THE WEISS CONJECTURE AND WEAK NORMS

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ABSTRACT. In this note we show that for analytic semigroups the so-called Weiss condition of uniform boundedness of the operators

$$Re(\lambda)^{1/2} C(\lambda + A)^{-1}, \qquad Re(\lambda) > 0$$

on the complex right half plane and weak Lebesgue $L^{2,\infty}$–admissibility are equivalent. Moreover, we show that the weak Lebesgue norm is best possible in the sense that it is the endpoint for the 'Weiss conjecture' within the scale of Lorentz spaces $L^{p,q}$.

1. INTRODUCTION

In this note we are concerned with linear control systems of the form

$$\left\{ \begin{array}{lcl} x'(t) + Ax(t) & = & 0 \quad (t > 0), \\ x(0) & = & x_0, \\ y(t) & = & Cx(t) \quad (t > 0), \end{array} \right. \tag{1.1}$$

where $-A$ is the generator of a strongly continuous semigroup $(T(\cdot))$ on a Banach space X. The function $x(\cdot)$ takes values in X, and the function $y(\cdot)$ takes values in a Banach space Y. The observation operator C may be an unbounded operator from X to Y. A commonly used minimal assumption on C is that C is bounded $X_1 \to Y$ where X_1 denotes the domain $D(A)$ of A equipped with the graph norm. We refer, e.g., to [11, 13, 16, 15].

Definition 1.1. Let $\mathscr{Y}_\infty$ be a space of functions $\mathbb{R}_+ \to Y$ and let, for each $\tau > 0$, denote $\mathscr{Y}_\tau$ the restricted space of functions on $[0, \tau]$. Then the system (1.1) is called *$\mathscr{Y}_\tau$-admissible* for $\tau \in (0, \infty]$ if the output of the system (1.1) depends continuously on initial state, i.e. if the mapping

$$X \to \mathscr{Y}_\tau, \qquad x_0 \mapsto y(\cdot)$$

is continuous.

If X and Y are Hilbert spaces, a natural choice for $\mathscr{Y}_\tau$ is $\mathscr{Y}_\tau = L^2([0, \tau], Y)$. We mainly focus on infinite-time admissibility, that is the case $\tau = \infty$. We will write shorthand L^2–admissibility instead of $L^2(\mathbb{R}_+; Y)$–admissibility. In other situations, other norms such as the L^p–norm may be useful (see e.g. [5]).

In this note we discuss the failure of the Weiss conjecture. In order to do so, we treat the case where $\mathscr{Y}$ is a Lorentz-Bochner space $L^{p,q}(0, \tau; Y)$. We recall some basic definitions and properties of these spaces. We refer to [3, 6] for references and further results if $Y = \mathbb{C}$. The vector-valued spaces are discussed e.g. in [1]. We recall the definitions. Let Y be a Banach space. For a measurable, Y-valued function f on a measure space (Ω, μ), define the distribution function

$$d_f(\alpha) = \mu\left(\{\omega \in \Omega : \|f(\omega)\| > \alpha\}\right)$$

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of f and the non-decreasing rearrangement f^* of f as

$$f^*(t) = \inf\{s > 0 :\ d_f(s) \le t\}.$$

If f is a continuous, real-valued, positive and decreasing function on $[0, \infty)$, it is easy to see that $f^* = f$. For $1 \le q \le \infty$ and $p > 1$, the Lorentz-Bochner space $L^{p,q}(\Omega; Y)$ is defined as the set of all measurable functions such that the (quasi)-norm

$$\|f\|_{L^{p,q}(o,\tau;X)} = \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} & \text{if } 1 \le q < \infty \\ \sup\{\alpha^{1/p} d_f(\alpha) :\ \alpha > 0\} & \text{if } q = \infty \end{cases}$$

is finite. If $Y = \mathbb{C}$ we will simply write $L^{p,q}(\Omega)$. Notice that by Fubinis theorem, $L^{p,p}(\Omega) = L^p(\Omega)$. The space $L^{p,\infty}$ is also called weak-Lebesgue space. A typical weak-$L^p(\mathbb{R}_+)$ function that is not in $L^p(\mathbb{R}_+)$ is $f(x) = |x|^{-1/p}$. Lorentz spaces are "natural" function spaces in since they are real interpolation spaces between usual Lebesgue spaces, see [14]. As such, they appear in the context of the Weiss conjecture, as will be explained in the next section in detail.

Recall the definition of a sectorial operator: a densely defined operator A on a Banach space X is called sectorial of angle $\omega \in [0, \pi)$ if the spectrum of A is contained in the open sector $S_\omega = \{z \in \mathbb{C}^* :\ |\arg(z)| < \omega\}$ and if for all larger angles $\theta \in (\omega, \pi)$, the operators

$$\{\lambda(\lambda + A)^{-1} :\ \lambda \notin S_\theta\}$$

are uniformly bounded. Negative generators of bounded C_0–semigroups are sectorial of angle $\pi/2$. Bounded analytic C_0–semigroups are characterised by uniform boundedness of the operators $tAT(t)$, $t > 0$. They are precisely those semigroups whose (negative) generator is sectorial of angle $< \pi/2$ (see e.g. [2] for details).

2. The Weiss condition and decay rates of observed semigroup orbits

Definition 2.1. Let $-A$ be the generator of a bounded strongly continuous semigroup on a Banach space X and $C \in \mathcal{B}(\mathcal{D}(A), Y)$ be an observation operator. We say that (A, C) satisfies the *Weiss condition* if the operators

$$Re(\lambda)^{1/2} C(\lambda + A)^{-1}, \qquad Re(\lambda) > 0 \tag{2.1}$$

are uniformly bounded in $B(X, Y)$ when λ runs through the open right half plane.

By the Laplace transform,

$$C(\lambda+A)^{-1}x = \int_0^\infty e^{-\lambda t}\, CT(t)x \,\mathrm{d}t \tag{2.2}$$

for $x \in \mathcal{D}(A)$. If C is L^2–admissible, i.e. if $CT(\cdot)x \in L^2(\mathbb{R}_+; Y)$, taking norms in (2.2) and using Cauchy-Schwarz inequality (2.1) follows. The *Weiss conjecture* [17] states that if X and Y are Hilbert spaces, L^2–admissibility and the Weiss condition (2.1) are equivalent. This equivalence is known to be true in a certain number of cases, for instance for normal semigroups [17] or semigroups of contractions with scalar output [7]. However, the conjecture is wrong, even in a restricted version were the output space is $Y = \mathbb{C}$ [9]. For more results and references concerning the Weiss conjecture we refer to the survey [8]. Recall the following result of Le Merdy.

Theorem 2.2 ([10, Theorem 4.1])**.** *Let $T(t)$ be a bounded analytic semigroup on a Banach space X. Assume that its generator $-A$ is injective and that C satisfies the Weiss condition* (2.1)*. Then C is (infinite time) L^2–admissible provided that A admits upper square function estimates.*

It is known [10] that the extra assumption of upper square function estimates in Le Merdy's theorem cannot dropped: it suffices to observe that $C = A^{1/2}$ satisfies (2.1), and that admissibility of $A^{1/2}$ actually *is* an upper square function estimate. What can be obtained instead of L^2–admissibility when dropping the assumption of upper square function estimates? We give an answer in the next two theorems.

Theorem 2.3. *Let $T(t)$ be an exponentially stable analytic semigroup on a Banach space X. Then the following conditions are equivalent.*

(a) *The Weiss condition* (2.1)

(b) *There is a constant $K > 0$ such that*

$$\|CT(t)x\|_Y \le K t^{-1/2} \|x\|_X, \qquad t > 0 \tag{2.3}$$

(c) *C is $L^{2,\infty}$ admissible.*

The next result tells us that the weak Lebesgue norm is optimal in the sense that it is the endpoint for the 'Weiss conjecture' within the scale of Lorentz spaces $L^{p,q}$.

Theorem 2.4. *There exists an exponentially stable analytic semigroup $T(t)$ on a Hilbert space H and a scalar valued observation operator C such that $CT(\cdot)x$ is not in any $L^{2,q}(0,\tau;Y)$ for whatsoever choice of $q < \infty$ or $\tau > 0$.*

Proof of Theorem 2.3. We first show the equivalence of (a) and (b). In [4, Corollary 4.7] it is shown that the Weiss condition is equivalent to C being bounded from Z to Y, where $Z := (X, \dot{X}_1)_{1/2,1}$ is the real interpolation space between X and the homogeneous domain space $\dot{X}_1$ of A. Since A is invertible, $\dot{X}_1 = X_1$. On the other hand, for analytic semigroups, the space Z is characterised by the fact that

$$\|CT(t)\|_{Z\to Y} \lesssim t^{-1/2}.$$

This is essentially shown in [5, Proposition 3.9]. For the sake of completeness we repeat the short argument. Indeed, by analyticity of the semigroup,

$$\|T(t)\|_{X\to \dot{X}_1} \lesssim t^{-1} \quad \text{and} \quad \|T(t)\|_{\dot{X}_1\to \dot{X}_1} \lesssim 1$$

and from the boundedness of C on X_1 we deduce

$$\|CT(t)\|_{X\to Y} \lesssim t^{-1} \quad \text{and} \quad \|CT(t)\|_{\dot{X}_1\to Y} \lesssim 1.$$

The estimate $Z \to Y$ then follows by interpolation. For the converse, notice that

$$\begin{aligned}\|Cx\|_Y = c\left\|\int_0^\infty CAT(2t)x\,\mathrm{d}t\right\|_Y &\le c\int_0^\infty \|CT(t)AT(t)x\|_Y\,\mathrm{d}t\\ &\le c'\int_0^\infty t^{-1/2}\,\|AT(t)x\|_Y\,\mathrm{d}t \sim c'\|x\|_Z.\end{aligned}$$

Putting both results together we obtain that the Weiss condition (2.1) is equivalent to (2.3). It is clear that (b) implies (c). The remaining implication can be shown in Hilbert spaces by a dyadic decomposition argument and Fourier transform. The following quicker and more general argument has been pointed out to us by Peer Kunstmann: let $\mathrm{Re}(\lambda) > 0$ and $f(t) := e^{-Re(\lambda)t}$ on $[0,\infty)$. Then $f^*(t) = f(t)$ since f is decreasing and continuous. One has therefore

$$\|f\|_{L^{2,1}(\mathbb{R}_+)} = \int_0^\infty t^{1/2} f^*(t)\tfrac{\mathrm{d}t}{t} = \mathrm{Re}(\lambda)^{-1/2}\int_0^\infty s^{-1/2}e^{-s}\,\mathrm{d}s = \mathrm{Re}(\lambda)^{-1/2}\Gamma(1/2).$$

Now use the duality $(L^{2,1}(\mathbb{R}_+))' = L^{2,\infty}(\mathbb{R}_+)$ (see [3, Theorem 1.4.17]) and the Laplace transform (2.2) to conclude that

$$\|C(\lambda + A)^{-1}x\|_Y \le \int_0^\infty \|CT(t)x\|_Y f(t)\,\mathrm{d}t$$

$$\le \|f\|_{L^{2,1}(0,\infty)} \big\|CT(\cdot)x\big\|_{L^{2,\infty}(\mathbb{R}_+;Y)}$$
$$\le C\mathrm{Re}(\lambda)^{-1/2}\|x\|. \qquad \square$$

Proof of Theorem 2.4. Recall that Lorentz spaces satisfy $L^{2,p} \subset L^{2,q}$ for $p \le q$. It suffices therefore to show that C is not $L^{2,q}$–admissible for any finite $q > 2$. To this end we fix $2 < q < \infty$. The idea is to extend the counterexample of Jacob and Zwart [9] in the following sense: not only do we pass from $q = 2$ to $q \ge 2$ (recall $L^{2,2} = L^2$), but, in contrast with their abstract argument referencing to interpolation sequences, we simply provide an explicit element $x \in H$ for which the observed orbit $CT(t)x$ does not lie in the Lorentz space $L^{2,q}(0,\tau;Y)$. The main idea however is identical to the construction of Jacob and Zwart: on Hilbert spaces there exist conditional bases (e_n) that satisfy

(a) (e_n) is not Besselian, i.e. there is no constant $c_B > 0$ such that
$$\sum |\alpha_k|^2 \le c_B \Big\|\sum \alpha_k e_k\Big\|.$$
(b) (e_n) is Hilbertian, i.e. there is a constant $c_H > 0$ such that
$$\Big\|\sum \alpha_k e_k\Big\| \le c_H \sum |\alpha_k|^2.$$
(c) $\inf \|e_n\| > 0$

A concrete example is given in [12, Example II.11.2]: let $\beta = \frac{1}{2q'} \in (\frac14, \frac12)$ and consider
$$e_{2n}(s) = |s|^\beta e^{ins} \quad \text{and} \quad e_{2n+1}(s) = |s|^\beta e^{-ins}$$
on $H = L^2(-\pi,\pi)$. Let $T(t)$ be the exponentially stable and analytic semigroup given by $T(t)e_n = e^{-4^n t}e_n$ and consider an observation operator $C : H \to \mathbb{C}$ given by $Ce_n = 2^n$ for $n \in \mathbb{N}$. Thus, if $x = \sum \xi_k e_k$,
$$|CT(t)x| = \Big|\sum_k 2^k e^{-4^k t}\xi_k\Big|.$$
Let $x(s) = |s|^{-\beta}\mathbf{1}_{[-\pi,\pi]}$. Since x is square integrable, there exist (ξ_n) such that
$$x(s) = \sum_{n\ge 0} \xi_n e_n(s).$$
However, necessarily,
$$\xi_{2n} = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} x(s)|s|^{-\beta}e^{-ins}\,\mathrm{d}s \quad \text{and} \quad \xi_{2n+1} = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} x(s)|s|^{-\beta}e^{ins}\,\mathrm{d}s.$$
Let us determine the growth order of the coefficients: by symmetry, $\xi_{2n} = \xi_{2n+1}$. Let $\gamma > 0$. Then
$$\int_0^\pi s^\gamma e^{ins}\tfrac{\mathrm{d}s}{s} = n^{-\gamma}\int_0^{\pi n} s^\gamma e^{is}\tfrac{\mathrm{d}s}{s}.$$
The function $\varphi(z) = z^{\gamma-1}e^{iz}$ is holomorphic on the open right half plane. By deplacing the integral from $[0,n\pi]$ to the positive imaginary axis,
$$\int_0^\pi s^\gamma e^{ins}\tfrac{\mathrm{d}s}{s} = n^{-\gamma}\left[e^{i\pi\gamma/2}\int_0^{\pi n} s^\gamma e^{-s}\tfrac{\mathrm{d}s}{s} - \int_0^{\pi/2}(n\pi e^{i\theta})^\gamma e^{in\pi e^{i\theta}}\,d\theta\right].$$
The first integral behaves as $n^{-\gamma}e^{i\pi\gamma/2}\Gamma(\gamma) + O(\frac1n)$; the second is $O(\frac1n)$ as can be seen by estimating $e^{-n\pi\sin(t)}$ using $\frac{2}{\pi}t \le \sin(t) \le t$ on $[0,\pi/2]$. By complex conjugation,
$$\int_{-\pi}^0 |s|^\gamma e^{ins}\tfrac{\mathrm{d}s}{s} = \int_0^\pi s^\gamma e^{-ins}\tfrac{\mathrm{d}s}{s} = \overline{\int_0^\pi s^\gamma e^{ins}\tfrac{\mathrm{d}s}{s}}.$$

so that

$$\xi_n = 2n^{-\gamma}\cos(\gamma\pi/2)\Gamma(\gamma) + O(\tfrac{1}{n}).$$

In our case, $\gamma = 1 - 2\beta$, and so $\xi_n \sim n^{2\beta-1} = n^{-1/q}$ when $n \to +\infty$. Finally notice that $\xi_n \geq 0$ since

$$\xi_n = 2\sum_{l=0}^{n-1}\int_0^{2\pi}(2\pi l + x)^{\gamma-1}\cos(x)\,dx$$

and each term in the sum is positive : let $f(x) = (2\pi l + x)^{\gamma-1}$. Then $f \in C^\infty(\mathbb{R}_+)$ is positive, decreasing and convex. Therefore, $g(x) = f(x) - f(x+\pi)$ satisfies $g'(x) = f'(x) - f'(x+\pi) = \pi f''(\eta) \leq 0$. So g is decreasing and positive. Thus,

$$\int_0^{2\pi} f(x)\cos(x)\,dx = \int_0^{\pi}\big(f(x) - f(x+\pi)\big)\cos(x)\,dx \geq 0.$$

Now let us come back to the observed semigroup orbits: recall that for exponentially stable semigroups admissibility in arbitrary small finite time or in infinite time are equivalent. The lack of $L^{2,q}$–integrability must therefore happen near the origin. For our choice of $x \in H$ and $t \in [4^{-n-1}, 4^{-n})$, one has

$$|CT(t)x| = \sum_{k=0}^{\infty}\xi_k 2^k e^{-4^k t} \geq \xi_n 2^n e^{-1} \sim n^{-1/q}2^n \sim (1+|\log(t)|)^{-1/q}t^{-1/2} =: f(t).$$

But $f \notin L^{2,q}(0,\tau)$ howsoever we choose $\tau \in (0,1)$. □

Open problem: An inspection of the proof of Theorem 2.3 shows that the implication (b) ⇒ (c) ⇒ (a) always holds. It is only for (a) ⇒ (b) that analyticity is used. This hypothesis is not optimal since the implication is trivially true in all cases when C is even L^2–admissible (i.e. when the Weiss conjecture holds). It follows that on Hilbert spaces the cases of (a) norm-continuous semigroups, (b) normal semigroups, (c) exponentially stable right invertible semigroups and (d) in case $Y = \mathbb{C}$ contraction semigroups and (e) diagonal semigroups on a Riesz basis are covered (see [8] for references). Our analysis of an non-Riesz basis example underlines further the idea that the Weiss condition (2.1) and $L^{2,\infty}$–admissibility should be equivalent on Hilbert spaces.

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