ON KATO'S METHOD FOR NAVIER–STOKES EQUATIONS

BERNHARD H. HAAK AND PEER CHR. KUNSTMANN

ABSTRACT. We investigate Kato's method for parabolic equations with a quadratic non-linearity in an abstract form. We extract several properties known from linear systems theory which turn out to be the essential ingredients for the method. We give necessary and sufficient conditions for these conditions and provide new and more general proofs, based on real interpolation. In application to the Navier-Stokes equations, our approach unifies several results known in the literature, partly with different proofs. Moreover, we establish new existence and uniqueness results for rough initial data on arbitrary domains in $\mathbb{R}^3$ and irregular domains in $\mathbb{R}^n$.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a domain, i.e. an open and connected subset. In this paper we study the Navier–Stokes equation in the form

$$\text{(NSE)} \quad \begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= f, \quad (t > 0) \\ \nabla \cdot u &= 0 \\ u(0, \cdot) &= v_0 \\ u|_{\partial\Omega} &= 0. \end{cases}$$

The equation (NSE) describes the motion of an incompressible fluid filling the region Ω under "no slip" boundary conditions, where $u = u(t, x) \in \mathbb{R}^n$ denotes the velocity vector at time t at the point x, $p = p(t, x) \in \mathbb{R}$ denotes the pressure, and v_0 denotes the initial velocity field which is also assumed to be divergence–free, i.e. $\nabla \cdot v_0 = 0$. Of course, the boundary condition is not present in case $\Omega = \mathbb{R}^n$. Observe already that $\nabla \cdot u = 0$ allows to rewrite $(u \cdot \nabla)u = \nabla \cdot (u \otimes u)$.

Initiated perhaps by Cannone's work ([5]) there has been a lot of interest in the last decade in mild solutions of (NSE) (see e.g. [2, 19, 21, 24, 29]) for initial data in so-called critical spaces. All these results rely on variations of Kato's method ([9]) which allows to obtain global solutions if the initial data is small by a fixed point argument (which is based on Banach's fixed point principle or, equivalently, on a direct fixed point iteration).

The fixed point equation is obtained from (NSE) by first applying the Helmholtz projection $\mathbb{P}$ to get rid of the pressure term

$$\text{(1)} \quad \begin{cases} u_t - \mathbb{P}\Delta u + \mathbb{P}\nabla \cdot (u \otimes u) &= \mathbb{P}f, \quad (t > 0) \\ \nabla \cdot u &= 0 \\ u(0, \cdot) &= v_0 \\ u|_{\partial\Omega} &= 0. \end{cases}$$

The operator $-\mathbb{P}\Delta$ with Dirichlet boundary conditions is, basically, the *Stokes operator* A which – hopefully – is the negative generator of a bounded analytic semigroup $T(\cdot)$, the *Stokes semigroup*, in the divergence–free function space X

under consideration. Then the solution to (1) is formally given by the variation-of-constants formula

$$(2) \qquad u = T(\cdot)v_0 - T(\cdot) * \mathbb{P}\nabla \cdot (u \otimes u) + T(\cdot) * \mathbb{P}f.$$

If one can give sense to the Helmholtz projection $\mathbb{P}$, the Stokes operator A and the Stokes semigroup $T(\cdot)$, this is a fixed point equation for u. A *mild solution* to (NSE) is a solution to (2).

The nonlinearity is quadratic and may be rewritten using the bilinear map $F(u,v) := \mathbb{P}\nabla \cdot (u \otimes v)$. The natural space for a fixed point argument yielding global solutions would be $C([0,\infty), X)$, but this rarely works for critical spaces. The idea of Kato's method for the critical space $X = L^3$ on $\Omega = \mathbb{R}^3$ is to use an auxiliary space $Z = L^q$ with $q \in (3,6]$ and a weighted sup-norm with a polynomial weight t^α and to carry out the iteration scheme in a suitable function space with norm

$$\|t \mapsto u(t)\|_{L^\infty(\mathbb{R}_+, L^3)} + \|t \mapsto t^{\frac{1}{2} - \frac{3}{2q}} u(t)\|_{L^\infty(\mathbb{R}_+, L^q)}.$$

In our paper we use the notation $L^p_\alpha((0,\tau), X)$ for the space of all X-valued measurable functions f such that

$$\|f\|_{L^p_\alpha((0,\tau), X)} := \|t \mapsto t^\alpha f(t)\|_{L^p((0,\tau), X)} < \infty.$$

As Cannone observed ([5]), Kato's approach leads to Besov spaces in a natural way. On suitable domains $\Omega \neq \mathbb{R}^n$, Amann's work ([2]) underlined the fundamental role of real interpolation and of abstract extrapolation and interpolation scales. The present paper takes up this point of view.

We start our main results with an abstract version of Kato's method for parabolic equations with quadratic non-linearity (Theorem 3.1), which clearly isolates the properties one has to check for in order to obtain local solutions and global solutions for small initial data. These properties [A1], [A2], and [A3] only concern linear problems.

In the literature, there is an abstract version of Kato's method due to Weissler [38], formulated for parabolic equations with quadratic non-linearity. The approach, however, is different already for the bilinear term (see Remark 3.4), and in extension to Weissler's result we do not only consider weighted sup-norms for functions with values in an auxiliary space, but also weighted L^p–spaces with polynomial weights t^α for $p \in [2, \infty]$ (the restriction $p \geq 2$ is due to the quadratic nature of the non-linearity). Moreover, in our second main result (Theorem 3.6) we give necessary and sufficient conditions for the properties [A1], [A2], and [A3]. We were led to these results by our previous work on linear systems of the form

$$(3) \qquad \begin{cases} x'(t) + Ax(t) &= Bu(t), \quad t > 0 \\ y(t) &= Cx(t), \quad t > 0 \\ x(0) &= x_0 \end{cases}$$

Theorem 3.1 is actually a result on a quadratic feedback law $u(t) = F(y(t), y(t))$ for (3). In (3), C and B are unbounded linear operators (in the application to (NSE) they are the identity on suitable spaces, see below), and [A1] and [A2] simply mean that they are *admissible* for the corresponding weighted Bochner spaces. The conditions in Theorem 3.6 (a) and (b) are generalisations of our results in [12] to the case of not necessarily densely defined operators A. Moreover, we give here new and very transparent proofs based on real interpolation (see Section 5) whereas the proofs in [12] relied on H_0^∞–functional calculus arguments.

In Section 4 we apply our abstract results to obtain mild solutions to (NSE). On $\mathbb{R}^n$ we reobtain Cannone's result ([5]) on initial values in Besov spaces. In Subsection 4.2 we give a variant of a result due to Sawada ([29]) on time-local solutions for initial values in Besov spaces $B_{\infty,p}^{-1+\epsilon}$ with $p \in (n, \infty)$, but with a quite different proof.

For the quadratic term we simply use the product inequality for Hölder continuous functions whereas the keystone of the proof in [29] was a Hölder type inequality for functions in general Besov spaces. In Subsection 4.3 we show that, in close analogy to Subsection 4.1 one may likewise use weak Lebesgue spaces as auxiliary spaces Z which leads to mild solutions for initial values in Besov spaces that are based on weak Lebesgue spaces. This result is new. Subsection 4.4 studies mild solutions for arbitrary domains $\Omega \subset \mathbb{R}^3$, and we improve results due to H. Sohr ([32, Theorem V.4.2.2]) and S. Monniaux ([25, Theorem 3.5]). Moreover, our approach allows to compare both results. In Subsection 4.5 we assume that Helmholtz projection and Stokes semigroup act in a scale of L^q-spaces, $q \in [q_0', q_0]$, and investigate how the value of $q_0 > 2$ influences existence of mild solutions for certain initial values. It becomes apparent that, already under these relatively weak assumptions, a larger q_0 allows for more initial values, where the case $q_0 > \max(4, n)$ needs an additional gradient estimate for the Stokes semigroup. In any case, these new results make very clear which properties one has to check for the Stokes semigroup in order to obtain mild solutions for "rough" initial values, i.e.for initial values in certain extrapolation spaces.

The paper is organised as follows. In Section 2 we collect basic facts on the Helmholtz decomposition and the Stokes semigroup for arbitrary domains. Those are the basis for applications of the abstract results to (NSE) in Section 4. In Section 3 we present our abstract results, a part of the proofs is relegated to Section 5. In an appendix we have gathered facts on Besov spaces based on weak Lebesgue spaces that are needed in Subsection 4.3.

2. Preliminaries

Let $n \geq 2$ and let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open and connected subset. We start with basics on the Helmholtz decomposition in $L^q(\Omega)^n$ where $q \in (1, \infty)$. To this end we define

$$\dot{W}_q^1(\Omega) := \left\{ [u] = u + \mathbb{C} : \ u \in L_{loc}^q(\Omega) \text{ and } \nabla u \in L^q(\Omega)^n \right\}$$

with norm $\|u\|_{\dot{W}_q^1(\Omega)} := \|\nabla u\|_{L^q(\Omega)^n}$. Since the Navier–Stokes equations involve real valued functions we only consider real function spaces in the sequel.

The space $\dot{W}_q^1(\Omega)$ is a Banach space and the linear map $\nabla_q : \dot{W}_q^1(\Omega) \to L^q(\Omega)^n$, $u \mapsto \nabla u$, is isometric. We also define

$$(\dot{W}_q^1(\Omega))' := \{\phi : \ \dot{W}_q^1(\Omega) \to \mathbb{R} : \ \phi \text{ is linear and continuous}\}$$

with the usual operator norm. Then $(\dot{W}_q^1(\Omega))'$ is a Banach space and the dual map $(\nabla_{q'})' : L^q(\Omega)^n \to (\dot{W}_{q'}^1(\Omega))'$ of ∇_q is surjective with norm ≤ 1.

We recall the space $\mathscr{D}(\Omega) = C_c^\infty(\Omega)$ of test functions and the dual space $\mathscr{D}'(\Omega)$ of distributions on Ω.

Remark 2.1. If $u \in \mathscr{D}'(\Omega)$ satisfies $\nabla u \in L^q(\Omega)^n$ then u belongs to $L_{loc}^q(\Omega)$ ([26]).

Now let $G^q(\Omega) := \mathrm{Im}\nabla_q = \nabla_q \dot{W}_q^1(\Omega)$ denote the space of gradients in $L^q(\Omega)^n$ and $L_\sigma^q(\Omega) := \mathrm{Ker}(\nabla_{q'})'$ denote the space of divergence-free vector fields.

Remark 2.2. It is clear from the construction that

$$
\begin{aligned}
G^q(\Omega) &= \{f \in L^q(\Omega)^n : \ \forall g \in L_\sigma^{q'}(\Omega) : \ \langle f, g \rangle = 0\} \quad \text{and} \\
L_\sigma^q(\Omega) &= \{f \in L^q(\Omega)^n : \ \forall g \in G^{q'}(\Omega) : \ \langle f, g \rangle = 0\}.
\end{aligned}
$$

Let $\mathscr{D}_\sigma(\Omega) := \{\phi \in \mathscr{D}(\Omega)^n : \ \nabla \cdot \phi = 0\}$ denote the space of divergence-free test functions. The following theorem holds.

Theorem 2.3 (de Rham). *Let $T \in \mathscr{D}'(\Omega)^n$. There is an $S \in \mathscr{D}'(\Omega)$ with $T = \nabla S$ if and only if T vanishes on $\mathscr{D}_\sigma(\Omega)$.*

Remark 2.4. This was first noticed by J.L. Lions [22, p.67] who resorted to a result due to G. de Rham [7, th. 17', p.114]. We refer to [31] for more details and an elementary proof.

Now we are able to prove the following representation of the space $L_\sigma^q(\Omega)$ which is often taken as the definition. The argument in the proof is the same as in [22, p.67].

Proposition 2.5. *For any $q \in (1, \infty)$, the space $L_\sigma^q(\Omega)$ is the closure of $\mathscr{D}_\sigma(\Omega)$ in $L^q(\Omega)^n$.*
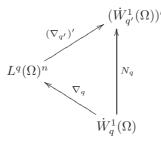
Proof. Let $\phi \in \mathscr{D}_\sigma(\Omega)$ and $u \in \dot{W}_{q'}^1(\Omega)$. Then

$$\langle (\nabla_{q'})' \phi, u \rangle = \langle \phi, \nabla_{q'} u \rangle = -\langle \nabla \cdot \phi, u \rangle = 0,$$

and $\phi \in L_\sigma^q(\Omega)$. To show density of $\mathscr{D}_\sigma(\Omega)$ in $L_\sigma^q(\Omega)$ we take $g \in L^{q'}(\Omega)^n$ such that $\langle g, \cdot \rangle$ vanishes on $\mathscr{D}_\sigma(\Omega)$ and have to show that $\langle g, \cdot \rangle$ vanishes on $L_\sigma^q(\Omega)$. By Theorem 2.3 we find $v \in \mathscr{D}'(\Omega)$ such that $\nabla v = g$. By Remark 2.1 we have $v \in L_{loc}^{q'}(\Omega)$, i.e. $g \in G^{q'}(\Omega)$. Now we use Remark 2.2. $\qquad \square$

Concerning the Helmholtz projection we quote the following theorem, which is the essence of the approach in [30].

Theorem 2.6. *Let $q \in (1, \infty)$. Then $L^q(\Omega)^n = L_\sigma^q(\Omega) \oplus G^q(\Omega)$ if and only if the operator $N_q := (\nabla_{q'})'\nabla_q : \dot{W}_q^1(\Omega) \to (\dot{W}_{q'}^1(\Omega))'$ is bijective.*

If the operator N_q has a bounded inverse $N_q^{-1} : (\dot{W}_{q'}^1(\Omega))' \to \dot{W}_q^1(\Omega)$, then the projection from $L^q(\Omega)^n$ onto $L_\sigma^q(\Omega)$ with kernel $G^q(\Omega)$ is given by $\mathbb{P}_q := I - \nabla_q N_q^{-1}(\nabla_{q'})'$. This projection is called the Helmholtz projection in $L^q(\Omega)^n$.

Proof. If N_q is bijective then its inverse N_q^{-1} is bounded by the open mapping theorem, the projection $\mathbb{P}_q$ has the desired properties and we obtain $L^q(\Omega)^n = L_\sigma^q(\Omega) \oplus G^q(\Omega)$.

Conversely, if $L^q(\Omega)^n = L_\sigma^q(\Omega) \oplus G^q(\Omega)$ then $\nabla_q \dot{W}_q^1(\Omega) \cap \mathrm{Ker}(\nabla_{q'})' = \{0\}$ and N_q is injective. Moreover $(\dot{W}_{q'}^1(\Omega))' = (\nabla_{q'})'G^q(\Omega)$, thus N_q is surjective by $G^q(\Omega) = \nabla_q \dot{W}_q^1(\Omega)$. $\qquad \square$

In [30], the operator $-N_q$ is interpreted as a weak version of the Neumann–Laplacian on Ω. For $q = 2$, N_2 is always bijective, and $\mathbb{P}_2$ is the orthogonal projection from $(L^2(\Omega))^n$ onto $L_\sigma^2(\Omega)$. This follows from Remark 2.2 or from Theorem 2.6 via Lax–Milgram.

Remark 2.7. Since $\mathscr{D}(\Omega) \subset \dot{W}_q^1(\Omega)$, Remark 2.2 shows that $u \in L_\sigma^q(\Omega)$ implies $\nabla \cdot u = 0$ in the sense of distributions. But $L_\sigma^q(\Omega)$ also contains information on the behaviour of u at the boundary (see [30], [32, Lemma II.2.5.3]): for example for bounded Lipschitz domains one has $u \in L_\sigma^q(\Omega)$ if, and only if $\nabla \cdot u = 0$ on Ω and $\nu \cdot u = 0$ on $\partial\Omega$.

Function spaces. For $q \in (1, \infty)$, we use the usual notation and write $W_q^{-1}(\Omega) := (W_{q',0}^1(\Omega))'$. Moreover, we let

$$\dot{W}_{q,0}^1(\Omega) := (W_{q,0}^1(\Omega), \|\nabla \cdot \|_q)^{\sim} \quad \text{and} \quad \dot{W}_q^{-1}(\Omega) := (\dot{W}_{q',0}^1(\Omega))',$$

where $^{\sim}$ denotes the completion. Then $\dot{W}_q^{-1}(\Omega)$ consists of all $\phi \in W_q^{-1}(\Omega)$ satisfying

$$\|\phi\|_{\dot{W}_q^{-1}} = \sup\{|\phi(v)| : \ v \in W_{q',0}^1(\Omega), \|\nabla v\|_{q'} \le 1\} < \infty.$$

The corresponding spaces of "divergence-free" vectors are

$$\mathbb{V}_q(\Omega) := W_{q,0}^1(\Omega)^n \cap L_\sigma^q(\Omega) \quad \text{and} \quad \dot{\mathbb{V}}_q(\Omega) := (\mathbb{V}_q(\Omega), \|\nabla \cdot \|_q)^{\sim}.$$

$$\dot{\mathbb{V}}_q^{-1}(\Omega) := (\dot{\mathbb{V}}_{q'}(\Omega))' = \{\phi : \ \dot{\mathbb{V}}_{q'}(\Omega) \to \mathbb{R} \ \text{linear} : \ \phi \ \text{is continuous for} \ \|\nabla \cdot \|_{q'}\}$$

with the natural operator norm. Then $\dot{\mathbb{V}}_q(\Omega)$ is a Banach space and $\mathbb{V}_q(\Omega)$ is Banach space for the norm of $W_q^1(\Omega)^n$ and a dense subset of $\dot{\mathbb{V}}_q(\Omega)$.

Lemma 2.8. *The set $\mathscr{D}_\sigma(\Omega)$ is dense in $(\mathbb{V}_q(\Omega), \|\cdot\|_{W_q^1})$ and in $(\dot{\mathbb{V}}_q(\Omega), \|\nabla \cdot \|_q)$.*

Proof. By definition it suffices to consider $\mathbb{V}_q(\Omega)$. It is clear that $\mathscr{D}_\sigma(\Omega) \subset \mathbb{V}_q(\Omega)$. Now take $\phi \in (W_{q,0}^1(\Omega)^n)'$ such that ϕ vanishes on $\mathscr{D}_\sigma(\Omega)$. Notice that ϕ is a distribution on Ω. By Theorem 2.3 there exists $h \in \mathscr{D}'(\Omega)$ satisfying $\phi = \nabla h$ and h is unique up to a constant. Since ϕ can be represented as a sum of partial derivatives of $L^{q'}$-functions, we conclude that we can assume $h \in L^{q'}(\Omega)$. For $u \in \mathbb{V}_q$ we choose a sequence (u_k) in $\mathscr{D}(\Omega)^n$ such that $u_k \to u$ in $W_{q,0}^1(\Omega)^n$, and we finally obtain

$$\phi(u) = \langle \nabla h, u \rangle = \lim_k \langle \nabla h, u_k \rangle = -\lim_k \langle h, \nabla \cdot u_k \rangle = -\langle h, \nabla \cdot u \rangle = 0$$

by $\nabla \cdot u = 0$ (see Remark 2.7). This ends the proof. $\qquad \square$

Coming back to the Navier-Stokes equation we notice that, for $u \in L^q(\Omega)^n$, we have $u \otimes u \in L^{q/2}(\Omega)^{n \times n}$ and $\nabla \cdot (u \otimes u) \in \dot{W}_{q/2}^{-1}(\Omega)^n$. Applying the Helmholtz projection to get rid of the pressure term ∇p in (NSE) thus needs extensions P_q of the Helmholtz projection $\mathbb{P}_q$ to $\dot{W}_q^{-1}(\Omega)^n$, $q \in (1, \infty)$. Those are defined by restriction (as in, e.g., [32], [25]):

$$P_q : \dot{W}_q^{-1}(\Omega)^n \to \dot{\mathbb{V}}_q^{-1}(\Omega), \qquad P_q\phi(v) := \phi|_{\dot{\mathbb{V}}_{q'}(\Omega)}.$$

Observe that this is meaningful since $\dot{\mathbb{V}}_{q'}(\Omega) \subset \dot{W}_{q',0}^1(\Omega)^n$. Moreover, P_q is linear and continuous. We show that P_q and $\mathbb{P}_q$ are consistent.

Lemma 2.9. *We have $P_q\phi = \mathbb{P}_q f$ for each $\phi \in \dot{W}_q^{-1}(\Omega)^n$ and $f \in L^q(\Omega)^n$ such that $\phi(v) = \langle f, v \rangle$ for all $v \in W_{q',0}^1(\Omega)^n$.*

Proof. It suffices to check equality on $\mathbb{V}_{q'}(\Omega) = W_{q',0}^1(\Omega)^n \cap L_\sigma^{q'}(\Omega)$. For $v \in \mathbb{V}_{q'}(\Omega)$ we have

$$P_q\phi(v) = \phi(v) = \langle f, v \rangle = \langle f, \mathbb{P}_{q'}v \rangle = \langle \mathbb{P}_q f, v \rangle$$

by $\mathbb{P}_{q'}v = v$ and $(\mathbb{P}_{q'})' = \mathbb{P}_q$. $\qquad \square$

The Stokes operator. We define the Stokes operator in $L_\sigma^2(\Omega)$ by the form method. To this end we let $\mathbb{V} := \mathbb{V}_2 = L_\sigma^2(\Omega) \cap (W_{2,0}^1(\Omega))^n$ and define the closed sesquilinear form

$$\mathfrak{a} : \mathbb{V} \times \mathbb{V} \to \mathbb{C}, \qquad \mathfrak{a}(u, v) := \int_\Omega \overline{\nabla v} \cdot \nabla u \, dx.$$

The operator A associated with $\mathfrak{a}$ is the *Stokes operator on Ω* (with Dirichlet boundary conditions). It is well-known that $D(A^{1/2}) = \mathbb{V}$ with equivalent norms (see [18]; for the definition of fractional domain spaces see Section 3). Hence $\dot{\mathbb{V}} := \dot{\mathbb{V}}_2 = (\mathbb{V}, \|\nabla \cdot \|_2)^{\sim}$ equals the homogeneous space $\dot{D}(A^{1/2})$ and the dual space

$\dot{\mathbb{V}}_2^{-1} := (\dot{\mathbb{V}}_2)'$ can be identified with the homogeneous space $(L^2_\sigma(\Omega), \|A^{-\frac{1}{2}} \cdot \|_2)^\sim$. Observe that, by Lax–Milgram, a suitable extension $\tilde{A}$ of the operator A acts as an isomorphism $\dot{\mathbb{V}}_2 \to \dot{\mathbb{V}}_2^{-1}$. The operator $-A$ generates the bounded analytic semigroup $(T(t)) = (e^{-tA})$ in $L^2_\sigma(\Omega)$, the *Stokes semigroup*.

L^q**-theory.** If there is $q_0 \in (2, \infty)$ such that the Helmholtz projection $\mathbb{P}_{q_0}$ is bounded in $L^{q_0}(\Omega)^n$ and there is a bounded analytic semigroup $T_{q_0}(\cdot)$ in L^{q_0} which is consistent with the Stokes semigroup in the sense that

$$T_{q_0}(t)f = T(t)f, \qquad \text{for all } f \in L^2_\sigma(\Omega) \cap L^{q_0}(\Omega),$$

then $T_{q_0}(\cdot)$ is called *Stokes semigroup in* $L^{q_0}_\sigma(\Omega)$ or simply *in* L^{q_0} and its negative generator A_{q_0} is called the *Stokes operator in* L^{q_0}. Observe that by interpolation and self-duality of the Stokes semigroup we then obtain for any $q \in [q_0', q_0]$ that the Helmholtz projection is L^q–bounded and that the Stokes semigroup extends to a bounded analytic semigroup in $L^q_\sigma(\Omega)$.

3. Abstract Kato method

Sectorial operators. For $0 < \omega \le \pi$ we denote by

$$S(\omega) := \{z = re^{i\phi} : r > 0, \ |\phi| < \omega\}$$

the open sector of angle 2ω in the complex plane, symmetric about the positive real axis. In addition we define $S(0) := (0, \infty)$. Let A be linear operator on a Banach space X. The resolvent set of A is denoted by $\varrho(A)$ and its spectrum by $\sigma(A)$. The operator A is called sectorial of type ω, if $\sigma(A) \subseteq \overline{S(\omega)}$ and if for all $\nu \in (\omega, \pi)$ there is a constant M with $\|\lambda R(\lambda, A)\| \le M$ for all $\lambda \in S(\pi - \nu)$. The infimum of all such angles ω is referred to as the sectoriality angle of A.

Inter- and Extrapolation spaces. Given a sectorial operator A on a Banach space X, for each $n \in \mathbb{N}$, the space $X_n := (\mathcal{D}(A^n), \|(I + A)^n \cdot \|_X)$ is a Banach space. There are other scales of inter– and extrapolation spaces. We give the definitions we need in the sequel, resorting to a construction in [11]: Let A be an injective sectorial operator on X. As above, endow $\mathcal{D}(A^k)$ with the norm $\|(I + A)^k \cdot \|$ and $\mathcal{R}(A^k)$ with the corresponding norm $\|(I + A^{-1})^k \cdot \|$. Let $L := A(I + A)^{-2}$. Then $L(X) = \mathcal{D}(A) \cap \mathcal{R}(A)$ and the sum norm on $\mathcal{D}(A) \cap \mathcal{R}(A)$ is equivalent to the norm $\|L^{-1} \cdot \| = \|(2 + A + A^{-1}) \cdot \|$. Endowing $\mathbb{X}_1 := \mathcal{D}(A) \cap \mathcal{R}(A)$ with this norm and letting $\mathbb{X}_0 := X$ makes $L : \mathbb{X}_0 \to \mathbb{X}_1$ an isometric isomorphism. Hence, by abstract nonsense we can construct a Banach space $\mathbb{X}_{-1}$ and an embedding $\iota : \mathbb{X}_0 \to \mathbb{X}_{-1}$ together with an isometric isomorphism $L_{-1} : \mathbb{X}_{-1} \to \mathbb{X}_0$ making the diagram

$$
\begin{array}{ccc}
\mathbb{X}_{-1} & \xdashrightarrow{\ L_{-1}\ } & \mathbb{X}_0 = X \\
\uparrow{\scriptstyle \iota} & & \uparrow \\
X = \mathbb{X}_0 & \xrightarrow{\ \ L\ \ } & \mathbb{X}_1 = \mathcal{D}(A) \cap \mathcal{R}(A)
\end{array}
$$

commute. Identifying $\mathbb{X}_0$ and $\iota \mathbb{X}_0$ we regard L as a restriction of L_{-1}. The operator $\mathbb{A}_{-1} := L_{-1}^{-1} A L_{-1}$ is an extension of A and again injective and sectorial of the same type in $\mathbb{X}_{-1}$. We define recursively for $k \in \mathbb{N}$ spaces $\mathbb{X}_{-k}$ and injective sectorial operators $\mathbb{A}_{-k}$ in $\mathbb{X}_{-k}$, and obtain isometric isomorphisms $L_{-k} : \mathbb{X}_{-k} \to \mathbb{X}_{-k+1}$, $k \ge 1$. In this framework, we now define homogeneous inter– and extrapolation spaces for $k \in \mathbb{N}$: let $\dot{X}_k := \mathbb{A}_{-k}^{-k}(X)$ and $\dot{X}_{-k} := \mathbb{A}_{-k}^{k}(X)$ with the natural induced norms. Then we have a scale of spaces

$$\ldots \hookrightarrow \dot{X}_n \hookrightarrow \ldots \hookrightarrow \dot{X}_1 \hookrightarrow \dot{X}_0 := X \hookrightarrow \dot{X}_{-1} \hookrightarrow \ldots \hookrightarrow \dot{X}_{-n} \hookrightarrow \ldots$$

where, for each $n \in \mathbb{Z}$, a suitable restriction of $\mathbb{A}_{-n-1}$ acts as an isometric isomorphism $\dot{X}_{n+1} \to \dot{X}_n$. For each $k \in \mathbb{N}$, we also let $X_{-k} := (I + \mathbb{A}_{-k})^k(X)$ with natural norm and denote by A_{-k} the part of $\mathbb{A}_{-k}$ in X_{-k}. This gives rise to a scale

$$\ldots \hookrightarrow X_n \hookrightarrow \ldots \hookrightarrow X_1 \hookrightarrow X_0 := X \hookrightarrow X_{-1} \hookrightarrow \ldots \hookrightarrow X_{-n} \hookrightarrow \ldots$$

where, for each $n \in \mathbb{Z}$, the operator $I + A_{-n}$ acts as an isometric isomorphism $X_{n+1} \to X_n$.

Notice that $X_{-k} = X + \dot{X}_{-k}$ and $X_k = X \cap \dot{X}_k$, $k \in \mathbb{N}$. Moreover, $\dot{X}_k + \dot{X}_{-k} = \mathbb{X}_{-k}$ and $\dot{X}_k \cap \dot{X}_{-k} = \mathcal{D}(A^k) \cap \mathcal{R}(A^k) := \mathbb{X}_k$, $k \in \mathbb{N}$ (see [11, 13] for more details). We remark that $\dot{X}_k = X_k$ for all $k \in \mathbb{Z}$ if $0 \in \varrho(A)$. In any case we have $\|x\|_{\dot{X}_k} = \|A^k x\|_X$ for $x \in \mathcal{D}(A^k)$ and $\|x\|_{\dot{X}_{-k}} = \|A^{-k} x\|_X$ for $x \in \mathcal{R}(A^k)$.

Finally we mention that, if A is densely defined with dense range, we can define the spaces above by completion, i.e.

$$\mathbb{X}_{-k} := (X, \|A^k(I+A)^{-2k} \cdot \|)^\sim, \qquad \dot{X}_k := (\mathcal{D}(A^k), \|A^k \cdot \|)^\sim,$$

$$\dot{X}_{-k} := (\mathcal{R}(A^k), \|A^{-k} \cdot \|)^\sim, \qquad X_{-k} := (X, \|(I+A)^{-k} \cdot \|)^\sim,$$

for each $k \in \mathbb{N}$, (see [16, 17, 20]). In this case, we shall also use the notation $\dot{D}(A)$ in place of $\dot{X}_1$ to make clear with respect to which operator the homogeneous domain space is taken.

Abstract Kato method. Let X, Z, W be Banach spaces and let $\tau \in (0, \infty]$. Let $-A$ generate a (not necessarily strongly continuous) bounded analytic semigroup $T(\cdot)$ on X.

Let $B \in B(W, X_{-1})$ and $C \in B(X_1, Z)$ be such that C is a closed linear operator $X \to Z$. Finally let $F : Z \times Z \to W$ be a bilinear map satisfying $\|F(y, \widetilde{y})\| \leq K\|y\| \|\widetilde{y}\|$ for some $K > 0$. We consider the abstract problem

$$(4) \qquad \begin{cases} x'(t) + Ax(t) &= Bu(t), & t > 0, \\ x(0) &= x_0, \\ y(t) &= Cx(t), & t > 0 \\ u(t) &= F(y(t), y(t)) & t > 0 \end{cases}$$

We seek for mild solutions $x(\cdot)$ in the space $C([0, \tau), X)$, i.e. for functions x satisfying

$$(5) \qquad x(t) = T(t)x_0 + \int_0^t T(t-s)BF(Cx(s), Cx(s)) \, ds.$$

Theorem 3.1. *Let $\tau \in (0, \infty]$ and $p \in (2, \infty)$ or $p = \infty$ in case $\tau < \infty$. For $\alpha \in (0, \frac{1}{2} - \frac{1}{p})$ we assume*

    [A1] *The map $x \mapsto CT(\cdot)x$ is bounded $X \to L^p_\alpha((0, \tau), Z)$.*

    [A2] *The map $(T_{-1}(\cdot)B)*$ is bounded $L^{p/2}_{2\alpha}((0, \tau), W) \to L^\infty((0, \tau), X)$.*

    [A3] *The map $(CT_{-1}(\cdot)B)*$ is bounded $L^{p/2}_{2\alpha}((0, \tau), W) \to L^p_\alpha((0, \tau), Z)$.*

Then, under the above assumptions on the operators B, C and F, for any initial value $x_0 \in X^\flat := \overline{\mathcal{D}(A)}$ (the closure being taken in X) there exits $\eta \in (0, \tau]$ such that the abstract problem (4) has a unique local mild solution x in $C([0, \eta), X^\flat)$ satisfying $Cx \in L^p_\alpha((0, \eta), Z)$. Moreover, if $\|x_0\|_X$ is sufficiently small, then the solution exists globally.

An essential ingredient for the proof is the following lemma taking care of the non-linearity (see e.g. [5, Lemma 1.2.6]).

Lemma 3.2. *Let E be a Banach space and $\mathbb{B} : E \times E \to E$ a bilinear map with $\|B(e_1, e_2)\| \leq \eta\|e_1\| \|e_2\|$ for all $e_1, e_2 \in E$. Then, for all $y \in E$ with $\|y\| < \frac{1}{4\eta}$ there exists $z \in E$ verifying $z = y + \mathbb{B}(z, z)$ and $\|z\|_E \leq 2\|y\|_E$.*

The lemma is shown by resorting to Banach's fixed-point theorem on a small ball within E.

Proof of Theorem 3.1. (Existence) Let $\eta > 0$ and $E := L_\alpha^p((0, \eta), Z)$ and consider the bilinear map

$$\mathbb{B} := \begin{cases} E \times E & \to & E \\ (x, \widetilde{x}) & \mapsto & (CT_{-1}(\cdot)B) * F(x, \widetilde{x}). \end{cases}$$

For $x_0 \in X$ we have $y := CT(\cdot)x_0 \in E$ by [A1]. For $x, \widetilde{x} \in E$ we have $F(x, \widetilde{x}) \in L_{2\alpha}^{p/2}((0, \eta), W)$. Moreover, $\mathbb{B}$ is bounded by [A3]. If $p < \infty$,

$$\|y\|_E = \left(\int_0^\eta \|t^\alpha CT(t)x_0\|_Z^p \, dt \right)^{1/p}$$

becomes small if $\eta > 0$ is small enough. For $p = \infty$, notice that by [A1], $t^\alpha T(t)x_0$ is bounded in Z and that for $x_0 \in \mathcal{D}(A)$ we have $t^\alpha T(t)x_0 \to 0$ in X_1 for $t \to 0+$ since $\alpha > 0$ and $T(\cdot)$ is strongly continuous on $\mathcal{D}(A)$. Thus, for all $p \in [2, \infty]$ and $x_0 \in X^\flat$, we can make $\|y\|_E$ arbitrary small choosing $\eta > 0$ small enough. If, on the other hand, $\tau = \infty$ and $\|x_0\|$ is small enough, assumption [A1] allows to take $\eta = \infty$.

In any case Lemma 3.2 applies and shows existence of a solution $z \in E$ satisfying $z = y + \mathbb{B}(z, z)$. Now put

$$x(t) := T(t)x_0 + \int_0^t T(t-s)BF(z(s), z(s)) \, ds$$

Then, by [A2] $x \in L^\infty((0, \eta), X)$. By definition of x and the fixed–point equation satisfied by z, $z(t) = Cx(t)$ (recall that C is closed as operator $X \to Z$). Thus, $x(\cdot)$ is a mild solution of the abstract problem (3.1) as claimed. To see that x is continuous with values in $X^\flat$ we need an additional argument: By [12, Theorem 1.8], [A2] implies $\mathcal{R}(B) \subseteq (\dot{X}_{-1}, X)_{\theta, \infty} =: \widetilde{W}$ with $\theta = 2/p + 2\alpha \in (0, 1)$. Let $b \in \mathcal{R}(B)$. Since $T(\cdot)$ is analytic, $T(t)b \in \mathcal{D}^\infty(A)$ for $t > 0$. Moreover, $\|T(t)\|_{\widetilde{W} \to X} \leq C \, t^{\theta-1}$ whence the integral $\int_0^t T(s)b \, ds$ converges absolutely within X. Since $X^\flat$ is a closed subspace of X, $\int_0^t T(s)b \, ds \in X^\flat$. This implies that the convolution of the semigroup with $\mathcal{R}(B)$–valued simple functions is continuous in $X^\flat$ and by density (recall $(p, T) \neq (\infty, \infty)$), the solution x found above is indeed continuous with values in $X^\flat$.

(Uniqueness) Assume the existence of two solutions u, v to (4) in $C([0, \eta), X^\flat)$ satisfying both $Cu, Cv \in L_\alpha^p((0, \eta), Z)$ and therefore satisfying both the fixed point equation (5). Let $\eta_0 \in (0, \eta)$. Using bilinearity and continuity of F and assumption [A3] we obtain

$$
\begin{aligned}
& \|C(u - v)\|_{L_\alpha^p((0,\eta_0),Z)} \\
= \ & \|(CT(\cdot)B) * (F(Cu, Cu) - F(Cv, Cv))\|_{L_\alpha^p((0,\eta_0),Z)} \\
\leq \ & \|(CT(\cdot)B) * (F(Cu, C(u-v)))\|_{L_\alpha^p((0,\eta_0),Z)} \\
& + \|(CT(\cdot)B) * (F(C(u-v), Cu))\|_{L_\alpha^p((0,\eta_0),Z)} \\
\leq \ & M \big(\|F(Cu, C(u-v))\|_{L_{2\alpha}^{p/2}((0,\eta_0),W)} + \|F(C(u-v), Cu)\|_{L_{2\alpha}^{p/2}((0,\eta_0),W)} \big) \\
\leq \ & M\|F\| \big(\|Cu\|_{L_\alpha^p((0,\eta_0),Z)} \|C(u-v)\|_{L_\alpha^p((0,\eta_0),Z)} \\
& \qquad + \|Cv\|_{L_\alpha^p((0,\eta_0),Z)} \|C(u-v)\|_{L_\alpha^p((0,\eta_0),Z)} \big) \\
= \ & M\|F\| \big(\|Cu\|_{L_\alpha^p((0,\eta_0),Z)} + \|Cv\|_{L_\alpha^p((0,\eta_0),Z)} \big) \|C(u-v)\|_{L_\alpha^p((0,\eta_0),Z)}.
\end{aligned}
$$

Now choosing $\eta_0 > 0$ small enough makes $\|Cu\|_{L_\alpha^p((0,\eta_0),Z)}$ and $\|Cv\|_{L_\alpha^p((0,\eta_0),Z)}$ arbitrary small which allows to conclude $Cu = Cv$ in $L_\alpha^p((0, \eta_0), Z)$. For $p < \infty$

this smallness is immediate; for $p = \infty$ we argue with $u \in C([0, \eta), X^\flat)$ as above. Thus,

$$
\begin{aligned}
u(t) &= T(t)x_0 + \int_0^t T(t-s)BF(Cu(s), Cu(s))\, ds \\
&= T(t)x_0 + \int_0^t T(t-s)BF(Cv(s), Cv(s))\, ds = v(t)
\end{aligned}
$$

for $t \in [0, \eta_0)$. Repeating the argument with $x_0 := u(\eta_0) \in X^\flat$ yields uniqueness of the solution as claimed. $\qquad\square$

Remark 3.3. Notice that in a setting of linear systems theory, assumptions [A1] and [A2] of the theorem mean weighted admissibility conditions for the observation operator C and the control operator B. We refer to [12] for more details.

Remark 3.4. In the applications to (NSE), the operators C and B are suitable identity operators. Weissler's result [38] assumes continuity of the bilinearity $Z \times X \to W$. Observe that, for general operators B and C, this leads to a different setting, whereas we are working entirely with the space Z for the fixed point argument. In applications to (NSE) this has the advantage that (tensor) products need only be defined for elements of Z, and that we can allow for spaces X with very rough initial data (see Section 4).

Corollary 3.5. *Let additionally to the situation in Theorem 3.1 Banach spaces $W^{(1)}, \ldots, W^{(m)}$ and operators $B_j \in B(W^{(j)}, X_{-1})$ be given and consider the abstract problem*

(6)
$$
\begin{cases}
x'(t) + Ax(t) &= Bu(t) + \sum_{j=1}^m B_j f_j(t), & t > 0, \\
x(0) &= x_0, \\
y(t) &= Cx(t), & t > 0 \\
u(t) &= F(y(t), y(t)) & t > 0
\end{cases}
$$

where the 'inhomogeneities' f_j satisfy $f_j \in L^{p_j}_{\beta_j}((0, \tau), W^{(j)})$ for some p_j, β_j with $1/p_j + \beta_j \in (0, 1)$. Moreover we require

[A_j2] *The maps $(T_{-1}(\cdot)B_j)*$ are bounded $L^{p_j}_{\beta_j}((0, \tau), W^{(j)}) \to L^\infty((0, \tau), X)$.*

[A_j3] *The maps $(CT(\cdot)B_j)* : L^{p_j}_{\beta_j}((0, \tau), W^{(j)}) \to L^p_\alpha((0, \tau), Z)$ are bounded.*

for all $j = 1, \ldots, m$. Then time-local mild solutions always exist in case $p < \infty$. In case $p = \infty$ or in order to obtain global solutions a smallness condition on the norms of the functions f_j has to be imposed $(j = 1, \ldots, m)$.

Proof. As for Theorem 3.1 but with $y = CT(\cdot)x_0 + \sum_{j=1}^m (CT(\cdot)B_j) * f_j$. $\qquad\square$

Before coming to applications in Section 4 we sum up necessary and sufficient conditions for the assumptions [A1] – [A3] in Theorem 3.1. Notice that [A_j2] and [A2] are of the same type whereas [A3] is a special case of [A_j3].

Theorem 3.6. *Assume $p \in (2, \infty]$ and $\alpha \in \mathbb{R}$ such that $\alpha + 1/p \in (0, 1/2)$. Then*

(a) *If [A1] holds, then C is bounded in the norm $(X, \dot{X}_1)_{\alpha+1/p, 1} \to Z$. The converse is true provided that $X \hookrightarrow (\dot{X}_{-1}, \dot{X}_1)_{1/2, p}$.*

(b) *If [A2] holds, then B is bounded in the norm $W \to (\dot{X}_{-1}, X)_{2(\alpha+1/p), \infty}$. The converse is true in case $\alpha > 0$ or in case $\alpha = 0$ and $(\dot{X}_{-1}, \dot{X}_1)_{1/2, 1/2} \hookrightarrow X$.*

(c) *Condition [A_j3] holds provided that $\|CT(t)B\|_{W \to Z} \leq c\,t^{-\gamma}$ for $t \in (0, \tau)$ and that $\beta + \gamma + 1/q = 1 + \alpha + 1/p$ where $\gamma \in (0, 1)$ and $\alpha, \beta > 0$.*

A proof of the theorem and some additional results will be provided in Section 5. As the proof will actually show, the restriction $p > 2$ (instead of $p > 1$) and $\alpha + \frac{1}{p} < \frac{1}{2}$ (instead of < 1) in the above formulation is only due to the bilinear structure which forces to consider the parameters $\frac{p}{2}$ and 2α in part (b). We mention that in part (a) (and in part (b) in case $\alpha = 0$) of the Theorem the embedding assumption on the space X is optimal. This follows by choosing $C = A^{\alpha + \frac{1}{p}}$ and $B = A^{1 - \frac{2}{p}}$, see also the discussion in [12, Section 1].

Remark 3.7. As mentioned above, condition $[A_j 3]$ contains condition $[A3]$ as special cases letting $p_j = \frac{p}{2}$ and $\beta_j = 2\alpha$. Here, and for one direction of part (b) in case $\alpha > 0$, the proof bases on a classical one-dimensional convolution estimate due to HARDY and LITTLEWOOD, see Lemma 5.7.

Remark 3.8. Suppose that $\tau < \infty$ and $\beta + \gamma + \frac{1}{q} \leq 1 + \alpha + \frac{1}{p}$, one finds $\widetilde{\gamma} > \gamma$ such that $\|CT(t)B\|_{W \to Z} \leq \widetilde{c}\, t^{-\widetilde{\gamma}}$ and $\beta + \widetilde{\gamma} + \frac{1}{q} = 1 + \alpha + \frac{1}{p}$. Moreover, when $\tau < \infty$, one can without loss of generality assume A to be boundedly invertible (see also [12, Lemma 1.3]).

The rôle of maximal regularity for recovering the pressure terms. From now on we shall always use $C = \mathrm{Id}_Z$ and $B = \mathrm{Id}_W$, i.e. we suppose $X_1 \hookrightarrow Z$ and $W \hookrightarrow X_{-1}$. Consequently, the semigroup $T(t)$ acts (pointwise as a bounded operator) $X \to Z$ and $W \to X$. In this setting, the abstract Kato problem (4) takes the form

$$
(7) \qquad \begin{cases} x'(t) + Ax(t) &= F(x(t), x(t)) \\ x(0) &= x_0 \end{cases}
$$

Remark 3.9. The embedding assumptions in Theorem 3.6 (a) and (b) concerning the spaces Z and W are equivalent to pointwise growth estimates for the semigroup. Indeed, for $\sigma, \theta \in (0, 1)$ one has

$$
(X, \dot{X}_1)_{\sigma, 1} \hookrightarrow Z \quad \text{if, and only if} \quad \|T(t)\|_{X \to Z} \leq c\, t^{-\sigma}
$$

and

$$
W \hookrightarrow (\dot{X}_{-1}, X)_{\theta, \infty} \quad \text{if, and only if} \quad \|T(t)\|_{W \to X} \leq c\, t^{\theta - 1}
$$

Proof. Since the semigroup is bounded and analytic, one has the elementary estimates

$$
\|T(t)\|_{\dot{X}_n \to \dot{X}_{n+1}} \leq C\, t^{-1} \quad \text{and} \quad \|T(t)\|_{\dot{X}_n \to \dot{X}_n} \leq M
$$

for $n \in \mathbb{Z}$. Therefore, the embedding properties for Z and W imply the growth estimates of the semigroup acting $X \to Z$ and $W \to X$ by interpolation. Conversely, the estimate $\|T(t)\|_{X \to Z} \leq c\, t^{-\sigma}$ for $t > 0$ implies

$$
\|x\|_Z = \widetilde{c} \left\| \int_0^\infty AT(2t)x\, dt \right\|_Z \leq \widetilde{c} \int_0^\infty \left\| AT(2t)x \right\|_Z dt
$$

$$
\leq c \int_0^\infty t^{-\sigma} \left\| AT(t)x \right\|_X dt = c\, \|x\|_{(X, \dot{X}_1)_{\sigma, 1}}
$$

for $x \in X_1 = \dot{X}_1 \cap X$ which is dense in $(X, \dot{X}_1)_{\sigma, 1}$. Finally, the estimate $\|T(t)\|_{W \to X} \leq c\, t^{\theta - 1}$ for $t > 0$ implies

$$
\|x\|_{(\dot{X}_{-1}, X)_{\theta, \infty}} = \sup_{t > 0} t^{-\theta} \|tAT(t)x\|_{\dot{X}_{-1}} = \sup_{t > 0} t^{1 - \theta} \|T(t)x\|_X \leq c\, \|x\|_W
$$

which finishes the proof. $\qquad\square$

Given an abstract Cauchy problem of the form

$$
(8) \qquad\qquad x'(t) + Ax(t) = f(t), \qquad x(0) = 0
$$

on a Banach space W, we say that A has *maximal L^p–regularity*, $p \in [1, \infty]$ if the mild solution x to (8) satisfies $x', Ax \in L^p((0, \tau), W)$ whenever $f \in L^p((0, \tau), W)$. We refer to [1, 6, 8, 13, 20, 37] for this relation, the problem of maximal regularity, characterisation results, and further references on the subject.

Theorem 3.10. *Suppose $\tau \in (0, \infty]$, $p \in (2, \infty]$ and $\alpha \in \mathbb{R}$ such that $\alpha + \frac{1}{p} \in (0, \frac{1}{2})$. Let X, Z, W be Banach spaces satisfying*

$$(9) \qquad\qquad W \hookrightarrow (\dot{X}_{-1}, X)_{2(\alpha + \frac{1}{p}), \infty},$$

$$(10) \qquad\qquad (X, \dot{X}_1)_{\alpha + \frac{1}{p}, 1} \hookrightarrow Z, \qquad and$$

$$(11) \qquad\qquad X \hookrightarrow (\dot{X}_{-1}, \dot{X}_1)_{\frac{1}{2}, p}$$

and assume that $-A$ is injective and generates consistent bounded analytic semigroups on X and W. Let A have maximal $L^{\frac{p}{2}}$–regularity on W. Then for every $x \in X^\flat$, the abstract problem (7) has a unique time-local mild solution

$$x \in C([0, \eta), X) \cap L^p_\alpha((0, \eta), Z) \cap L^{\frac{p}{2}}_{2\alpha}((0, \eta), W)$$

that satisfies $x', Ax \in L^{\frac{p}{2}}_{2\alpha}((0, \eta), W) + L^p_{\alpha+1}((0, \eta), Z)$.

Proof. By Theorem 3.6, equation (7) has a time-local mild solution x as claimed for some $\eta \in (0, \tau)$. The Prüß-Simonett theorem (see [28, Theorem 2.4], also [12, Theorem 1.13]) shows that maximal $L^{\frac{p}{2}}$–regularity of A transfers to maximal $L^{\frac{p}{2}}_{2\alpha}$–regularity in W (recall that $\alpha + \frac{1}{p} < \frac{1}{2}$); this result is also true in case $p = \infty$, as an inspection of the proof shows (it is actually even more easy to prove than for finite p). Let x be the mild solution to (7). Writing

$$x(t) = T(t)x_0 + \big(T * F(x, x)\big)(t) = x_1(t) + x_2(t)$$

one deduces from maximal $L^{\frac{p}{2}}_{2\alpha}(W)$–regularity that $x_2 \in W_1 = \mathcal{D}(A_W)$ a.e. and that x_2 is a.e. differentiable satisfying $x_2', Ax_2 \in L^{\frac{p}{2}}_{2\alpha}((0, \eta), W)$.

Using Proposition 5.2, we have $x_1' = -Ax_1 \in L^p_{\alpha+1}((0, \eta), (X, \dot{X}_1)_{\alpha + \frac{1}{p}, 1})$ for

$$x_0 \in \big((\dot{X}_{-1}, X)_{\alpha + \frac{1}{p}, 1}, (\dot{X}_1, \dot{X}_2)_{\alpha + \frac{1}{p}, 1}\big)_{1 - \frac{1}{2}(\alpha + \frac{1}{p} + 1), p} = (\dot{X}_{-1}, \dot{X}_1)_{\frac{1}{2}, p}$$

where the equality is due to reiteration. By (11), this condition holds for $x_0 \in X^\flat$. Finally, assumption (10) finishes the proof. $\square$

Observe if one has $X \hookrightarrow (\dot{X}_{-1}, \dot{X}_1)_{\frac{1}{2}, \frac{p}{2}}$ in place of (11) one obtains $x', Ax \in L^{\frac{p}{2}}_{2\alpha}((0, \eta), W)$ by similar arguments. This applies in particular in case $p = \infty$.

For $p \in [2, \infty)$, the da Prato-Grisvard theorem ([6], see also [13, Theorem 9.3.9]) provides several function spaces in which negative generators of analytic semigroups have maximal $L^{\frac{p}{2}}$–regularity. In our situation, a particularly import class are real interpolation spaces of the form $(\dot{X}_{-k}, \dot{X}_k)_{\theta, r}$ for some $k \in \mathbb{N}$ and $\theta \in (0, 1)$, $r \in [1, \infty]$. When $1 < p < \infty$, maximal L^p–regularity is independent of p (see [3]). In case $p = \infty$, the following lemma may be used to verify L^∞–maximal regularity.

Lemma 3.11. *Let the injective operator $-A$ generate a (not necessarily densely defined) bounded analytic semigroup on W and let U be a Banach space, such that $\|T(t)\|_{W \to U} \le c\, t^{-1}$ for some $c > 0$. If $(W, \dot{W}_2)_{\frac{1}{2}, \infty} \hookrightarrow U$, then*

$$\operatorname{ess.\,sup}_{t > 0} \left\| \int_0^t T(t-s)w(s)\, ds \right\|_U \le C\, \|w\|_{L^\infty(\mathbb{R}_+, W)}$$

for all $w \in L^\infty(\mathbb{R}_+, W)$.

Proof. It is clearly sufficient to verify

$$\left\|\int_0^\infty T(s)w(s)\,ds\right\|_{(W,\dot{W}_2)_{1/2,\infty}} \le C\|w\|_{L^\infty(\mathbb{R}_+,W)}$$

for all $w \in L^\infty(\mathbb{R}_+,W)$. Using Proposition 5.2 one has

$$
\begin{aligned}
\left\|\int_0^\infty T(s)w(s)\,ds\right\|_{(W,\dot{W}_2)_{1/2,\infty}} &\sim \left\|t \mapsto tA^2T(t)\int_0^\infty T(s)w(s)\,ds\right\|_{L^\infty(\mathbb{R}_+,W)} \\
&= \left\|t \mapsto \int_0^\infty tA^2T(t+s)w(s)\,ds\right\|_{L^\infty(\mathbb{R}_+,W)} \\
&\le \operatorname*{ess\,sup}_{t>0}\int_0^\infty \tfrac{t}{(t+s)^2}\|w(s)\|_W\,ds \\
&\le \|w\|_{L^\infty(\mathbb{R}_+,W)}\int_0^\infty \tfrac{1}{(1+\sigma)^2}\,d\sigma
\end{aligned}
$$

by substituting $s = t\sigma$. $\qquad\square$

Remark 3.12. By Theorem 3.10 one can give a sense to the first line in (8) for a.e. $t > 0$ in the time interval under consideration. In applications to the Navier-Stokes (NSE) equations this means that

$$u'(t) + Au(t) + P\nabla \cdot (u(t) \otimes u(t)) - Pf(t) = 0$$

for a.e. $t > 0$. Interpreting A as $-P\Delta$ and the operator P as restriction $P : \mathscr{D}'(\Omega)^n \to \mathscr{D}'_\sigma(\Omega)$ (see e.g. [25, 32]) we are led to

$$P\big(u_t - \Delta u(t) + \dot{\nabla}(u \otimes u) - f\big) = 0$$

if $u_t - \Delta u(t) + \nabla(u \otimes u) - f \in \mathscr{D}'(\Omega)$ at a fixed time $t > 0$. Now the pressure term ∇p can be recovered by Theorem 2.3 which passes from (1) back to (NSE).

The equality $A = -P\Delta$ is no problem in case $\Omega = \mathbb{R}^n$ since then Δ commutes with P. If $\Omega \subseteq \mathbb{R}^n$ is bounded or an exterior domain with $\partial\Omega \in C^{1,1}$, equality $A = -\mathbb{P}\Delta_D$ holds on $\mathcal{D}(A_q) = W_q^2(\Omega)^n \cap W_{q,0}^1(\Omega)^n \cap L_\sigma^q(\Omega)$ for $q \in (1,\infty)$ where Δ_D denotes the Dirichlet Laplacian on Ω. On arbitrary domains $\Omega \subseteq \mathbb{R}^3$, equality $A = -P\Delta$ holds on $\mathbb{V} = \mathbb{V}_2$, see [25].

4. Application to the Navier-Stokes equations

In this section we apply the abstract result to the Navier–Stokes equations (NSE), where (4) corresponds to (1) and (5) corresponds to (2). In these applications we always have $C = \operatorname{Id}_Z$ and $B = \operatorname{Id}_W$ which means that the necessary conditions in Theorem 3.6 (a) and (b) boil down to continuous embeddings or via Remark 3.9 to decay estimates for the semigroup.

It turns out that the choice of the "auxiliary space" Z is most significant. The structure of the map F then determines the space W, and one can calculate the exponent γ for which $\|T(t)\|_{W \to Z} \le C\,t^{-\gamma}$. Depending on the context, this may hold on $(0,\infty)$ or on bounded time intervals $(0,\tau)$ where $\tau < \infty$. Observe that an application of Theorem 3.6 (c) (and Remark 3.7) requires $\gamma \in (1/2, 1)$ and restricts α and p to $\alpha + 1/p \le 1 - \gamma$ for local solutions and to $\alpha + 1/p = 1 - \gamma$ for global solutions. Nevertheless, we have some freedom for the choice of α and p. Once α and p are fixed, Theorem 3.6 (a) and (b) allow to adjust the space X for initial values appropriately.

In the sequel we discuss various choices of Z on $\mathbb{R}^n$ and on domains. The common approach covers some known results, provides new proofs for other known results, but it also yields new results on $\mathbb{R}^n$ and on domains.

4.1. **Lebesgue spaces on $\mathbb{R}^n$.** Here and in the following subsections we consider the Navier–Stokes equations (NSE) on $\mathbb{R}^n$, $n \geq 2$. For simplicity we shall omit $\mathbb{R}^n$ and superscripts n or $n \times n$ in notation.

Let $q \in (n, \infty)$ and consider the case $Z = L^q$. For $u, v \in Z$ we then have $\nabla \cdot (u \otimes v) \in \dot{H}^{-1}_{q/2} =: W$. Notice that $\|T(t)\|_{\dot{H}^{-1}_{q/2} \to \dot{H}^{\delta}_{q/2}} \leq c\, t^{-(1+\delta)/2}$, $t > 0$, and that $\dot{H}^{\delta}_{q/2} \hookrightarrow L^q$ provided $1/q = 2/q - \delta/n$, i.e. provided $\delta = n/q$ (see e.g. [35, Theorems 2.7.1 and 5.2.5]). We obtain $\|T(t)\|_{W \to Z} \leq c\, t^{-\gamma}$, $t > 0$, where $\gamma = \frac{1}{2} + \frac{n}{2q} \in (1/2, 1)$. Hence we should have $\alpha + 1/p = 1 - \gamma = \frac{1}{2} - \frac{n}{2q}$ which restricts p to $p \in [\frac{2q}{q-n}, \infty]$. For such a p, consider the space $X = \dot{B}^{-1+n/q}_{q,p}$. Then

$$(\dot{X}_{-1}, \dot{X}_1)_{1/2, p} = X \quad \text{and} \quad (\dot{X}_{-1}, \dot{X}_1)_{1/2, n/2} = \dot{B}^{-1+n/q}_{q, n/2} \hookrightarrow X$$

whence Theorem 3.6 allows to verify [A1] and [A2] of Theorem 3.1 easily. Indeed, we have $(X, \dot{X}_1)_{\alpha + 1/p, 1} = \dot{B}^s_{q,1}$ with $s = 2\alpha + 2/p + n/q - 1$ and $C = \mathrm{Id}_Z$ certainly satisfies the condition of Theorem 3.6 (a) if $\dot{B}^s_{q,1}$ embeds into Z which is the case if $s = 0$, i.e. if $\alpha + 1/p = \frac{1}{2} - \frac{n}{2q}$.

Moreover, $(\dot{X}_{-1}, X)_{2(\alpha + 1/p), \infty} = \dot{B}^t_{q,\infty}$ with $t = 4(\alpha + 1/p) + n/q - 3$ whence B satisfies the condition in Theorem 3.6 (b) if $W \hookrightarrow \dot{B}^t_{q,\infty}$ which happens by

$$\dot{H}^{-1}_{q/2} \hookrightarrow \dot{H}^{-1-n/q}_q \hookrightarrow \dot{B}^{-1-n/q}_{q,q} \hookrightarrow \dot{B}^t_{q,\infty}$$

if $-1 - n/q = t$, i.e. if $\alpha + 1/p = \frac{1}{2} - \frac{n}{2q}$ (see [35, Theorems 2.7.1 and 5.2.5]).

Finally, using Theorem 3.6 (c) and Remark 3.7 the values of γ and p determine α by

$$(12) \qquad \alpha + 1/p = \frac{1}{2} - \frac{n}{2q},$$

and then [A1] and [A2] are satisfied by the arguments above. We sum up the above considerations in the following theorem.

Theorem 4.1. *Let $n \geq 2$, $q \in (n, \infty)$, and let $\alpha \geq 0$ and $p \in (2, \infty]$ such that (12) holds. Let $X = \dot{B}^{-1+n/q}_{q,p}(\mathbb{R}^n)$. Then the Navier-Stokes equation (NSE) admits a time-local mild solution in $C([0, \tau), X)$ for every $u_0 \in X^\flat = \overline{\mathcal{D}(A)}$ satisfying $\nabla \cdot u_0 = 0$. The solution is unique in $C([0, \tau), X) \cap L^p_\alpha((0, \tau), L^q(\mathbb{R}^n))$. If the norm $\|u_0\|_X$ is sufficiently small, the solution exists globally.*

Remark 4.2. Notice that $X^\flat = X$ in case $p < \infty$ whereas in case $p = \infty$, $X^\flat$ equals the (homogeneous) *little Besov space* $\dot{b}^{-1+n/q}_{q,\infty}$ or the (homogeneous) *little Nikolski space* $\dot{n}^{-1+n/q}_q$ (see e.g. [2, 29] for the inhomogeneous counterparts).

Remark 4.3. If we are interested in time–local solutions and use Remark 3.8 we are led to $\alpha + 1/p \leq 1 - \gamma$ which is equivalent to $2\alpha + 2/p + n/q \leq 1$ since $\gamma = \frac{1}{2} + \frac{n}{2q}$. For $\alpha = 0$ we obtain Serrin's uniqueness condition $2/p + n/q \leq 1$ for weak solutions (see e.g. [32, V.1.5]). In this context we remark that the argument that proved uniqueness in Theorem 3.1 can be used to show uniqueness of weak solutions $u, w \in L^p_\alpha((0, \tau), L^q(\Omega)^n)$ with the same initial value, but that the assumptions in, e.g., [32, V.Thm.1.5.1] are somewhat weaker and involve energy inequalities.

Remark 4.4. (a) In case $n = 3$, $q > 3$, and $p = \infty$ we have $\alpha = \frac{1}{2} - \frac{3}{2q}$ and reobtain a result similar to Cannone [5, Theorem 3.3.4]. There smallness is measured in X but the initial value u_0 is taken in L^3 and the solution is required to belong to $C_b([0, \tau), L^3)$. Since the action $T(t) : W = \dot{H}^{-1}_{q/2} \to L^3$ is needed, this leads to the restriction $q < 6$ (see [5]).

(b) The general case $n \geq 2$, $q > n$ and $p = \infty$, $\alpha = \frac{1}{2} - \frac{n}{2q}$ is due to AMANN [2] whose proof is similar to taking $W = \dot{H}_q^{-1-n/q}$ for $Z = L^q$. However, [2] also covers the case of (sufficiently smooth) domains $\Omega \subset \mathbb{R}^n$, we shall come back to this in Section 4.5 below.

(c) In [29], SAWADA shows existence of time–local mild solutions for divergence–free initial values $u_0 \in \overline{\mathcal{D}(A)}$ for the inhomogeneous space $X = B_{q,p}^{-1+n/q+\epsilon}$ where $q \in (n, \infty]$, $p \in [1, \infty]$ and $\epsilon \in (0, 1)$. Observe that Theorem 4.1 yields, for $q \in (n, \infty)$ and $p \in [\frac{2q}{q-n}, \infty]$, local solutions for divergence–free initial values in the space $X = \dot{B}_{q,p}^{-1+n/q}$ (i.e. for $\epsilon = 0$) and global solutions for small initial data (which is not covered by the result in [29]). Moreover, the proof in [29] relied on a Hölder type inequality for products of Besov space functions whereas our proof simply uses the Hölder inequality for the product of two L^q–functions. We also remark that we can obtain time-local solutions for the inhomogeneous space $X = B_{q,p}^{-1+n/q}$ by considering $\tau < \infty$ and $W = H_{q/2}^{-1}$. We shall discuss the case $q = \infty$ of Sawada's result in the following Subsection 4.2.

4.2. **Hölder spaces on $\mathbb{R}^n$.** We seek for time-local solutions in this case. In view of Remark 3.8, we can assume A to be boundedly invertible which simplifies the calculation of inter- and extrapolation spaces.

For fixed $\epsilon \in (0, 1)$, we consider $Z := (C^\epsilon)^n = (B_{\infty,\infty}^\epsilon)^n$. Then for $u, v \in Z$, one has $u \otimes v \in (C^\epsilon)^{n \times n} \subset (\dot{C}^\epsilon)^{n \times n}$ and thus $\nabla \cdot (u \otimes v)$ belongs to the space

$$W := \nabla \cdot (\dot{C}^\epsilon)^{n \times n} := \{\nabla \cdot (v_{jk}) : (v_{jk}) \in (\dot{C}^\epsilon)^{n \times n}\},$$

which we equip with the natural quotient–like norm

$$\|(v_k)\|_{\nabla \cdot \dot{C}^\epsilon} := \inf\{\|(w_{jk})\|_{\dot{C}^\epsilon} : \nabla \cdot (w_{jk}) = (v_k)\}.$$

Observe that, in a canonical way, $\nabla \cdot (\dot{C}^\epsilon)^{n \times n}$ equals $(\nabla \cdot (\dot{C}^\epsilon)^n)^n$, and that W is a space of distributions although $\dot{C}^\epsilon$ is not. Since Riesz transforms are bounded on $\dot{C}^\epsilon$ (see, e.g. [15, Corollary 2.2] they are bounded on W, and therefore the Helmholtz projection is bounded on W (the basic idea is that the origin, in which the symbol $\xi_k/|\xi|$ is not differentiable, plays no rôle when considering *homogeneous* Besov spaces).

We claim that $\|T(t)\|_{W \to Z} \leq C \max(1, t^{-1/2})$, $t > 0$. Denoting by $(S(\cdot)) = (G(\cdot)*)$ the heat semigroup on $\mathbb{R}^n$, we have by translation invariance, for $t \in (0, \infty)$,

$$\|S(t)\|_{W \to W} \leq 1 \quad \text{and} \quad \|S(t)\|_{W \to \dot{C}^\epsilon} \leq C\, t^{-1/2},$$

the latter by writing $S(t) \sum_j \partial_j w_j = \sum_j (\partial_j G(t)) * w_j$ and using the fact that $\|\partial_j G(t)\|_{L^1} \leq C\, t^{-1/2}$. Consequently, $\|T(t)\|_{W \to W \cap \dot{C}^\epsilon} \leq C \max(t^{-1/2}, 1)$, and it rests to show $W \cap \dot{C}^\epsilon \hookrightarrow Z$, which in turn follows from $W \cap \dot{C}^\epsilon \hookrightarrow L^\infty$. To this end we observe that any $f \in W$ belongs to $\dot{B}_{\infty,\infty}^{\epsilon-1}$ and thus has a Littlewood–Paley decomposition $f = \sum_{k \in \mathbb{Z}} f_k$, for which we obtain

$$
\begin{aligned}
\left\| \sum_{k \in \mathbb{Z}} f_k \right\|_\infty &\leq \sum_{k \in \mathbb{Z}} \|f_k\|_\infty \\
&= \sum_{k \geq 0} 2^{-k\epsilon} \left(2^{k\epsilon} \|f_k\|_\infty\right) + \sum_{k < 0} 2^{-k(\epsilon-1)} \left(2^{k(\epsilon-1)} \|f_k\|_\infty\right) \\
&\leq \left(\sum_{k \geq 0} 2^{-k\epsilon}\right) \|f\|_{\dot{B}_{\infty,\infty}^\epsilon} + \left(\sum_{k \geq 0} 2^{-k(1-\epsilon)}\right) \|f\|_{\dot{B}_{\infty,\infty}^{\epsilon-1}}.
\end{aligned}
$$

Since we are on a finite time interval we can choose $\gamma = \frac{1+\delta}{2} > \frac{1}{2}$ where $\delta > 0$ is small. Then $\|T(t)\|_{W \to Z} \leq c\, t^{-\gamma}$ on $(0, \tau)$ for some $c = c_{\gamma,\tau} > 0$. Such choice of γ implies $\alpha + \frac{1}{p} = 1 - \gamma = \frac{1-\delta}{2} < \frac{1}{2}$ whence we obtain for p the range $[\frac{2}{1-\delta}, \infty]$. For such a p we let $X = B_{\infty,p}^{-2(\alpha+1/p)+\epsilon} = B_{\infty,p}^{-1+\delta+\epsilon}$. Then

$$(X_{-1}, X_1)_{1/2, p} = X \quad \text{and} \quad (X_{-1}, X_1)_{1/2, p/2} = B_{\infty, p/2}^{-1+\delta+\epsilon} \hookrightarrow X.$$

Moreover,

$$(X, X_1)_{\alpha+1/p, 1} = B_{\infty,1}^{\epsilon+\delta} \hookrightarrow Z \quad \text{and} \quad W \hookrightarrow \dot{B}_{\infty,\infty}^{-1+\epsilon} \hookrightarrow B_{\infty,\infty}^{-1+\epsilon} \hookrightarrow B_{\infty,\infty}^{-1-\delta+\epsilon}$$

where the latter space equals $(X_{-1}, X)_{2(\alpha+1/p), \infty}$ (recall $2(\alpha + \frac{1}{p}) = 1 - \delta$). We refer to, e.g., [36, Theorem 2.8.1]. Therefore, the assumptions of Theorem 3.6 (a) and (b) are satisfied, as well.

Theorem 4.5. *Let $n \geq 2$, $p \in (2, \infty]$, and $\epsilon \in (0, 1)$. Let $\alpha > 0$ be such that $\alpha + \frac{1}{p} < \frac{1}{2}$. Then the Navier-Stokes equation* (NSE) *admits a time-local mild solution in $C([0, \tau), X)$ for every divergence–free $u_0 \in X = B_{\infty,p}^{-2(\alpha+1/p)+\epsilon}(\mathbb{R}^n)$, which is unique in the space $C([0, \tau), X) \cap L_\alpha^p((0, \tau), C^\epsilon(\mathbb{R}^n))$.*

Remark 4.6. The result by Sawada [29] also covers time–local solutions for initial values in spaces $B_{\infty,p}^{-1+\epsilon}$ for p up to ∞. However, the space for uniqueness does not involve $L_\alpha^p(C^\epsilon)$ but L_β^∞–spaces with values in certain Besov spaces. This is due to the fact that the key stone in [29] is a Hölder type inequality for products in (inhomogeneous) Besov spaces which is proved there by means of Littlewood-Paley decomposition and paraproducts. Our proof uses the simple product inequality in C^ϵ instead, and we obtain the second index p in X by taking L^p in time. So, in our proof, improvement comes from a better understanding of the *linear* ingredients for the problem whereas in [29] it comes from a new insight for the *non-linearity*.

4.3. **Weak Lebesgue spaces on $\mathbb{R}^n$.** In this section we consider as space Z the weak Lebesgue space $L^{q,\infty}$ for a fixed $q \in (n, \infty)$. For the definition of weak Lebesgue spaces and subsequently used embedding and interpolation results, see the Appendix in Section 6. The analysis follows the lines of Section 4.1.
For $u, v \in L^{q,\infty}$, clearly $u \otimes v \in L^{q/2,\infty}$ and therefore $W := \dot{H}_{(q/2,\infty)}^{-1}$ guarantees $\nabla \cdot (u \otimes v) \in W$. Notice that

$$\|T(t)\|_{W \to \dot{H}_{(q/2,\infty)}^{\delta}} \leq c\, t^{-\gamma}, \quad t > 0,$$

with $\gamma = \frac{1+\delta}{2}$ by bounded analyticity of the semigroup. By (37) in the proof of Lemma 6.2 we have the embedding $\dot{H}_{(q/2,\infty)}^{n/q} \hookrightarrow L^{q,\infty}$. Thus $\delta = \frac{n}{q}$ yields the estimate $\|T(t)\|_{W \to Z} \leq c\, t^{-\gamma}$, $t > 0$, required in Theorem 3.6 (c) and Remark 3.7 with $\gamma = \frac{1}{2} + \frac{n}{2q}$. Choosing α and p such that

$$(13) \qquad\qquad \alpha + \frac{1}{p} = 1 - \gamma = \frac{1}{2} - \frac{n}{2q},$$

condition [A3] is satisfied. Moreover, letting $X := \dot{B}_{(q,\infty),p}^{-1+n/q}$, the embeddings

$$(14) \qquad X = (\dot{X}_1, \dot{X}_{-1})_{1/2, p} \quad \text{and} \quad (\dot{X}_1, \dot{X}_{-1})_{1/2, p/2} = \dot{B}_{(q,\infty), p/2}^{-1+n/q} \hookrightarrow X$$

hold (for a proof check the corresponding interpolation properties of vector-valued l_p^s–spaces [36, Theorem 1.18.2] and apply a retraction / coretraction argument). Thus, we can employ Theorem 3.6 in the above setting for verification of the assumptions of Theorem 3.1 [A1] and [A2].

The same arguments that proved (14) also show

$$(X, \dot{X}_1)_{\alpha+1/p,1} = \dot{B}^s_{(q,\infty),1} \quad \text{where} \quad s = -1 + n/q + 2(\alpha + 1/p),$$

$$(\dot{X}_{-1}, X)_{2(\alpha+1/p),\infty} = \dot{B}^t_{(q,\infty),\infty} \quad \text{where} \quad t = 4(\alpha + 1/p) + n/q - 3.$$

By the embedding property (36) (see Appendix, page 28) the first space embeds into $Z = L^{q,\infty}$ for $s = 0$ which holds by (13). Consequently [A1] is satisfied. Similarly, for verification of assumption [A2], we observe that W embeds into the second space by Lemma 6.2 provided that $\alpha + 1/p = \frac{1}{2} - \frac{n}{2q}$ which again holds by (13).

Theorem 4.7. *Let $n \geq 2$, $q \in (n,\infty)$ and let $\alpha \geq 0$ and $p \in (2,\infty]$ such that (13) holds. Let $X := \dot{B}^{-1+n/q}_{(q,\infty),p}$. Then the Navier-Stokes equation* (NSE) *admits a time-local mild solution in $C([0,\tau), X)$ for every $u_0 \in X^\flat = \overline{\mathcal{D}(A)}$ satisfying $\nabla \cdot u_0 = 0$. The solution is unique in $C([0,\tau), X) \cap L^p_\alpha((0,\tau), L^{q,\infty}(\mathbb{R}^n))$. If the norm of u_0 is sufficiently small, the solution exists globally.*

Remark 4.8. In the limit case $q=n$ one has $X=Z=L^{n,\infty}(\mathbb{R}^n)$ and $W := \dot{H}^{-1}_{(n/2,\infty)}$. In this setting, existence and uniqueness of solutions in $L^\infty((0,\tau), L^{n,\infty}(\mathbb{R}^n))$ has been shown by Y. MEYER [24, Theorem 18.2]. In our abstract setting we need boundedness of the convolution $T(\cdot)* : L^\infty(\mathbb{R}_+, W) \to L^\infty(\mathbb{R}_+, Z)$, which holds by Lemma 3.11 if $(W, \dot{W}_2)_{1/2,\infty} \hookrightarrow Z$. By reiteration, the latter condition is equivalent to $(\dot{H}^{1-\delta}_{(n/2,\infty)}, \dot{H}^{1+\delta}_{(n/2,\infty)})_{1/2,\infty} \hookrightarrow L^{n,\infty}$. This embedding however holds by $\dot{H}^{1\pm\delta}_{(n/2,\infty)} \hookrightarrow L^{n/(1\mp\delta),\infty}$ (see (37) in the proof of Lemma 6.2) and another reiteration identity: $(L^{n/(1+\delta),\infty}, L^{n/(1-\delta),\infty})_{1/2,\infty} = L^{n,\infty}$.

4.4. **Arbitrary domains in $\mathbb{R}^3$.** To our knowledge, there are two results in the literature on mild solutions of the Navier–Stokes equations on arbitrary domains $\Omega \subseteq \mathbb{R}^3$, due to H. Sohr [32, Theorem V.4.2.2] and S. Monniaux [25, Theorem 3.5]. Our results allow to discuss both approaches, to compare them, and to improve them.

Let $\Omega \subseteq \mathbb{R}^3$ be an arbitrary domain. Since there is no regularity assumed for $\partial\Omega$, existence of the Stokes semigroup $(T(t)) = (e^{-tA})$ is only guaranteed in $L^2_\sigma(\Omega)$ or in interpolation and extrapolation spaces that are associated to $L^2_\sigma(\Omega)$ and the Stokes operator A.

Since we need the action of $(T(t))$ in W we take $W := \dot{D}(A^{-1/2}) = \dot{\mathbb{V}}^{-1}_2(\Omega)$ (see Section 2). On $\mathbb{R}^n$, this would correspond to the space $\dot{H}^{-1}_2$, but now we have to pay more attention to the Helmholtz projection and W has to be a space of divergence–free vectors. We observe that $u,v \in L^4(\Omega)^3$ implies $u \otimes v \in L^2(\Omega)^{3\times 3}$, $\nabla \cdot (u \otimes v) \in \dot{W}^{-1}_2(\Omega)^3$, and finally $P\nabla \cdot (u \otimes v) \in \dot{\mathbb{V}}^{-1}_2(\Omega) = W$ by Section 2. Since we have Dirichlet boundary conditions, $\dot{D}(A^{1/2}) = V\!V_2 \subseteq \dot{W}^1_{2,0}(\Omega)^3$ embeds into $L^6(\Omega)^3$, and by self adjointness of A and (complex) interpolation we obtain $\dot{D}(A^{1/4}) \hookrightarrow L^3(\Omega)^3$ and $\dot{D}(A^{3/8}) \hookrightarrow L^4(\Omega)^3$. Thus, also $u,v \in \dot{D}(A^{3/8})$ implies $P\nabla \cdot (u \otimes v) \in W$, and $\dot{D}(A^{1/4})$ might be the right space of initial values if we seek for global solutions.

For $Z \in \{\dot{D}(A^{3/8}), L^4(\Omega)^3\}$ we now clearly have

$$\|T(t)\|_{W \to Z} \leq c\|T(t)\|_{\dot{D}(A^{-1/2}) \to \dot{D}(A^{3/8})} \leq c\,t^{-1/2-3/8}, \quad t > 0,$$

i.e. $\gamma = 7/8$. By Theorem 3.6 we hence should have $\alpha + 1/p = 1/8$. For inhomogeneities $f_j \in L^{p_j}_{\beta_j}(\mathbb{R}_+, W^{(j)})$ the condition $1+\alpha+1/p = \gamma_j + \beta_j + 1/p_j$ then reads $\gamma_j + \beta_j + 1/p_j = \frac{9}{8}$ where γ_j satisfies $\|T(t)\|_{W^{(j)} \to Z} \leq c\,t^{-\gamma_j}$, $t > 0$.

Suppose that X is a Banach space satisfying

$$(15) \qquad (L_\sigma^2(\Omega), \dot{\mathbb{V}})_{1/2,4} \hookrightarrow X \hookrightarrow (L_\sigma^2(\Omega), \dot{\mathbb{V}})_{1/2,8},$$

or, for some $p \in (8, \infty]$,

$$(16) \qquad (L_\sigma^2(\Omega), \dot{\mathbb{V}})_{1/2,1} \hookrightarrow X \hookrightarrow (L_\sigma^2(\Omega), \dot{\mathbb{V}})_{1/2,p},$$

and in which the Stokes semigroup acts as a bounded analytic semigroup. One obtains the pointwise norm estimates $\|T(t)\|_{X \to Z} \le c\, t^{-1/8}$ and $\|T(t)\|_{W \to X} \le c\, t^{-3/4}$, see e.g. [12, Lemma 1.12]. By reiteration, equation (15) can be reformulated as

$$(\dot{X}_{-1}, \dot{X}_1)_{1/2,4} \hookrightarrow X \hookrightarrow (\dot{X}_{-1}, \dot{X}_1)_{1/2,8}$$

and a similar reformulation is possible for (16). Thus, we obtain

Theorem 4.9. *Let $Z \in \{\dot{D}(A^{3/8}), L^4(\Omega)^3\}$ and $p \in [8, \infty]$. Suppose that X is a Banach space satisfying* (15) *for $p = 8$ and* (16) *for $p > 8$. For any initial value $u_0 \in X^\flat$ and any $f = f_0 + \nabla \cdot F$ with $f_0 \in L_{\beta_1}^{p_1}(\mathbb{R}_+, L^2(\Omega)^3)$ and $F \in L_{\beta_2}^{p_2}(\mathbb{R}_+, L^2(\Omega)^{3 \times 3})$, where $\beta_j \ge 0$, $p_j \in [1, \infty]$ with $\beta_1 + 1/p_1 = 3/4$ and $\beta_2 + 1/p_2 = 1/4$, there exists a unique mild solution u to the Navier–Stokes equation* (NSE) *satisfying*

$$(17) \qquad u \in C\big([0, \tau), X\big) \cap L_\alpha^p\big((0, \tau), Z\big)$$

where $\alpha + 1/p = 1/8$ and τ depends only on the norms $\|u_0\|_X$, $\|f_0\|_{L_{\beta_1}^{p_1}(L^2)}$, $\|F\|_{L_{\beta_2}^{p_2}(L^2)}$. We have $\tau = \infty$ if these norms are sufficiently small.

Proof. We have $f_1 = \mathbb{P}f_0$, $f_2 = P\nabla \cdot F$. Taking $W^{(1)} = L_\sigma^2$ and $W^{(2)} = W$ and observing $1 + \alpha + 1/p = 9/8$, we only have to check $\|T(t)\|_{L_\sigma^2 \to Z} \le c\, t^{-3/8}$, i.e. $\gamma_1 = 3/8$, recall $\|T(t)\|_{W \to Z} \le c\, t^{-7/8}$, i.e. $\gamma_2 = 7/8$, and observe $9/8 - \gamma_1 = 3/4 = \beta_1 + 1/p_1$, $9/8 - \gamma_2 = 1/4 = \beta_2 + 1/p_2$. $\square$

Remark 4.10. The following, which takes up an observation from [5] shows that, for the choice of $Z = \dot{D}(A^{3/8})$, the space $X = (L_\sigma^2(\Omega), \dot{\mathbb{V}})_{1/2,p}$ is maximal for the result:

$$\begin{aligned} \|T(\cdot)x\|_{L_\alpha^p(\dot{D}(A^{3/8}))} &= \|t \mapsto t^{1/8} AT(t)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}, \dot{D}(A^{-5/8}))} \\ &\sim \|x\|_{(\dot{D}(A^{-5/8}), \dot{D}(A^{3/8}))_{7/8,p}} \sim \|x\|_{(L_\sigma^2(\Omega), \dot{\mathbb{V}})_{1/2,p}}. \end{aligned}$$

Remark 4.11. Sohr's result ([32, Theorem V.4.2.2]) has $Z = L^4(\Omega)^3$, $p = 8$, $\alpha = 0$, $\beta_1 = \beta_2 = 0$ and $p_1 = 4/3$, $p_2 = 4$. It is remarkable that (15) does not allow to take $X = \dot{D}(A^{1/4})$. In fact, Sohr takes *weak* solutions u of (NSE) which always satisfy $u \in L_{loc}^\infty([0, \eta), L_\sigma^2(\Omega)) \cap L_{loc}^2([0, \eta), \mathbb{V}_2(\Omega))$. Observe that the space $X = \dot{D}(A^{1/4})$ becomes admissible if we choose $p > 8$.

Remark 4.12. Taking $Z = \dot{D}(A^{3/8})$ and $p = \infty$, $\alpha = 1/8$ in Theorem 4.9 we obtain an improvement of Monniaux's result ([25, Theorem 3.5], cf. also the discussion below). Here we may choose $X = (L_\sigma^2(\Omega), \dot{\mathbb{V}})_{1/2,s}$ for $s \in [1, \infty]$. Thus the maximal space for initial values is $X = (L_\sigma^2(\Omega), \dot{\mathbb{V}})_{1/2,\infty}$. Observe that $X = \dot{D}(A^{1/4})$ for $s = 2$ since A is self adjoint.
In [25] the right hand side is $f = 0$. Moreover, the assertion there only covers time–local solutions. Actually, the space $\mathscr{E}_T$ in [25] is not a Banach space, in general. Since only time–local solutions are considered in [25], the proof can be corrected by replacing A in the definition of the norm of $\mathscr{E}_T$ with $\delta + A$ in case $0 \in \sigma(A)$. In this context, we remark that $\dot{\mathbb{V}} = \mathbb{V}$, $W = \mathbb{V}'$ and $\dot{D}(A^r) = D(A^r)$ for $r > 0$ if $0 \in \rho(A)$ which happens, e.g., if Ω is bounded.

We want to discuss the result in [25] a bit further. The approach there corresponds to taking $Z = \mathbb{V} = D(A^{1/2})$. For $u, v \in Z$, one has $u \cdot \nabla v \in L^{3/2}(\Omega)^3$. Dualising $D(A^{1/4}) \hookrightarrow L^3(\Omega)^3$ yields $P : L^{3/2}(\Omega)^3 \to (D(A^{1/4}))'$, and the latter space equals $W := (L^2_\sigma(\Omega), \|(\delta + A)^{-1/4} \cdot \|)^\sim$ (the embedding $\dot{D}(A^{3/8}) \hookrightarrow L^3(\Omega)^3$ might be used as well; then $P : L^{3/2}(\Omega)^3 \to \dot{D}(A^{-1/4})$, and one could choose $W := \dot{D}(A^{-1/4})$, but the other choice is closer to what is actually happening in [25]). Now clearly $\|T(t)\|_{W \to Z} \leq c\, t^{-3/4}$ on bounded time–intervals. In order to satisfy $\alpha + 1/p \leq 1 - 3/4 = 1/4$ choose $p = \infty$ and $\alpha = 1/4$. If X is a Banach space satisfying

$$(L^2_\sigma(\Omega), \mathbb{V})_{1/2,1} \hookrightarrow X \hookrightarrow (L^2_\sigma(\Omega), \mathbb{V})_{1/2,\infty}$$

in which the Stokes semigroup acts as a bounded analytic semigroup, then we obtain

Theorem 4.13. *For any initial value $u_0 \in X^\flat$ and $f = f_0 + \nabla \cdot F$ with $f_0 \in L^{p_1}_{\beta_1}(\mathbb{R}_+, L^2(\Omega)^3)$ and $F \in L^p_\alpha(\mathbb{R}_+, L^2(\Omega)^{3\times 3})$, where $\alpha, \beta_1 \geq 0$, $p_1 \leq \infty$, $p < \infty$ and $\alpha + 1/p = 1/4$, $\beta_1 + 1/p_1 = 3/4$, there is a unique mild solution u to the Navier–Stokes equation (NSE) satisfying*

$$(18) \qquad u \in C_b([0, \tau), X) \cap L^p_\alpha((0, \tau), \mathbb{V})$$

where τ depends only on the norms $\|u_0\|_X$, $\|f_0\|_{L^{\beta_1}_1(L^2)}$, $\|F\|_{L^p_\alpha(L^2)}$.

Proof. We have $f_1 = \mathbb{P}f_0$, $f_2 = P\nabla \cdot F$, $W^{(1)} = L^2_\sigma$, $W^{(2)} = \mathbb{V}_2^{-1}$. Observing $\|T(t)\|_{L^2_\sigma \to \mathbb{V}} \leq ct^{-1/2}$, $\|T(t)\|_{\mathbb{V}_2^{-1} \to \mathbb{V}} \leq ct^{-1}$ on finite time intervals, this leads to $\gamma_1 = 1/2$ and $\gamma_2 = 1$. Notice that $1 + \alpha + 1/p - 1/2 = 5/4 - 1/2 = 3/4 = \beta_1 + 1/p_1$. However, now that $\gamma_2 = 1$ we need maximal L^p_α-regularity (on finite time intervals, see discussion in Section 3) for the Stokes operator A in $\mathbb{V}_2^{-1}$, which holds since $\mathbb{V}_2^{-1}$ is a Hilbert space, $p < \infty$, and $0 \leq \alpha \leq 1/4 - 1/p$. $\qquad \square$

Remark 4.14. As mentioned before, [25] has $f = 0$. Observe that, although the space Z is different, the conditions on the right hand side f are the same as in Theorem 4.9, but that $\gamma_2 = 1$ led to the restriction $\beta_2 = \alpha$, $p_2 = p < \infty$, since we need the continuous action $T(\cdot)* : L^{p_2}_{\beta_2}(\mathbb{V}_2^{-1}) \to L^p_\alpha(\mathbb{V}_2)$. If $F = 0$ we can admit $p = \infty$ in the assertion. Observe also that it was essential for the argument to use the *inhomogeneous* space $\mathbb{V}$, which in turn restricts the result to time-local solutions.

4.5. **Domains which admit an L^q-theory.** In this subsection $\Omega \subseteq \mathbb{R}^n$ is a domain for which we assume additionally that, for some $q_0 \in (2, \infty)$, the Helmholtz projection is bounded in $L^{q_0}(\Omega)^n$ and the Stokes semigroup is bounded analytic in $L^{q_0}_\sigma(\Omega)$ (cf. the end of Section 2). We distinguish two cases.

Case I, $n = 3$ and $q_0 \in (2, 4]$: We start with a preparation. By interpolating the semigroup estimates $\|T(t)\| \leq c\, t^{-1/2}$ for the action $T(t) : L^2_\sigma \to L^6$ and $\|T(t)\| \leq c$ for the action $T(t) : L^{q_0}_\sigma \to L^{q_0}$ one obtains

$$(19) \qquad \|T(t)\|_{L^q_\sigma \to L^4} \leq c\, t^{-3/2(1/q - 1/4)}, \quad t > 0,$$

where θ is determined by $\frac{1}{4} = \frac{1-\theta}{6} + \frac{\theta}{q_0}$ and q satisfies $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{q_0}$ (observe that the negative t-exponent equals $\delta := \frac{3}{2}(\frac{1}{q} - \frac{1}{4}) = \frac{3}{2}\frac{1-\theta}{3} = \frac{1-\theta}{2} + \theta \cdot 0$).

Choose $W := \dot{\mathbb{V}}_2^{-1}(\Omega)$ and $Z := L^4(\Omega)^3$ as in the previous subsection for Theorem 4.9. Then still $\alpha + 1/p = 1/8$. We want to find spaces X associated to $L^q_\sigma(\Omega)$ and A_q. To this end we let $\widetilde{Z} := (L^q_\sigma(\Omega), \dot{D}(A_q))_{\delta,1}$. By (19) and Remark 3.9 we know that $\widetilde{Z} \hookrightarrow L^4(\Omega)^3 = Z$. Using the Stokes semigroup in $L^q_\sigma(\Omega)$ we calculate (as in Remark 4.10)

$$\|T(\cdot)x\|_{L^p_\alpha(\widetilde{Z})} = \|t \mapsto t^{1/8} AT(t)x\|_{L^p(\frac{dt}{t}, (\widetilde{Z})^{\cdot}_{-1})} \sim \|x\|_{((\widetilde{Z})^{\cdot}_{-1}, \widetilde{Z})_{7/8, p}}.$$

Clearly, $(\widetilde{Z})^{\cdot}_{-1} = (\dot{D}(A_q^{-1}), L_\sigma^q(\Omega))_{\delta,1}$, and by reiteration,

$$((\widetilde{Z})^{\cdot}_{-1}, \widetilde{Z})_{7/8,p} = (\dot{D}(A_q^{-1/2}), \dot{D}(A_q^{1/2}))_{\frac{3}{2q},p}.$$

Indeed, observe that $\frac{1}{8}(\delta-1)+\frac{7}{8}\delta = \delta-\frac{1}{8} = \frac{3}{2q}-\frac{1}{2}$ and $(1-\frac{3}{2q})(-\frac{1}{2})+\frac{3}{2q}\cdot\frac{1}{2} = -\frac{1}{2}+\frac{3}{2q}$. As an illustration we remark that, for $\Omega = \mathbb{R}^3$, this space equals the homogeneous divergence–free Besov space $\dot{B}_{q,p,\sigma}^{-\frac{1}{2}+\frac{3}{2q}}$.

Theorem 4.15. *Suppose q_0 and q are as above. Let $\alpha \geq 0$, $p \in [8,\infty]$ such that $\alpha + 1/p = 1/8$, and let $X := (\dot{D}(A_q^{-1/2}), \dot{D}(A_q^{1/2}))_{\frac{3}{2q},p}$. For any initial value $u_0 \in X^\flat$ and any $f = f_0 + \nabla \cdot F$ with $f_0 \in L_{\beta_1}^{p_1}(\mathbb{R}_+, L^2(\Omega)^3)$ and $F \in L_{\beta_2}^{p_2}(\mathbb{R}_+, L^2(\Omega)^{3\times 3})$, where $\beta_j \geq 0$, $p_j \in [1,\infty]$ with $\beta_1 + 1/p_1 = 3/4$ and $\beta_2 + 1/p_2 = 1/4$, there is a unique mild solution to the Navier-Stokes equation (NSE) satisfying*

$$u \in C_b([0,\tau), X) \cap L_\alpha^p((0,\tau), L^4(\Omega)^3)$$

where τ depends only on the norms $\|u_0\|_X$, $\|f_0\|_{L(L^2)}$, $\|F\|_{L(L^2)}$, and $\tau = \infty$ if these norms are sufficiently small.

Remark 4.16. Concerning the relation of q and q_0 we remark that a calculation shows $\frac{1}{q} = \frac{1}{4}(2 - \frac{q_0-2}{6-q_0}) = \frac{1}{4}\frac{12-3q_0+2}{6-q_0}$. Hence we have $q = q_0$ for $q_0 \in \{2,4\}$, and the special cases $q = 12/5$ for $q_0 = 3$ and $q = 3$ for $q_0 = 18/5$. We did not use any further properties besides boundedness of the Helmholtz projection in L^{q_0} and bounded analyticity of the Stokes semigroup in $L_\sigma^{q_0}$. Once one has

$$(20) \qquad \|T(t)\|_{L_\sigma^{q_1} \to L^4} \leq c\, t^{-3/2(1/q_1 - 1/4)}, \quad t > 0,$$

for some $q_1 \in (q, q_0]$, in Theorem 4.15 the spaces $(\dot{D}(A_{q_1}^{-1/2}), \dot{D}(A_{q_1}^{1/2}))_{\frac{3}{2q_1},p}$ can be used in place of the spaces $(\dot{D}(A_q^{-1/2}), \dot{D}(A_q^{1/2}))_{\frac{3}{2q},p}$. In this context, we remark that the spaces $(\dot{D}(A_q^{-1/2}), \dot{D}(A_q^{1/2}))_{\frac{3}{2q},p}$ actually grow with q. Since $A^{-1/2}$ is an isometry, it is sufficient to show that $(L_\sigma^q(\Omega), \dot{D}(A_q))_{\frac{3}{2q},p}$ grows with q. To see this, we let $q_1 \in (q, q_0]$, write out the norms and use the semigroup property

$$\|x\|_{(L_\sigma^{q_1}(\Omega), \dot{D}(A_{q_1}))_{\frac{3}{2q_1},p}} \quad \sim \quad \|t \mapsto t^{1-\frac{3}{2q_1}} AT(t)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}, L^{q_1})}$$

$$\sim \quad \|t \mapsto t^{1-\frac{3}{2q_1}} T(t)AT(t)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}, L^{q_1})}$$

$$\leq \quad c\|t \mapsto t^{1-\frac{3}{2q}} AT(t)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}, L^q)},$$

where we used

$$(21) \qquad \|T(t)\|_{L_\sigma^q \to L^{q_1}} \leq ct^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{q_1})}, \quad t > 0,$$

in the last step, which in turn follows by interpolation of the action $T(t) : L_\sigma^{q_1} \to L^{q_1}$ and $T(t) : L_\sigma^2 \to L^{q_1}$, recall that $q_1 \leq 4 < 6$. It is clear that, besides bounded analytic action of the Stokes semigroup in $L_\sigma^q(\Omega)$ and $L_\sigma^{q_1}(\Omega)$, the estimate (21) is all that is needed to prove the desired inclusion in more general cases.

Case II, $q_0 > \max\{n,4\}$: We let $q := q_0$ for simplicity of notation. One can choose $Z := L^q(\Omega)^n$. For $u, v \in Z$ then $u \otimes v \in L^{q/2}(\Omega)^{n\times n}$, $\nabla \cdot (u \otimes v) \in \dot{W}_{q/2}^{-1}(\Omega)^n$, and $P\nabla \cdot (u \otimes v) \in \dot{\mathbb{V}}_{q/2}^{-1}(\Omega) =: W$. Notice that $q/2 > 2$ by $q > 4$. We now aim at

$$(22) \qquad \|T(t)\|_{\dot{\mathbb{V}}_{q/2}^{-1}(\Omega) \to L^q(\Omega)^n} \leq c\, t^{-\frac{1}{2}-\frac{n}{2q}}, \quad t > 0,$$

i.e. $\gamma = \frac{1}{2} - \frac{n}{2q}$. Here, $L^q(\Omega)^n$ can be replaced by $L^q_\sigma(\Omega)$, in which space we can use the Stokes semigroup to obtain the equivalent condition

$$(23) \qquad \dot{\mathbb{V}}^{-1}_{q/2}(\Omega) \hookrightarrow (\dot{D}(A_q^{-1}), L^q_\sigma(\Omega))_{\frac{1}{2} - \frac{n}{2q}, \infty}$$

by Remark 3.9. Dualising (22) (with $L^q_\sigma(\Omega)$ in place of $L^q(\Omega)^n$) yields as another equivalent condition

$$(24) \qquad \|T(t)\|_{L^{q'}_\sigma(\Omega) \to \dot{\mathbb{V}}_{(q/2)'}(\Omega)} \le c\, t^{-\frac{1}{2} - \frac{n}{2q}}, \quad t > 0,$$

which in turn can be reformulated by Remark 3.9 as

$$(25) \qquad (L^{q'}_\sigma(\Omega), \dot{D}(A_{q'}))_{\frac{1}{2} + \frac{n}{2q}, 1} \hookrightarrow \dot{\mathbb{V}}_{(q/2)'}(\Omega),$$

where we used the Stokes semigroup in $L^{q'}_\sigma(\Omega)$. Another reformulation of (24) is the following gradient estimate

$$(26) \qquad \|\nabla T(t)f\|_{(q/2)'} \le c\, t^{-\frac{1}{2} - \frac{n}{2q}} \|f\|_{q'}, \quad t > 0, f \in L^{q'}_\sigma(\Omega).$$

We thus obtain the following new result.

Theorem 4.17. *Suppose that $q = q_0$ is as above and assume that one of the equivalent conditions (22), (23), (24), (25), (26) holds. Let $\alpha \ge 0$, $p \in (2, \infty]$ such that $\alpha + \frac{1}{p} = \frac{1}{2} - \frac{n}{2q}$, and let $X := (\dot{D}(A_q^{-1/2}), \dot{D}(A_q^{1/2}))_{\frac{n}{2q}, p}$. For any initial value $u_0 \in X^\flat$ and $f = f_0 + \nabla \cdot F$ with $f_0 \in L^{p_1}_{\beta_1}(\mathbb{R}_+, L^2(\Omega)^3)$ and $F \in L^{p_2}_{\beta_2}(\mathbb{R}_+, L^2(\Omega)^{3 \times 3})$, where $\beta_j \ge 0$, $p_j \in [1, \infty]$ with $\beta_1 + \frac{1}{p_1} = \frac{3}{2} - \frac{n}{q}$ and $\beta_2 + \frac{1}{p_2} = 1 - \frac{n}{q}$ there is a unique mild solution to the Navier-Stokes equation (NSE) satisfying*

$$u \in C_b([0, \tau), X) \cap L^p_\alpha((0, \tau), L^q(\Omega)^n)$$

where τ depends only on the norms $\|u_0\|_X$, $\|f_0\|_{L(L^2)}$, $\|F\|_{L(L^2)}$, and $\tau = \infty$ if these norms are sufficiently small.

Proof. As mentioned above we have $Z = L^q(\Omega)^n$ and $W = \dot{\mathbb{V}}^{-1}_{q/2}(\Omega)$. We have $\gamma = \frac{1}{2} + \frac{n}{2q}$ which explains the condition on $\alpha + \frac{1}{p}$. Notice that $W^{(1)} = L^{q/2}_\sigma(\Omega)$ yields $\gamma_1 = \frac{n}{2q}$ and since $W^{(2)} = W$, $\gamma_2 = \gamma$. To verify $[A_j 3]$, by 3.6 requires $\beta_1 + \frac{1}{p_1} + \gamma_1 = \alpha + \frac{1}{p} + 1$ and $\alpha + \frac{1}{p} + \gamma = 1$ – both are guaranteed by the assumptions on $\beta_{1/2}$ and $p_{1/2}$. Moreover, we have $(\dot{X}_{-1}, \dot{X}_1)_{1/2, p/2} \hookrightarrow (\dot{X}_{-1}, \dot{X}_1)_{1/2, p} = X$ by reiteration, and an argument as in Remark 4.10 shows that X satisfies $[A1]$. Finally, assumption $[A2]$ follows from (23) by reiteration. $\qquad\square$

Remark 4.18. If Ω is bounded and $\partial\Omega$ is of class $C^{1,1}$ then Theorem 4.17 may be applied to any $q \in (n, \infty)$. It is well-known that the Stokes semigroup is bounded analytic in all L^q_σ, $q \in (1, \infty)$. Moreover, condition (26) is satisfied for any $q \in (n, \infty)$. This follows from

$$\|T(t)\|_{L^{q'}_\sigma \to L^{(q/2)'}_\sigma} \le c t^{-\frac{n}{2q}}, \quad t > 0,$$

and

$$(27) \qquad \|\nabla T(t)\|_{L^{(q/2)'}_\sigma \to L^{(q/2)'}} \le c t^{-1/2}, \quad t > 0,$$

where the latter is due to the fact that $A_{(q/2)'}$ has a bounded H^∞-calculus in $L^{(q/2)'}_\sigma(\Omega)$ ([16, Thm.9.17]), hence has bounded imaginary powers, which leads to $\dot{D}(A^{1/2}_{(q/2)'}) = [L^{(q/2)'}_\sigma(\Omega), \dot{D}(A_{(q/2)'})]_{1/2} = \dot{\mathbb{V}}_{(q/2)'}$ (observe that $0 \in \rho(A_{(q/2)'})$ and thus homogeneous and inhomogeneous domain spaces coincide). The result on mild solutions in this case is due to Amann [2].

5. Proof of Theorem 3.6

For a sectorial operator in a Banach space X, real interpolation spaces between X and inhomogeneous spaces X_k are well-studied, see [4, 23, 36]. In this section we provide the results on real interpolation of *homogeneous* spaces needed for the proof of Theorem 3.6. The following result is an analogue of [36, Theorem 1.14.2].

Proposition 5.1. *Let X be a Banach space and A be an injective, sectorial operator on X. Then, for $m \in \mathbb{N}$, $(X, \dot{X}_m)$ is a quasi-linearisable interpolation couple in the sense of* [36, Definition 1.8.3]. *Moreover, for $p \in [1, \infty]$ and $\theta \in (0, 1)$, an equivalent norm on $(X, \dot{X}_m)_{\theta,p}$ is given by $\|\lambda \mapsto \lambda^{\theta m} A^m (\lambda + A)^{-m} x\|_{L^p(\mathbb{R}_+, d\lambda/\lambda, X)}$.*

Proof. Let $E_m := X + \dot{X}_m$. Clearly, $E_m = (I + \mathbb{A})^m(\dot{X}_m)$ and $\|e\|_{E_m} = \|\mathbb{A}^m(I + \mathbb{A})^{-m} e\|_X$. We borrow a decomposition technique inspired by [20, Proposition 15.26]: let a_j be defined by $\sum_{j=1}^{2m-1} a_j z^j = (1+z)^{2m} - (1+z^m)(1+z)^m$. Therefore, setting $z = \lambda^{1/m} A$, we obtain for all $x \in E_m$,

$$
x = \left[\lambda A^m (1 + \lambda^{1/m} A)^{-m} x + \sum_{j=m}^{2m-1} a_j \lambda^{j/m} A^j (1 + \lambda^{1/m} A)^{-2m} x \right]
$$
$$
+ \left[(1 + \lambda^{1/m} A)^{-m} x + \sum_{j=1}^{m-1} a_j \lambda^{j/m} A^j (1 + \lambda^{1/m} A)^{-2m} x \right]
$$

(all operators are bounded). Call the first term in brackets $V_0(\lambda)x$ and the second one $V_1(\lambda)x$. A direct calculation shows quasi-linearisability. Hence,

$$
K(\lambda, x, X, \dot{X}_m) \sim \|V_0(\lambda)x\|_X + \lambda \|V_1(\lambda)x\|_{\dot{X}_m}.
$$

Notice that

$$
\lambda^{-\theta} (\lambda^{1/m} A)^m (I + \lambda^{1/m} A)^{-m} x = \lambda^{-\theta} A^m (\lambda^{-1/m} + A)^{-m} x \stackrel{\tau = \lambda^{-1/m}}{=} \tau^{\theta m} A^m (\tau + A)^{-m} x.
$$

Thus, it remains to show

$$
\|V_0(\lambda)x\|_X + \lambda \|V_1(\lambda)x\|_{\dot{X}_m} \quad \sim \quad \|(\lambda^{1/m} A)^m (I + \lambda^{1/m} A)^{-m} x\|.
$$

For the estimate "$\leq$", notice that

$$
\lambda^{\frac{j+m}{m}} A^{m+j} (I + \lambda^{1/m} A)^{-2m} = (\lambda^{1/m} A)^j (I + \lambda^{1/m} A)^{-m} \cdot \lambda A^m (I + \lambda^{1/m} A)^{-m},
$$

where the first expression is bounded by sectoriality of A.

Finally, let $f(\lambda) := \lambda A^m (1 + \lambda^{1/m} A)^{-m}$. Then for $x \in E_m$ and $x = y + z$ with $y \in X$ and $z \in \dot{X}_m$,

$$
f(\lambda)x = f(\lambda)y + f(\lambda)z = f(\lambda)y + \lambda(\lambda^{-1} A^{-m}) f(\lambda) A^m z,
$$

whence by sectoriality of A,

$$
\|f(\lambda)x\|_X \leq \|f(\lambda)y\|_X + M\lambda \|z\|_{\dot{X}_m}.
$$

Taking the infimum over all such decomposition yields $\|f(\lambda)x\|_X \leq c\, K(\lambda, x, X, \dot{X}_m)$ and the proof is finished. $\square$

The following result corresponds to [36, Theorem 1.14.5] and gives another equivalent norm on $(X, \dot{X}_m)_{\theta,p}$ in the case of analytic semigroups. We omit the proof since it is identical to the non–homogeneous case.

Proposition 5.2. *Let $T(\cdot)$ be a bounded and analytic semigroup on a Banach space X and $-A$ its generator. If A is injective, then for $p \in [1, \infty]$ and $\theta \in (0, 1)$, an equivalent norm on $(X, \dot{X}_m)_{\theta,p}$ is given by $\|t^{m(1-\theta)} A^m T(t)x\|_{L^p(\mathbb{R}_+, dt/t, X)}$.*

The following result is an analogue of [36, Theorem 1.14.3 (a)].

Lemma 5.3. *Let X be a Banach space and A be an injective, sectorial operator on X. Then, for $k, j, m \in \mathbb{Z}$ with $k < j < m$, we have*

$$\left(\dot{X}_k, \dot{X}_m\right)_{\frac{j-k}{m-k},1} \hookrightarrow \dot{X}_j \hookrightarrow \left(\dot{X}_k, \dot{X}_m\right)_{\frac{j-k}{m-k},\infty}.$$

Proof. We can assume $k = 0$. Let $x \in \mathcal{D}(A^m) = X \cap \dot{X}_m \subseteq (X, \dot{X}_m)_{j/m,1}$. Then, by [36, 1.14.2/(1)], for some constant c_m

$$x = c_m \int_0^\infty (tA)^m (t+A)^{-2m} x \, \frac{dt}{t}.$$

Therefore, by sectoriality of A,

$$
\begin{aligned}
\left\|A^j x\right\|_X & \leq c_m \int_0^\infty t^{m-1} \left\|A^{j+m}(t+A)^{-2m}x\right\|_X dt \\
& = c_m \int_0^\infty \left\|t^{m-j} A^j (t+A)^{-m} \, t^j \left[A(t+A)^{-1}\right]^m x\right\| \frac{dt}{t} \\
& \leq M c_m \int_0^\infty \left\|t^j \left[A(t+A)^{-1}\right]^m x\right\| \frac{dt}{t} \leq \widetilde{c}_m \|x\|_{(X,\dot{X}_m)_{j/m,1}}.
\end{aligned}
$$

The second embedding follows also by sectoriality of A from

$$\left\|t^j [A(t+A)^{-1}]^m x\right\|_X = \left\|t^j A^{m-j}(t+A)^{-m} A^j x\right\|_X \leq \widetilde{M} \|x\|_{\dot{X}_j},$$

which is true for all $x \in \dot{X}_j$. $\qquad\square$

The assertion does hold for arbitrary interpolation indices $\theta \in (0,1)$ but we shall not introduce fractional homogeneous spaces (see [11, 13]) since the above version is sufficient for our purposes.

5.1. **Results on assumption [A1].** In this section we discuss boundedness of the map

$$(28) \qquad \Psi_\tau : X \to L^p_\alpha((0,\tau), Z), \qquad \Psi_\tau(x) = CT(t)x$$

for $p \in [1, \infty]$ and $\alpha \in (-1/p, 1 - 1/p)$. We start our considerations with a simple necessary condition for boundedness of Ψ_τ: boundedness of the set

$$(29) \qquad \{\lambda^{1-\alpha-1/p} C(\lambda + A)^{-1} : \lambda > 0\}$$

in $B(X, Z)$. Indeed, writing the resolvent of A as Laplace transform of the semigroup and using Hölder's inequality we have for $x \in X_1$ and $\lambda > 0$

$$
\begin{aligned}
\left\|C(\lambda + A)^{-1}x\right\| & \leq \int_0^\infty \left\|t^\alpha CT(t)\right\| t^{-\alpha} e^{-\lambda t} \, dt \\
& \leq \left\|t \mapsto t^\alpha CT(t)x\right\|_{L^p(\mathbb{R}_+, Z)} \left(\int_0^\infty t^{-\alpha p'} e^{-\lambda p' t} \, dt\right)^{1/p'} \\
& \overset{s=\lambda p' t}{\leq} M \|x\|_X \left((p'\lambda)^{-1+\alpha p'} \Gamma\left(1 + (k-1-\alpha)p'\right)\right)^{1/p'} \\
& = K \lambda^{-1+\alpha+1/p} \|x\|_X,
\end{aligned}
$$

where Γ is the usual Gamma function and the number K depends only on p and the norm M of the map $x \mapsto CT(\cdot)x$. Next we treat the special case $p = \infty$.

Proposition 5.4. *Let A be an injective sectorial operator of type $\omega < \pi/2$ on a Banach space X and let $C \in B(X_1, Z)$. Then the following assertions are equivalent:*

(i) *The map Ψ_τ is bounded $X \to L^\infty_\alpha((0,\tau), Z)$*
(ii) *The set $\{\lambda^{1-\alpha} C(\lambda + A)^{-1} : \lambda > 0\}$ is bounded in $B(X, Z)$.*
(iii) *C is bounded in the norm $(X, \dot{X}_k)_{\alpha,1} \to Z$.*

Proof. From the necessary condition (29) it follows directly that (i) implies (ii). Assume that (ii) holds. For $x \in \mathcal{D}(A)$ we have

$$(30) \qquad Cx = \int_0^\infty \lambda C A (\lambda + A)^{-2} x \, \tfrac{d\lambda}{\lambda}.$$

Indeed, convergence follows from (ii) since for small $\lambda > 0$,

$$\|\lambda^\alpha A (\lambda + A)^{-1} x\| \leq c_1 \lambda^\alpha \|x\|$$

by sectoriality of A. For $\lambda \to \infty$, we have

$$\|\lambda^\alpha A (\lambda + A)^{-1} x\| \leq c_2 \lambda^{\alpha - 1} \|Ax\|$$

also by sectoriality of A. Equality in (30) follows immediately from formula [36, 1.14.2/(1)]. Therefore, for $x \in \mathcal{D}(A)$, we obtain

$$
\begin{aligned}
\|Cx\|_Z &\leq c^{-1} \int_0^\infty \left\| \lambda^{1-\alpha} C (\lambda + A)^{-1} \lambda^\alpha A (\lambda + A)^{-1} x \right\|_Z \tfrac{d\lambda}{\lambda} \\
&\leq c^{-1} M \int_0^\infty \left\| \lambda^\alpha A (\lambda + A)^{-1} x \right\|_X \tfrac{d\lambda}{\lambda} \sim \|x\|_{(X, \dot{X}_1)_{\alpha,1}},
\end{aligned}
$$

where we used Proposition 5.1. So (ii) implies (iii).
Finally, let (iii) hold. By Proposition 5.2 we then have

$$
\begin{aligned}
\left\| t^\alpha C T(t) x \right\|_Z &\leq M \left\| t^\alpha T(t) x \right\|_{(X, \dot{X}_1)_{\alpha,1}} \\
&\sim M \int_0^\infty \left\| t^\alpha s^{1-\alpha} A T(t+s) x \right\|_X \tfrac{ds}{s} \\
&\leq \widetilde{M} \|x\| \int_0^\infty t^\alpha s^{1-\alpha} (s+t)^{-1} \tfrac{ds}{s} \\
&\stackrel{s = \sigma t}{=} \widetilde{M} \|x\| \int_0^\infty \sigma^{1-\alpha} (1 + \sigma)^{-1} \tfrac{d\sigma}{\sigma}.
\end{aligned}
$$

The second estimate used the fact that, for bounded analytic semigroups, the operators $(tA)\,T(t)$, $t > 0$ are uniformly bounded. $\qquad \square$

Theorem 5.5. *Let $1 \leq p \leq q \leq \infty$ and A be an injective sectorial operator of type $\omega < \pi/2$ on a Banach space X and let $C \in B(X_1, Z)$. Let $\alpha \in (-1/p, 1 - 1/p)$. Then the following assertions hold:*

(a) *If Ψ_τ is bounded $X \to L^p_\alpha(Z)$, then it is also bounded $X \to L^q_{\alpha + 1/p - 1/q}(Z)$.*

(b) *If Ψ_τ is bounded $L^q_{\alpha + 1/p - 1/q}(Z)$ and if $X \hookrightarrow (\dot{X}_{-1}, \dot{X}_1)_{1/2, p}$, then it is also bounded $X \to L^p_\alpha(Z)$.*

Theorem 3.6 (a) is a corollary of this result letting $q = \infty$. Before giving a proof we point out its main argument, a simple reiteration observation.

Key Observation 5.6. Let numbers $p, q \in [1, \infty]$ and $\theta \in (0, 1)$ be given. Then,

$$
\begin{aligned}
(31) \qquad \left((\dot{X}_{-1}, X)_{\theta, q}, (X, \dot{X}_1)_{\theta, q} \right)_{1-\theta, p} &= \left((\dot{X}_{-1}, \dot{X}_1)_{\theta/2, q}, (\dot{X}_{-1}, \dot{X}_1)_{1/2 + \theta/2, q} \right)_{1-\theta, p} \\
&= (\dot{X}_{-1}, \dot{X}_1)_{1/2, p}.
\end{aligned}
$$

The first equality in (31) holds by Lemma 5.3 and reiteration for the real method. The second equality is the reiteration formula, see [36, Theorem 1.10.2].

Proof of Theorem 5.5. (a). If Ψ_τ is bounded $X \to L^p_\alpha(Z)$, then by the necessary condition (29) and Proposition 5.4, Ψ_τ is bounded for $X \to L^\infty_{\alpha+1/p}(Z)$. Therefore,

$$\left(\int_0^\infty \left\| t^{1/p - 1/q + \alpha} CT(t)x \right\|^q dt \right)^{1/q}$$

$$= \left(\int_0^\infty \left\| t^\alpha CT(t)x \right\|^p \left\| t^{\alpha + 1/p} CT(t)x \right\|^{q-p} dt \right)^{1/q}$$

$$\leq \left(\int_0^\infty \left\| t^\alpha CT(t)x \right\|^p dt \right)^{1/q} \cdot \left(\sup_{t>0} t^{\alpha + 1/p} \left\| CT(t)x \right\| \right)^{1 - p/q}$$

$$\leq c \left\| x \right\|^{p/q} \cdot \left\| x \right\|^{1 - p/q} = c \left\| x \right\|$$

Now let (b) hold. By the necessary condition (29) and Proposition 5.4, C is bounded in norm $E \to Z$, where $E := (X, \dot{X}_1)_{\theta,1}$ with $\theta := \alpha + 1/p$. The part A_E of $\mathbb{A}_{-1}$ in E is injective and sectorial in E. Therefore, we can define $\dot{E}_{-1} := \dot{E}_1(A_E)$ and obtain $\dot{E}_{-1} = (\dot{X}_{-1}, X)_{\theta,1}$ whence $(\dot{E}_{-1}, E)_{1-\theta,p} = (\dot{X}_{-1}, \dot{X}_1)_{1/2,p}$ by letting $q = 1$ in Observation 5.6. Taking this into account, we obtain by $C \in B(E, Z)$ and Proposition 5.2

$$\left\| t \mapsto t^\alpha CT(t)x \right\|_{L^p(\mathbb{R}_+, Z)} \leq M \left\| t \mapsto t^\alpha T(t)x \right\|_{L^p(\mathbb{R}_+, E)}$$

$$= M \left\| t \mapsto t^{\alpha + 1/p} T(t)x \right\|_{L^p(\mathbb{R}_+, dt/t, E)}$$

$$= M \left\| t \mapsto t^{\alpha + 1/p} AT(t)x \right\|_{L^p(\mathbb{R}_+, dt/t, \dot{E}_{-1})}$$

$$\leq \widetilde{M} \left\| x \right\|_{(\dot{E}_{-1}, E)_{1-\theta,p}} = \widetilde{M} \left\| x \right\|_{(\dot{X}_{-1}, \dot{X}_1)_{1/2,p}}$$

Thus Ψ_τ is bounded $X \to L^p_\alpha(Z)$ since by assumption $X \hookrightarrow (\dot{X}_{-1}, \dot{X}_1)_{1/2,p}$. □

5.2. **Results on assumption [A2].** Theorem 3.6 (b) is in fact covered by [12, Theorems 1.8 and 1.9]. We shall shortly sketch basic idea of proof. Necessity of the boundedness of B as stated in Theorem 3.6 (b) follows from

$$(32) \qquad\qquad \mathcal{R}(B) \subseteq (\dot{X}_{-1}, X)_{2(\alpha+1/p), \infty}$$

Indeed, consider the function $u(s) := \mathbb{1}_{(1/2, t)} u_0$. Then assumption [A2] shows

$$\left\| [T(t) - T(t/2)] Bu_0 \right\|_{\dot{X}_{-1}} \leq c_{\alpha,p} t^\theta \qquad t > 0.$$

with $\theta = \frac{2}{p}(\alpha p + 1)$. By [13, Theorem 6.4.2] we therefore also have

$$\left\| t^{1-\sigma} AT(t) Bu_0 \right\|_{\dot{X}_{-1}} \leq c_{\alpha,p}, \qquad t > 0$$

whence (32) follows from Proposition 5.2. Conversely, by analyticity of the semigroup $T(\cdot)$, we have the pointwise estimates

$$\|T_{-1}(t)\|_{X \to X} \leq c \quad \text{and} \quad \|T_{-1}(t)\|_{\dot{X}_{-1} \to X} \leq c \, t^{-1}.$$

Thus, by real interpolation, (32) implies $\|T(t)B\|_{W \to X} \leq M t^{-\gamma}$ for all $t > 0$ where $\gamma = 1 - 2\alpha - 2/p$ (see [12, Lemma 1.12]). Thus

$$\left\| \int_0^t T(t-s) Bu(s) \, ds \right\|_X \leq \int_0^t \left\| T(t-s)B \right\|_{W \to X} s^{-\alpha} \left\| s^\alpha u(s) \right\|_W ds$$

$$\leq c \int_0^t (t-s)^{-\gamma} s^{-\alpha} \| s^\alpha u(s) \|_W \, ds.$$

Let $k_{\alpha,\gamma}(t,s) = \mathbb{1}_{(0,t)}(s)(t-s)^{-\gamma} s^{-\alpha}$ for $s, t \in (0, \tau)$. Therefore, the study of the kernel $k_{\alpha,\gamma}$ which may or may not induce a bounded integral operator $K_{\alpha,\gamma}$:

$L^p(0,\tau) \to L^\infty(0,\tau)$. The following lemma is taken from [14, Theorem 7], see also [33, Theorem B]).

Lemma 5.7 (Hardy, Littlewood). *Let $1 < q < p \leq \infty$ and $\gamma \in (0,1)$. Then, for any numbers $0 \leq \alpha < \beta < 1$ satisfying $1 + \alpha - \beta - \gamma = \frac{1}{q} - \frac{1}{p} > 0$, the operator*

$$(T_\gamma f)(t) := \int\limits_0^\infty \frac{f(s)}{(t-s)^\gamma}\, ds$$

is bounded $L_\beta^q(\mathbb{R}_+) \to L_\alpha^p(\mathbb{R}_+)$. This also holds if $p = q = \infty$ or if $\alpha = \beta = 0$.

The original proof of Hardy and Littlewood is incorrect (in [14, displayed formula after (4.14) of p. 579], see also a comment and a corrected proof in [33, p. 504]). We provide here a short interpolation argument.

Proof. (1) Let $p = q = \infty$. It suffices to verify $k(t, \cdot) \in L^1(\mathbb{R}_+)$ with a uniform norm bound for $t \in (0, \tau)$ where $k(t, s) := \mathbb{1}_{[0,t]}(s)(t-s)^{-\gamma} t^\alpha s^{-\beta}$. A simple substitution shows that the characterising condition is

$$\beta, \gamma < 1 \quad \text{and} \quad 1 + \alpha = \beta + \gamma.$$

(2) Next we consider the case $\alpha = \beta = 0$. Since $t^{-\gamma} \in \text{weak–}L^{\frac{1}{\gamma}}$, a version of Young's inequality (see e.g. [10, Theorem 1.2.13]) yields $t^{-\gamma}* : L^s \to L^r$ for $1 + \frac{1}{r} = \frac{1}{s} + \gamma$, $r, s \notin \{1, \infty\}$.

(3) The general case now follows by complex interpolation: (see e.g. [36, Theorem 1.18.5]) of (1) and (2):

$$L_\beta^q = [L^s, L_{\beta/\theta}^\infty]_\theta \quad \text{and} \quad L_\alpha^p = [L^r, L_{\alpha/\theta}^\infty]_\theta$$

provided that $\frac{1}{q} = (1-\theta)\frac{1}{s}$ and $\frac{1}{p} = (1-\theta)\frac{1}{r}$. Moreover, by (2),

$$t^{-\gamma}* : L^s \to L^r \qquad \text{holds for } 1 + \frac{1}{r} = \frac{1}{s} + \gamma$$

and by (1),

$$t^{-\gamma}* : L_{\beta/\theta}^\infty \to L_{\alpha/\theta}^\infty \qquad \text{holds for } \beta, \gamma < \theta \quad \text{and} \quad \theta + \alpha = \beta + \theta\gamma.$$

Under the assumptions on $\alpha, \beta, \gamma, p, q$ all of the above conditions are satisfied. $\qquad \square$

Thus, $K_{\alpha,\gamma}$, given by $k_{\alpha,\gamma}(t, s) = \mathbb{1}_{(0,t)}(s)(t-s)^{-\gamma} s^{-\alpha}$ for $s, t \in (0, \tau)$ is bounded $L^p(0, \tau) \to L^\infty(0, \tau)$ provided that one of the following conditions holds:

$$(33) \quad
\begin{array}{lllll}
(i) & p = 1 & \tau < \infty & \alpha \leq 0 & \gamma \leq 0 \\
(ii) & p = 1 & \tau = \infty & \alpha = 0 & \gamma = 0 \\
(iii) & p > 1 & \tau < \infty & \alpha + \frac{1}{p} < 1 & \gamma + \frac{1}{p} < 1 \quad \alpha + \gamma + \frac{1}{p} \leq 1 \\
(iv) & p > 1 & \tau = \infty & \alpha + \frac{1}{p} < 1 & \gamma + \frac{1}{p} < 1 \quad \alpha + \gamma + \frac{1}{p} = 1.
\end{array}$$

Observe that condition (iv) implies $\alpha > 0$ and $\gamma > 0$. In case $\alpha > 0$ the assertion of Theorem 3.6 (b) now follows immediately. In case $\alpha = 0$ we follow a different strategy. Instead of analysing boundedness of the convolution operator $T(\cdot)*$ we can study boundedness of the map

$$(34) \qquad \Phi_\tau : L_\alpha^p((0, \tau), W) \to X, \qquad \Phi_\tau(u) = \int_0^\tau T(t)Bu(t)\, dt$$

for $p \in [1, \infty]$ and $\alpha = 0$. Therefore in some sense we are in dual situation to the discussion of assumption [A1] and indeed similar methods can be employed. We shall discuss boundedness of Φ_τ in (34) for general α and then deduce the remaining step for Theorem 3.6 (b) as a special case.

Proposition 5.8. *Let A be an injective sectorial operator of type $\omega < \frac{\pi}{2}$ on a Banach space X. Let $B \in B(W, X_{-1})$ and let $\alpha \in (0, 1)$. Then the following assertions are equivalent:*

(i) *The map Φ_τ is bounded $L^1_\alpha(W) \to X$.*
(ii) *The set $\{\lambda^{1-\alpha}(\lambda + A_{-1})^{-1}B : \lambda > 0\}$ is bounded in $B(W, X)$.*
(iii) *B is bounded in the norm $W \to (\dot{X}_{-1}, X)_{1-\alpha,\infty}$.*

Proof. The implication (i) $\Rightarrow$ (ii) is similar to Proposition 5.4.
Let (ii) hold, and let $u \in U$. By Proposition 5.1, we have

$$
\begin{aligned}
\|Bu\|_{(\dot{X}_{-1},X)_{1-\alpha,\infty}} &\sim \sup_{\lambda>0} \|\lambda^{1-\alpha}A_{-1}(\lambda+A)^{-1}Bu\|_{\dot{X}_{-1}} \\
&= \sup_{\lambda>0} \|\lambda^{1-\alpha}(\lambda+A)^{-1}Bu\|_X \leq K\|u\|.
\end{aligned}
$$

Hence (ii) implies (iii). Finally, if (iii) holds, then, by Proposition 5.2,

$$
\begin{aligned}
\left\|\int_0^\infty t^\alpha T_{-1}(t)Bu(t)\,dt\right\|_X &\leq \int_0^\infty \|t^\alpha A_{-1}T_{-1}(t)Bu(t)\|_{\dot{X}_{-1}}\,dt \\
&\leq \int_0^\infty \sup_{s>0} \|s^\alpha A_{-1}T_{-1}(s)Bu(t)\|_{\dot{X}_{-1}}\,dt \\
&\leq c\int_0^\infty \|Bu(t)\|_{(\dot{X}_{-1},X)_{1-\alpha,\infty}}\,dt \\
&= c\|u\|_{L^1(\mathbb{R}_+,U)}\|B\|_{U \to (\dot{X}_{-1},X)_{1-\alpha,\infty}},
\end{aligned}
$$

which proves the last implication. □

Theorem 5.9. *Let $1 \leq r \leq p \leq \infty$ and A be an injective sectorial operator of type $\omega < \pi/2$ on a Banach space X. Let $B \in B(W, X_{-1})$ and let $\alpha \in (-1/p', 1 - 1/p')$. Then the following assertions hold:*

(a) *If Φ_τ is bounded $L^p_\alpha(W) \to X$, then it is also bounded $L^r_{\alpha+1/r-1/p}(W) \to X$.*
(b) *If Φ_τ is bounded $L^r_{\alpha+1/r-1/p}(W) \to X$ and if $(\dot{X}_{-1}, \dot{X}_1)_{1/2,p} \hookrightarrow X$, then it is also bounded $L^p_\alpha(W) \to X$.*

Proof. (a). By assumption we have

$$
\left\|\int_0^\infty t^{1/r-1/p+\alpha}T_{-1}(t)Bu(t)\,dt\right\| \leq c_1 \|t^{1/r-1/p}u(t)\|_{L^p(\mathbb{R}_+,W)}.
$$

Proposition 5.8 shows that Φ_τ is bounded $L^1_{\alpha+1/p'}(W) \to X$ whence

$$
\left\|\int_0^\infty t^{1/r-1/p+\alpha}T_{-1}(t)Bu(t)\,dt\right\| \leq c_2 \|t^{1/r-1}u(t)\|_{L^1(\mathbb{R}_+,W)}.
$$

These two estimates allow interpolation by the complex method with $\theta \in (0,1)$ chosen such that $1/r = \theta \cdot 1 + (1-\theta)1/p$. Applying [4, Theorem 5.5.3], one obtains

$$
\left\|\int_0^\infty t^{1/r-1/p+\alpha}T_{-1}(t)Bu(t)\,dt\right\| \leq c_3 \|t^\tau u(t)\|_{L^r(\mathbb{R}_+,W)},
$$

where $\tau = \theta(1/r - 1) + (1-\theta)(1/r - 1/p) = 0$, and the assertion is proved.
(b). If Φ_τ is bounded $L^r_{\alpha+1/r-1/p}(W) \to X$, then B is bounded in norm $W \to F$ where $F := (\dot{X}_{-1}, X)_{1-\sigma,\infty}$ with $\sigma = (\alpha + 1/p')/1$. Notice that $\dot{F}_1 = (X, \dot{X}_1)_{1-\sigma,\infty}$, and that we have $(F, \dot{F}_1)_{\sigma,p} = (\dot{X}_{-1}, \dot{X}_1)_{1/2,p}$ by letting $q = \infty$ and $\theta = 1 - \sigma$ in

Observation 5.6. By $(\dot{X}_{-1}, \dot{X}_1)_{1/2,p} \hookrightarrow X$ we thus have

$$\left\| \int_0^\infty s^\alpha T_{-1}(s) Bu(s)\, ds \right\|_X$$

$$\leq c \left\| \int_0^\infty s^\alpha T_{-1}(s) Bu(s)\, ds \right\|_{(F, \dot{F}_1)_{\sigma,p}}$$

$$= c \left\| t \mapsto \int_0^\infty s^\alpha t^{1-\sigma} A T_{-1}(s+t) Bu(s)\, ds \right\|_{L^p(\mathbb{R}_+, dt/t, F)}$$

$$= c \left\| t \mapsto \int_0^\infty s^\alpha t^{1-\alpha-1/p'-1/p} A T_{-1}(s+t) Bu(s)\, ds \right\|_{L^p(\mathbb{R}_+, F)}.$$

Notice that the operator-valued kernel $K(s,t) := s^\alpha t^{-\alpha} A T_{-1}(s+t)$ satisfies

$$\|K(s,t)\| \leq M \frac{s^\alpha t^{-\alpha}}{t+s} =: k(s,t)$$

since $T(\cdot)$ is bounded analytic. The scalar kernel $k(\cdot, \cdot)$ is homogeneous of degree -1 and, by $\alpha \in (-1/p', 1 - 1/p')$, the function

$$s \mapsto s^{-1/p} k(s,1) = \frac{s^{\alpha - 1/p}}{1+s}$$

is integrable over $\mathbb{R}_+$. By [34, Lemma A.3], we thus obtain

$$\left\| \int_0^\infty s^\alpha T_{-1}(s) Bu(s)\, ds \right\|_X \leq c' \|Bu\|_{L^p(\mathbb{R}_+, F)} \leq c'' \|u\|_{L^p(\mathbb{R}_+, W)},$$

as desired. $\qquad\square$

6. APPENDIX

Weak Lebesgue-spaces, weak homogeneous Besov- and Triebel-Lizorkin spaces. Recall the following defining norm of weak Lebesgue (or Marcinkiewicz) spaces $L^{q,\infty}$ for $q \in [1, \infty)$

$$\|f\|L^{q,\infty}(X) := \sup_{t>0} \quad t \cdot \mu\big(\{x \in X : \|f(x)\| > t\}\big)^{1/q},$$

and set $L^{\infty,\infty} = L^\infty$. It is well known (see [36, 1.18.6]), that the spaces $L^{q,\infty}$, $q \in (1, \infty)$ can be realised as real interpolation spaces between L^q–spaces:

$$\left(L^{q_0}, L^{q_1}\right)_{\theta,\infty} = L^{q,\infty} \qquad \text{where } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

Consequently lifting with $(I - \delta)^{-s/2}$ allows to define the spaces $H^s_{q,\infty}$ as corresponding real interpolation spaces of H^s_q–spaces:

(35)
$$
\begin{array}{ccccc}
L^{q_0} & {-}{-}{-}{-}{-}{-} & L^{q,\infty} & {-}{-}{-}{-}{-}{-} & L^{q_1} \\
{\scriptstyle(1-\delta)^{-s/2}}\downarrow & & \vdots & & \downarrow{\scriptstyle(1-\delta)^{-s/2}} \\
H^s_{q_0} & {-}{-}{-}{-}{-}{-} & H^s_{q,\infty} & {-}{-}{-}{-}{-}{-} & H^s_{q_1}
\end{array}
$$

We have thus $\|x\|_{H^s_{q,\infty}} \sim \|(1-\delta)^{s/2} x\|_{L^{q,\infty}}$. By [36, Theorem 2.4.2], $H^s_{q,\infty} = F^s_{(q,\infty),2}$ in our notation and $H^s_{q,\infty} = F^s_{q,2,(\infty)}$ the notation of Triebel's book [36]. Indeed,

$$F^s_{(p,\infty),2} = (F^s_{p_0,2}, F^s_{p_1,2})_{\theta,\infty} = (H^s_{p_0}, H^s_{p_1})_{\theta,\infty}$$

$$= (-\delta)^{s/2} (L^{p_0}, L^{p_1})_{\theta,\infty} = (-\delta)^{s/2} L^{p,\infty} = H^s_{q,\infty}$$

To define homogeneous spaces $\dot{H}^s_{p,\infty}$ and $\dot{B}^s_{p,q}$ and to prove analogous properties we employ the same technique but lift with $\delta^{-s/2}$ instead of $(I - \delta)^{-s/2}$. The analogue of the above identity for homogeneous spaces holds as well.

Lemma 6.1. $B_{q_j,1}^s \hookrightarrow H_{q_j}^s \hookrightarrow B_{q_j,r}^s$, $j = 0, 1$ *with* $r = \max(q_0, q_1, 2)$

Proof. For $q_j \geq 2$ this follows from

$$H_{q_j}^s = F_{q_j,2}^s \hookrightarrow F_{q_j,q_j}^s = B_{q_j,q_j}^s \hookrightarrow B_{q_j,r}^s$$

and for $q < 2$ this follows from $H_{q_j}^s = F_{q_j,2}^s \overset{(*)}{\hookrightarrow} B_{q_j,2}^s \hookrightarrow B_{q_j,r}^s$ where we use Minkowski's inequality in $(*)$. $\square$

Real interpolation of the embedding in Lemma 6.1 with (θ, p) and using a result of Peetre [27, Theorem 1] (see also [36, Remark 4,Section 2.4.1]) yields the following embedding which is valid for all $s \in \mathbb{R}$

$$(36) \qquad B_{(q,p),\min(p,r^*)}^s \hookrightarrow H_{q,p}^s \hookrightarrow B_{(q,p),\max(p,r^*)}^s.$$

Here, $\frac{1}{r^*} = 1 - \theta + \frac{\theta}{r}$. Notice that Peetre's proof holds in case of homogeneous spaces as well since it is essentially an embedding result for interpolation spaces of vector-valued l_q^s–spaces. Precisely, one uses that the mappings

$$S : B_{(q,r),p}^s \to l_p^s(L^{q,r}) \qquad Sf := (f * \phi_j)_{j=-0}^\infty$$

$$\dot{S} : \dot{B}_{(q,r),p}^s \to l_p^s(L^{q,r}) \qquad \dot{S}f := (f * \phi_j)_{j=-\infty}^\infty$$

are coretractions in the sense of [36, Definition 1.2.4] (see also [36, (2.3.2/12)] for more details).

Lemma 6.2. $\dot{H}_{q_2,\infty}^{-1} \hookrightarrow \dot{B}_{(q,\infty),\infty}^t$ *provided that* $-1 - {}^n/q = t$.

Proof. Notice that by the well-known Sobolev embedding, $H_q^s \hookrightarrow L^{q^*}$ where $s > 0$ and $\frac{n}{q^*} = \frac{n}{q} - s$. A scaling argument (i.e. regarding norm estimates for $u(\lambda\cdot)$ with $\lambda > 0$) yields the estimate

$$\lambda^{-n/q^*} \|u\|_{q^*} \leq C(\lambda^{-n/q} \|u\|_q + \lambda^{s-n/q} \|\mathcal{F}^{-1}(\xi \mapsto |\xi|^s \mathcal{F}(u)\|_q)$$

Now multiplication with $\lambda^{n/q^*} = \lambda^{n/q - s}$ and letting $\lambda \to \infty$ gives the embedding of the homogeneous space $\dot{H}_q^s \hookrightarrow L^{q^*}$. In particular, we have $\dot{H}_{q_{j/2}}^{n/q_j} \hookrightarrow L^{q_j}$. Real interpolation of this embedding for adequate values q_0, q_1 and θ yields

$$(37) \qquad \dot{H}_{(q_2,\infty)}^{n/q} = (\dot{H}_{q_{0/2}}^{n/q_0}, \dot{H}_{q_{1/2}}^{n/q_1})_{\theta,\infty} \hookrightarrow (L^{q_0}, L^{q_1})_{\theta,\infty} = L^{q,\infty}$$

Now apply (36) with $s = 0$ in $(*)$ to conclude $\dot{H}_{(q_2,\infty)}^{n/q} \hookrightarrow L^{q,\infty} \overset{(*)}{\hookrightarrow} \dot{B}_{(q,\infty),\infty}^0$. Finally, lifting by $-{}^n/q - 1$ proves the lemma. $\square$

References

[1] Herbert Amann, *Linear and quasilinear parabolic problems. Vol. I*, Monographs in Mathematics, vol. 89, Birkhäuser Boston Inc., Boston, MA, 1995, Abstract linear theory.

[2] ———, *On the strong solvability of the Navier-Stokes equations*, J. Math. Fluid Mech. **2** (2000), no. 1, 16–98.

[3] A. Benedek, A.-P. Calderón, and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365.

[4] Jöran Bergh and Jörgen Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, No. 223.

[5] Marco Cannone, *Ondelettes, paraproduits et Navier-Stokes*, Diderot Editeur, Paris, 1995, With a preface by Yves Meyer.

[6] G. Da Prato and P. Grisvard, *Sommes d'opérateurs linéaires et équations différentielles opérationnelles*, J. Math. Pures Appl. (9) **54** (1975), no. 3, 305–387.

[7] Georges de Rham, *Variétés différentiables. formes, courants, formes harmoniques. 2 éd.*, Actualites Scientifiques et Industrielles. 1222. Publications de l'Institut de Mathématique de l'Université de Nancago. III. Paris: Hermann & Cie. XII, 196 p., 1960 (French).

[8] Giovanni Dore, *L^p regularity for abstract differential equations*, Functional analysis and related topics, 1991 (Kyoto), Lecture Notes in Math., vol. 1540, Springer, Berlin, 1993, pp. 25–38.

[9] Hiroshi Fujita and Tosio Kato, *On the Navier-Stokes initial value problem. I*, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[10] Loukas Grafakos, *Classical and Modern Fourier Analysis*, Prentice Hall, 2004.

[11] Bernhard H. Haak, Markus Haase, and Peer Christian Kunstmann, *Perturbation, Interpolation, and Maximal Regularity*, Advances in Differential Equations **11** (2006), no. 2, 201–240.

[12] Bernhard H. Haak and Peer Christian Kunstmann, *Weighted admissibility and wellposedness of linear systems in Banach spaces*, To appear in SIAM Journal of Control and Optimization.

[13] Markus Haase, *The functional calculus for sectorial operators*, Operator Theory, Advances and Applications, vol. 169, Birkhäuser Verlag, 2006.

[14] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals (I)*, M. Z. **27** (1928), 565–606 (English).

[15] Cornelia Kaiser, *Wavelet transform for vector-valued homogeneous Besov spaces*, Submitted, 2007.

[16] Nigel Kalton, Peer Kunstmann, and Lutz Weis, *Perturbation and interpolation theorems for the h^∞-calculus with applications to differential operators.*, Math. Ann. **336** (2006), no. 4, 747–801.

[17] Nigel Kalton and Lutz Weis, *Euclidian structures*, In preparation .

[18] Tosio Kato, *Perturbation theory for linear operators*, second ed., Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, Band 132.

[19] Herbert Koch and Daniel Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math. **157** (2001), no. 1, 22–35.

[20] Peer C. Kunstmann and Lutz Weis, *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, Functional analytic methods for evolution equations, Lecture Notes in Math., vol. 1855, Springer, Berlin, 2004, pp. 65–311.

[21] P. G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 431, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[22] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[23] Alessandra Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications, 16, Birkhäuser Verlag, Basel, 1995.

[24] Yves Meyer, *Wavelets, paraproducts, and Navier-Stokes equations*, Current developments in mathematics, 1996 (Cambridge, MA), Int. Press, Boston, MA, 1997, pp. 105–212.

[25] Sylvie Monniaux, *Navier-Stokes equations in arbitrary domains: the Fujita-Kato scheme*, Math. Res. Lett. **13** (2006), no. 2-3, 455–461.

[26] Jindřich Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris, 1967.

[27] Jaak Peetre, *Sur les espaces de Besov*, C. R. Acad. Sci. Paris Sér. A-B **264** (1967), A281–A283.

[28] Jan Prüss and Gieri Simonett, *Maximal regularity for evolution equations in weighted L_p-spaces*, Arch. Math. (Basel) **82** (2004), no. 5, 415–431.

[29] Okihiro Sawada, *On time-local solvability of the Navier-Stokes equations in Besov spaces*, Adv. Differential Equations **8** (2003), no. 4, 385–412.

[30] Christian G. Simader and Hermann Sohr, *A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains*, Mathematical problems relating to the Navier-Stokes equation, Ser. Adv. Math. Appl. Sci., vol. 11, World Sci. Publishing, River Edge, NJ, 1992, pp. 1–35.

[31] Jacques Simon, *Primitives de distributions et applications*, Séminaire d'Analyse, 6, 7, 1990–1991, 1991–1992 (Aubière, 1990–1991 and 1991–1992), Univ. Blaise Pascal Lab. Math. Pures Appl., Clermont, 1995, pp. Exp. No. 6.20, 21.

[32] Hermann Sohr, *The Navier-Stokes equations*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2001, An elementary functional analytic approach.

[33] E. M. Stein and Guido Weiss, *Fractional integrals on n-dimensional Euclidean space*, J. Math. Mech. **7** (1958), 503–514.

[34] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[35] Hans Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.

[36] ———, *Interpolation theory, function spaces, differential operators*, second ed., Johann Ambrosius Barth, Heidelberg, 1995.

[37] Lutz Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann. **319** (2001), no. 4, 735–758.

[38] Fred B. Weissler, *The Navier-Stokes initial value problem in L^p*, Arch. Rational Mech. Anal. **74** (1980), no. 3, 219–230.

MATHEMATISCHES INSTITUT I, UNIVERSITÄT KARLSRUHE, ENGLERSTRASSE 2, 76128 KARLSRUHE, GERMANY

E-mail address: Bernhard.Haak@math.uni-karlsruhe.de, Peer.Kunstmann@math.uni-karlsruhe.de