ON THE KERNEL OF THE BRAUER-MANIN PAIRING

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Abstract. Let $\mathcal{X}$ be a regular scheme, flat and proper over the ring of integers of a p-adic field, with generic fiber X and special fiber $\mathcal{X}_s$. We study the left kernel $\mathrm{Br}(\mathcal{X})$ of the Brauer-Manin pairing $\mathrm{Br}(X)\times \mathrm{CH}_0(X)\to \mathbb{Q}/\mathbb{Z}$. Our main result is that the kernel of the reduction map $\mathrm{Br}(\mathcal{X})\to \mathrm{Br}(\mathcal{X}_s)$ is the direct sum of $(\mathbb{Q}/\mathbb{Z}[\frac{1}{p}])^s\oplus(\mathbb{Q}/\mathbb{Z})^t$ and a finite p-group, where $s+t=\rho_{\mathcal{X}_s}-\rho_X-I+1$, for $\rho_{\mathcal{X}_s}$ and ρ_X the Picard numbers of $\mathcal{X}_s$ and X, and I the number of irreducible components of $\mathcal{X}_s$. Moreover, we show that $t>0$ implies $s>0$.

1. Introduction

The Brauer group plays an important role in arithmetic geometry. Over a finite field, Artin conjectured that the Brauer group of any proper scheme is finite [7, Rem. 2.5c)]; this was proved by Grothendieck for curves [7, Rem. 2.5b)]. If X is smooth and proper, then the finiteness of $\mathrm{Br}(X)$ is equivalent to Tate's conjecture on the surjectivity of the cycle map for divisors on X, and for a normal crossing scheme the finiteness follows from Tate's conjecture for all (smooth) intersections of the components. The next interesting case are varieties over a p-adic field K. It is a classical result of Hasse that the Brauer group $\mathrm{Br}(K)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$. Lichtenbaum [11] proved that if X is a curve, then the Brauer group $\mathrm{Br}(X)$ is Pontrjagin dual to the Chow group of zero cycles $\mathrm{CH}_0(X)$. In particular, it is the direct sum of a finite group, of $\mathbb{Q}/\mathbb{Z}$, and of a divisible p-torsion group of corank the genus of X times the degree of $[K:\mathbb{Q}_p]$.

This result was generalized by Colliot-Thélène and Saito [3], and Saito and Sato [16]. They show that if X has a proper and regular model $\mathcal{X}$, then the Brauer-Manin pairing between $\mathrm{CH}_0(X)$ and $\mathrm{Br}(X)$ has left kernel $\mathrm{Br}(\mathcal{X})$. Moreover, for $l\neq p$, the l-part of $\mathrm{Br}(X)/\operatorname{im}(\mathrm{Br}(\mathcal{X})\oplus \mathrm{Br}(K))$ is finite and vanishes for almost all l. However, not much is known about $\mathrm{Br}(\mathcal{X})$. We prove the following:

Theorem 1.1. *Let $\mathcal{X}$ be a regular scheme, flat and proper over the ring of integers of a p-adic field, and let $\mathcal{X}_s$ be the closed fiber. Then the kernel of the reduction map* $\mathrm{Br}(\mathcal{X})\to\mathrm{Br}(\mathcal{X}_s)$ *is the direct sum of* $(\mathbb{Q}/\mathbb{Z}[\frac{1}{p}])^s\oplus(\mathbb{Q}/\mathbb{Z})^t$ *and a finite p-group, where* $s+t=r:=\rho_{\mathcal{X}_s}-\rho_X-I+1$ *for* $\rho_{\mathcal{X}_s}$ *and* ρ_X *the Picard numbers of*

Date: October 30, 2021.
2010 *Mathematics Subject Classification.* Primary: 14F22; Secondary: 14F40, 14F42, 11G40.
Key words and phrases. Brauer group; Brauer Manin pairing; local fields; Picard number.
The first named author is Supported by JSPS Grant-in-Aid (C) 18K03258, and the second named author by grant ANR-15-CE40-0002.

$\mathcal{X}_s$ and X, and I the number of irreducible components of $\mathcal{X}_s$. Moreover, if $t > 0$ then $s > 0$.

Note that $\mathrm{Br}(\mathcal{X}_s)$ is conjecturally finite. The statement on the l-corank follows from the proper base change theorem, and the non-trivial part of the theorem is that the p-corank is strictly smaller than the l-corank unless both vanish.

Corollary 1.2. *1) The kernel of $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_s)$ is finite if and only if $r = 0$.*
2) If $r = 1$, then the p-part of the kernel of $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_s)$ is finite.

Our construction together with a theorem of Flach-Siebel gives a map $b : \mathrm{Pic}\,\mathcal{X}_s \to H^2(X, \mathcal{O}_X)$ which is related to the Chern class map. We show in Theorem 6.2 that s is the dimension of the $\mathbb{Q}_p$-vector space spanned by image of $\mathrm{Pic}(\mathcal{X}_s)$ in $H^2(X, \mathcal{O}_X)$. In particular, $H^2(X, \mathcal{O}_X) = 0$ implies that the kernel of $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_s)$ is finite. We use this to give some explicit calculations.

Theorem 1.3. *Let $\mathcal{X}$ be a family of abelian or K3 surfaces over $\mathrm{Spec}\,\mathbb{Z}_p$. If $r = 0$, then $\mathrm{Br}(\mathcal{X})$ is finite. If $r > 0$, then*

$$\mathrm{Br}(\mathcal{X}) \cong (\mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}]) \oplus (\mathbb{Q}/\mathbb{Z})^{r-1} \oplus P,$$

where P is a finite p-group.

We give an explicit example of an abelian surface with

$$\mathrm{Br}(\mathcal{X}) \cong (\mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}]) \oplus (\mathbb{Q}/\mathbb{Z})^2 \oplus P.$$

Finally, we briefly discuss the intermediate groups

$$\mathrm{Br}(\mathcal{X}) \to \lim \mathrm{Br}(\mathcal{X}_n) \to \mathrm{Br}(\mathcal{X}_s)$$

for $\mathcal{X}_n = \mathcal{X} \times_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$.

Notation: Throughout the paper, K is a finite extension of $\mathbb{Q}_p$ of degree f with Galois group G_K, and X is a smooth and proper scheme over K of dimension d. We let $h^{0,i} = \dim_K H^i(X, \mathcal{O}_X)$, and $\rho_X = \mathrm{rank}\,\mathrm{NS}(X)$ the Picard number.

We let $\mathcal{O}_K$ be the ring of integers of K and assume that X has a proper regular model $\mathcal{X}/\mathcal{O}_K$, which we can (by Stein factorization) assume to have geometrically connected fibers. Let $i : \mathcal{X}_s \to \mathcal{X}$ be the special fiber, $\rho_{\mathcal{X}_s} = \mathrm{rank}\,\mathrm{Pic}(\mathcal{X}_s)$ its Picard number and I the number of irreducible components of $\mathcal{X}_s$. The number

$$r = \rho_{\mathcal{X}_s} - \rho_X - I + 1$$

plays an important role in this paper.

The Brauer group $\mathrm{Br}(S)$ of a scheme S is the cohomological Brauer group, i.e., the group $H^2_{\mathrm{et}}(S, \mathbb{G}_m)$. By a theorem of Gabber [4], the Brauer group defined using Azumaya algebras is isomorphic to $\mathrm{Br}(S)_{\mathrm{tor}}$ if S is projective over an affine scheme.

For an abelian group A we let $A^{\wedge l} = \lim_i A/l^i$ be the l-adic completion, ${}_mA$ be the subgroup of m-torsion elements, $T_l = \lim_i {}_{l^i}A$ the l-adic Tate module, and $V_l = T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$.

Acknowledgements: We would like to thank T. Suzuki for helpful discussions. The second named author thanks Rikkyo University for their hospitality.

2. The Brauer group

We start by recalling some known facts on the cohomology of $\mathbb{G}_m$. Recall that $f = [K : \mathbb{Q}_p]$.

Proposition 2.1. *1) We have $H^0_{\text{et}}(\mathcal{X}, \mathbb{G}_m) \cong \mathcal{O}(\mathcal{X})^\times$, a direct sum of a $\mathbb{Z}_p$-module of rank f and a finite group, and $H^0_{\text{et}}(X, \mathbb{G}_m) \cong \mathcal{O}(X)^\times \cong \mathcal{O}(\mathcal{X})^\times \times \mathbb{Z}$.*

2) The group $H^1_{\text{et}}(X, \mathbb{G}_m) \cong \operatorname{Pic}(X)$ is an extension of a finitely generated group of rank ρ_X by a finitely generated $\mathbb{Z}_p$-module of rank $f \cdot h^{0,1}$, and there is an exact sequence

$$0 \to \mathbb{Z}^{I-1} \to \operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(X) \to 0, \tag{1}$$

where I is the number of irreducible components of $\mathcal{X}_s$.

3) The groups $\operatorname{Br}(\mathcal{X})$ and $\operatorname{Br}(X)$ are torsion groups with finite m-torsion for every m. Moreover, $\operatorname{Br}(X)$ contains $\operatorname{Br}(\mathcal{X})$ and $\mathbb{Q}/\mathbb{Z} \cong \operatorname{im}\operatorname{Br}(K)$ as subgroups with trivial intersection, and $\operatorname{Br}(X)/(\operatorname{im}\operatorname{Br}(K) \oplus \operatorname{Br}(\mathcal{X}))$ is isomorphic to the sum of a finite group and finitely many copies of $\mathbb{Q}_p/\mathbb{Z}_p$.

Proof. 1) This follows from $\mathcal{O}(\mathcal{X}) \cong \mathcal{O}_K$ and $\mathcal{O}(X) \cong K$ because of geometric connectedness.

2) Consider the low degree terms of the Hochschild-Serre spectral sequence:

$$0 \to \operatorname{Pic}(X) \to \operatorname{Pic}(\bar{X})^{G_K} \xrightarrow{d_2} \operatorname{Br}(K) \to \operatorname{Br}(X).$$

Since $\mathbb{Q}/\mathbb{Z} \cong \operatorname{Br}(K) \to \operatorname{Br}(X)$ has finite kernel (as one sees with a K'-rational point for K'/K finite), the image of $\operatorname{Br}(K)$ in $\operatorname{Br}(X)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$ and $\operatorname{Pic}(X)$ and $\operatorname{Pic}(\bar{X})^{G_K}$ differ by a finite group.

We know that $\operatorname{Pic}(\bar{X})^{G_K}$ is an extension of the finitely generated Néron-Severi group (of rank ρ_X) and the rational points of an abelian variety of dimension $h^{0,1}$, which has a subgroup of finite index isomorphic to $\mathcal{O}_K^{h^{0,1}} \cong \mathbb{Z}_p^{fh^{0,1}}$ by Mattuck's theorem [13]. In view of $H^i_{\text{et}}(\mathcal{X}, \mathbb{G}_m) \cong CH^1(\mathcal{X}, 1-i)$ and $H^i_{\text{et}}(X, \mathbb{G}_m) \cong CH^1(X, 1-i)$ for $i \leq 1$, the sequence is the localization sequence for higher Chow groups

$$0 \to \mathcal{O}_K^\times \to K^\times \to \mathbb{Z}^I \to \operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(X) \to 0,$$

where we use the identification $CH^1(\mathcal{X}, 1) = \mathcal{O}_K^\times$, $CH^1(X, 1) = K^\times$, as well as $CH_d(\mathcal{X}_s) \cong \mathbb{Z}^I$, the free abelian group on the irreducible components of $\mathcal{X}_s$.

3) The Brauer groups are torsion because they are contained in the corresponding cohomology groups of their function fields. To prove finiteness of the m-torsion, it suffices to show finiteness of $H^2_{\text{et}}(X, \mu_m)$, because this group surjects onto ${}_m\operatorname{Br}(X)$ and $\operatorname{Br}(\mathcal{X}) \subseteq \operatorname{Br}(X)$. The finiteness of $H^2_{\text{et}}(X, \mu_m)$ follows from the Hochschild-Serre spectral sequence

$$H^s(K, H^t_{\text{et}}(\bar{X}, \mu_m)) \Rightarrow H^{s+t}_{\text{et}}(X, \mu_m)$$

because the coefficients $H^t_{\mathrm{et}}(\bar{X},\mu_m)$ are finite, and Galois cohomology of a local field of characteristic 0 of a finite module is finite.

The prime to p-part of the statement about $\mathrm{Br}(X)/(\operatorname{im}\mathrm{Br}(K)\oplus\mathrm{Br}(\mathcal{X}))$ is proven in [3], see also [16, Prop. 5.2.1]. The p-part follows from the finiteness of the p-torsion of $\mathrm{Br}(X)$. □

For later use we note the following facts about the cohomology of the special fiber [14].

Proposition 2.2. *The group of units $H^0_{\mathrm{et}}(\mathcal{X}_s,\mathbb{G}_m)$ is a finite group, and the Picard group $H^1_{\mathrm{et}}(\mathcal{X}_s,\mathbb{G}_m)$ is finitely generated.*

Estimates using l-adic cohomology. For any prime l (including $l=p$), we have the short exact coefficient sequence

$$0\to \mathrm{Pic}(X)^{\wedge l}\otimes_{\mathbb{Z}_l}\mathbb{Q}_l\to H^2_{\mathrm{et}}(X,\mathbb{Q}_l(1))\to V_l\,\mathrm{Br}(X)\to 0.$$

The left $\mathbb{Q}_l$-vector space has dimension equal rank ρ_X if $l\neq p$, and equal to $\rho_X+fh^{0,1}$ for $l=p$ by Proposition 2.1(2). Thus in order to understand $V_l\,\mathrm{Br}(X)$, we calculate $H^2_{\mathrm{et}}(X,\mathbb{Q}_l(1))$. The spectral sequence

$$E_2^{s,t}=H^s(K,H^t_{\mathrm{et}}(\bar{X},\mathbb{Q}_l(1)))\Rightarrow H^{s+t}_{\mathrm{et}}(X,\mathbb{Q}_l(1)) \tag{2}$$

degenerates at $E_2^{s,t}$ by [5]. Since $H^2(K,H^0_{\mathrm{et}}(\bar{X},\mathbb{Q}_l(1)))\cong V_l\,\mathrm{Br}(K)\cong\mathbb{Q}_l$, we obtain

$$\dim H^2_{\mathrm{et}}(X,\mathbb{Q}_l(1))=1+\dim H^1(K,H^1_{\mathrm{et}}(\bar{X},\mathbb{Q}_l(1)))+\dim H^2_{\mathrm{et}}(\bar{X},\mathbb{Q}_l(1))^{G_K}.$$

From the divisibility of $\mathrm{Pic}^0(\bar{X})$ we obtain a short exact sequence

$$0\to \mathrm{NS}(\bar{X})\otimes\mathbb{Q}_l\to H^2_{\mathrm{et}}(\bar{X},\mathbb{Q}_l(1))\to V_l\,\mathrm{Br}(\bar{X})\to 0.$$

This sequence splits as a sequence of Galois-modules, hence $\mathrm{NS}(X)_{\mathbb{Q}_l}\cong\mathrm{NS}(\bar{X})_{\mathbb{Q}_l}^{G_K}$ implies

$$\dim H^2_{\mathrm{et}}(\bar{X},\mathbb{Q}_l(1))^{G_K}=\dim\mathrm{NS}(X)_{\mathbb{Q}_l}+\dim V_l\,\mathrm{Br}(\bar{X})^{G_K}=\rho_X+\dim V_l\,\mathrm{Br}(\bar{X})^{G_K}.$$

The remaining direct summand of $H^2_{\mathrm{et}}(X,\mathbb{Q}_l(1))$ is calculated in the following proposition.

Proposition 2.3. *The vector space $H^1(K,H^1_{\mathrm{et}}(\bar{X},\mathbb{Q}_l(1)))$ is zero-dimensional for $l\neq p$ and has dimension $2fh^{0,1}$ for $l=p$.*

Proof. We have $H^1_{\mathrm{et}}(\bar{X},\mathbb{Q}_l(1))\cong V_l\,\mathrm{Pic}^0(\bar{X})$ is a vector space of dimension $2h^{0,1}$ for any l. Using Euler-Poincaré characteristic,

$$\sum_i(-1)^i\dim H^i(K,V)=\begin{cases}0, & V \text{ a } \mathbb{Q}_l\text{-vector space;}\\ -f\dim V, & V \text{ a } \mathbb{Q}_p\text{-vector space,}\end{cases}$$

it suffices to show that $H^1(\bar{X}, \mathbb{Q}_l(1))^{G_K}$ and $H^2(K, H^1_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l(1)))$ vanish. To show the vanishing of $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l(1))^{G_K}$, we note that $V_l \operatorname{Pic}(X) = 0$ implies that the two left groups in the short exact sequence arising from (2),

$$0 \to H^1(K, \mathbb{Q}_l(1)) \to H^1_{\mathrm{et}}(X, \mathbb{Q}_l(1)) \to H^1_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l(1))^{G_K} \to 0$$

are both isomorphic to $(K^\times)^{\wedge l} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. By local duality

$$H^2(K, H^1_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l(1))) \cong H^1_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l)_{G_K}$$

and by Poincaré duality the right hand term is dual to $H^{2d-1}_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l(d))^{G_K}$, which, by the hard Lefschetz theorem, is isomorphic to $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l(1))^{G_K} = 0$. □

Comparing the two expressions for $H^2_{\mathrm{et}}(X, \mathbb{Q}_l(1))$, we obtain

Theorem 2.4. *We have*

$$\dim_{\mathbb{Q}_l} V_l \operatorname{Br}(X) = \begin{cases} 1 + \dim V_l \operatorname{Br}(\bar{X})^{G_K} & \text{for } l \neq p; \\ 1 + \dim V_p \operatorname{Br}(\bar{X})^{G_K} + f h^{0,1} & \text{for } l = p. \end{cases}$$

Note that in general $\dim_{\mathbb{Q}_p} H^2_{\mathrm{et}}(\bar{X}, \mathbb{Q}_p(1))^{G_K} \leq \dim_{\mathbb{Q}_l} H^2_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l(1))^{G_K}$, or equivalently, $\dim_{\mathbb{Q}_p} V_p \operatorname{Br}(\bar{X})^{G_K} \leq \dim_{\mathbb{Q}_l} V_l \operatorname{Br}(\bar{X})^{G_K}$ for $l \neq p$, so that it is not clear which of $\dim_{\mathbb{Q}_l} V_l \operatorname{Br}(X)$ and $\dim_{\mathbb{Q}_p} V_p \operatorname{Br}(X)$ is larger.

If $\mathcal{X}$ is a regular proper model, then by Proposition 2.1(3) and the proper base change theorem, we have

$$\dim_{\mathbb{Q}_l} V_l \operatorname{Br}(X) = 1 + \dim_{\mathbb{Q}_l} V_l \operatorname{Br}(\mathcal{X}) = 1 + r + \dim_{\mathbb{Q}_l} V_l \operatorname{Br}(\mathcal{X}_s)$$

for $l \neq p$, hence

$$\dim V_l \operatorname{Br}(\bar{X})^{G_K} = r + \dim_{\mathbb{Q}_l} V_l \operatorname{Br}(\mathcal{X}_s)$$

In particular, r is independent of the model if we assume Artin's conjecture on the finiteness of $\operatorname{Br}(\mathcal{X}_s)$.

3. Completions

We give some facts about completions needed below; the reader familiar with completions can skip this section. For a complex A of abelian groups and integer m, $A \otimes^{\mathbb{L}} \mathbb{Z}/m$ is represented by the total complex of the double complex $A \xrightarrow{m} A$ concentrated in (cohomological) degrees -1 and 0. The canonical map $A \to A \otimes^{\mathbb{L}} \mathbb{Z}/m$ is induced by mapping A to the component in degree 0. The cohomology of $A \otimes^{\mathbb{L}} \mathbb{Z}/m$ can be calculated by the exact sequence

$$0 \to H^i(A)/m \to H^i(A \otimes^{\mathbb{L}} \mathbb{Z}/m) \to {}_mH^{i+1}(A) \to 0. \tag{3}$$

For a prime number p, the p-completion of $A^\wedge$ is the pro-system $\{A \otimes^{\mathbb{L}} \mathbb{Z}/p^j\}_j$, where the transition maps in the system are multiplication by p in degree -1 and the identity in degree 0. We define continuous cohomology $H^i_{\mathrm{cont}}(A^\wedge)$ to be the cohomology of $R\lim A^\wedge$. We have a short exact sequence

$$0 \to \lim^1_j H^{i-1}(A \otimes^{\mathbb{L}} \mathbb{Z}/p^j) \to H^i_{\mathrm{cont}}(A^\wedge) \to \lim_j H^i(A \otimes^{\mathbb{L}} \mathbb{Z}/p^j) \to 0.$$

With the inverse limit of the sequence (3) and using $\lim^1_j H^i(A\otimes^{\mathbb{L}}\mathbb{Z}/p^j)\cong \lim^1_j {}_{p^j}H^{i+1}(A)$ we obtain a diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & H^i(A) & \xrightarrow{d^i} & \lim_j H^i(A)/p^j & & \\
 & & & & {\scriptstyle c^i}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \lim^1_j {}_{p^j}H^i(A) & \longrightarrow & H^i_{\mathrm{cont}}(A^\wedge) & \xrightarrow{e^i} & \lim_j H^i(A\otimes^{\mathbb{L}}\mathbb{Z}/p^j) & \longrightarrow & 0. \\
 & & & & & & \downarrow & & \\
 & & & & & & T_pH^{i+1}(A) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

If ${}_{p^j}H^i(A)$ is finite for one (or, equivalently, for all) j, then $\lim^1_j {}_{p^j}H^i(A)$ vanishes and $\ker c^i=\ker d^i$. The kernel of d^i consists of the p-divisible elements of $H^i(A)$ but we are interested in the (smaller) kernel of c^i. For an abelian group G let (G,p) be the inverse system with constant group G and multiplication by p as transition maps.

Proposition 3.1. *The kernel of $c^i: H^i(A)\to H^i_{\mathrm{cont}}(A^\wedge)$ is the maximal p-divisible subgroup of $H^i(A)$. The cokernel of c^i is an extension of $T_pH^{i+1}(A)$ by the uniquely p-divisible group $\lim^1(H^i(A),p)$. In particular, it is p-torsion free.*

We will discuss $\lim^1 A_i$ for finitely generated A_i of constant rank r in Theorem 7.5.

Proof. The diagram gives an exact sequence

$$0\to\ker c^i\to\ker d^i\to\lim^1_j {}_{p^j}H^i(A)\to\operatorname{coker} c^i\to\operatorname{coker} e^ic^i\to 0. \tag{4}$$

For any abelian group G we have a sequence of inverse systems

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_{p^{j+1}}G & \longrightarrow & G & \xrightarrow{p^{j+1}} & p^{j+1}G & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle p}\downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_{p^j}G & \longrightarrow & G & \xrightarrow{p^j} & p^jG & \longrightarrow & 0
\end{array}$$

and taking the 6-term exact derived lim sequence we get

$$0\to T_pG\to\lim(G,p)\xrightarrow{\xi}\lim p^jG\to\lim^1 {}_{p^j}G\to\lim^1(G,p)\to\lim^1 p^jG\to 0. \tag{5}$$

Applying this to $G = H^i(A)$ and comparing to (4) we see that $\lim p^j H^i(A) = \bigcap_j p^j H^i(A) = \ker d^i$ implies $\ker c^i \cong \operatorname{im} \xi$, which is the maximal p-divisible group of G by [8, Lemma 4.3a)].

By definition, the cone of the completion map c of complexes is isomorphic to the cohomology of $R\lim(A,p)$, where the system is the complex A with transition map multiplication by p. We obtain a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \lim^1(H^i(A),p) & \longrightarrow & H^i(\mathrm{cone}(c)) & \longrightarrow & \lim(H^{i+1}(A),p) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \xi} & & \\
0 & \longrightarrow & \operatorname{coker} c^i & \longrightarrow & H^i(\mathrm{cone}(c)) & \longrightarrow & \ker c^{i+1} & \longrightarrow & 0
\end{array}$$

By (5), the kernel of the right map is $T_p H^{i+1}(A)$ and we see by the snake Lemma that $\operatorname{coker} c^i$ is an extension of $T_p H^{i+1}(A)$ by $\lim^1(H^i(A),p)$. $\square$

Corollary 3.2. *If $H^i(A)$ is a $\mathbb{Z}_{(p)}$-module, then $\operatorname{coker} c^i$ is the direct sum of $T_p H^{i+1}(A)$ and the uniquely divisible group $\lim^1(H^i(A),p)$.*

A similar discussion applies to a bounded complex of sheaves B on X by applying the above to $A = R\Gamma_{\mathrm{et}}(X,B)$. Since $R\Gamma_{\mathrm{et}}(X, B\otimes^{\mathbb{L}}\mathbb{Z}/m) \cong R\Gamma_{\mathrm{et}}(X,B)\otimes^{\mathbb{L}} \mathbb{Z}/m$, we define $H^i_{\mathrm{cont}}(X, B^\wedge)$ as the cohomology of $R\lim_j(R\Gamma_{\mathrm{et}}(X,B)\otimes^{\mathbb{L}}\mathbb{Z}/p^j)$ and obtain the sequence

$$0 \to H^i_{\mathrm{et}}(X,B)^{\wedge p} \to H^i_{\mathrm{cont}}(X,B^\wedge) \to T_p H^{i+1}_{\mathrm{et}}(X,B) \to 0 \tag{6}$$

if $H^{i-1}_{\mathrm{et}}(X, B\otimes^{\mathbb{L}}\mathbb{Z}/p)$ is finite.

4. The p-adic completion of $\mathcal{X}$

We have an exact sequence of etale sheaves on $\mathcal{X}$,

$$0 \to \mathcal{K} \to \mathbb{G}_{m,\mathcal{X}} \to i_*\mathbb{G}_{m,\mathcal{X}_s} \to 0. \tag{7}$$

Since $\mu_m = \ker(\mathbb{G}_m \overset{\times m}{\to} \mathbb{G}_m)$ is locally constant on $\mathcal{X}$ for m prime to p, the proper base change theorem implies that the cohomology of $\mathcal{K}$ is uniquely l-divisible for any $l \neq p$, i.e., it consists of $\mathbb{Z}_{(p)}$-modules.

As the map $\mathcal{O}(\mathcal{X})^\times \to \mathcal{O}(\mathcal{X}_s)^\times$ is surjective with p-adically complete kernel $H^0(\mathcal{X},\mathcal{K})$, we obtain a long exact sequence

$$0 \to H^1_{\mathrm{et}}(\mathcal{X},\mathcal{K}) \to \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_s) \to H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K}) \to \mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_s). \tag{8}$$

The group $\mathrm{Pic}(\mathcal{X}_s)$ is finitely generated [14], and we let $N \subseteq H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K})$ be the cokernel of the map $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_s)$.

Proposition 4.1. *The group N is finitely generated of rank $r = \rho_{\mathcal{X}_s} - \rho_X - I + 1$, and $H^1_{\mathrm{et}}(\mathcal{X},\mathcal{K})$ is a finitely generated $\mathbb{Z}_p$-module of rank $fh^{0,1}$.*

Proof. Since N is finitely generated it suffices to calculate $N\otimes\mathbb{Q}_l/\mathbb{Z}_l$ for $l\neq p$. We have a short exact sequence

$$0\to \mathrm{Pic}(\mathcal{X})\otimes\mathbb{Q}_l/\mathbb{Z}_l\to \mathrm{Pic}(\mathcal{X}_s)\otimes\mathbb{Q}_l/\mathbb{Z}_l\to N\otimes\mathbb{Q}_l/\mathbb{Z}_l\to 0,$$

where the left map is injective by the proper base change theorem. The Lemma follows from Proposition 2.1(2) by counting coranks.

The same short exact sequence shows that the subgroup $\mathbb{Z}^{I-1}\subseteq \mathrm{Pic}(\mathcal{X})$ injects into $\mathrm{Pic}(\mathcal{X}_s)$. Hence $H^1_{\mathrm{et}}(\mathcal{X},\mathcal{K})$ can be viewed a subgroup of $\mathrm{Pic}(X)$. Since it is uniquely l-divisible, it maps to zero in $\mathrm{NS}(X)/l$ for all $l\neq p$, hence it has trivial image in the finitely generated group $\mathrm{NS}(X)$. Thus $H^1_{\mathrm{et}}(\mathcal{X},\mathcal{K})$ can be viewed as a subgroup of $\mathrm{Pic}^0(X)$. Since the quotient is finitely generated, we conclude that it is a finitely generated $\mathbb{Z}_p$-module of the same rank $fh^{0,1}$. □

The $\mathbb{Z}_{(p)}$-module $H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K})$ has finite p-torsion because N is finitely generated and $\mathrm{Br}(\mathcal{X})$ has finite p-torsion. Since $\mathrm{Br}(\mathcal{X})$ is torsion, $N_{\mathbb{Q}}\cong H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K})_{\mathbb{Q}}$ has dimension r.

Consider the formal completion of $\mathcal{X}$ at p, i.e., the direct system of the reductions $\mathcal{X}_n=\mathcal{X}\times_{\mathcal{O}}\mathcal{O}/p^n$, of $\mathcal{X}$ modulo p^n. We obtain a short exact sequence of pro-sheaves on the topological space $\mathcal{X}_s$,

$$0\to\mathcal{K}_n\to i_n^*\mathbb{G}_{m,\mathcal{X}_n}\to\mathbb{G}_{m,\mathcal{X}_s}\to 0, \tag{9}$$

where $i_n:\mathcal{X}_s\to\mathcal{X}_n$ is the closed embedding. Let $H^i_{\mathrm{cont}}(\mathcal{X}_s,\mathbb{G}_{m,\bullet})$ and $H^i_{\mathrm{cont}}(\mathcal{X}_s,\mathcal{K}_\bullet)$ be the continuous cohomology of the pro-sheaves $(i_n^*\mathbb{G}_{m,\mathcal{X}_n})_n$ and $(\mathcal{K}_n)_n$ on $\mathcal{X}_s$, respectively.

Since i_n is a universal homeomorphism, the cohomology can be calculated with the short exact sequence

$$0\to\lim{}^1 H^{i-1}_{\mathrm{et}}(\mathcal{X}_n,\mathbb{G}_{m,\mathcal{X}_n})\to H^i_{\mathrm{cont}}(\mathcal{X}_s,\mathbb{G}_{m,\bullet})\to\lim H^i_{\mathrm{et}}(\mathcal{X}_n,\mathbb{G}_{m,\mathcal{X}_n})\to 0. \tag{10}$$

If $i=1$, then the left term vanishes because the groups $H^0_{\mathrm{et}}(\mathcal{X}_n,\mathbb{G}_{m,\mathcal{X}_n})$ are finite by properness of $\mathcal{X}_n$. Moreover, the natural map $\mathrm{Pic}(\mathcal{X})\to\lim\mathrm{Pic}(\mathcal{X}_n)$ is an isomorphism by Grothendiecks formal existence theorem [6, Cor. 5.1.6, Scholie 5.1.7], so that we have an isomorphism

$$\mathrm{Pic}(\mathcal{X})\cong H^1_{\mathrm{cont}}(\mathcal{X}_s,\mathbb{G}_{m,\bullet}). \tag{11}$$

This implies that the long exact cohomology sequence associated to the short exact sequence (9) takes the form

$$\begin{aligned}0\to H^1_{\mathrm{cont}}(\mathcal{X}_s,\mathcal{K}_\bullet)\to\mathrm{Pic}(\mathcal{X})\to\mathrm{Pic}(\mathcal{X}_s)&\\ \to H^2_{\mathrm{cont}}(\mathcal{X}_s,\mathcal{K}_\bullet)\to H^2_{\mathrm{cont}}(\mathcal{X}_s,\mathbb{G}_{m,\bullet})&\to H^2_{\mathrm{et}}(\mathcal{X}_s,\mathbb{G}_m).\end{aligned} \tag{12}$$

Let $H^i_{\mathrm{cont}}(\mathcal{X},\mathcal{K}^\wedge)$ be the cohomology of the p-adic completion $\mathcal{K}^\wedge$ of $\mathcal{K}$ as in Section 3.

Proposition 4.2. *We have* $H^i_{\mathrm{cont}}(\mathcal{X}_s,\mathcal{K}_\bullet)\cong H^i_{\mathrm{cont}}(\mathcal{X},\mathcal{K}^\wedge)$ *for all* i.

To prove the proposition, we compare both sides to the p-completion $(\mathcal{K}_n^\wedge)_n$, i.e., the double inverse system $i_n^*\mathcal{K}_n \otimes^{\mathbb{L}} \mathbb{Z}/p^t$, and show that we have isomorphisms

$$H^i_{\text{cont}}(\mathcal{X}, \mathcal{K}^\wedge) \xrightarrow{\sim} H^i_{\text{cont}}(\mathcal{X}_s, i^*\mathcal{K}^\wedge) \xrightarrow{\sim} H^i_{\text{cont}}(\mathcal{X}_s, \mathcal{K}_\bullet^\wedge) \xleftarrow{\sim} H^i_{\text{cont}}(\mathcal{X}_s, \mathcal{K}_\bullet).$$

The first isomorphism follows from the proper base change because $H^i(\mathcal{O}_K, \mathcal{F}) \xrightarrow{\sim} H^i(s, i^*\mathcal{F})$ for any etale sheaf $\mathcal{F}$ on $\operatorname{Spec} \mathcal{O}_K$. The third and second isomorphism follow from the following Proposition.

Proposition 4.3. *1) For fixed t, we have an isomorphism of pro-sheaves on $\mathcal{X}_s$*

$$i^*\mathcal{K} \otimes^{\mathbb{L}} \mathbb{Z}/p^t \xrightarrow{\sim} \{i_n^*\mathcal{K}_n \otimes^{\mathbb{L}} \mathbb{Z}/p^t\}_n.$$

2) For fixed n, we have an isomorphism of pro-sheaves on $\mathcal{X}_n$,

$$\mathcal{K}_n \xrightarrow{\sim} \{\mathcal{K}_n \otimes^{\mathbb{L}} \mathbb{Z}/p^t\}_t.$$

Proof. 1) This is proven in [2, Lemma 2].

2) The pro-system $\{\mathcal{K}_n \otimes^{\mathbb{L}} \mathbb{Z}/p^t\}_t$ is quasi-isomorphic to the pro-complex $\{\mathcal{K}_n \xrightarrow{p^t} \mathcal{K}_n\}_t$, where the transition maps in the left system are multiplication by p. It suffices to show that the left-system is Artin-Rees zero. For this we fix s such that $p^s \geq n$ and show that the $(n+s)$-fold transition map in the system is the zero map.

The stalks of $\mathcal{K}_n$ are sections over strictly local $\mathbb{Z}/p^n$-algebras, and every section of $\mathcal{K}_n$ over a local $\mathbb{Z}/p^n$-algebra A can be written as $1+x$ with $x \in pA$. It suffices to show that $(1+x)^{p^{n+s}} = 1$ for all $x \in pA$. The monomials $\binom{p^{n+s}}{j}x^j$ vanish for $j \geq p^s \geq n$ because $x \in pA$ and $p^n = 0$ in A, and they vanish for $0 < j < p^s$ by the following lemma. □

Lemma 4.4. *For fixed s we have $v_p(\binom{p^{n+s}}{u}) > n$ for all $0 < u < p^s$.*

Proof. From Legendre's formula we get

$$v_p(\binom{z}{u}) = \sum_{i=1}^{\infty} \Big(\Big\lfloor \frac{z}{p^i} \Big\rfloor - \Big\lfloor \frac{u}{p^i} \Big\rfloor - \Big\lfloor \frac{z-u}{p^i} \Big\rfloor \Big),$$

where for a real number x, $\lfloor x \rfloor$ denotes the largest integer which is not larger than x. For real numbers x and y we have $\lfloor x+y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor \geq 0$, and $\lfloor x+y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor = 1$, if $x+y$ is an integer but x, y are not. If $z = p^{n+s}$ and $0 < u < p^s$, then $\frac{z}{p^i}$ is an integer but $\frac{u}{p^i}$ is not for $i = s, \dots, n+s$, and the Lemma follows. □

5. Comparison of sequences

The natural maps $i^*\mathbb{G}_{m,\mathcal{X}} \to i_n^*\mathbb{G}_{m,\mathcal{X}_n}$ induce maps $H^i_{\text{et}}(\mathcal{X}, \mathbb{G}_m) \to H^i_{\text{cont}}(\mathcal{X}_s, \mathbb{G}_{m,\bullet})$ and $H^i_{\text{et}}(\mathcal{X}, \mathcal{K}) \to H^i_{\text{cont}}(\mathcal{X}_s, \mathcal{K}_\bullet)$, hence a map between the sequences (8) and (12).

By Proposition 4.2 we obtain a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\mathrm{et}}(\mathcal{X},\mathcal{K}) & \longrightarrow & \mathrm{Pic}(\mathcal{X}) & \longrightarrow & \mathrm{Pic}(\mathcal{X}_s) & \longrightarrow & H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K}) & \longrightarrow \\
 & & \downarrow & & \| & & \| & & \downarrow a & \\
0 & \longrightarrow & H^1_{\mathrm{cont}}(\mathcal{X},\mathcal{K}^\wedge) & \longrightarrow & \mathrm{Pic}(\mathcal{X}) & \longrightarrow & \mathrm{Pic}(\mathcal{X}_s) & \longrightarrow & H^2_{\mathrm{cont}}(\mathcal{X},\mathcal{K}^\wedge) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow & \mathrm{Br}(\mathcal{X}) & \longrightarrow & H^2_{\mathrm{et}}(\mathcal{X}_s,\mathbb{G}_m) & \xrightarrow{u} & H^3_{\mathrm{et}}(\mathcal{X},\mathcal{K}) \\
 & \downarrow b & & \| & & \downarrow c \\
\longrightarrow & H^2_{\mathrm{cont}}(\mathcal{X}_s,\mathbb{G}_{m,\bullet}) & \longrightarrow & H^2_{\mathrm{et}}(\mathcal{X}_s,\mathbb{G}_m) & \xrightarrow{v} & H^3_{\mathrm{cont}}(\mathcal{X},\mathcal{K}^\wedge).
\end{array} \tag{13}$$

Thus $H^1_{\mathrm{et}}(\mathcal{X},\mathcal{K}) \cong H^1_{\mathrm{cont}}(\mathcal{X},\mathcal{K}^\wedge)$, we have an isomorphism $\ker a \cong \ker b$, and a sequence

$$0 \to \operatorname{coker} a \to \operatorname{coker} b \to \ker c \cap \operatorname{im} u \to 0.$$

Lemma 5.1. *We have $T_p H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K}) = 0$, and the composition*

$$N \to H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K}) \to \lim H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K})/p^r$$

is injective.

Proof. Since ${}_{p^r}H^i_{\mathrm{et}}(\mathcal{X},\mathcal{K})$ is finite for $i \le 2$, we obtain a short exact sequence

$$0 \to \lim H^i_{\mathrm{et}}(\mathcal{X},\mathcal{K})/p^r \to H^i_{\mathrm{cont}}(\mathcal{X},\mathcal{K}^\wedge) \to T_p H^{i+1}_{\mathrm{et}}(\mathcal{X},\mathcal{K}) \to 0.$$

For $i = 1$, $H^1_{\mathrm{et}}(\mathcal{X},\mathcal{K}) \cong H^1_{\mathrm{cont}}(\mathcal{X},\mathcal{K}^\wedge)$ implies that this group is p-adically complete and that $T_p H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K})$ vanishes. For $i = 2$, the sequence implies that $\lim H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K})/p^r \subseteq H^2_{\mathrm{cont}}(\mathcal{X},\mathcal{K}^\wedge)$, and that the map from N to the latter group is injective by diagram (13). ☐

Proposition 5.2. *The torsion subgroup $\mathrm{Tor} H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K})$ is a finite p-group, and $H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K})/\mathrm{tor}$ is an extension of $\mathbb{Q}^t$ by $\mathbb{Z}^s_{(p)}$ with $s + t = r$.*

Proof. Since $C = H^2_{\mathrm{et}}(\mathcal{X},\mathcal{K})$ is a $\mathbb{Z}_{(p)}$-module, C_{tor} consists of p-power torsion. But every torsion group contains a basic subgroup [15, Thm. 10.36], i.e., C has a pure subgroup B which is a direct sum of cyclic groups and such that C/B is divisible. Since T_pC vanishes, we have $C = B$, and then the finiteness of ${}_pC$ implies finiteness of C_{tor}.

Now $\bar{C} = C/C_{\mathrm{tor}}$ is a $\mathbb{Z}_{(p)}$-submodule of $C_{\mathbb{Q}} \cong N_{\mathbb{Q}} \cong \mathbb{Q}^r$ because $\mathrm{Br}(\mathcal{X})$ is torsion. It thus suffices to prove the following lemma.

Lemma 5.3. *Every $\mathbb{Z}_{(p)}$-submodule M of $\mathbb{Q}^r$ with $M_{\mathbb{Q}} \cong \mathbb{Q}^r$ is an extension of $\mathbb{Q}^t$ by $\mathbb{Z}^s_{(p)}$ with $s + t = r$.*

Proof. We proceed by induction on r. If $r = 1$, then M is a $\mathbb{Z}_{(p)}$- submodule of $\mathbb{Q}$. Every element of M can be written as ap^u, where $a \in \mathbb{Z}^\times_{(p)}$ and $u \in \mathbb{Z}$. If there exist elements with arbitrary large negative u, then $M = \mathbb{Q}$, and if not,

$M = p^{-v}\mathbb{Z}_{(p)}$ for some v, hence M is isomorphic to $\mathbb{Z}_{(p)}$. If $r > 1$, let $\pi : \mathbb{Q}^r \to \mathbb{Q}$ be a non-trivial homomorphism with kernel $\mathbb{Q}^{r-1}$, and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M \cap \mathbb{Q}^{r-1} & \longrightarrow & M & \longrightarrow & \pi(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}^{r-1} & \longrightarrow & \mathbb{Q}^r & \longrightarrow & \mathbb{Q} & \longrightarrow & 0. \end{array}$$

By induction hypothesis $M \cap \mathbb{Q}^{r-1}$ has a free $\mathbb{Z}_{(p)}$-submodule with uniquely divisible quotient, and $\pi(M)$ is either isomorphic to $\mathbb{Z}_{(p)}$ or to $\mathbb{Q}$. In the former case, we have $M \cong M \cap \mathbb{Q}^{r-1} \oplus \mathbb{Z}_{(p)}$, and in the latter case, M still has a free $\mathbb{Z}_{(p)}$-submodule with uniquely divisible quotient. $\square$

Corollary 5.4. *We have*

$$\ker\left(\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_s)\right) \cong (\mathbb{Q}/\mathbb{Z}')^s \oplus (\mathbb{Q}/\mathbb{Z})^t \oplus P$$

with $s + t = r$, P a finite p-group, and $\mathbb{Q}/\mathbb{Z}' = \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$. Moreover $s = 0$ is equivalent to $r = 0$.

Note that Artin's conjecture states that $\mathrm{Br}(\mathcal{X}_s)$ is finite.

Proof. The kernel of $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_s)$ is isomorphic to $H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K})/N$. Let A be the kernel of the composition $N \hookrightarrow H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K}) \to \mathbb{Q}^t$ of Proposition 5.2, and let B be its image. By Proposition 5.2 we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & N & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathbb{Z}_{(p)})^s \oplus F & \longrightarrow & H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K}) & \longrightarrow & \mathbb{Q}^t & \longrightarrow & 0 \end{array}$$

where F is a finite p-group and all vertical maps are injective. Tensoring with $\mathbb{Q}$ we see that A is finitely generated of rank s and B is finitely generated of rank t. We obtain a short exact sequence of cokernels

$$0 \to (\mathbb{Q}/\mathbb{Z}')^s \oplus F' \to H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K})/N \to (\mathbb{Q}/\mathbb{Z})^t \to 0,$$

where F' is a quotient of F. Because $(\mathbb{Q}/\mathbb{Z}')^s$ is injective, $H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K})/N$ is an extension of $(\mathbb{Q}/\mathbb{Z})^t \oplus (\mathbb{Q}/\mathbb{Z}')^s$ by F', and this is isomorphic to $(\mathbb{Q}/\mathbb{Z})^t \oplus (\mathbb{Q}/\mathbb{Z}')^s \oplus \tilde{F}$ with $\tilde{F}$ a quotient of F' by the following Lemma. $\square$

Lemma 5.5. *Let E be an extension of a divisible group D by a finite group F. Then $E \cong D \oplus F'$ for F' a quotient of F.*

Proof. Let E' be the maximal divisible subgroup of E, and let K and D' be the kernel and image of the composition $E' \to E \to D$, respectively, so that we obtain a diagram with vertical inclusions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & D' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & D & \longrightarrow & 0. \end{array}$$

Finiteness of F implies finiteness of K, which implies that the divisible groups E', D', and D all have the same l-corank for all l. Hence the injection $D' \to D$ is an isomorphism, and $E = E' \oplus F/K$. □

6. Relationship to the Chern class map, examples

We have the following theorem of Flach-Siebel [2, Lemma 1].

Theorem 6.1. *We have* $H^i(X, \mathcal{O}_X) \cong H^i_{\mathrm{cont}}(\mathcal{X}, \mathcal{K}^\wedge) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

The Theorem together with the sequence (12) and Proposition 4.2 induces an injection

$$\beta : N_{\mathbb{Q}} \to H^2_{\mathrm{cont}}(\mathcal{X}_s, \mathcal{K}_\bullet)_{\mathbb{Q}} \cong H^2(X, \mathcal{O}_X)$$

which we can extend to a map

$$\beta_{\mathbb{Q}_p} : N \otimes \mathbb{Q}_p \to H^2(X, \mathcal{O}_X).$$

Theorem 6.2. *We have* $s = \dim_{\mathbb{Q}_p} \operatorname{im} \beta_{\mathbb{Q}_p}$ *and* $t = \dim_{\mathbb{Q}_p} \ker \beta_{\mathbb{Q}_p}$.

In other words, s is the dimension of the $\mathbb{Q}_p$-vector space spanned by the abelian group N of rank r.

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1_{\mathrm{cont}}(\mathcal{X}_s, \mathcal{K}_\bullet) & \longrightarrow & \operatorname{Pic} \mathcal{X}^{\wedge p} & \xrightarrow{f} & \operatorname{Pic} \mathcal{X}_s^{\wedge p} & \xrightarrow{g} & H^2_{\mathrm{cont}}(\mathcal{X}_s, \mathcal{K}_\bullet) \\ & & \| & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & H^1_{\mathrm{cont}}(\mathcal{X}_s, \mathcal{K}_\bullet) & \longrightarrow & H^2_{\mathrm{et}}(\mathcal{X}, \mathbb{Z}_p(1)) & \longrightarrow & H^2_{\mathrm{et}}(\mathcal{X}_s, \mathbb{Z}_p(1)) & \longrightarrow & H^2_{\mathrm{cont}}(\mathcal{X}_s, \mathcal{K}_\bullet) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & T_p \operatorname{Br} \mathcal{X} & \xrightarrow{\alpha'} & T_p \operatorname{Br} \mathcal{X}_s & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

The upper (non-exact) row is obtained by completing the cohomology groups in (12), the exact middle row is obtained by p-completing the coefficients in (9). The columns are exact coefficient sequences. The middle left horizontal map is injective because $H^1_{\mathrm{et}}(\mathcal{X},\mathbb{Z}_p(1)) \to H^1_{\mathrm{et}}(\mathcal{X}_s,\mathbb{Z}_p(1))$ is surjective, hence so is the upper left horizontal map. A diagram chase shows that $\ker g/\operatorname{im} f \cong \ker\alpha'$ (which has rank t by Theorem 1.1). On the other hand, the diagram

$$\begin{array}{ccccccc}
\operatorname{Pic}\mathcal{X}\otimes\mathbb{Z}_p & \longrightarrow & \operatorname{Pic}\mathcal{X}_s\otimes\mathbb{Z}_p & \longrightarrow & N\otimes\mathbb{Z}_p & \longrightarrow & 0\\
\downarrow & & \| & & \| & & \\
\operatorname{Pic}\mathcal{X}^{\wedge p} & \xrightarrow{f} & \operatorname{Pic}\mathcal{X}_s^{\wedge p} & \longrightarrow & N^{\wedge p} & \longrightarrow & 0
\end{array}$$

shows that $\operatorname{Pic}\mathcal{X}_s^{\wedge p}/\operatorname{im} f \cong N\otimes\mathbb{Z}_p$, a $\mathbb{Z}_p$-module of rank r, and that $(\operatorname{im} g)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = \operatorname{im}\beta_{\mathbb{Q}_p}$. Combining this with the canonical short exact sequence

$$0\to \ker g/\operatorname{im} f \to \operatorname{Pic}\mathcal{X}_s^{\wedge}/\operatorname{im} f \to \operatorname{Pic}\mathcal{X}_s^{\wedge}/\ker g \to 0$$

we see that $\operatorname{Pic}\mathcal{X}_s^{\wedge p}/\ker g \cong \operatorname{im} g$ has rank s. □

If W denotes the Witt vectors of the residue field, then in the good reduction case we have a commutative diagram of Berthelot-Ogus [1, Cor. 3.7],

$$\begin{array}{ccccccc}
\operatorname{Pic}\mathcal{X} & \xrightarrow{c_{dR}} & H^2(X,F^1\Omega^\bullet_X) & \longrightarrow & H^2(X,\Omega^\bullet_X) & \longrightarrow & H^2(X,\mathcal{O}_X)\\
\downarrow & & & & \| & & \\
\operatorname{Pic}\mathcal{X}_s & \xrightarrow{c_{cris}} & H^2_{crys}(\mathcal{X}_s/W)^{F=p} & \longrightarrow & H^2_{crys}(\mathcal{X}_s/W)\otimes_W K, & &
\end{array}$$

and we obtain β as the composition from the southwest to the northeast corner, which vanishes on $\operatorname{Pic}\mathcal{X}$ because the upper row is the zero-map.

Examples: Abelian and K3 surfaces. We calculate $\operatorname{Br}(\mathcal{X})$ for $\mathcal{X}$ an abelian scheme or a family of K3 surfaces over $\mathbb{Z}_p$.

Theorem 6.3. *Let $\mathcal{X}$ be an abelian scheme or a family of K3 surfaces over $\mathbb{Z}_p$. If $r=0$, then $\operatorname{Br}(\mathcal{X})$ is finite. If $r>0$, then*

$$\operatorname{Br}(\mathcal{X}) \cong (\mathbb{Q}/\mathbb{Z}') \oplus (\mathbb{Q}/\mathbb{Z})^{r-1} \oplus (finite).$$

Proof. Since Tate's conjecture is known for abelian varieties [17] and K3 surfaces over a finite field [12], [10], we know that $\operatorname{Br}(\mathcal{X}_s)$ is finite. Then $r=0$ implies that $\operatorname{Br}(\mathcal{X})$ is finite. If $r>0$, then since $H^2(\mathcal{X},\mathcal{O}_{\mathcal{X}})\cong\mathcal{O}$ we obtain that $s\le 1$ by Theorem 6.2, but since $s=0$ implies $r=s+t=0$ we must have $s=1$. □

We give an example of an abelian surface with $r>1$, showing that $\operatorname{Br}(\mathcal{X})$ contains a divisible p-group.

Proposition 6.4. *If $\mathcal{X}_s$ is a simple abelian surface over $\mathbb{F}_p$, then* $\operatorname{rank}\operatorname{Pic}(\mathcal{X}_s)=2$. *If $\mathcal{X}_s$ is the product of two elliptic curves E_1 and E_2 over $\mathbb{F}_p$, then* $\operatorname{rank}\operatorname{Pic}(\mathcal{X}_s)=4$ *if E_1 are E_2 are isogenous, and* $\operatorname{rank}\operatorname{Pic}(\mathcal{X}_s)=2$ *if they are not.*

Proof. This follows by considering Weil numbers. □

The Picard numbers of X can be calculated explicitly in many cases.

Example 6.5. *Let $\mathcal{X}/\mathbb{Z}_p$ be the Jacobian of the smooth projective curve of genus two defined by the equation $y^2 = x^5 - 1$. The 5th roots of unity μ_5 act on $\mathcal{X}$ in the obvious way, so that $\mathbb{Z}[\zeta_5]$ acts on any factor of $X_{\bar{K}}$. By the classification of endomorphism algebras of abelian varieties we see that $X_{\bar{K}}$ is simple and that $\mathrm{End}(X_{\bar{K}})_{\mathbb{Q}} = \mathbb{Q}(\zeta_5)$. The rank of the Néron-Severi group of X is 1, because it is a subgroup of $\mathrm{End}(X)_{\mathbb{Q}}$ which is $\mathbb{Q}$ by [18].*

If $p \equiv -1 \pmod 5$, then $\mathcal{X}_s$ has good reduction at p and $\mathcal{X}_s$ is isogenous to E^2, where E is an elliptic curve over $\mathbb{F}_p$ satisfying $|E(\mathbb{F}_p)| = p+1$. Hence the rank of the Néron-Severi group of the special fiber is 4, $r = 3$, and we obtain

$$\mathrm{Br}(\mathcal{X}) \cong (\mathbb{Q}/\mathbb{Z}') \oplus (\mathbb{Q}/\mathbb{Z})^2 \oplus (finite).$$

7. The inverse system of Brauer groups

We discuss the maps in the diagram

(14)
$$\begin{array}{ccccccccc} & & & & \mathrm{Br}(\mathcal{X}) & \longrightarrow & H^2_{\mathrm{et}}(\mathcal{X}_s, \mathbb{G}_m) & & \\ & & & & b\downarrow & & \uparrow & & \\ 0 & \longrightarrow & \lim^1 \mathrm{Pic}(\mathcal{X}_n) & \longrightarrow & H^2_{\mathrm{cont}}(\mathcal{X}_s, \mathbb{G}_{m,\bullet}) & \longrightarrow & \lim H^2_{\mathrm{et}}(\mathcal{X}_n, \mathbb{G}_m) & \longrightarrow & 0, \end{array}$$

where the lower sequence is (10) for $i = 2$. As a first result we have

Proposition 7.1. *The kernel of $\lim H^2_{\mathrm{et}}(\mathcal{X}_n, \mathbb{G}_m) \to H^2_{\mathrm{et}}(\mathcal{X}_s, \mathbb{G}_m)$ is a finitely generated $\mathbb{Z}_p$-module of rank at most $f \cdot h^{0,2}$.*

Proof. From the exact sequences

(15) $$H^2(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}) \to H^2_{\mathrm{et}}(\mathcal{X}_n, \mathbb{G}_m) \to H^2_{\mathrm{et}}(\mathcal{X}_{n-1}, \mathbb{G}_m) \to H^3(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s})$$

we inductively see that the kernel K_n of $H^2_{\mathrm{et}}(\mathcal{X}_n, \mathbb{G}_m) \to H^2_{\mathrm{et}}(\mathcal{X}_s, \mathbb{G}_m)$ is a finite p-group. Taking the limit we obtain

$$0 \to \lim K_n \to \lim H^2_{\mathrm{et}}(\mathcal{X}_n, \mathbb{G}_m) \to H^2_{\mathrm{et}}(\mathcal{X}_s, \mathbb{G}_m).$$

This shows that the kernel is a pro-p group. But the finitely generated $\mathbb{Z}_p$-module $H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ surjects onto the kernel of the composition

$$H^2_{\mathrm{cont}}(\mathcal{X}_s, \mathbb{G}_{m,\bullet}) \to \lim H^2_{\mathrm{et}}(\mathcal{X}_n, \mathbb{G}_m) \to H^2_{\mathrm{et}}(\mathcal{X}_s, \mathbb{G}_m),$$

hence it surjects onto the kernel of the second map because the first map is surjective. □

Theorem 7.2. *The map $\mathrm{Br}(\mathcal{X}) \xrightarrow{b} H^2_{\mathrm{cont}}(\mathcal{X}_s, \mathbb{G}_{m,\bullet})$ is injective.*

Grothendieck [7, Lemma 3.3] showed that the natural map $\mathrm{Br}(\mathcal{X}) \to \lim \mathrm{Br}(\mathcal{X}_n)$ is injective if the system $(\mathrm{Pic}(\mathcal{X}_n))_n$ is Mittag-Leffler. But if $(\mathrm{Pic}(\mathcal{X}_n))_n$ is Mittag-Leffler, then $H^2_{\mathrm{cont}}(\mathcal{X}_s, \mathbb{G}_{m,\bullet}) \cong \lim H^2_{\mathrm{et}}(\mathcal{X}_n, \mathbb{G}_m)$, hence the Theorem is a generalization of [7, Lemma 3.3]. The theorem follows by diagram (13) from the following proposition.

Proposition 7.3. *The map* $a : H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K}) \to H^2_{\mathrm{cont}}(\mathcal{X}, \mathcal{K}^\wedge)$ *is injective, and its cokernel is the finitely generated free* $\mathbb{Z}_p$*-module* $T_pH^3_{\mathrm{et}}(\mathcal{X}, \mathcal{K})$ *if* $r = 0$*, and the direct sum of* $T_pH^3_{\mathrm{et}}(\mathcal{X}, \mathcal{K})$ *and an uncountable, uniquely divisible group if* $r > 0$*. In particular, the cokernel of* a *is a torsion free* $\mathbb{Z}_{(p)}$*-module.*

Proof. We showed that $C = H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K})$ is a $\mathbb{Z}_{(p)}$-module with finite torsion and $\bar{C} = C/C_{\mathrm{tor}}$ is an extension of $\mathbb{Q}^t$ by $\mathbb{Z}^s_{(p)}$ with $s + t = r$. Since $\mathrm{Br}(\mathcal{X})$ is torsion,

$$\ker a \cong \ker\big(\mathrm{Br}(\mathcal{X}) \to H^2_{\mathrm{cont}}(\mathcal{X}_\bullet, \mathbb{G}_m)\big) \subseteq C_{\mathrm{tor}}$$

is finite. Moreover, a factors as

$$a : C \to C^\wedge \to H^2_{\mathrm{cont}}(\mathcal{X}, \mathcal{K}^\wedge),$$

where the second map is injective with cokernel $T_pH^3_{\mathrm{et}}(\mathcal{X}, \mathcal{K})$. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{\mathrm{tor}} & \longrightarrow & C & \longrightarrow & \bar{C} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^\wedge_{\mathrm{tor}} & \longrightarrow & C^\wedge & \longrightarrow & \bar{C}^\wedge & \longrightarrow & 0. \end{array}$$

The lower row is exact on the left because $\bar{C}$ is torsion free and on the right because $\lim^1 C_{\mathrm{tor}}/p^r = 0$. The injectivity of $C_{\mathrm{tor}} \xrightarrow{\sim} C^\wedge_{\mathrm{tor}} \to C^\wedge$ implies that a is injective. By Corollary 3.1, the cokernel is the direct sum of $T_pH^3_{\mathrm{et}}(\mathcal{X}, \mathcal{K})$ and the uniquely divisible group $\lim^1(H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K}), p)$. By Proposition 5.2, and the fact that $\lim(\mathbb{Z}_{(p)}, p) = \lim^1(\mathbb{Q}^t, p) = 0$, we have a sequence

$$0 \to \lim(H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K}), p) \to \mathbb{Q}^t \to \lim{}^1(\mathbb{Z}^s_{(p)}, p) \to \lim{}^1(H^2_{\mathrm{et}}(\mathcal{X}, \mathcal{K}), p) \to 0.$$

Taking the long exact derived lim-sequence of the sequence of inverse systems

$$0 \to (\mathbb{Z}^s_{(p)}, p) \xrightarrow{(p^n)} (\mathbb{Z}^s_{(p)}, \mathrm{id}) \to (\mathbb{Z}/p^n)^s \to 0$$

we obtain $(\mathbb{Z}_p/\mathbb{Z}_{(p)})^s \cong \lim^1(\mathbb{Z}^s_{(p)}, p)$, hence the result. □

Corollary 7.4. *Assuming finiteness of* $\mathrm{Br}(\mathcal{X}_s)$, $\mathrm{Br}(\mathcal{X})$ *agrees with the torsion subgroup of* $\lim^1 \mathrm{Pic}(\mathcal{X}_n)$ *up to finite groups.*

Proof. Finiteness of $\mathrm{Br}(\mathcal{X}_s)$ implies finiteness of $\mathrm{Br}(\mathcal{X}_n)$ by the sequence (15), and this implies that $\lim H^2_{\mathrm{et}}(\mathcal{X}_n, \mathbb{G}_m)$ is a pro-finite group. It follows that any divisible group maps to zero in $\lim H^2_{\mathrm{et}}(\mathcal{X}_n, \mathbb{G}_m)$, hence the divisible part of $\mathrm{Br}(\mathcal{X})$ injects into $\lim^1 \mathrm{Pic}(\mathcal{X}_n)$. On the other hand, the torsion free group $\mathrm{coker}\, a$ is a subgroup of the cokernel of $\mathrm{Br}(\mathcal{X}) \to H^2_{\mathrm{cont}}(\mathcal{X}_\bullet, \mathbb{G}_m)$ of finite index by (13) and finiteness of $\mathrm{Br}(\mathcal{X}_s)$. □

We are comparing our results to results about $\lim^1 \mathrm{Pic}(\mathcal{X}_n)$. The groups $\mathrm{Pic}(\mathcal{X}_n)$ are finitely generated of constant rank, and as the derived limit of finite groups vanishes, we can consider the torsion free quotients $\overline{\mathrm{Pic}(\mathcal{X}_n)}$ instead. Since $\mathrm{Pic}(\mathcal{X}_n) \to \mathrm{Pic}(\mathcal{X}_{n-1})$ has finite kernel and cokernel, the maps $\overline{\mathrm{Pic}(\mathcal{X}_n)} \to \overline{\mathrm{Pic}(\mathcal{X}_{n-1})}$ are injective, hence the images T of $\mathrm{Pic}(\mathcal{X})$ in each group are isomorphic. We obtain an exact sequence of pro-systems

$$0 \to T \to \overline{\mathrm{Pic}(\mathcal{X}_n)} \to Q_n \to 0,$$

hence $\lim^1 \mathrm{Pic}(\mathcal{X}_n) \cong \lim^1 \overline{\mathrm{Pic}(\mathcal{X}_n)} \cong \lim^1 Q_n$, where each Q_n is finitely generated of rank r.

By Jensen [9, Thms. 2.5, 2.7], if the groups A_i in a countable pro-system are finitely generated, then $\lim^1 A_i \cong \mathrm{Ext}(M, \mathbb{Z})$, where $M = \operatorname{colim} \mathrm{Hom}(A_i, \mathbb{Z})$ is a countable torsion free group. Moreover, if $\lim^1 A_i$ does not vanish, then

$$\lim^1 A_i \cong \mathbb{Q}^{n_0} \oplus \bigoplus_p (\mathbb{Q}_p/\mathbb{Z}_p)^{n_p} \tag{16}$$

where n_0 is the cardinality of the continuum $2^{\aleph_0}$, and n_p is either $2^{\aleph_0}$ or finite (possibly zero). We give a more precise statement in a special situation.

Theorem 7.5. *Let A_i be an inverse system of finitely generated groups of constant rank r and transition maps with finite cokernel. If $\lim^1 A_i$ does not vanish, then $0 \le n_p \le r$ in (16). If the cokernels of the maps in the system are finite p-groups, then $\lim^1 A_i$ vanishes or $n_p < n_l$ for all $l \ne p$, and $n_l = n_{l'}$ for $l, l' \ne p$.*

Proof. We can assume that each group A_i is a free abelian group of rank r and proceed by induction on r. If $r = 1$, we let $M = \operatorname{colim} \mathrm{Hom}(A_i, \mathbb{Z})$. Choosing any non-zero element of M identifies $M \otimes \mathbb{Q}$ with $\mathbb{Q}$, hence the inclusion $M \to M \otimes \mathbb{Q} \cong \mathbb{Q}$ identifies M with a subgroup of $\mathbb{Q}$, which is of the form $\mathbb{Z}[\{p^{-e_p}\}_p]$, where p runs through the primes and e_p is an integer or infinity. A different choice of an element of M changes finitely many e_p by a finite amount.

Lemma 7.6. *Let $M \cong \mathbb{Z}[\{p^{-e_p}\}_p] \subseteq \mathbb{Q}$, where p runs through the primes and e_p is a non-negative integer or infinity. If all e_i are finite and almost all vanish, then $M \cong \mathbb{Z}$ and $\mathrm{Ext}(M, \mathbb{Z}) = 0$. Otherwise*

$$\mathrm{Ext}(M, \mathbb{Z}) \cong \mathbb{Q}^{n_0} \oplus \bigoplus_p (\mathbb{Q}_p/\mathbb{Z}_p)^{n_p},$$

where $n_0 = 2^{\aleph_0}$, $n_p = 0$ if e_p is infinity, and $n_p = 1$ if e_p is finite.

For example,

$$\mathrm{Tor}\,\mathrm{Ext}(M, \mathbb{Z}) = \begin{cases} (\mathbb{Q}/\mathbb{Z})', & M = \mathbb{Z}[p^{-\infty}] \\ \mathbb{Q}_p/\mathbb{Z}_p, & M = \mathbb{Z}_{(p)} \\ \mathbb{Q}/\mathbb{Z}, & M = \mathbb{Z}[p^{-1} | \text{infinitely many } p] \end{cases}$$

Proof. The case $M \cong \mathbb{Z}$ is easy, so that we assume $M \not\cong \mathbb{Z}$. The long exact $\mathrm{Ext}^i(-, \mathbb{Z})$ sequence associated to the short exact sequence $0 \to \mathbb{Z} \to M \to$

$M/\mathbb{Z} \to 0$ together with $\mathrm{Hom}(M, \mathbb{Z}) = 0$ because $M \not\cong \mathbb{Z}$ gives

$$0 \to \mathbb{Z} \to \mathrm{Ext}(M/\mathbb{Z}, \mathbb{Z}) \to \mathrm{Ext}(M, \mathbb{Z}) \to 0.$$

Let us first consider the torsion subgroup $\mathrm{Tor}\,\mathrm{Ext}(M, \mathbb{Z})$. The six term sequence associated to derived tensor product $- \otimes \mathbb{Z}/p^r$ together with the fact that $\mathrm{Ext}(M, \mathbb{Z})$ is divisible gives

$$0 \to {}_{p^r}\mathrm{Ext}(M/\mathbb{Z}, \mathbb{Z}) \to {}_{p^r}\mathrm{Ext}(M, \mathbb{Z}) \to \mathbb{Z}/p^r\mathbb{Z} \to \mathrm{Ext}(M/\mathbb{Z}, \mathbb{Z})/p^r \to 0. \tag{17}$$

Now $M/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}/p^{e_p}$, where we set $\mathbb{Z}/p^{e_p} = \mathbb{Q}_p/\mathbb{Z}_p$ if e_p is infinity, and then

$$\mathrm{Ext}(M/\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Hom}(M/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \prod_p \mathbb{Z}_p/p^{e_p}, \tag{18}$$

where we set $p^{e_p} = 0$ if e_p is infinity. In particular, if e_p is finite, then the left and right groups in (17) have the same cardinality, so that ${}_{p^r}\mathrm{Ext}(M, \mathbb{Z})$ has cardinality p^r for all r, hence the p-primary torsion of $\mathrm{Ext}(M, \mathbb{Z})$ is $\mathbb{Q}_p/\mathbb{Z}_p$. On the other hand, if e_p is infinity, then the left group in (17) vanishes whereas the right group is isomorphic to $\mathbb{Z}_p/p^r\mathbb{Z}_p$, hence we obtain ${}_{p^r}\mathrm{Ext}(M, \mathbb{Z}) = 0$.

Finally, note that $\mathrm{Ext}(M/\mathbb{Z}, \mathbb{Z})$, hence $\mathrm{Ext}(M, \mathbb{Z})$, has the same cardinality as the continuum by our hypothesis on e_p and the description in (18). Since $\mathrm{Ext}(M, \mathbb{Z})$ is divisible and $\mathrm{Tor}\,\mathrm{Ext}(M, \mathbb{Z})$ is countable, we get the statement of the Proposition. □

We continue the proof of Theorem 7.5. The case $r = 1$ follows from the Lemma because $\lim^1 A_i \cong \mathrm{Ext}(M, \mathbb{Z})$ for $M = \mathrm{colim}\,\mathrm{Hom}(A_i, \mathbb{Z})$. If the transition maps have p-groups as the cokernel, then $M \cong \mathbb{Z}$ or $M \cong \mathbb{Z}[p^{-\infty}]$ in which case we get the claimed statement on the n_p.

For general r, we can, by performing elementary column operations (which corresponds to changing the basis of the next group in the inverse system), assume that the transition maps are given by matrices $M_i = \begin{pmatrix} a_i & 0 & 0 \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix}$. Thus there is a subsystem (A_i') consisting of free groups of rank $r-1$ and a quotient system (A_i'') consisting of free groups of rank 1. By hypothesis $a_i \neq 0$. If the a_i are ± 1 for almost all i, then $\lim A_i'' \cong \mathbb{Z}$ and $\lim^1 A_i'' = 0$, and we have a sequence

$$0 \to \lim A_i' \to \lim A_i \to \mathbb{Z} \xrightarrow{\delta} \lim^1 A_i' \to \lim^1 A_i \to 0.$$

If δ has finite image, then the parameters n_p for (A_i) and (A_i') agree. If δ has infinite image, then the parameters n_p of A_i are one larger than the parameters for A_i'. If a_i is different from ± 1 for infinitely many i, then $\lim A_i'' = 0$ and we obtain a sequence

$$0 \to \lim^1 A_i' \to \lim^1 A_i \to \lim^1 A_i'' \to 0.$$

In this case the parameters n_p of A_i are the sum of the parameters n_p of A'_i and of A''_i. □

References

[1] Berthelot, P.; Ogus, A., F-isocrystals and de Rham cohomology. I. Invent. Math. 72 (1983), no. 2, 159–199.

[2] Flach, M., Siebel, D., Special values of the zeta function of an arithmetic scheme, Preprint 2019

[3] Colliot-Thélène, J.L., Saito, S., Zéro-cycles sur les variétés p-adiques et groupe de Brauer. Internat. Math. Res. Notices 1996, no. 4, 151–160.

[4] de Jong, A. J., A result of Gabber.
`https://www.math.columbia.edu/~dejong/papers/2-gabber.pdf`

[5] Deligne, P. Théorème de Lefschetz et critères de dégńérescence de suites spectrales, Publications Math ématiques de l'IHES 35.1 (1968), 107-126.

[6] Grothendieck, A., Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. Inst. Hautes Études Sci. Publ. Math. No. 11 (1961), 167 pp.

[7] Grothendieck, A. Le groupe de Brauer. III. Exemples et compléments. Dix exposés sur la cohomologie des schémas, 88–188, Adv. Stud. Pure Math., 3, North-Holland, Amsterdam, 1968.

[8] Jannsen, U., Continuous étale cohomology. Math. Ann. 280 (1988), no. 2, 207–245.

[9] Jensen, C.U., Les foncteurs dérivés de *lim* et leurs applications en théorie des modules. Lecture Notes in Mathematics, Vol. 254. Springer-Verlag, Berlin-New York, 1972, iv+103 pp.

[10] Kim, W.; Madapusi Pera, K., 2-adic integral canonical models. Forum Math. Sigma 4 (2016), Paper No. e28.

[11] Lichtenbaum, S., Duality theorems for curves over p-adic fields. Invent. Math. 7 (1969), 120–136.

[12] Madapusi Pera, K., The Tate conjecture for K3 surfaces in odd characteristic. Invent. Math. 201 (2015), no. 2, 625–668.

[13] Mattuck, A., Abelian varieties over p-adic ground fields. Ann. of Math. (2) 62 (1955), 92–119.

[14] Roquette, P., Einheiten und Divisorklassen in endlich erzeugbaren Körpern. Jber. Deutsch. Math.-Verein. 60 (1957), no. Abt., Abt. 1, 1–21.

[15] Rotman, J., An introduction to the theory of groups. Graduate Texts in Mathematics, 148. Springer-Verlag, New York, (1995).

[16] Saito, S., Sato, K., Zero-cycles on varieties over p-adic fields and Brauer groups. Ann. Sci. Éc. Norm. Supér. (4) 47 (2014), no. 3, 505–537.

[17] Tate, J., Endomorphisms of abelian varieties over finite fields, Inv. Math., 2 (1966), 134–144.

[18] The L-functions and Modular Forms Database,
https://www.lmfdb.org/Genus2Curve/Q/160000/c/800000/1

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