissible and exactly observable BFC-system, then $\varepsilon(t) > 0$.*

Proof. By the definition of BFC-system, there exist $0 < \eta < \tau$ such that

$$\delta := m(\tau)^2 - M(\eta)^2 > 0.$$

By the semigroup property,

$$\begin{aligned} m(\tau)^2 \|x\|^2 &\leq \int_0^\tau \|CT(t)x\|_Y^2 dt \\ &= \int_0^\eta \|CT(t)x\|_Y^2 dt + \int_0^{\tau-\eta} \|CT(t)T(\eta)x\|_Y^2 dt \\ &\leq M(\eta)^2 \|x\|^2 + M(\tau-\eta)^2 \|T(\eta)x\|^2 \end{aligned}$$

which immediately yields

$$\|T(\eta)x\|^2 \geq M(\tau-\eta)^{-2} (m(\tau)^2 - M(\eta)^2) \|x\|^2 \geq M(\tau-\eta)^{-2} \delta \|x\|^2.$$

Therefore, $\varepsilon(\eta) > 0$, and hence $\varepsilon(t) > 0$ for all $t > 0$. \square

Lemma 3.2. *Suppose that (A, C) is an admissible and exactly observable BFC-system. Then $T(t)^*$ is injective for one (and thus all) $t > 0$ if and only if $T(t)$ extends to a group on X .*

Proof. We know by Lemma 3.1 that $\varepsilon(t) > 0$. This implies that $T(t)$ is injective and has closed image for all $t \geq 0$. Thus, $T(t)$ is bijective if and only if $T(t)^*$ is injective. The latter is clearly independent of $t > 0$ by the semigroup law. Indeed, if $T(t_0)^*$ is injective for some $t_0 > 0$, so are all $T(s)^*$ for $s < t_0$ since $T(t_0)^* = T(t_0-s)^* T(s)^*$. If $s > t_0$ then we find $n \in \mathbb{N}$, $\delta \in [0, t_0[$ such that $T(s)^* = (T(t_0)^*)^n T(\delta)^*$, and the injectivity of $T(s)^*$ follows from that of $T(t_0)^*$ and $T(\delta)^*$. We saw that $T(t)^*$ is injective for one (and thus all) $t > 0$ if and only if $T(t)$ is bijective which in turn by

$$S(t) := \begin{cases} T(t) & \text{if } t \geq 0 \\ T(t)^{-1} & \text{if } t < 0 \end{cases}$$

is equivalent to a group extension of $T(t)$ on X . \square

For a closed operator S on X recall the notions of point spectrum

$$\sigma_P(S) = \{\lambda \in \mathbb{C} : \ker(\lambda I - S) \neq \{0\}\},$$

the approximate point spectrum

$$\sigma_A(S) = \{\lambda \in \mathbb{C} : \inf_{x \in \mathcal{D}(S), \|x\|=1} \|\lambda x - Sx\| = 0\}$$

and the residual spectrum

$$\sigma_R(S) = \{\lambda \in \mathbb{C} : \text{range}(\lambda - S) \text{ is not dense in } X\}.$$

It is easy to see that $\sigma_R(S) = \sigma(S) \setminus \sigma_A(S)$. Of course, $\sigma_P(S) \subseteq \sigma_A(S)$.

Proposition 3.3. *Let (A, C) be an admissible and exactly observable BFC-system. Then there exist no approximate point spectrum of A with arbitrary large real parts.*

In particular, if C is an admissible zero-class operator and A has a sequence of approximate point spectrum with arbitrary large real parts then C is not exactly observable.

Proof. Recall that $\exp(-t\sigma_A(A)) \subseteq \sigma_A(T(t))$ (see [5, p. 276], note that $-A$ is the generator). If we find a sequence $\lambda_n \in \sigma_A(A)$ with $\text{Re}(\lambda_n) \rightarrow +\infty$, then $e^{-t\lambda_n} \in \sigma_A(T(t))$. Hence, $\inf_{\|x\|=1} \|T(t)x - e^{-t\lambda_n}x\| = 0$. By

$$\varepsilon(t)\|x\| \leq \|T(t)x\| \leq \|T(t)x - e^{-t\lambda_n}x\| + e^{-t\text{Re}(\lambda_n)}\|x\|$$

we get $\varepsilon(t) = 0$ which is incompatible with exact observability by Lemma 3.1. \square

Since $-A$ is the generator of a strongly continuous semigroup, $\sigma(A)$ is contained in a right-half plane. On the other hand, it is well known that the boundary of the spectrum $\partial\sigma(A)$ is contained in $\sigma_A(A)$. We obtain from the previous proposition that $\text{Re}(\partial\sigma(A))$ is bounded, i.e., $\partial\sigma(A)$ is contained in a vertical strip. Thus

Corollary 3.4. *Let (A, C) be an admissible BFC-system. Then*

$$\text{Re}(\partial\sigma(A)) := \{\text{Re}(\lambda) : \lambda \in \partial\sigma(A)\}$$

is bounded.

The following lemma is known.

Lemma 3.5. *Let $S \in \mathcal{B}(X)$ satisfy $\|Sx\| \geq \gamma\|x\|$ for some $\gamma > 0$ and all $x \in X$ and assume $0 \in \sigma(S)$. Then there exists $\delta > 0$ such that $B(0, \delta) \subseteq \sigma_R(S)$.*

The main result in [21] which states that if C is zero-class admissible and $\sigma_R(A)$ is empty, then $T(t)$ extends to a group. The next propositions extend this result.

Proposition 3.6. *Let (A, C) be an admissible BFC-system. If $\text{Re}(\sigma_R(A)) := \{\text{Re}\lambda : \lambda \in \sigma_R(A)\}$ is bounded, then $(T(t))_{t \geq 0}$ extends to a group on X .*

Proof. We know by Lemma 3.1 that $\varepsilon(t) > 0$. If $T(t)$ was not boundedly invertible for some $t > 0$, then $0 \in \sigma(T(t))$. By Lemma 3.5, there exists $\delta_t > 0$ such that $B(0, \delta_t) \subseteq \sigma_R(T(t))$. Since $\sigma_R(T(t)) \setminus \{0\} = \exp(-t\sigma_R(A))$ (see [5, p. 276]) we obtain

$$B(0, \delta_t) \setminus \{0\} \subseteq \exp(-t\sigma_R(A)),$$

Therefore, there exists a real sequence $(\lambda_n) \in \sigma_R(A)$ such that $\text{Re}\lambda_n \rightarrow +\infty$. This contradicts the assumption. \square

Proposition 3.7. *Assume that (A, C) is an admissible BFC-system. If $\sigma(A)$ does not contain a half-plane then $(T(t))_{t \geq 0}$ extends to a group on X .*

Proof. By Corollary 3.4 we see that $\sigma(A)$ is either contained in vertical strip or contains a half-plane. Now we apply Proposition 3.6 to conclude. \square

Considering the right shift semigroup $T(t)$ on $L^2(\mathbb{R}_+)$ with the identity observation $C = \text{Id}$ provides an example of a BFC-system (even a zero-class admissible one, see [21, Remark 3.1]) for which no group extension is possible. In this example, it is not difficult to check the spectrum of A satisfies $\sigma(A) = \sigma_R(A) = \mathbb{C}^+$ (the right half-plane). This shows that the spectral condition in Proposition 3.7 cannot be omitted.

Assume that (A, C) is an admissible BFC-system. If $T(t)$ is analytic, differentiable or merely eventually continuous then by [5, p. 113] $\sigma(A)$ does not contain a half-plane. Thus, we conclude by Proposition 3.7 that $(T(t))_{t \geq 0}$ extends to group on X .

Proposition 3.8. *Assume that (A, C) is an admissible BFC-system. If $T(t)$ is compact for some $t > 0$, then X has finite dimension.*

Proof. If $T(t)$ is compact for some $t > 0$ then $\sigma(A) = \sigma_P(A)$ is discrete. It follows from Proposition 3.3 that $\sigma(A)$ is bounded. We conclude by Proposition 3.6 that $(T(t))_{t \geq 0}$ extends to a group on X . Thus, $I = T(t)T(-t)$ is compact on X and therefore X has finite dimension. \square

4. SUFFICIENT CONDITIONS FOR EXACT OBSERVABILITY

Our aim in this section is to derive conditions on C and A which imply exact observability. Our condition reads as follows

$$(4.1) \quad \|CA^{-(1-\beta)}x\|_Y \geq \delta\|x\|$$

for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Here $\beta \in (0, 1)$ and $\delta > 0$ are constants. Since we shall assume that A is injective, it may be convenient to understand (4.1) in the sense

$$\|CA^{-1}x\|_Y \geq \delta\|A^{-\beta}x\|.$$

Of course, $\mathcal{R}(A) \cap \mathcal{R}(A^\beta) = \mathcal{R}(A)$.

In the sequel we need some basic properties of the H^∞ functional calculus for sectorial operators. This functional calculus goes back to the work of McIntosh[12]. More recent publications of the meanwhile rich theory can be found in [6] or [11] and the references given therein. We briefly sketch the needed results and definitions.

Definition 4.1. We denote by S_ω the open sector $\{z \in \mathbb{C}^* : |\arg(z)| < \omega\}$ and by $\overline{S_\omega}$ the closure of S_ω in \mathbb{C} .

We call a closed operator A on X *sectorial of angle ω* if A is densely defined having its spectrum in $\overline{S_\omega}$ such that $\lambda R(\lambda, A) := \lambda(\lambda - A)^{-1}$ of A is uniformly bounded on the complement of each strictly larger sectors S_θ , $\theta > \omega$.

Notice that if $-A$ generates a bounded semigroup $T(t)_{t \geq 0}$, then A is sectorial of angle $\pi/2$ by the Hille-Yosida theorem. Moreover, the semigroup is (bounded) analytic if and only if A is sectorial of angle $< \pi/2$.

Let $H^\infty(S_\omega)$ denote the holomorphic and bounded functions on S_ω and Let $H^\infty(\overline{S_\omega})$ denote the holomorphic and bounded functions on S_ω that are continuous and bounded on $\overline{S_\omega}$. We further consider the ideal $H_0^\infty(S_\omega)$ (respectively $H_0^\infty(\overline{S_\omega})$) of all functions $f \in H^\infty(S_\omega)$ (respectively $H_0^\infty(\overline{S_\omega})$) that allow an estimate $|f(z)| \leq M \max(|z|^\varepsilon, |z|^{-\varepsilon})$. The class of $H_0^\infty(S_\omega)$ functions admits a natural functional calculus for sectorial operators A of angle ω . Indeed, if $f \in H_0^\infty(S_\theta)$ for some $\theta > \omega$

and if $\Gamma = \partial S_\theta$ denotes the orientated path with strictly decreasing imaginary part, the Cauchy integral

$$f(A) := \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda, A) d\lambda$$

converges absolutely in norm and defines therefore a bounded operator $f(A)$. If A has, say, dense range, one obtains then a functional calculus for all functions $f \in H_0^\infty(S_\theta)$.

Definition 4.2. We say that A admits a *bounded $H^\infty(S_\omega)$ (respectively $H^\infty(\overline{S_\omega})$) functional calculus* if $f(A)$ is a bounded operator on X and there exists a constant M such that

$$(4.2) \quad \|f(A)\| \leq M \|f\|_\infty$$

for all $f \in H^\infty(S_\theta)$ and $\theta > \omega$ (respectively for all $f \in H^\infty(\overline{S_\omega})$).

Let us mention that by an approximation argument, if (4.2) holds for all $f \in H_0^\infty(S_\theta)$ then it holds for all $f \in H^\infty(S_\theta)$. Therefore, it is enough to check the validity of (4.2) for all $f \in H_0^\infty(S_\theta)$ to obtain bounded $H^\infty(S_\omega)$ functional calculus. Similarly, if (4.2) holds for for all $f \in H_0^\infty(\overline{S_\omega})$ we obtain a $H^\infty(\overline{S_\omega})$ functional calculus.

We say that A admits upper square function estimates on S_θ if there is a constant $M > 0$ such that

$$(4.3) \quad \forall x \in X : \quad \int_0^\infty \|\varphi(tA)x\|^2 \frac{dt}{t} \leq M^2 \|x\|^2$$

for all $\varphi \in H_0^\infty(S_\theta)$. In the same way we speak of lower square function estimates on S_θ if one has

$$(4.4) \quad \forall x \in X : \quad \|x\|^2 \leq K^2 \int_0^\infty \|\varphi(tA)x\|^2 \frac{dt}{t}$$

If $X = H$ is a Hilbert space, then upper square function estimates for A and for A^* for $H_0^\infty(S_\mu)$ functions for all $\mu > \omega$ are equivalent to a bounded $H_0^\infty(S_\mu)$ functional calculus for all $\mu > \omega$. Moreover, by an approximate identity argument and a duality estimate, lower square function estimates for A follow from upper estimate of its adjoint A^* . We will go into some details on the Hilbert space theory of the functional calculus in the last section and refer at this point to [12, 3] for more details.

Before stating our first result of this section we discuss the following estimate for A

$$(SQ_\beta) \quad \|x\|^2 \leq K^2 \int_0^\infty \|(tA)^{-\beta}(T(2t^{2\beta}) - T(t^{2\beta}))x\|^2 \frac{dt}{t}$$

that we will need to formulate the theorem. Here, K is a positive constant and $\beta \in (0, 1)$. In case $\beta = 1/2$, letting $\varphi(z) := z^{-1/2}(e^{-2z} - e^{-z})$, this corresponds to a lower square function estimate (4.4) for $\varphi \in H_0^\infty(\overline{S_{\pi/2}})$. As mentioned above, (4.4) follows from H^∞ -functional calculus when X is a Hilbert space. We will discuss again this in the next section.

Assume that $-A$ is the generator of bounded strongly continuous semigroup on X and is injective. If we let $\varphi(z) := z^{-\beta}(e^{-2z} - e^{-z})$, then (SQ_β) can be seen as

$$\begin{aligned} \|x\|^2 &\leq K^2 \int_0^\infty \|(tA)^{-\beta}(T(2t^{2\beta}) - T(t^{2\beta}))x\|^2 \frac{dt}{t} \\ &= K^2 \int_0^\infty \|t^{2\beta^2-\beta}\varphi(t^{2\beta}A)x\|^2 \frac{dt}{t} \\ (\text{letting } s = t^{2\beta}) &= \frac{K^2}{2\beta} \int_0^\infty \|s^{-(\frac{1}{2}-\beta)}\varphi(sA)x\|^2 \frac{ds}{s}, \end{aligned}$$

i.e. as 'weighted' lower square function estimate for $\varphi \in H_0^\infty(\overline{S_{\pi/2}})$.

By [6, Theorem 6.4.6], the completion $X_{\frac{1}{2}-\beta,2}$ of X with respect to the seminorm

$$[x]_\beta = \left(\int_0^\infty \|s^{-(\frac{1}{2}-\beta)}\varphi(sA)x\|^2 \frac{ds}{s} \right)^{\frac{1}{2}}$$

is independent of the choice of φ and coincides (with equivalent norms) with the real interpolation space:

$$(4.5) \quad \left(\dot{X}_{-1}(A), \dot{X}_1(A) \right)_{\frac{3}{4}-\frac{\beta}{2},2} = X_{\frac{1}{2}-\beta,2}.$$

Here $\dot{X}_{-1}(A)$ is the completion of $\mathcal{R}(A)$ with respect to $\|A^{-1}x\|$ and $\dot{X}_1(A)$ is the completion of $\mathcal{D}(A)$ with respect to $\|Ax\|$. From (4.5), it follows that (SQ_β) is equivalent to the continuous embedding

$$(4.6) \quad \left(\dot{X}_{-1}(A), \dot{X}_1(A) \right)_{\frac{3}{4}-\frac{\beta}{2},2} \hookrightarrow X.$$

For the rest of this discussion, we assume for simplicity that A is invertible. In this case, $\dot{X}_1(A) = \mathcal{D}(A)$ and $\dot{X}_{-1}(A) = X_{-1}$.

It is a known fact that the semigroup $(T(t))$ extends to a strongly continuous semigroup $(T_{-1}(t))$ on the extrapolation space X_{-1} , whose (negative) generator A_{-1} is an extension of A (see [5] Chapter II, Section 5). In addition A is the part of A_{-1} on X and hence $\mathcal{D}(A_{-1}^2) = \mathcal{D}(A)$ with equivalent norms. Indeed,

$$\begin{aligned} x \in \mathcal{D}(A_{-1}^2) &\Leftrightarrow x \in \mathcal{D}(A_{-1}) = X, A_{-1}x \in \mathcal{D}(A_{-1}) \\ &\Leftrightarrow x \in X, A_{-1}x \in \mathcal{D}(A_{-1}) \\ &\Leftrightarrow x \in \mathcal{D}(A). \end{aligned}$$

The fact that $A_{-1} : X \rightarrow X_{-1}$ is an isometry implies that $\|Ax\|_X = \|A_{-1}^2x\|_{X_{-1}}$. Assume now that $\beta < \frac{1}{2}$. We have

$$\begin{aligned} (X_{-1}, \mathcal{D}(A))_{\frac{3}{4}-\frac{\beta}{2},2} &= (X_{-1}, \mathcal{D}(A_{-1}^2))_{\frac{3}{4}-\frac{\beta}{2},2} \\ &= (\mathcal{D}(A_{-1}), \mathcal{D}(A_{-1}^2))_{\frac{1}{2}-\beta,2} \\ &= (X, \mathcal{D}(A))_{\frac{1}{2}-\beta,2} \hookrightarrow X. \end{aligned}$$

Note that the second equality follows from [19, p. 105]. Hence for $\beta < \frac{1}{2}$

$$(4.7) \quad X_{\frac{1}{2}-\beta,2} = (X_{-1}, \mathcal{D}(A))_{\frac{3}{4}-\frac{\beta}{2},2} \hookrightarrow X,$$

which means that (SQ_β) always holds for $\beta < \frac{1}{2}$.

Let us finally mention that for $\beta > \frac{1}{2}$, (SQ_β) never holds for non-negative self-adjoint operators with compact resolvent in infinite dimension separable Hilbert

spaces. Indeed, consider such an operator A . The spectrum is discrete $\sigma(A) = \{\lambda_n\}$ with $\lambda_n \rightarrow +\infty$. Applying (SQ_β) to a normalised eigenvector $x = \varphi_n$ (associated with λ_n) yields

$$\begin{aligned} 1 &\leq K^2 \int_0^\infty \|(tA)^{-\beta} \varphi_n\|^2 (e^{-2t^{2\beta}} - e^{-t^{2\beta}})^2 \frac{dt}{t} \\ &= \frac{K^2}{\lambda_n^{2\beta}} \int_0^\infty t^{-2\beta} (e^{-2t^{2\beta} \lambda_n} - e^{-t^{2\beta} \lambda_n})^2 \frac{dt}{t} \\ &= \frac{K^2 \lambda_n}{2\beta \lambda_n^{2\beta}} \int_0^\infty (e^{-2s} - e^{-s})^2 \frac{ds}{s^2} \\ &= C \lambda_n^{1-2\beta}. \end{aligned}$$

For $\beta > \frac{1}{2}$, the last term goes to 0 as n tends to $+\infty$. This shows that (SQ_β) cannot hold.

Now we come to our main result of this section concerning exact observability.

Theorem 4.3. *Let $-A$ be the generator of a bounded semigroup on the Banach space X and assume that A is injective and has dense range. Let $C : \mathcal{D}(A) \rightarrow Y$ be bounded and suppose that there exists $\beta \in (0, 1)$ such that the lower square function estimate (SQ_β) and (4.1) are satisfied. Then there exists a constant $m > 0$ such that*

$$(4.8) \quad m^2 \|x\|^2 \leq \int_0^\infty \|CT(t)x\|_Y^2 dt$$

for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$.

Proof. Fix $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ and apply the lower square function estimate (SQ_β) to obtain

$$\begin{aligned} \|x\|^2 &\leq K^2 \int_0^\infty \|(tA)^{-\beta} (T(2t^{2\beta}) - T(t^{2\beta}))x\|^2 \frac{dt}{t} \\ &\leq \frac{K^2}{\delta^2} \int_0^\infty \|CA^{-(1-\beta)} (tA)^{-\beta} (T(2t^{2\beta}) - T(t^{2\beta}))x\|_Y^2 \frac{dt}{t} \\ &= \frac{K^2}{\delta^2} \int_0^\infty \|CA^{-1} (T(2t^{2\beta}) - T(t^{2\beta}))x\|_Y^2 \frac{dt}{t^{1+2\beta}}. \end{aligned}$$

Using the fact that $x \in \mathcal{D}(A)$, we can write

$$CA^{-1} (T(2t^{2\beta}) - T(t^{2\beta}))x = -CA^{-1} \int_{t^{2\beta}}^{2t^{2\beta}} AT(s)x ds = - \int_{t^{2\beta}}^{2t^{2\beta}} CT(s)x ds.$$

Using this and the previous estimates yields

$$\begin{aligned} \|x\|^2 &\leq \frac{K^2}{\delta^2} \int_0^\infty \left\| \int_{t^{2\beta}}^{2t^{2\beta}} CT(s)x ds \right\|_Y^2 \frac{dt}{t^{1+2\beta}} \\ &\leq \frac{K^2}{\delta^2} \int_0^\infty \left(\int_{t^{2\beta}}^{2t^{2\beta}} 1 \cdot \|CT(s)x\|_Y ds \right)^2 \frac{dt}{t^{1+2\beta}} \\ &\leq \frac{K^2}{\delta^2} \int_0^\infty \int_{t^{2\beta}}^{2t^{2\beta}} \|CT(s)x\|_Y^2 ds \frac{dt}{t} \\ &= \frac{K^2}{\delta^2} \int_1^2 \int_0^\infty \|CT(t^{2\beta}u)x\|_Y^2 t^{2\beta-1} dt du \end{aligned}$$

$$= \log(2) \frac{K^2}{2\beta\delta^2} \int_0^\infty \|CT(r)x\|_Y^2 dr.$$

This shows (4.8) with $m = \sqrt{\log(2)/2\beta\frac{K}{\delta}}$. \square

Notice that if $0 \in \varrho(A)$ then $\mathcal{R}(A) = X$ and hence $\mathcal{D}(A) \cap \mathcal{R}(A) = \mathcal{D}(A)$. In this case (4.8) holds for all $x \in \mathcal{D}(A)$ and this means that C is exactly observable for A .

Corollary 4.4. *Let $-A$ be the generator of a bounded semigroup on the Banach space X and assume that A is boundedly invertible. Let $C : \mathcal{D}(A) \rightarrow Y$ be a bounded operator and assume that there exists $\beta \in (0, 1)$ such that the lower square function estimate (SQ_β) and (4.1) are satisfied. Then C is exactly observable for A .*

Corollary 4.5. *Let $-A$ be the generator of a bounded semigroup on the Banach space X and assume that A is injective and has dense range. Let $C : \mathcal{D}(A) \rightarrow Y$ be infinite-time admissible for A and assume that there exists $\beta \in (0, 1)$ such that the lower square function estimate (SQ_β) and (4.1) are satisfied. Then C is exactly observable for A .*

Proof. It remains now to extend the estimate (4.8) from Theorem 4.3 for all $x \in \mathcal{D}(A)$. This is an easy task. For $x \in \mathcal{D}(A)$ the sequence

$$x_n := n(n+A)^{-1}x - n^{-1}(n^{-1}+A)^{-1}x \in \mathcal{D}(A) \cap \mathcal{R}(A)$$

converges to x in $\mathcal{D}(A)$ (for the graph norm). Therefore, Cx_n converges in Y to Cx and $\int_0^\infty \|CT(t)x_n\|_Y^2 dt$ converges to $\int_0^\infty \|CT(t)x\|_Y^2 dt$ since C is supposed to be infinite-time admissible. We obtain (4.8) for all $x \in \mathcal{D}(A)$. \square

In the next corollary we obtain a criterion for finite time exact observability.

Corollary 4.6. *Let $-A$ be the generator of a bounded semigroup on the Banach space X and assume that there exists a constant $\omega > 0$ such that $A + \omega$ satisfies (SQ_β) . Assume that $C : \mathcal{D}(A) \rightarrow Y$ is bounded and that*

$$\|C(\omega + A)^{-(1-\beta)}x\|_Y \geq \delta\|x\|$$

for $x \in \mathcal{D}(A)$. Then C is exactly observable in finite time.

Proof. We apply the previous theorem to $\omega + A$ and obtain

$$\|x\|^2 \leq M \int_0^\infty \|Ce^{-\omega t}T(t)x\|_Y^2 dt$$

for all $x \in \mathcal{D}(A)$. We split the right hand side into two parts and write

$$\begin{aligned} \int_0^\infty \|Ce^{-\omega t}T(t)x\|_Y^2 dt &= \int_0^\tau \|Ce^{-\omega t}T(t)x\|_Y^2 dt + \int_\tau^\infty \|Ce^{-\omega t}T(t)x\|_Y^2 dt \\ &\leq \int_0^\tau \|CT(t)x\|_Y^2 dt + \int_0^\infty \|Ce^{-\omega(\tau+t)}T(t+\tau)x\|_Y^2 dt \\ &= \int_0^\tau \|CT(t)x\|_Y^2 dt + e^{-\omega\tau} \int_0^\infty \|Ce^{-\omega t}T(t)T(\tau)x\|_Y^2 dt. \end{aligned}$$

Since C is infinite-time admissible for $e^{-\omega t}T(t)$ and the semigroup $T(t)$ is bounded we have for some constants M', M''

$$\int_0^\infty \|Ce^{-\omega t}T(t)T(\tau)x\|_Y^2 dt \leq M'\|T(\tau)x\|^2 \leq M''\|x\|^2.$$

Therefore,

$$\|x\|^2 \leq M \int_0^\tau \|CT(t)x\|_Y^2 dt + MM''e^{-\omega\tau}\|x\|^2.$$

If we choose τ large enough such that $MM''e^{-\omega\tau} < 1$ we obtain the desired inequality. \square

As explained at the beginning of this section, (SQ_β) holds for all $\beta < \frac{1}{2}$ and all generators of bounded semigroup (see (4.7)). Therefore, applying Theorem 4.3 with $\beta = \frac{1}{2} - \varepsilon$, we obtain the following corollary.

Corollary 4.7. *Let $-A$ be the generator of a bounded semigroup $(T(t))_{t \geq 0}$ on a X and assume that A is invertible. Then, if $\|CA^{-1/2+\varepsilon}x\| \geq \delta\|x\|$ for some $\varepsilon, \delta > 0$ and all $x \in \mathcal{D}(A)$, C is infinite-time exactly observable for A .*

5. EXACT OBSERVABILITY ON HILBERT SPACES

Proposition 5.1. *Let X and Y be Hilbert spaces. Then, if (A, C) is exactly observable and admissible in infinite time, $T(t)_{t \geq 0}$ is similar to a contraction semigroup.*

Proof. Denote by $\langle x, y \rangle_Y$ the scalar product of Y and define for $x, y \in \mathcal{D}(A)$

$$\langle x, y \rangle_X^\sim := \int_0^\infty \langle CT(t)x, CT(t)y \rangle_Y dt.$$

This is clearly a bilinear (or sesquilinear) form on $\mathcal{D}(A) \times \mathcal{D}(A)$. Admissibility and exact observability imply that $\|x\|_X$ and $\|x\|_X^\sim$ are equivalent. By density, we extend this to all $x \in X$ and $\|\cdot\|_X^\sim$ is associated with a scalar product on X . With respect to the new norm,

$$\|T(t)x\|_X^\sim = \left(\int_0^\infty \|CT(t+s)x\|_Y^2 ds \right)^{1/2} = \left(\int_t^\infty \|CT(s)x\|_Y^2 ds \right)^{1/2} \leq \|x\|_X^\sim$$

and so $T(t)_{t \geq 0}$ is a contraction semigroup with respect to $\|\cdot\|_X^\sim$. \square

Now we turn back to Theorem 4.3. As mentioned at the beginning of the previous section, the lower square function estimate (SQ_β) holds for small β for all generators of bounded strongly continuous semigroups. It is then tempting to use the theorem for small β in order to include a large class of semigroups $T(t)$. On the other hand, if we assume that $0 \in \varrho(A)$ and $\|CA^{-\alpha-\varepsilon}x\| \geq \delta\|x\|$, one also has

$$\|CA^{-\alpha}x\| = \|CA^{-\alpha-\varepsilon}A^\varepsilon x\| \geq \delta\|A^\varepsilon x\| \geq \delta'\|x\|.$$

That is, the invertibility condition on $CA^{-(1-\beta)}$ becomes more restrictive when β decreases. To admit more observation operators C one therefore seeks for values of β large enough. Combining both conditions forces to play with different values of β in different situations. In the following corollary we choose $\beta = \frac{1}{2}$.

Corollary 5.2. *Let $-A$ be the generator of a semigroup of contractions $(T(t))_{t \geq 0}$ on a Hilbert space X . If A has dense range and $\|CA^{-1/2}x\| \geq \delta\|x\|$ for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, then*

$$(5.1) \quad m^2\|x\|^2 \leq \int_0^\infty \|CT(t)x\|_Y^2 dt$$

for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. In addition, if either A is invertible or C is infinite-time admissible for A then C is infinite-time exactly observable for A .

Notice that in view of Proposition 5.1, the hypothesis of a semigroup of contractions is necessary to be able to conclude in the case that C is admissible.

Proof. By the Lumer-Phillips theorem, A is an accretive operator, i.e. $\operatorname{Re}\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{D}(A)$. Since A has dense range and X is reflexive, A is actually injective (cf. [3, Theorem. 3.8]). We need to verify that A admits lower square function estimates (4.4) with functions φ that are bounded holomorphic on $S_{\pi/2}$ and continuous on $\overline{S_{\pi/2}}$. We shall explain how this follows from the functional calculus.

First notice that A has a bounded $H^\infty(\overline{S_{\pi/2}})$ functional calculus. This means that for every bounded holomorphic function φ on $S_{\pi/2}$ and continuous on $\overline{S_{\pi/2}}$, $\varphi(A)$ is well defined and

$$\|\varphi(A)\|_{\mathcal{L}(X)} \leq M \sup_{z \in \overline{S_{\pi/2}}} |\varphi(z)|.$$

This is essentially von Neumann's inequality for contractions on a Hilbert space. Indeed, if A is accretive, $T := (A - 1)(A + 1)^{-1}$ is a contraction and von Neumann's inequality states $\|p(T)\| \leq \|p\|_{H^\infty(\mathbb{D})}$ for every polynomial p . From this, the H^∞ -calculus can be derived by approximation arguments. Two different direct proofs for the boundedness of the $H^\infty(\overline{S_{\pi/2}})$ calculus are given in [6, Theorem 7.1.7]. One uses a dilation theorem of the semigroup into a unitary C_0 -group due to Sz.-Nagy. The second exploits accretivity of A from a 'numerical range' viewpoint and can be seen as the most simple case of the Crouzeix-Delyon theorems [4, 2].

Having the boundedness of the functional calculus on $\overline{S_{\pi/2}}$ in hands we certainly have upper square function estimates for functions in $H_0^\infty(S_\theta)$ when $\theta > \pi/2$. This is well known and proved by McIntosh [12]. For the particular functions $\psi_\alpha(z) := z^\alpha/(1+z)$ for $\alpha \in (0, 1)$ this means that for some positive constants k_α

$$(5.2) \quad k_\alpha \int_0^\infty \|\psi_\alpha(tA)x\|^2 \frac{dt}{t} \leq \|x\|^2.$$

Given now a function $\psi \in H_0^\infty(\overline{S_{\pi/2}})$, we choose $\varepsilon > 0$ small such that $|\psi(z)| \leq M \max(|z|^{2\varepsilon}, |z|^{-2\varepsilon})$ and write

$$\psi(z) = \psi_\varepsilon(z) \times z^{-\varepsilon} \psi(z) + \psi_{1-\varepsilon}(z) \times z^\varepsilon \psi(z).$$

Notice that $z^{\pm\varepsilon} \psi(z) \in H_0^\infty(\overline{S_{\pi/2}})$. Therefore, upper square function estimate for A with the function ψ follow from (5.2) using the boundedness of the $H^\infty(\overline{S_{\pi/2}})$ calculus.

All what we explain here works also for the adjoint A^* . By a duality argument as in [12] or [3], we pass from upper square function estimates for A^* to lower square function estimates for A . As a particular case, (SQ_β) holds with $\beta = \frac{1}{2}$. We then apply Theorem 4.3. \square

Again as in Corollary 4.6, we can obtain observability in finite time by adding a constant w to A . That is

Corollary 5.3. *Let $-A$ be the generator of a semigroup $(T(t))$ on the Hilbert space X . Suppose that there exists a constant $\omega > 0$ such that $A + \omega$ is accretive and that $C : \mathcal{D}(A) \rightarrow Y$ is bounded. If $\|C(A + \omega)^{-1/2}x\| \geq \delta\|x\|$ for all $x \in \mathcal{D}(A)$ and some $\delta > 0$, then C is exactly observable in finite time.*

Examples 5.4. 1- Consider the Schrödinger equation

$$\begin{cases} \frac{\partial u}{\partial t} &= i\Delta u \\ u &= 0 \text{ on } \partial\Omega \\ y(t) &= Cu(t) = \nabla u(t) \end{cases}$$

with $X = L^2(\Omega)$, $Y = (L^2(\Omega))^d$ and Ω is any open subset of \mathbb{R}^d with boundary $\partial\Omega$. In this problem, Δ denotes the Laplacian with Dirichlet boundary conditions. It is (the negative of) the associated operator with the symmetric form

$$\mathfrak{a}(u, v) := \int_{\Omega} \nabla u \nabla v dx, \quad \mathcal{D}(\mathfrak{a}) = W_0^{1,2}(\Omega).$$

Since

$$\|(-i\Delta)^{1/2}u\|_2^2 = \|(-\Delta)^{1/2}u\|_2^2 = \|Cu\|_2^2$$

we may conclude from Corollary 5.2 that

$$(5.3) \quad \delta \|f\|_2^2 \leq \int_0^\infty \|\nabla e^{it\Delta} f\|_2^2 dt$$

for all $f \in \mathcal{D}(\Delta) \cap \mathcal{R}(\Delta)$. Note that we can replace here the Dirichlet boundary condition by the Neumann one. The arguments are the same, we just need to replace $\mathcal{D}(\mathfrak{a}) = W_0^{1,2}(\Omega)$ by $\mathcal{D}(\mathfrak{a}) = W^{1,2}(\Omega)$. The same method applies for more general boundary conditions.

2- Let A be the uniformly elliptic operator

$$A = - \sum_{j,k=1}^d \frac{\partial}{\partial x_k} \left(a_{jk} \frac{\partial}{\partial x_j} \right)$$

with bounded measurable coefficients $a_{jk} \in L^\infty(\mathbb{R}^d)$. The operator A is defined by sesquilinear form techniques (see for example [15]) and note that A is not necessarily self-adjoint.

It is a standard fact that $-A$ generates a contraction semigroup on $L^2(\mathbb{R}^d)$. Consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} &= -Au \\ y(t) &= Cu(t) = \nabla u(t) \end{cases}$$

with $X = L^2(\mathbb{R}^d)$ and $Y = (L^2(\mathbb{R}^d))^d$. By the solution of the Kato's square root problem (see [1]), it is known that

$$(5.4) \quad \|\nabla u\|_2 \approx \|A^{1/2}u\|_2 \quad \forall u \in W^{1,2}(\mathbb{R}^d).$$

On the other hand, the accretivity of A implies an H^∞ functional calculus on $L^2(\mathbb{R}^d)$. Hence it satisfies upper and lower square function estimate (4.3) and (4.4). In particular,

$$(5.5) \quad \|f\|_2^2 \approx \int_0^\infty \|A^{1/2} e^{-tA} f\|_2^2 dt.$$

This implies that $C = \nabla$ is both admissible and exactly observable for A in infinite time.

Note that the semigroup e^{-tA} is bounded holomorphic on some sector S_ω and

$e^{-te^{iw}A}$ is a contraction on $L^2(\mathbb{R}^d)$ (see [15]). Taking the maximal angle w , we obtain a contraction semigroup $e^{-te^{iw}A}$ which is not holomorphic. Since $(e^{iw}A)^{1/2} = e^{iw/2}A^{1/2}$ we see that (5.4) holds with $e^{iw}A$ in place of A . By Corollary 5.2 we have

$$(5.6) \quad \delta \|f\|_2^2 \leq \int_0^\infty \|\nabla e^{-te^{iw}A} f\|_2^2 dt$$

for $f \in \mathcal{D}(A) \cap \mathcal{R}(A)$. We may consider the same problem on a bounded Lipschitz domain instead of \mathbb{R}^d . In this case, A is invertible. Hence $e^{iw}A$ is also invertible and we obtain (5.6) for all $f \in \mathcal{D}(A)$. This means that C is exactly observable for $e^{iw}A$.

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