Compatibility of special value conjectures with the functional equation of Zeta functions

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Abstract. We prove that the special value conjecture for the Zeta function $\zeta(\mathcal{X},s)$ of a proper, regular arithmetic scheme $\mathcal{X}$ that we formulated in a previous article is compatible with the functional equation of $\zeta(\mathcal{X},s)$ provided that the rational factor $C(\mathcal{X},n)$ we were not able to compute previously has the simple explicit form given in the introduction below.

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1 Introduction

This article is a continuation of our previous article [8] in which we formulated a conjecture describing the leading Taylor coefficient of the Zeta function $\zeta(\mathcal{X},s)$ of a proper regular arithmetic scheme $\mathcal{X}$ at integer arguments $n \in \mathbb{Z}$. Our conjecture involved a rather inexplicit correction factor $C(\mathcal{X},n) \in \mathbb{Q}^\times$, defined in terms of p-adic Hodge theory at all primes p, which we could only compute for $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ where F is a number field all of whose completions F_v are absolutely abelian. Based on this example a general formula for $C(\mathcal{X},n)$ in terms of factorials was suggested in [9] and proven for $n = 1$ in [10]. More precisely, for $n \geq 1$ we conjecture

$$C(\mathcal{X},n)^{-1} \stackrel{?}{=} C_\infty(\mathcal{X},n) \tag{1}$$

where

$$C_\infty(\mathcal{X},n) := \prod_{i\leq n-1;\, j} (n-1-i)!^{(-1)^{i+j}\dim_{\mathbb{Q}} H^j(\mathcal{X}_{\mathbb{Q}},\Omega^i)}. \tag{2}$$

For $n \leq 0$ one has $C_\infty(\mathcal{X},n) = C(\mathcal{X},n) = 1$ by definition.

In this article we prove that our special value conjecture is compatible with the conjectured functional equation of the Zeta function if $C(\mathcal{X},n)^{-1}$ is replaced by $C_\infty(\mathcal{X},n)$. We refer to Thm. 1.4 below in this introduction for a precise statement. We regard this result as convincing evidence that $C_\infty(\mathcal{X},n)$ is indeed the right factor, even though we cannot yet prove that identity (1) holds true for $n\geq 2$ and arbitrary $\mathcal{X}$. The definition of $C(\mathcal{X},n)$ was made in such a way that our conjecture is compatible with the Tamagawa number conjecture of Bloch, Kato, Fontaine and Perrin-Riou [6], [11]. By replacing $C(\mathcal{X},n)^{-1}$ with $C_\infty(\mathcal{X},n)$ we are in effect making a special value conjecture which is independent of p-adic Hodge theory and which is compatible with the functional equation of $\zeta(\mathcal{X},s)$. Note that compatibility with the functional equation of motivic L-functions is not in general known for the Tamagawa number conjecture. Even for Tate motives over a number field F it is only known if all F_v are absolutely abelian.

1.1 Statement of the main result

We begin with a brief statement of our special value conjecture, Conjecture 1.1 below, even though it is not needed for the rest of this article. Neither do any of the results in sections 2-4 of this article depend on unproven conjectures. Our main result Thm. 1.4 follows from an unconditional theorem, Thm. 1.2, both of which are stated below in this subsection.
Let $\mathcal{X}$ be a regular scheme of dimension d, proper over $\mathrm{Spec}(\mathbb{Z})$. Associated to $\mathcal{X}$ and $n\in\mathbb{Z}$ is an invertible $\mathbb{Z}$-module ("fundamental line")

$$\Delta(\mathcal{X}/\mathbb{Z},n):=\det{}_{\mathbb{Z}}R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))\otimes_{\mathbb{Z}}\det{}_{\mathbb{Z}}R\Gamma(\mathcal{X}_{Zar},L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$$

where $L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}}$ is the derived de Rham complex [14] modulo the n-th step in the Hodge filtration and $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))$ is a perfect complex of abelian groups whose definition is dependent on assumptions (finite generation of étale motivic cohomology, Artin-Verdier duality for torsion motivic cohomology) denoted by $\mathbf{L}(\overline{\mathcal{X}}_{et},n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et},d-n)$, $\mathbf{AV}(\overline{\mathcal{X}}_{et},n)$ in [8][Conj. 3.2, Conj. 3.1]. Also assuming the Beilinson conjectures in the form of conjecture $\mathbf{B}(\mathcal{X},n)$ of [8][Conj. 2.5] one can construct a natural trivialization

$$\lambda_\infty:\mathbb{R}\xrightarrow{\sim}\Delta(\mathcal{X}/\mathbb{Z},n)\otimes_{\mathbb{Z}}\mathbb{R}. \tag{3}$$

For each prime number p a factor

$$C_p(\mathcal{X},n)\in p^{\mathbb{Z}}$$

was defined in [8][Def. 5.6] under yet another assumption $\mathbf{D}_p(\mathcal{X},n)$ [8][Conj. 5.5] as well as assumption $\mathbf{R}(\mathbb{F}_p,\dim(\mathcal{X}_{\mathbb{F}_p}))$ (resolution of singularities) borrowed from [12]. Conjecture $\mathbf{D}_p(\mathcal{X},n)$ can be regarded as a syntomic description of $R\Gamma_{et}(\mathcal{X}_{\mathbb{Z}_p},\mathbb{Q}_p(n))$ (p-adically completed, rational, étale motivic cohomology of $\mathcal{X}_{\mathbb{Z}_p}$) and is proven in the cases where it is known using techniques from p-adic Hodge theory. We then define

$$C(\mathcal{X},n) := \prod_{p<\infty} C_p(\mathcal{X},n).$$

Let $\zeta(\mathcal{X},s)$ be the Zeta function of $\mathcal{X}$ and $\zeta^*(\mathcal{X},n) \in \mathbb{R}^\times$ its leading Taylor coefficient at $s=n$. Our special value conjecture [8][Conj. 5.12] is the assertion

$$\lambda_\infty(\zeta^*(\mathcal{X},n)^{-1} \cdot C(\mathcal{X},n) \cdot \mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z},n).$$

As was explained earlier in this introduction, for the purposes of this article we replace this conjecture by the following

Conjecture 1.1. *Let $\mathcal{X}$ be a regular scheme, proper and flat over* $\mathrm{Spec}(\mathbb{Z})$, *which satisfies assumptions* $\mathbf{L}(\overline{\mathcal{X}}_{et},n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et},d-n)$, $\mathbf{AV}(\overline{\mathcal{X}}_{et},n)$ *and* $\mathbf{B}(\mathcal{X},n)$ *in* [8]. *Then*

$$\lambda_\infty(\zeta^*(\mathcal{X},n)^{-1} \cdot C_\infty(\mathcal{X},n)^{-1} \cdot \mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z},n)$$

where $C_\infty(\mathcal{X},n)$ is defined in (2).

This conjecture determines the real number $\zeta^*(\mathcal{X},n) \in \mathbb{R}$ up to sign. It is independent of Conjectures $\mathbf{D}_p(\mathcal{X},n)$ and $\mathbf{R}(\mathbb{F}_p,\dim(\mathcal{X}_{\mathbb{F}_p}))$ and does not involve p-adic Hodge theory at any point in its formulation. Our previous conjecture [8][Conj. 5.12] and Conjecture 1.1 are equivalent if and only if identity (1) holds true but at this point we feel unable to judge the difficulty of proving (1). Under assumptions $\mathbf{L}(\overline{\mathcal{X}}_{et},n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et},d-n)$ and $\mathbf{AV}(\overline{\mathcal{X}}_{et},n)$, we defined in [8][Def. 3.26] an exact triangle of perfect complexes of abelian groups

$$R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}(n)) \to R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)). \tag{4}$$

Here $\overline{\mathcal{X}}$ is an Artin-Verdier compactification, $\mathcal{X}_\infty$ is the quotient topological space $\mathcal{X}(\mathbb{C})/G_\mathbb{R}$ and

$$R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)) := R\Gamma(\mathcal{X}_\infty, i_\infty^*\mathbb{Z}(n))$$

where $i_\infty^*\mathbb{Z}(n)$ is a certain complex of sheaves on $\mathcal{X}_\infty$, which is unconditionally defined. In [8][5.7] we defined (unconditionally) the invertible $\mathbb{Z}$-module

$$\begin{aligned}\Xi_\infty(\mathcal{X}/\mathbb{Z},n) \quad := \quad & \det\nolimits_\mathbb{Z} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)) \otimes \det\nolimits_\mathbb{Z}^{-1} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}}) \\ & \otimes \det\nolimits_\mathbb{Z}^{-1} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n)) \otimes \det\nolimits_\mathbb{Z} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<d-n}_{\mathcal{X}/\mathbb{Z}})\end{aligned}$$

and a canonical trivialization

$$\xi_\infty : \mathbb{R} \xrightarrow{\sim} \Xi_\infty(\mathcal{X}/\mathbb{Z},n) \otimes \mathbb{R}$$

which will be recalled in the proof of Theorem 1.2 in section 5 below. We denote by

$$x_\infty(\mathcal{X},n)^2 \in \mathbb{R}_{>0}$$

the strictly positive real number such that

$$\xi_\infty(x_\infty(\mathcal{X},n)^{-2}\cdot\mathbb{Z})=\Xi_\infty(\mathcal{X}/\mathbb{Z},n)$$

and prove the following unconditional

Theorem 1.2. *Let $\mathcal{X}$ be a regular scheme of dimension d, proper and flat over* $\mathrm{Spec}(\mathbb{Z})$. *We have*

$$x_\infty(\mathcal{X},n)^2=\pm A(\mathcal{X})^{n-d/2}\cdot\frac{\zeta^*(\mathcal{X}_\infty,n)}{\zeta^*(\mathcal{X}_\infty,d-n)}\cdot\frac{C_\infty(\mathcal{X},d-n)}{C_\infty(\mathcal{X},n)}$$

where $A(\mathcal{X})$ is the Bloch conductor (see Definition 3.2) and $\zeta(\mathcal{X}_\infty,s)$ is the archimedean Euler factor of $\mathcal{X}$ (see Section 4).

We explain the significance of this result. Let $\zeta(\overline{\mathcal{X}},s):=\zeta(\mathcal{X},s)\cdot\zeta(\mathcal{X}_\infty,s)$ be the completed Zeta-function of $\mathcal{X}$.

Conjecture 1.3. *(Functional Equation) Let $\mathcal{X}$ be a regular scheme of dimension d, proper and flat over* $\mathrm{Spec}(\mathbb{Z})$. *Then $\zeta(\mathcal{X},s)$ has a meromorphic continuation to all $s\in\mathbb{C}$ and*

$$A(\mathcal{X})^{(d-s)/2}\cdot\zeta(\overline{\mathcal{X}},d-s)=\pm A(\mathcal{X})^{s/2}\cdot\zeta(\overline{\mathcal{X}},s).$$

Assume that $\mathbf{L}(\overline{\mathcal{X}}_{et},n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et},d-n)$, $\mathbf{AV}(\overline{\mathcal{X}}_{et},n)$, $\mathbf{B}(\mathcal{X},n)$ and $\mathbf{B}(\mathcal{X},d-n)$ hold, so that Conjecture 1.1 for $(\mathcal{X},n)$ and $(\mathcal{X},d-n)$ makes sense. By [8][Prop. 5.29], the exact triangle (4) and Weil-étale duality [8][Thm. 3.22] induce a canonical isomorphism

$$\Delta(\mathcal{X}/\mathbb{Z},n)\otimes\Xi_\infty(\mathcal{X}/\mathbb{Z},n)\xrightarrow{\sim}\Delta(\mathcal{X}/\mathbb{Z},d-n)$$

compatible with ξ_∞ and the trivializations (3) of $\Delta(\mathcal{X}/\mathbb{Z},n)$ and $\Delta(\mathcal{X}/\mathbb{Z},d-n)$. As was shown in [8][Cor. 5.31] and will be recalled in the proof of Theorem 1.4 in section 5 below, this leads to compatibility of Conjecture 1.1 with the functional equation of $\zeta(\mathcal{X},s)$ in the following sense.

Theorem 1.4. *Assume $\mathcal{X}$ is a regular scheme of dimension d, proper and flat over* $\mathrm{Spec}(\mathbb{Z})$ *which satisfies Conjectures* $\mathbf{L}(\overline{\mathcal{X}}_{et},n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et},d-n)$, $\mathbf{AV}(\overline{\mathcal{X}}_{et},n)$, $\mathbf{B}(\mathcal{X},n)$ *and* $\mathbf{B}(\mathcal{X},d-n)$ *in* [8]. *Assume that $\zeta(\overline{\mathcal{X}},s)$ satisfies Conjecture 1.3. Then Conjecture 1.1 for $(\mathcal{X},n)$ is equivalent to Conjecture 1.1 for $(\mathcal{X},d-n)$.*

1.2 Cyclic homology and $C_\infty(\mathcal{X},n)$

In this section we briefly discuss two suggestions for a more conceptual origin of the numerical factor $C_\infty(\mathcal{X},n)$ both of which were discovered by the second author. First, it was already shown in [9][Remark 5.2] that there is a fairly natural modification $\tilde{L}\Omega^{<n}_{\mathcal{X}/\mathbb{Z}}$ of the derived deRham complex such that

$$\det_{\mathbb{Z}}R\Gamma(\mathcal{X}_{Zar},\tilde{L}\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})=C_\infty(\mathcal{X},n)\cdot\det_{\mathbb{Z}}R\Gamma(\mathcal{X}_{Zar},L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$$

inside $\det_{\mathbb{Q}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})_{\mathbb{Q}} \cong \det_{\mathbb{Q}} R\Gamma(\mathcal{X}_{\mathbb{Q}}, \Omega^{<n}_{\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}})$. This leads to a statement of Conjecture 1.1

$$\lambda_\infty(\zeta^*(\mathcal{X},n)^{-1}\cdot\mathbb{Z}) = \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))\otimes_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, \tilde{L}\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$$

without any correction factor. Another such modification of derived deRham cohomology that is perhaps even more natural than the definition of $\tilde{L}\Omega^{<n}_{\mathcal{X}/\mathbb{Z}}$ was very recently outlined by the second author in [20]. Recall from [1] that there is a motivic filtration on cyclic homology $\mathrm{Fil}^*_{Mot}\, HC(\mathcal{X})$ with graded pieces given by derived deRham cohomology modulo the n-th step in the Hodge filtration

$$gr^n_{Mot} HC(\mathcal{X}) \cong R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})[2n-2].$$

The corresponding spectral sequence already appears in [19]. Cyclic homology arises as S^1-homotopy-coinvariants on Hochschild homology $HC(\mathcal{X}) \cong HH(\mathcal{X})_{S^1}$. One can consider the topological analogue and define

$$TC^+(\mathcal{X}) := THH(\mathcal{X})_{S^1}$$

where THH denotes topological Hochschild homology (see for example [21] for a review). Note that $TC^+(\mathcal{X})$ is *not* what is usually called topological cyclic homology. The main result of [20] is that there exists a motivic filtration $\mathrm{Fil}^*_{Mot}\, TC^+(\mathcal{X})$ that maps to $\mathrm{Fil}^*_{Mot}\, HC(\mathcal{X})$ inducing an isomorphism

$$\mathrm{Fil}^*_{Mot}\, TC^+(\mathcal{X})_{\mathbb{Q}} \cong \mathrm{Fil}^*_{Mot}\, HC(\mathcal{X})_{\mathbb{Q}}$$

and such that

$$\det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}}) = C_\infty(\mathcal{X},n)\cdot \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$$

where

$$R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}}) := gr^n_{Mot} TC^+(\mathcal{X})[-2n+2].$$

We therefore again obtain a version of Conjecture 1.1

$$\lambda_\infty(\zeta^*(\mathcal{X},n)^{-1}\cdot\mathbb{Z}) = \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))\otimes_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})$$

without correction factor. Here the determinant $\det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})$ makes sense since it was also shown in [20][Cor. 1.6] that $R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})$ is a $H\mathbb{Z}$-module spectrum.

For example, if $\mathcal{X}$ is smooth and proper over $\mathbb{F}_p$, the motivic filtration was already defined in [3] and one verifies that both $R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$ and $R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})$ have finite multiplicative Euler characteristic given by Milne's correction factor [8][Def. 5.4], [20][Cor. 1.7] (even though the natural map

$$R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}}) \to R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$$

is not a quasi-isomorphism). And indeed one has $C_\infty(\mathcal{X},n)=1$ by formula (2).
If F is a number field with ring of integers $\mathcal{O}_F$ and $\mathcal{X}=\mathrm{Spec}(\mathcal{O}_F)$ the motivic filtration on $HC(\mathcal{O}_F)$ and $TC^+(\mathcal{O}_F)$ is given respectively by

$$\mathrm{Fil}^n_{Mot}\, HC(\mathcal{O}_F)=\tau_{\geq 2n-3}HC(\mathcal{O}_F)$$

and [20][Cor. 1.4]

$$\mathrm{Fil}^n_{Mot}\, TC^+(\mathcal{O}_F):=\tau_{\geq 2n-3}TC^+(\mathcal{O}_F).$$

Denote by $\mathcal{D}_F$ the different ideal of $\mathcal{O}_F$ and by $|D_F|=N\mathcal{D}_F$ the absolute value of the discriminant. As was shown in [8][1.6] there is an exact sequence

$$0\to HC_{2n-2}(\mathcal{O}_F)\to\mathcal{O}_F\to\Omega_{\mathcal{O}_F/\mathbb{Z}}(n)\to HC_{2n-3}(\mathcal{O}_F)\to 0$$

where $\Omega_{\mathcal{O}_F/\mathbb{Z}}(n)$ is a finite abelian group of cardinality $|D_F|^{n-1}$, i.e. we have

$$|HC_{2n-3}(\mathcal{O}_F)|\cdot[\mathcal{O}_F:HC_{2n-2}(\mathcal{O}_F)]=|D_F|^{n-1}.$$

By a theorem of Lindenstrauss and Madsen [18] one has

$$THH_i(\mathcal{O}_F)=\begin{cases}\mathcal{O}_F & i=0\\ \mathcal{D}_F^{-1}/j\cdot\mathcal{O}_F & i=2j-1\\ 0 & \text{else.}\end{cases}$$

An easy analysis of the spectral sequence

$$H_i(BS^1,THH_j(\mathcal{O}_F))\Rightarrow TC^+_{i+j}(\mathcal{O}_F)$$

then shows that $TC^+_{2n-3}(\mathcal{O}_F)$ is finite and $TC^+_{2n-2}(\mathcal{O}_F)\subseteq\mathcal{O}_F$ is a sublattice so that

$$|TC^+_{2n-3}(\mathcal{O}_F)|\cdot[\mathcal{O}_F:TC^+_{2n-2}(\mathcal{O}_F)]=(n-1)!^{[F:\mathbb{Q}]}\cdot|D_F|^{n-1}.$$

And indeed one has $C_\infty(\mathrm{Spec}(\mathcal{O}_F),n)=(n-1)!^{[F:\mathbb{Q}]}$ by formula (2).

1.3 Outline of this article

In section 2 we study Verdier duality on the locally compact space $\mathcal{X}_\infty:=\mathcal{X}(\mathbb{C})/G_\mathbb{R}$ and how it applies to the complexes of sheaves $i_\infty^*\mathbb{Z}(n)$ introduced in [8][Def. 3.23]. The key result in terms of relevance for the following sections is Prop. 2.23 which provides the correct power of 2 appearing in the functional equation.
In section 3 we review duality results for the exterior powers of the cotangent complex $L_{\mathcal{X}/\mathbb{Z}}$ due to T. Saito [24] and deduce duality for derived de Rham cohomology of $\mathcal{X}$. It turns out that the Bloch conductor $A(\mathcal{X})$ of $\mathcal{X}$ introduced

in [4] measures the failure of a perfect duality for these theories, see Thm. 3.3 and Prop. 3.5. Corollary 3.9 then provides the correct power of $A(\mathcal{X})$ appearing in the functional equation.
In section 4 we recall the archimedean Euler factors for $\zeta(\mathcal{X}, s)$ and make some preliminary computations towards the main result.
Finally, in section 5 we prove Thm. 1.2 and Thm. 1.4, also employing the results already established in [8][Cor. 5.31] towards compatibility with the functional equation.

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We would also like to thank Spencer Bloch for interesting discussions related to $C(\mathcal{X}, n)$.

2 Verdier duality on $\mathcal{X}_\infty = \mathcal{X}(\mathbb{C})/G_\mathbb{R}$

2.1 Statement of the duality theorem

Let $\mathcal{X}$ be a regular, flat and proper scheme over $\mathrm{Spec}(\mathbb{Z})$. Assume that $\mathcal{X}$ is connected of dimension d. We denote by $\mathcal{X}_\infty := \mathcal{X}(\mathbb{C})/G_\mathbb{R}$ the quotient topological space, where $\mathcal{X}(\mathbb{C})$ is endowed with the complex topology. Let

$$p : \mathcal{X}(\mathbb{C}) \to \mathcal{X}_\infty$$

be the quotient map and let

$$\pi : \mathrm{Sh}(G_\mathbb{R}, \mathcal{X}(\mathbb{C})) \to \mathrm{Sh}(\mathcal{X}_\infty)$$

be the canonical morphism of topoi, where $\mathrm{Sh}(G_\mathbb{R}, \mathcal{X}(\mathbb{C}))$ is the category of $G_\mathbb{R}$-equivariant sheaves on $\mathcal{X}(\mathbb{C})$. We have the formula

$$\pi_*(\mathcal{F}) \simeq (p_*\mathcal{F})^{G_\mathbb{R}}.$$

Let $\mathbb{Z}(n) := (2i\pi)^n \cdot \mathbb{Z}$ be the abelian sheaf on $\mathrm{Sh}(G_\mathbb{R}, \mathcal{X}(\mathbb{C}))$ defined by the obvious $G_\mathbb{R}$-action on $(2i\pi)^n \cdot \mathbb{Z}$. In [8][Def. 3.23], we defined the complex of sheaves on $\mathcal{X}_\infty$

$$i_\infty^*\mathbb{Z}(n) := \mathrm{Fib}(R\pi_*\mathbb{Z}(n) \to \tau^{>n} R\widehat{\pi}_*\mathbb{Z}(n))$$

for any $n \in \mathbb{Z}$. We define similarly

$$Ri_\infty^!\mathbb{Z}(n+1)[3] := \mathbb{Z}'(n) := \mathrm{Fib}(R\pi_*\mathbb{Z}(n) \to \tau^{\geq n} R\widehat{\pi}_*\mathbb{Z}(n))$$

and we set

$$e := d-1.$$

If Z is a locally compact topological space, we denote by $\mathcal{D}_Z := Rf^!\mathbb{Z}$ the dualizing complex, where $f : Z \to \{*\}$ is the map from Z to the point.

Theorem 2.1. *There is an equivalence* $\mathbb{Z}'(e) \xrightarrow{\sim} \mathcal{D}_{\mathcal{X}_\infty}[-2e]$ *and a perfect pairing*

$$i_\infty^*\mathbb{Z}(n) \otimes^L \mathbb{Z}'(e-n) \longrightarrow \mathbb{Z}'(e) \xrightarrow{\sim} \mathcal{D}_{\mathcal{X}_\infty}[-2e]$$

in the derived category of abelian sheaves over $\mathcal{X}_\infty$*, for any* $n \in \mathbb{Z}$.

Proof. We set $\mathbb{Z}(n) := i_\infty^*\mathbb{Z}(n)$, we denote by $\iota : \mathcal{X}(\mathbb{R}) \to \mathcal{X}_\infty$ the closed immersion and by j the complementary open immersion. By Proposition 2.5 there is a product map

$$\mathbb{Z}(n) \otimes^L \mathbb{Z}'(e-n) \to \mathcal{D}_{\mathcal{X}_\infty}[-2e]$$

inducing

$$\mathbb{Z}(n) \to R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e]). \tag{5}$$

Then (5) induces an equivalence

$$j^*\mathbb{Z}(n) \xrightarrow{\sim} j^*R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

by Proposition 2.7. Similarly, (5) induces an equivalence

$$R\iota^!\mathbb{Z}(n) \xrightarrow{\sim} R\iota^!R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

by Proposition 2.17. It follows that (5) is an equivalence. Applying $R\underline{\mathrm{Hom}}(-, \mathcal{D}_{\mathcal{X}_\infty}[-2e])$, we get an equivalence

$$\mathbb{Z}'(e-n) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(\mathbb{Z}(n), \mathcal{D}_{\mathcal{X}_\infty}[-2e]).$$

Since $\mathbb{Z}(0)$ is the constant sheaf $\mathbb{Z}$, we have

$$\mathbb{Z}'(e) \xrightarrow{\sim} \mathcal{D}_{\mathcal{X}_\infty}[-2e].$$

□

We immediately obtain

Corollary 2.2. *There is a trace map* $R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(e)) \to \mathbb{Z}[-2e]$ *and a perfect pairing*

$$R\Gamma(\mathcal{X}_\infty, i_\infty^*\mathbb{Z}(n)) \otimes^L R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(e-n)) \to R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(e)) \to \mathbb{Z}[-2e]$$

of perfect complexes of abelian groups, for any $n \in \mathbb{Z}$.

The following corollaries also follow easily from Theorem 2.1. We state them in order to justify the notation $Ri^!_\infty\mathbb{Z}(n)$.

Corollary 2.3. *There is a trace map*

$$R\Gamma(\mathcal{X}_\infty, Ri^!_\infty\mathbb{Z}(d)) \to \mathbb{Z}[-2d-1]$$

and a perfect pairing

$$R\Gamma(\mathcal{X}_\infty, i_\infty^*\mathbb{Z}(n)) \otimes^L R\Gamma(\mathcal{X}_\infty, Ri^!_\infty\mathbb{Z}(d-n)) \to R\Gamma(\mathcal{X}_\infty, Ri^!_\infty\mathbb{Z}(d)) \to \mathbb{Z}[-2d-1]$$

of perfect complexes of abelian groups, for any $n \in \mathbb{Z}$.

Corollary 2.4. *Assume that* $\mathcal{X}$ *satisfies the assumptions* $\mathbf{L}(\overline{\mathcal{X}}_{et}, n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et}, d-n)$ *and* $\mathbf{AV}(\overline{\mathcal{X}}_{et}, n)$ *of* [8]*[3.2]. We define*

$$R\Gamma_W(\mathcal{X}, \mathbb{Z}(n)) := R\mathrm{Hom}(R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(d-n)), \mathbb{Z}[-2d-1]).$$

Then we have an exact triangle

$$R\Gamma(\mathcal{X}_\infty, Ri^!_\infty\mathbb{Z}(n)) \to R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n)) \to R\Gamma_W(\mathcal{X}, \mathbb{Z}(n)).$$

2.2 Proof of the duality theorem

The proof of Theorem 2.1 relies on the results proven below.

2.2.1 Notations

We denote by $\iota : \mathcal{X}(\mathbb{R}) \to \mathcal{X}_\infty$ the closed immersion and by $j : \mathcal{X}^\circ_\infty \to \mathcal{X}_\infty$ the complementary open immersion, where $\mathcal{X}^\circ_\infty := \mathcal{X}_\infty - \mathcal{X}(\mathbb{R})$. We set $\mathcal{X}(\mathbb{C})^\circ := \mathcal{X}(\mathbb{C}) - \mathcal{X}(\mathbb{R})$. We denote by

$$p^\circ : \mathcal{X}(\mathbb{C})^\circ \to \mathcal{X}^\circ_\infty$$

the quotient map, and by

$$\pi^\circ : \mathrm{Sh}(G_\mathbb{R}, \mathcal{X}(\mathbb{C})^\circ) \xrightarrow{\sim} \mathrm{Sh}(\mathcal{X}^\circ_\infty)$$

the morphism of topoi induced by π, which is an equivalence since $G_\mathbb{R}$ has no fixed point on $\mathcal{X}(\mathbb{C})^\circ$. If $x \in \mathcal{X}(\mathbb{R})$ then we denote $\iota_x : x \to \mathcal{X}(\mathbb{R})$ (or $\iota_x : x \to \mathcal{X}_\infty$) the inclusion. The complex of sheaves over $\mathcal{X}_\infty$ denoted by $\mathbb{Z}(n)$ always refers to $i^*_\infty\mathbb{Z}(n)$.

We denote by $C^*(G_{\mathbb{R}},\mathbb{Z}(n)) := R\Gamma(G_{\mathbb{R}},\mathbb{Z}(n))$ the cohomology of $G_{\mathbb{R}}$ with coefficients in $(2i\pi)^n\mathbb{Z}$, by $\widehat{C}^*(G_{\mathbb{R}},\mathbb{Z}(n)) := R\widehat{\Gamma}(G_{\mathbb{R}},\mathbb{Z}(n))$ Tate cohomology, and by $C_*(G_{\mathbb{R}},\mathbb{Z}(n))$ the homology of $G_{\mathbb{R}}$ with coefficients in $(2i\pi)^n\mathbb{Z}$. We have a fiber sequence

$$C_*(G_{\mathbb{R}},\mathbb{Z}(n)) \to C^*(G_{\mathbb{R}},\mathbb{Z}(n)) \to \widehat{C}^*(G_{\mathbb{R}},\mathbb{Z}(n)).$$

Recall that, if Z is a locally compact topological space, we denote by $\mathcal{D}_Z := Rf^!\mathbb{Z}$ the dualizing complex, where $f : Z \to \{*\}$ is the map from Z to the point.

2.2.2 The duality map

PROPOSITION 2.5. *For any $n \in \mathbb{Z}$, there is a canonical map*

$$i_\infty^*\mathbb{Z}(n) \otimes^L \mathbb{Z}'(e-n) \to \mathcal{D}_{\mathcal{X}_\infty}[-2e]$$

in the derived category of abelian sheaves over $\mathcal{X}_\infty$.

Proof. Let f be the map from $\mathcal{X}_\infty$ to the point. We start with the morphism

$$i_\infty^*\mathbb{Z}(n) \otimes^L \mathbb{Z}'(e-n) \to R\pi_*((2i\pi)^n\mathbb{Z}) \otimes^L R\pi_*((2i\pi)^{e-n}\mathbb{Z}) \to R\pi_*((2i\pi)^e\mathbb{Z}).$$

Then the map

$$R\pi_*((2i\pi)^e\mathbb{Z}) \to \mathcal{D}_{\mathcal{X}_\infty}[-2e] := f^!\mathbb{Z}[-2e]$$

is given by

$$Rf_!R\pi_*((2i\pi)^e\mathbb{Z}) \simeq R\Gamma(G_{\mathbb{R}},\mathcal{X}(\mathbb{C}),(2i\pi)^e\mathbb{Z}) \to \mathbb{Z}[-2e]$$

where the last map is

$$\begin{aligned} R\Gamma(G_{\mathbb{R}},\mathcal{X}(\mathbb{C}),(2i\pi)^e\mathbb{Z}) \to & R\Gamma(\mathcal{X}(\mathbb{C}),(2i\pi)^e\mathbb{Z}) \\ \to & \tau^{\geq 2e}R\Gamma(\mathcal{X}(\mathbb{C}),(2i\pi)^e\mathbb{Z}) \to \mathbb{Z}[-2e]. \end{aligned}$$

Note that $Rf_! = Rf_*$ since $\mathcal{X}_\infty$ is compact. $\square$

DEFINITION 2.6. *For any $n \in \mathbb{Z}$, we consider the morphism*

$$\mathbb{Z}(n) \to R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n),\mathcal{D}_{\mathcal{X}_\infty}[-2e]) \tag{6}$$

induced by the product map above.

2.2.3 The map $j^*\mathbb{Z}(n) \to j^*R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n),\mathcal{D}_{\mathcal{X}_\infty}[-2e])$

PROPOSITION 2.7. *The canonical map*

$$j^*\mathbb{Z}(n) \to j^*R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n),\mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

is an equivalence.

Proof. We replace n by $e-n$. We have

$$\begin{array}{rcl} j^*R\underline{\mathrm{Hom}}(\mathbb{Z}'(n),\mathcal{D}_{\mathcal{X}_\infty}[-2e]) & \simeq & R\underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{X}_\infty^\circ)}(j^*\mathbb{Z}'(n),\mathcal{D}_{\mathcal{X}_\infty^\circ}[-2e]) \\ & \simeq & R\underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{X}_\infty^\circ)}(\pi_*^\circ(2i\pi)^n\mathbb{Z},\mathcal{D}_{\mathcal{X}_\infty^\circ})[-2e]. \end{array}$$

Similarly, we have $j^*\mathbb{Z}(e-n)=\pi_*^\circ(2i\pi)^{e-n}\mathbb{Z}$. So we need to check that the map

$$\pi_*^\circ(2i\pi)^{e-n}\mathbb{Z}\to R\underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{X}_\infty^\circ)}(\pi_*^\circ(2i\pi)^n\mathbb{Z},\mathcal{D}_{\mathcal{X}_\infty^\circ})[-2e]$$

is an equivalence. The map $p^\circ:\mathcal{X}(\mathbb{C})^\circ\to\mathcal{X}_\infty^\circ$ is a finite étale Galois cover, hence $p^{\circ,*}$ is conservative. Hence it is enough to check that

$$p^{\circ,*}\pi_*^\circ(2i\pi)^{e-n}\mathbb{Z}\to R\underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{X}(\mathbb{C})^\circ)}(p^{\circ,*}\pi_*^\circ(2i\pi)^n\mathbb{Z},\mathcal{D}_{\mathcal{X}(\mathbb{C})^\circ})[-2e]$$

is an equivalence. But we have

$$p^{\circ,*}\pi_*^\circ(2i\pi)^n\mathbb{Z}\simeq(2i\pi)^n\mathbb{Z},$$

hence one is reduced to observe that

$$(2i\pi)^{e-n}\mathbb{Z}\to R\underline{\mathrm{Hom}}_{\mathrm{Sh}(\mathcal{X}(\mathbb{C})^\circ)}((2i\pi)^n\mathbb{Z},\mathcal{D}_{\mathcal{X}(\mathbb{C})^\circ})[-2e]$$

is an equivalence by Verdier duality on the complex (hence orientable) manifold $\mathcal{X}(\mathbb{C})^\circ$. □

2.2.4 The complex $\iota_x^*R\iota^!\mathbb{Z}(n)$

Lemma 2.8. *For any $n\in\mathbb{Z}$ and any $x\in\mathcal{X}(\mathbb{R})$, we have a fiber sequence*

$$R\Gamma(G_{\mathbb{R}},\mathbb{Z}(n))\to\iota_x^*Rj_*j^*\mathbb{Z}(n)\to R\Gamma(G_{\mathbb{R}},\mathbb{Z}(n-e))[-(e-1)]$$

and $\iota_x^*Rj_*j^*\mathbb{Z}(n)$ is cohomologically concentrated in degrees $\in[0,e-1]$.

Proof. For $e=0$, the map j is both a closed and an open immersion hence $\iota_x^*Rj_*j^*\mathbb{Z}(n)=0$. So the result is obvious in that case, hence we may assume $e\geq 1$.

Note first that $j^*\mathbb{Z}(n)\simeq R\pi_*^\circ((2i\pi)^n\mathbb{Z})$. Let $x\in\mathcal{X}(\mathbb{R})\subset\mathcal{X}(\mathbb{C})$. For a point $z\in\mathcal{X}(\mathbb{C})$ in the neighbourhood of x, we have

$$z=(a_1,b_1,\cdots,a_e,b_e)\in\mathbb{C}^e=(\mathbb{R}\oplus i\cdot\mathbb{R})^e$$

where σ acts as follows

$$(a_1,\cdots,a_e,b_1,\cdots,b_e)\mapsto(a_1,\cdots,a_e,-b_1,\cdots,-b_e)\in\mathbb{R}^e\oplus i\cdot\mathbb{R}^e.$$

So a basic open neighborhood of $x\in\mathcal{X}(\mathbb{R})$ in $\mathcal{X}(\mathbb{C})$ is of the form $B^e\times B^e$ where B^e denotes an open ball in $\mathbb{R}^e$, and σ acts trivially on the first ball and by multiplication by -1 on the second ball. We have

$$\mathcal{X}(\mathbb{R})\cap(B^e\times B^e)=B^e\times 0$$

and a $G_{\mathbb{R}}$-equivariant homotopy equivalence

$$\mathcal{X}(\mathbb{C})^{\circ} \cap (B^e \times B^e) = B^e \times (B^e - 0) \simeq B^e \times \mathbf{S}^{e-1} \simeq \mathbf{S}^{e-1}$$

where $G_{\mathbb{R}}$ acts by its antipodal action on the $(e-1)$-sphere $\mathbf{S}^{e-1}$. We obtain

$$\begin{aligned} \iota_x^* Rj_*j^*\mathbb{Z}(n) &\simeq \mathrm{colim}_{x\in U\subset \mathcal{X}_\infty} R\Gamma(U - \mathcal{X}(\mathbb{R}), \mathbb{Z}(n)) \\ &\simeq \mathrm{colim}_{x\in U\subset \mathcal{X}_\infty} R\Gamma(G_{\mathbb{R}}, p^{-1}(U - \mathcal{X}(\mathbb{R})), \mathbb{Z}(n)) \\ &\simeq R\Gamma(G_{\mathbb{R}}, \mathbf{S}^{e-1}, \mathbb{Z}(n)) \end{aligned}$$

where $G_{\mathbb{R}}$ acts both on $\mathbf{S}^{e-1}$ and $\mathbb{Z}(n) := (2i\pi)^n\mathbb{Z}$. But we have a fiber sequence in the derived category of $\mathbb{Z}[G_{\mathbb{R}}]$-modules

$$\mathbb{Z}(n) \to R\Gamma(\mathbf{S}^{e-1}, \mathbb{Z}(n)) \to \mathbb{Z}(n-e)[-(e-1)]$$

where the boundary map $\mathbb{Z}(n-e))[-(e-1)] \to \mathbb{Z}(n)[1]$ is the non-trivial class in

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}[G_{\mathbb{R}}]}(\mathbb{Z}(n-e))[-(e-1)], \mathbb{Z}(n)[1]) &\simeq \mathrm{Hom}_{\mathbb{Z}[G_{\mathbb{R}}]}(\mathbb{Z}, \mathbb{Z}(e)[e]) \\ &\simeq H^e(G_{\mathbb{R}}, \mathbb{Z}(e)) \\ &\simeq \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Indeed, it must be the non-trivial class because

$$R\Gamma(G_{\mathbb{R}}, \mathbf{S}^{e-1}, \mathbb{Z}(n)) \simeq R\Gamma(\mathbf{S}^{e-1}/\{\pm 1\}, \mathbb{Z}(n))$$

is cohomologically concentrated in degrees $\in [0, e-1]$ since $\mathbf{S}^{e-1}/\{\pm 1\}$ is a $(e-1)$-manifold. $\square$

LEMMA 2.9. *For any $n \in \mathbb{Z}$, we have*

$$\iota_x^* R\iota^! \mathbb{Z}(n) \simeq \mathrm{Fib}\left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-e] \to \tau^{>n} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(n))\right).$$

Proof. First we assume $n \geq 0$, so that $\iota_x^*\mathbb{Z}(n) \simeq \tau^{\leq n} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n))$. Then we have the following diagram with exact rows and columns:

$$\begin{tikzcd} \tau^{>n} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n))[-1] \arrow[r] \arrow[d] & \tau^{\leq n} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n)) \arrow[r] \arrow[d] & R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n)) \arrow[d] \\ \iota_x^* R\iota^! \mathbb{Z}(n) \arrow[r] \arrow[d] & \iota_x^* \mathbb{Z}(n) \arrow[r] \arrow[d] & \iota_x^* Rj_*j^*\mathbb{Z}(n) \arrow[d] \\ R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-e] \arrow[r] & 0 \arrow[r] & R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-(e-1)] \end{tikzcd}$$

Now we assume $n < 0$. By Lemma 2.11, we have an equivalence

$$\iota_x^* \mathbb{Z}(n) \simeq \tau_{\leq -n-2} C_*(G_{\mathbb{R}}, \mathbb{Z}(n))$$

where both sides vanish for $n = -1$. We obtain the following diagram with exact rows and columns:

$$\begin{tikzcd}
(\tau^{>n}\widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(n)))[-1] \arrow[r] \arrow[d] & \tau_{\leq -n-2}C_*(G_{\mathbb{R}}, \mathbb{Z}(n)) \arrow[r] \arrow[d] & C^*(G_{\mathbb{R}}, \mathbb{Z}(n)) \arrow[d] \\
\iota_x^* R\iota^! \mathbb{Z}(n) \arrow[r] \arrow[d] & \iota_x^*\mathbb{Z}(n) \arrow[r] \arrow[d] & \iota_x^* Rj_*j^*\mathbb{Z}(n) \arrow[d] \\
C^*(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-e] \arrow[r] & 0 \arrow[r] & C^*(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-(e-1)]
\end{tikzcd}$$

□

Proposition 2.10. *For $n < e$, we have*

$$\iota_x^* R\iota^! \mathbb{Z}(n) \simeq (\tau_{\leq e-n-2}C_*(G_{\mathbb{R}}, \mathbb{Z}(n-e)))[-e].$$

For $n \geq e$, we have

$$\iota_x^* R\iota^! \mathbb{Z}(n) \simeq (\tau^{\leq n-e} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e)))[-e].$$

Proof. We have

$$\tau^{>n} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(n)) \simeq (\tau^{>n-e} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(n-e)))[-e]$$

and an equivalence

$$\iota_x^* R\iota^! \mathbb{Z}(n) \simeq \mathrm{Fib}\left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e)) \to \tau^{>n-e} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(n-e))\right)[-e].$$

Hence the result follows from Lemma 2.11 below. □

Lemma 2.11. *For any $m \geq 0$, we have an equivalence*

$$\tau^{\leq m} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(m)) \simeq \mathrm{Fib}\left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(m)) \to \tau^{>m} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(m))\right).$$

Similarly, for any $m < 0$, we have

$$\tau_{\leq -m-2} C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) \simeq \mathrm{Fib}\left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(m)) \to \tau^{>m} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(m))\right).$$

Proof. The first assertion is obvious. The second equivalence holds for $m = -1$ since both side vanish. It remains to show that the second equivalence holds for $m \leq -2$. We have the following exact diagram

$$\begin{tikzcd}
(\tau^{\leq m}\widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(m)))[-1] \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \tau^{\leq m}\widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \arrow[d] \\
C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) \arrow[r] \arrow[d] & C^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \arrow[r] \arrow[d] & \widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \arrow[d] \\
F \arrow[r] & C^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \arrow[r] & \tau^{>m}\widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(m))
\end{tikzcd}$$

hence a cofiber sequence

$$(\tau^{\leq m}\widehat{C}(G_{\mathbb{R}},\mathbb{Z}(m)))[-1]\to C_*(G_{\mathbb{R}},\mathbb{Z}(m))\to F.$$

In view of the equivalences

$$(\tau^{\leq m}\widehat{C}(G_{\mathbb{R}},\mathbb{Z}(m)))[-1]\simeq\tau^{\leq m+1}(\widehat{C}(G_{\mathbb{R}},\mathbb{Z}(m))[-1])\simeq\tau^{\leq m+1}C_*(G_{\mathbb{R}},\mathbb{Z}(m))$$

we obtain

$$F\simeq\tau^{>m+1}C_*(G_{\mathbb{R}},\mathbb{Z}(m))=\tau^{\geq m+2}C_*(G_{\mathbb{R}},\mathbb{Z}(m))=\tau_{\leq -m-2}C_*(G_{\mathbb{R}},\mathbb{Z}(m)).$$

□

Lemma 2.12. *For any $m>0$, we have an equivalence*

$$\tau^{<m}R\Gamma(G_{\mathbb{R}},\mathbb{Z}(m))\simeq\mathrm{Fib}\left(R\Gamma(G_{\mathbb{R}},\mathbb{Z}(m))\to\tau^{\geq m}R\widehat{\Gamma}(G_{\mathbb{R}},\mathbb{Z}(m))\right).$$

Similarly, for any $m\leq 0$, we have

$$\tau_{\leq -m}C_*(G_{\mathbb{R}},\mathbb{Z}(m))\simeq\mathrm{Fib}\left(R\Gamma(G_{\mathbb{R}},\mathbb{Z}(m))\to\tau^{\geq m}R\widehat{\Gamma}(G_{\mathbb{R}},\mathbb{Z}(m))\right).$$

Proof. The first assertion is obvious. The second equivalence for $m=0$ follows from the exact sequence

$$0=\widehat{H}^{-1}(G_{\mathbb{R}},\mathbb{Z})\to H_0(G_{\mathbb{R}},\mathbb{Z})\to H^0(G_{\mathbb{R}},\mathbb{Z})\to\widehat{H}^0(G_{\mathbb{R}},\mathbb{Z})\to 0$$

and the isomorphism $H^i(G_{\mathbb{R}},\mathbb{Z})\xrightarrow{\sim}\widehat{H}^i(G_{\mathbb{R}},\mathbb{Z})$ for $i>0$.
It remains to show that the second equivalence holds for $m\leq -1$. We have the following exact diagram

$$\begin{tikzcd}
(\tau^{<m}\widehat{C}(G_{\mathbb{R}},\mathbb{Z}(m)))[-1] \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \tau^{<m}\widehat{C}^*(G_{\mathbb{R}},\mathbb{Z}(m)) \arrow[d] \\
C_*(G_{\mathbb{R}},\mathbb{Z}(m)) \arrow[r] \arrow[d] & C^*(G_{\mathbb{R}},\mathbb{Z}(m)) \arrow[r] \arrow[d] & \widehat{C}^*(G_{\mathbb{R}},\mathbb{Z}(m)) \arrow[d] \\
F \arrow[r] & C^*(G_{\mathbb{R}},\mathbb{Z}(m)) \arrow[r] & \tau^{\geq m}\widehat{C}^*(G_{\mathbb{R}},\mathbb{Z}(m))
\end{tikzcd}$$

hence a cofiber sequence

$$(\tau^{<m}\widehat{C}(G_{\mathbb{R}},\mathbb{Z}(m)))[-1]\to C_*(G_{\mathbb{R}},\mathbb{Z}(m))\to F.$$

In view of the equivalences

$$(\tau^{<m}\widehat{C}(G_{\mathbb{R}},\mathbb{Z}(m)))[-1]\simeq\tau^{<m+1}(\widehat{C}(G_{\mathbb{R}},\mathbb{Z}(m))[-1])\simeq\tau^{<m+1}C_*(G_{\mathbb{R}},\mathbb{Z}(m))$$

we obtain

$$F\simeq\tau^{\geq m+1}C_*(G_{\mathbb{R}},\mathbb{Z}(m))=\tau_{\leq -m-1}C_*(G_{\mathbb{R}},\mathbb{Z}(m))\simeq\tau_{\leq -m}C_*(G_{\mathbb{R}},\mathbb{Z}(m))$$

since

$$H_{-m}(G_{\mathbb{R}},\mathbb{Z}(m))=\widehat{H}^{m-1}(G_{\mathbb{R}},\mathbb{Z}(m))=0.$$

□

2.2.5 The complex $R\iota^! R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$

We denote by $f : \mathcal{X}(\mathbb{R}) \to \{*\}$ the map from $\mathcal{X}(\mathbb{R})$ to the point and we denote by $\omega_{\mathcal{X}(\mathbb{R})}$ the orientation sheaf on the e-manifold $\mathcal{X}(\mathbb{R})$. We have

$$\mathcal{D}_{\mathcal{X}(\mathbb{R})} := f^!\mathbb{Z} \simeq \omega_{\mathcal{X}(\mathbb{R})}[e].$$

Proposition 2.13. *For $e-n>0$ we have*

$$R\iota^! R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e]) \simeq f^*(\tau_{\leq e-n-2} C_*(G_\mathbb{R}, \mathbb{Z}(e-n))) \otimes^L \omega_{\mathcal{X}(\mathbb{R})}[-e].$$

Proof. Using Lemma 2.12 and Lemma 2.15, we obtain

$$\begin{aligned}
R\iota^! R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty})[-2e] &\simeq R\underline{\mathrm{Hom}}(\iota^*\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}(\mathbb{R})})[-2e] \\
&\simeq R\underline{\mathrm{Hom}}(f^*\tau^{<e-n} R\Gamma(G_\mathbb{R}, \mathbb{Z}(e-n)), \mathcal{D}_{\mathcal{X}(\mathbb{R})})[-2e] \\
&\simeq f^! R\mathrm{Hom}(\tau^{<e-n} R\Gamma(G_\mathbb{R}, \mathbb{Z}(e-n)), \mathbb{Z})[-2e] \\
&\simeq f^! R\underline{\mathrm{Hom}}(\tau^{\leq e-n-2} R\Gamma(G_\mathbb{R}, \mathbb{Z}(e-n)), \mathbb{Z})[-2e] \\
&\simeq f^!(\tau_{\leq e-n-2} C_*(G_\mathbb{R}, \mathbb{Z}(e-n)))[-2e] \\
&\simeq f^*(\tau_{\leq e-n-2} C_*(G_\mathbb{R}, \mathbb{Z}(e-n))) \otimes^L \omega_{\mathcal{X}(\mathbb{R})}[-e].
\end{aligned}$$

□

Proposition 2.14. *For $e-n\leq 0$ we have*

$$R\iota^! R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e]) \simeq f^*(\tau^{\leq n-e} R\Gamma(G_\mathbb{R}, \mathbb{Z}(n-e))) \otimes^L \omega_{\mathcal{X}(\mathbb{R})}[-e].$$

Proof. Using Lemma 2.12 and Lemma 2.15, we obtain

$$\begin{aligned}
R\iota^! R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty})[-2e] &\simeq R\underline{\mathrm{Hom}}(\iota^*\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}(\mathbb{R})})[-2e] \\
&\simeq R\underline{\mathrm{Hom}}(f^*\tau_{\leq n-e} C_*(G_\mathbb{R}, \mathbb{Z}(e-n)), \mathcal{D}_{\mathcal{X}(\mathbb{R})})[-2e] \\
&\simeq f^! R\mathrm{Hom}(\tau_{\leq n-e} C_*(G_\mathbb{R}, \mathbb{Z}(e-n)), \mathbb{Z})[-2e] \\
&\simeq f^!(\tau^{\leq n-e} R\Gamma(G_\mathbb{R}, \mathbb{Z}(n-e)))[-2e] \\
&\simeq f^*(\tau^{\leq n-e} R\Gamma(G_\mathbb{R}, \mathbb{Z}(n-e))) \otimes^L \omega_{\mathcal{X}(\mathbb{R})}[-e].
\end{aligned}$$

□

Lemma 2.15. *For any $n \in \mathbb{Z}$, the pairing*

$$C_*(G_\mathbb{R}, \mathbb{Z}(-n)) \otimes^L_\mathbb{Z} C^*(G_\mathbb{R}, \mathbb{Z}(n)) \to C_*(G_\mathbb{R}, \mathbb{Z}(0)) \to \mathbb{Z}[0]$$

induces a perfect pairing

$$\tau_{\leq n} C_*(G_\mathbb{R}, \mathbb{Z}(-n)) \otimes^L_\mathbb{Z} \tau^{\leq n} C^*(G_\mathbb{R}, \mathbb{Z}(n)) \to \mathbb{Z}[0]$$

of perfect complexes of abelian groups.

Proof. The result is trivial for $n<0$ and clear for $n=0$. So we assume $n>0$. The pairing induces an equivalence

$$C^*(G_{\mathbb{R}},\mathbb{Z}(n))\to R\mathrm{Hom}(C_*(G_{\mathbb{R}},\mathbb{Z}(-n)),\mathbb{Z})$$

hence it is enough to observe that

$$\tau^{\leq n}R\mathrm{Hom}(C_*(G_{\mathbb{R}},\mathbb{Z}(-n)),\mathbb{Z})\simeq R\mathrm{Hom}(\tau_{\leq n}C_*(G_{\mathbb{R}},\mathbb{Z}(-n)),\mathbb{Z}).$$

For any cohomological complex A^*, we have a short exact sequence

$$0\to \mathrm{Ext}(H^{-i+1}(A^*),\mathbb{Z})\to H^i(R\mathrm{Hom}(A^*,\mathbb{Z}))\to \mathrm{Hom}(H^{-i}(A^*),\mathbb{Z})\to 0.$$

We obtain

$$H^i(R\mathrm{Hom}(\tau_{\leq n}C_*(G_{\mathbb{R}},\mathbb{Z}(-n)),\mathbb{Z}))=H^i(R\mathrm{Hom}(C_*(G_{\mathbb{R}},\mathbb{Z}(-n)),\mathbb{Z}))$$

for $i\leq n$ and $i>n+1$. Since we have $H_n(G_{\mathbb{R}},\mathbb{Z}(-n))=0$ for any $n>0$, we get

$$H^{n+1}(R\mathrm{Hom}(\tau_{\leq n}C_*(G_{\mathbb{R}},\mathbb{Z}(-n)),\mathbb{Z}))=0.$$

□

Remark 2.16. *For $n>0$, we have $H_n(G_{\mathbb{R}},\mathbb{Z}(-n))=0$ hence*

$$\tau_{\leq n}C_*(G_{\mathbb{R}},\mathbb{Z}(-n))\simeq \tau_{<n}C_*(G_{\mathbb{R}},\mathbb{Z}(-n)).$$

2.2.6 The map $R\iota^!\mathbb{Z}(n)\to R\iota^!R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n),\mathcal{D}_{\mathcal{X}_\infty}[-2e])$

Proposition 2.17. *The map*

$$R\iota^!\mathbb{Z}(n)\to R\iota^!R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n),\mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

is an equivalence.

Proof. For $e-n>0$ and any $x\in\mathcal{X}(\mathbb{R})$, the map

$$\iota_x^*R\iota^!\mathbb{Z}(n)\to \iota_x^*R\iota^!R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n),\mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

can be identified with the identity

$$\tau_{\leq e-n-2}C_*(G_{\mathbb{R}},\mathbb{Z}(e-n))[-e]=\tau_{\leq e-n-2}C_*(G_{\mathbb{R}},\mathbb{Z}(e-n))[-e]$$

by Prop. 2.10 and Prop. 2.13.
For $e-n\leq 0$ and any $x\in\mathcal{X}(\mathbb{R})$, the map

$$\iota_x^*R\iota^!\mathbb{Z}(n)\to \iota_x^*R\iota^!R\underline{\mathrm{Hom}}(\mathbb{Z}'(e-n),\mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

can be identified with the identity

$$\tau^{\leq n-e}R\Gamma(G_{\mathbb{R}},\mathbb{Z}(n-e))[-e]=\tau^{\leq n-e}R\Gamma(G_{\mathbb{R}},\mathbb{Z}(n-e))[-e]$$

by Prop. 2.10 and Prop. 2.14.
The result follows since the family of functors $\{\iota_x^*,x\in\mathcal{X}(\mathbb{R})\}$ is conservative.

□

2.3 Comparison with $R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n))$

Recall that we define $G_{\mathbb{R}}$-equivariant sheaves

$$\mathbb{Z}(n) := (2i\pi)^n\mathbb{Z} \subset \mathbb{Q}(n) := (2i\pi)^n\mathbb{Q} \subset \mathbb{R}(n) := (2i\pi)^n\mathbb{R} \subset \mathbb{C}$$

on $\mathcal{X}(\mathbb{C})$. We abbreviate $C^* := R\operatorname{Hom}(C,\mathbb{Q})$ for a complex of $\mathbb{Q}$-vector spaces C and let $C^{\pm}$ be the image of the idempotent $(\sigma\pm 1)/2$ if C carries a $G_{\mathbb{R}} = \{1,\sigma\}$-action. Recall that

$$R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)) := R\Gamma(\mathcal{X}_\infty, i_\infty^*\mathbb{Z}(n))$$

and that $i_\infty^*\mathbb{Z}(n)\otimes\mathbb{Q} \cong \pi_*\mathbb{Q}(n) \cong R\pi_*\mathbb{Q}(n)$ in $\operatorname{Sh}(\mathcal{X}_\infty)$. We therefore have isomorphisms

$$\begin{aligned} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))_{\mathbb{Q}} \simeq & R\Gamma(\mathcal{X}_\infty, R\pi_*\mathbb{Q}(n)) \\ \simeq & R\Gamma(G_{\mathbb{R}};\mathcal{X}(\mathbb{C}),\mathbb{Q}(n)) \simeq R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(n))^+ \end{aligned}$$

and combining this with Poincaré duality

$$R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(r))\otimes R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(e-r)) \xrightarrow{\cup} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(e)) \xrightarrow{\mathrm{Tr}} \mathbb{Q}[-2e] \tag{7}$$

on the $2e$-manifold $\mathcal{X}(\mathbb{C})$ we obtain an isomorphism

$$R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n))_{\mathbb{Q}}^* \simeq R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(d-n))^{*,+} \simeq R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(n-1))^+[-2e]$$

using $e=d-1$. There is also a tautological isomorphism τ induced by multiplication by $2\pi i$ in the sense that the diagram

$$\begin{tikzcd} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(n-1))^+ \arrow[r] \arrow[d, "\tau", "\sim"'] & R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C}) \arrow[d, "\cdot 2\pi i", "\sim"'] \\ R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(n))^- \arrow[r] & R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C}) \end{tikzcd} \tag{8}$$

commutes. Combining the previous isomorphisms we obtain an isomorphism

$$\begin{aligned} &\left(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\otimes\det_{\mathbb{Z}}^{-1}R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n))\right)_{\mathbb{Q}} \\ \simeq\ & \det_{\mathbb{Q}}\left(R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(n))^+\oplus R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(n-1))^+\right) \\ \simeq\ & \det_{\mathbb{Q}}\left(R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(n))^+\oplus R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(n))^-\right) \\ \simeq\ & \det_{\mathbb{Q}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Q}(n)) \\ \simeq\ & (\det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n)))_{\mathbb{Q}} \end{aligned} \tag{9}$$

which we denote by λ_B.

Corollary 2.18. *We have*

$$\begin{aligned} \lambda_B &\left(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\otimes\det_{\mathbb{Z}}^{-1}R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n))\right) \\ &= \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n))\otimes\det_{\mathbb{Z}}^{(-1)^n}R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2\mathbb{Z}) \end{aligned}$$

Proof. We write $G_{\mathbb{R}} = \{1, \sigma\}$. We have an exact sequence of $\mathbb{Z}[G_{\mathbb{R}}]$-modules

$$0 \to \mathbb{Z} \cdot (\sigma - 1) \to \mathbb{Z}[G_{\mathbb{R}}] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where ϵ is the augmentation map. We have an isomorphism of $\mathbb{Z}[G_{\mathbb{R}}]$-modules $(\sigma - 1) \cdot \mathbb{Z} \simeq (2i\pi)\mathbb{Z}$ which maps $(\sigma - 1)$ to $(2i\pi)$. We write $\mathbb{Z}(n) := (2i\pi)^n\mathbb{Z}$, so that we have an exact sequence of $\mathbb{Z}[G_{\mathbb{R}}]$-modules

$$0 \to \mathbb{Z}(1) \to \mathbb{Z}[G_{\mathbb{R}}] \xrightarrow{\epsilon} \mathbb{Z} \to 0. \tag{10}$$

We denote by

$$p : \mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C})) \to \mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_\infty)$$

the morphism of topoi induced by the equivariant continuous map $p : \mathcal{X}(\mathbb{C}) \to \mathcal{X}_\infty$, where $G_{\mathbb{R}}$ acts trivially on $\mathcal{X}_\infty$. The category of abelian sheaves on $\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_\infty)$ is equivalent to the category of sheaves of $\mathbb{Z}[G_{\mathbb{R}}]$-modules over $\mathcal{X}_\infty$. For any sheaf $\mathcal{F}$ of $\mathbb{Z}[G_{\mathbb{R}}]$-modules over $\mathcal{X}_\infty$, and any $\mathbb{Z}[G_{\mathbb{R}}]$-module M, we define

$$R\underline{\mathrm{Hom}}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_\infty)}(M, \mathcal{F})$$

where M is seen as a constant sheaf of $\mathbb{Z}[G_{\mathbb{R}}]$-modules over $\mathcal{X}_\infty$. We have

$$R\pi_*\mathbb{Z}(n) \simeq R\underline{\mathrm{Hom}}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_\infty)}(\mathbb{Z}, Rp_*\mathbb{Z}(n)).$$

Moreover the functor

$$\begin{array}{ccc} \mathrm{Ab}(G_{\mathbb{R}}, \mathcal{X}_\infty) & \longrightarrow & \mathrm{Ab}(G_{\mathbb{R}}, \mathcal{X}_\infty) \\ \mathcal{F} & \longmapsto & \mathcal{F}(1) := \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Z}(1) \end{array}$$

is an equivalence of abelian categories with quasi-inverse $(-) \otimes_{\mathbb{Z}} \mathbb{Z}(-1)$. In particular we have

$$\begin{array}{rcl} R\pi_*\mathbb{Z}(n-1) & \simeq & R\underline{\mathrm{Hom}}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_\infty)}(\mathbb{Z}, Rp_*\mathbb{Z}(n-1)) \\ & \simeq & R\underline{\mathrm{Hom}}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_\infty)}(\mathbb{Z}(1), (Rp_*\mathbb{Z}(n-1))(1)) \\ & \simeq & R\underline{\mathrm{Hom}}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_\infty)}(\mathbb{Z}(1), Rp_*\mathbb{Z}(n)). \end{array}$$

Finally, we have

$$p_*\mathbb{Z}(n) \simeq Rp_*\mathbb{Z}(n) \simeq R\underline{\mathrm{Hom}}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_\infty)}(\mathbb{Z}[G_{\mathbb{R}}], Rp_*\mathbb{Z}(n)).$$

Therefore, (10) induces an exact triangle

$$R\pi_*\mathbb{Z}(n) \to Rp_*\mathbb{Z}(n) \to R\pi_*\mathbb{Z}(n-1)$$

and an exact diagram:

$$\begin{tikzcd} Rp_*\mathbb{Z}(n) \arrow[r] \arrow[d] & i_\infty^*\mathbb{Z}(n-1) \arrow[r] \arrow[d] & i_\infty^*\mathbb{Z}(n)[1] \arrow[d] \\ Rp_*\mathbb{Z}(n) \arrow[r] \arrow[d] & R\pi_*\mathbb{Z}(n-1) \arrow[r] \arrow[d] & R\pi_*\mathbb{Z}(n)[1] \arrow[d] \\ 0 \arrow[r] & \tau^{>n-1}R\widehat{\pi}_*\mathbb{Z}(n-1) \arrow[r] & (\tau^{>n}R\widehat{\pi}_*\mathbb{Z}(n))[1] \end{tikzcd}$$

In particular, there is an exact triangle

$$i_\infty^*\mathbb{Z}(n) \to Rp_*\mathbb{Z}(n) \to i_\infty^*\mathbb{Z}(n-1)$$

hence

$$R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n)) \to R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \to R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n-1)). \tag{11}$$

Moreover, we have the duality equivalence

$$R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(d-n)) \xrightarrow{\sim} R\mathrm{Hom}(R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(n-1)), \mathbb{Z}[-2e]). \tag{12}$$

Finally, we have the following exact diagram

$$\begin{tikzcd}
\iota_*\mathbb{Z}/2\mathbb{Z}_{\mathcal{X}(\mathbb{R})}[-n] \arrow[r] \arrow[d] & \mathbb{Z}'(n-1) \arrow[r] \arrow[d] & i_\infty^*\mathbb{Z}(n-1) \arrow[d] \\
0 \arrow[r] \arrow[d] & R\pi_*\mathbb{Z}(n-1) \arrow[r] \arrow[d] & R\pi_*\mathbb{Z}(n-1) \arrow[d] \\
\iota_*\mathbb{Z}/2\mathbb{Z}_{\mathcal{X}(\mathbb{R})}[-n+1] \arrow[r] & \tau^{\geq n-1}R\widehat{\pi}_*\mathbb{Z}(n-1) \arrow[r] & \tau^{>n-1}R\widehat{\pi}_*\mathbb{Z}(n-1)
\end{tikzcd}$$

where $\mathbb{Z}/2\mathbb{Z}_{\mathcal{X}(\mathbb{R})}$ is the constant sheaf $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{X}(\mathbb{R})$, hence an exact triangle

$$R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})[-n] \to R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(n-1)) \to R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n-1)). \tag{13}$$

Then (11), (12) and (13) induce the following canonical isomorphisms:

$$\begin{aligned}
& \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(d-n)) \\
\simeq\ & \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(n-1)) \\
\simeq\ & \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n-1)) \otimes \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})[-n] \\
\simeq\ & \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})[-n] \\
\simeq\ & \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{(-1)^n} R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}).
\end{aligned}$$

□

We now introduce some notation: we set

$$d_+(\mathcal{X}, n) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n))^+$$

and

$$d_-(\mathcal{X}, n) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n))^-.$$

If Z is a manifold and F a field, we set

$$\chi(Z, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_F H^i(Z, F).$$

Definition 2.19. *For a perfect complex of abelian groups C with finite cohomology groups we denote by*

$$\chi^{\times}(C) = \prod_{i\in\mathbb{Z}} |H^i(C)|^{(-1)^i}$$

its multiplicative Euler characteristic.

Proposition 2.20. *We have*

$$\chi^{\times}(R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2\mathbb{Z})[-n]) = 2^{d_+(\mathcal{X},n)-d_-(\mathcal{X},n)}.$$

Proof. We have

$$d_+(\mathcal{X},n) = d_-(\mathcal{X},n-1) = d_+(\mathcal{X},n-2)$$

hence

$$d_{\pm}(\mathcal{X},n) = (-1)^n \cdot d_{\pm}(\mathcal{X},0).$$

We obtain

$$2^{d_+(\mathcal{X},n)-d_-(\mathcal{X},n)} = \left(2^{d_+(\mathcal{X},0)-d_-(\mathcal{X},0)}\right)^{(-1)^n}.$$

Similarly, we have

$$\chi^{\times}(R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2\mathbb{Z})[-n]) := \chi^{\times}(R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2\mathbb{Z}))^{(-1)^n},$$

hence it is enough to show the result for $n = 0$. In view of Lemma 2.21 and Lemma 2.22, we have

$$\begin{aligned} d_+(\mathcal{X},0) - d_-(\mathcal{X},0) &= \sum_{i\in\mathbb{Z}} (-1)^i \cdot \left(\dim_{\mathbb{Q}} H^i(\mathcal{X}(\mathbb{C}),\mathbb{Q})^+ - \dim_{\mathbb{Q}} H^i(\mathcal{X}(\mathbb{C}),\mathbb{Q})^-\right) \\ &= \sum_{i\in\mathbb{Z}} (-1)^i \cdot \mathrm{Tr}\left(\sigma \mid H^i(\mathcal{X}(\mathbb{C}),\mathbb{Q})\right) \\ &= \chi(\mathcal{X}(\mathbb{R}),\mathbb{Q}) \\ &= \chi(\mathcal{X}(\mathbb{R}),\mathbb{F}_2). \end{aligned}$$

Hence the result follows from

$$\chi^{\times}(R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2\mathbb{Z})) = 2^{\chi(\mathcal{X}(\mathbb{R}),\mathbb{F}_2)}.$$

□

Lemma 2.21. *Let Y be a compact orientable manifold with an involution σ whose fixed points form a closed submanifold Z. Then we have*

$$\sum_{i\in\mathbb{Z}} (-1)^i \cdot \mathrm{Tr}\left(\sigma \mid H^i(Y,\mathbb{Q})\right) = \chi(Z,\mathbb{Q}).$$

Proof. Let $G_{\mathbb{R}} := \{1, \sigma\}$. If C is a perfect complex of $\mathbb{Q}$-vector spaces with $G_{\mathbb{R}}$-action, we set

$$\mathrm{Tr}\,(\sigma \mid C) := \sum_{i\in\mathbb{Z}} (-1)^i \cdot \mathrm{Tr}\left(\sigma \mid H^i(C)\right).$$

Let $Y^\circ := Y - Z$. The exact triangle

$$R\Gamma_c(Y^\circ, \mathbb{Q}) \to R\Gamma(Y, \mathbb{Q}) \to R\Gamma(Z, \mathbb{Q})$$

gives

$$\begin{aligned} \mathrm{Tr}\,(\sigma \mid R\Gamma(Y, \mathbb{Q})) &= \mathrm{Tr}\,(\sigma \mid R\Gamma_c(Y^\circ, \mathbb{Q})) + \mathrm{Tr}\,(\sigma \mid R\Gamma(Z, \mathbb{Q})) \\ &= \mathrm{Tr}\,(\sigma \mid R\Gamma_c(Y^\circ, \mathbb{Q})) + \chi(Z, \mathbb{Q}) \end{aligned}$$

since σ acts trivially on Z hence on $R\Gamma(Z, \mathbb{Q})$. Therefore the result follows from

$$\begin{aligned} \mathrm{Tr}\,(\sigma \mid R\Gamma_c(Y^\circ, \mathbb{Q})) &:= \sum_{i\in\mathbb{Z}} (-1)^i \cdot \mathrm{Tr}\left(\sigma \mid H^i_c(Y^\circ, \mathbb{Q})\right) \\ &= \sum_{i\in\mathbb{Z}} (-1)^i \cdot \mathrm{Tr}\left(\sigma \mid H^i_c(Y^\circ, \mathbb{Q})^*\right) \\ &= \sum_{i\in\mathbb{Z}} (-1)^i \cdot \mathrm{Tr}\left(\sigma \mid H^{d-i}(Y^\circ, \mathbb{Q})\right) \\ &= (-1)^d \sum_{i\in\mathbb{Z}} (-1)^i \cdot \mathrm{Tr}\left(\sigma \mid H^i(Y^\circ, \mathbb{Q})\right) \\ &= 0. \end{aligned}$$

where we use Poincaré duality and the Lefschetz fixed point theorem. Here $d = \dim(Y)$. □

Lemma 2.22. *Let Z be a topological space which is homotopy equivalent to a finite CW-complex. Then we have*

$$\chi(Z, F) = \chi(Z, F')$$

for any pair of fields F, F'.

Proof. The complex $R\Gamma(Z, \mathbb{Z})$ is quasi-isomorphic to a strictly perfect complex of abelian groups C^* and we have

$$\begin{aligned} \sum_{i\in\mathbb{Z}} (-1)^i \mathrm{rank}_{\mathbb{Z}} C^i &= \sum_{i\in\mathbb{Z}} (-1)^i \dim_F (C^i \otimes_{\mathbb{Z}} F) \\ &= \sum_{i\in\mathbb{Z}} (-1)^i \dim_F H^i(C^* \otimes_{\mathbb{Z}} F) = \chi(Z, F) \end{aligned}$$

for any field F. The result follows. □

Combining Corollary 2.18 with Prop. 2.20 we obtain.

Proposition 2.23. *We have*

$$\lambda_B\left(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n))\right) \\ = \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n))\cdot 2^{d_-(\mathcal{X},n)-d_+(\mathcal{X},n)}$$

Proof. Note that if C is as in Definition 2.19 then

$$\det_{\mathbb{Z}} C = \mathbb{Z}\cdot \chi^\times(C)^{-1}$$

under the canonical isomorphism

$$\det_{\mathbb{Q}} C_{\mathbb{Q}} \cong \mathbb{Q}$$

arising from the acyclicity of $C_{\mathbb{Q}}$. □

3 Duality for derived de Rham cohomology and the Bloch conductor

In this section $\mathcal{X}$ is a regular scheme of dimension d, proper and flat over Spec($\mathbb{Z}$). We denote by

$$L_{\mathcal{X}/\mathbb{Z}} \cong \Omega_{\mathcal{X}/\mathbb{Z}}[0] \tag{14}$$

the cotangent complex of $\mathcal{X}$ over $\mathbb{Z}$, a perfect complex of $\mathcal{O}_{\mathcal{X}}$-modules cohomologically concentrated in degree 0. For any $r\in\mathbb{Z}$ we let

$$L\wedge^r L_{\mathcal{X}/\mathbb{Z}} \cong L\wedge^r \Omega_{\mathcal{X}/\mathbb{Z}}[0]$$

be the r-th derived exterior power of $L_{\mathcal{X}/\mathbb{Z}}$ [13][4.2.2.6] which is again a perfect complex of $\mathcal{O}_{\mathcal{X}}$-modules. By definition $L\wedge^r L_{\mathcal{X}/\mathbb{Z}}=0$ for $r<0$ but $L\wedge^r L_{\mathcal{X}/\mathbb{Z}}$ is in general nonzero for $r>d-1=\mathrm{rank}_{\mathcal{O}_{\mathcal{X}}}\Omega_{\mathcal{X}/\mathbb{Z}}$.

3.1 Coherent duality for $L\wedge^r L_{\mathcal{X}/\mathbb{Z}}$

This subsection is a review of material from [23], [15] and [24] in the context of our global arithmetic scheme $\mathcal{X}$. The key result is Thm. 3.3 which is an immediate translation of [24][Cor. 4.9] to our context.

Lemma 3.1. *There is a canonical map*

$$L\wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}} \to \det_{\mathcal{O}_{\mathcal{X}}} L_{\mathcal{X}/\mathbb{Z}} \cong \omega_{\mathcal{X}/\mathbb{Z}} \tag{15}$$

where $\omega_{\mathcal{X}/\mathbb{Z}}$ *is the relative dualizing sheaf. Hence we get induced maps*

$$L\wedge^r L_{\mathcal{X}/\mathbb{Z}}\otimes^L L\wedge^{d-1-r} L_{\mathcal{X}/\mathbb{Z}} \to L\wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}} \to \omega_{\mathcal{X}/\mathbb{Z}}$$

and

$$L\wedge^r L_{\mathcal{X}/\mathbb{Z}} \to \underline{R\,\mathrm{Hom}}(L\wedge^{d-1-r} L_{\mathcal{X}/\mathbb{Z}},\omega_{\mathcal{X}/\mathbb{Z}}) \tag{16}$$

in the derived category of coherent sheaves on $\mathcal{X}$.

Proof. The multiplicative structure on derived exterior powers will be briefly recalled in the proof of Prop. 3.5 below, so it remains to show the existence of (15). Assume first there is a closed embedding $i : \mathcal{X} \to P$ of $\mathcal{X}$ into a smooth $\mathbb{Z}$-scheme P with ideal sheaf $\mathcal{I}$. The exact sequence of coherent sheaves on $\mathcal{X}$

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega_{P/\mathbb{Z}} \to \Omega_{\mathcal{X}/\mathbb{Z}} \to 0$$

can be viewed as a realization of (14) as a strictly perfect complex since $\mathcal{I}/\mathcal{I}^2$ and $i^*\Omega_{P/\mathbb{Z}}$ are locally free of ranks $n-d+1$ and n, respectively, where n is the relative dimension of P over $\mathbb{Z}$. The natural map

$$\wedge^{d-1}\Omega_{\mathcal{X}/\mathbb{Z}} \otimes \wedge^{n-d+1}(\mathcal{I}/\mathcal{I}^2) \to \wedge^n i^*\Omega_{P/\mathbb{Z}}$$

has adjoint

$$\wedge^{d-1}\Omega_{\mathcal{X}/\mathbb{Z}} \to \underline{\mathrm{Hom}}(\wedge^{n-d+1}(\mathcal{I}/\mathcal{I}^2), \wedge^n i^*\Omega_{P/\mathbb{Z}}) =: \omega^P_{\mathcal{X}/\mathbb{Z}}$$

and combined with the natural map

$$L\wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}} \to \mathcal{H}^0(L\wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}) \cong \wedge^{d-1}\Omega_{\mathcal{X}/\mathbb{Z}}$$

we obtain a morphism $(15)^P$ depending on $i : \mathcal{X} \to P$. If $i' : \mathcal{X} \to P'$ is another embedding into a smooth $\mathbb{Z}$-scheme P' an isomorphism

$$\epsilon^{P',P} : \omega^P_{\mathcal{X}/\mathbb{Z}} \xrightarrow{\sim} \omega^{P'}_{\mathcal{X}/\mathbb{Z}}$$

was constructed in [2][A.2] which satisfies the usual cocycle condition in the presence of a third embedding i''. Since embeddings into smooth schemes always exist Zariski locally on $\mathcal{X}$ the cocycle condition implies that one can define $\omega_{\mathcal{X}/\mathbb{Z}}$ by glueing the locally defined $\omega^P_{\mathcal{X}/\mathbb{Z}}$. It remains to show that likewise the locally obtained maps $(15)^P$ glue to a global map (15). By considering the fibre product $P'' := P \times_{\mathrm{Spec}(\mathbb{Z})} P'$ the construction of $\epsilon^{P',P}$ can be reduced to the case where there exists a smooth morphism $u : P' \to P$ over $\mathrm{Spec}(\mathbb{Z})$ and under $\mathcal{X}$. Namely one defines

$$\epsilon^{P',P} := \epsilon^{P'',P'}(q')^{-1} \circ \epsilon^{P'',P}(q)$$

where $q' : P'' \to P'$ and $q : P'' \to P$ are the projections and

$$\epsilon^{P',P}(u) : \omega^P_{\mathcal{X}/\mathbb{Z}} \xrightarrow{\sim} \omega^{P'}_{\mathcal{X}/\mathbb{Z}}$$

depends on u. More precisely, $\epsilon^{P',P}(u)$ is defined by the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \arrow[r] \arrow[d] & i^*\Omega_{P/\mathbb{Z}} \arrow[r] \arrow[d] & \Omega_{\mathcal{X}/\mathbb{Z}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{I}'/(\mathcal{I}')^2 \arrow[r] \arrow[d] & i'^*\Omega_{P'/\mathbb{Z}} \arrow[r] \arrow[d] & \Omega_{\mathcal{X}/\mathbb{Z}} \arrow[r] & 0 \\
 & i'^*\Omega_{P'/P} \arrow[r, equal] \arrow[d] & i'^*\Omega_{P'/P} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the columns are the transitivity triangles of the cotangent complex for $\mathcal{X} \to P' \xrightarrow{u} P$ and $P' \xrightarrow{u} P \to \mathrm{Spec}(\mathbb{Z})$, respectively, and we refer to [2][(A.2.2)] for the precise sign conventions. The above commutative diagram induces a commutative diagram

$$\begin{tikzcd}
\wedge^{d-1}\Omega_{\mathcal{X}/\mathbb{Z}} \arrow[r] \arrow[d, equal] & \omega^P_{\mathcal{X}/\mathbb{Z}} \arrow[d, "\epsilon^{P',P}(u)"] \\
\wedge^{d-1}\Omega_{\mathcal{X}/\mathbb{Z}} \arrow[r] & \omega^{P'}_{\mathcal{X}/\mathbb{Z}},
\end{tikzcd}$$

so that $(15)^P$ is indeed compatible with the isomorphisms $\epsilon^{P',P}(u)$ and therefore also with the isomorphisms $\epsilon^{P',P}$.

$\square$

Definition 3.2. *The* Bloch conductor *of the arithmetic scheme $\mathcal{X}$ is the positive integer*

$$A(\mathcal{X}) := \prod_p p^{(-1)^{d-1}d_p}$$

where the product is over all prime numbers p, $d_p := \deg c^{\mathcal{X}}_{d,\mathcal{X}_{\mathbb{F}_p}}(\Omega_{\mathcal{X}/\mathbb{Z}}) \in \mathbb{Z}$ and

$$c^{\mathcal{X}}_{d,\mathcal{X}_{\mathbb{F}_p}}(\Omega_{\mathcal{X}/\mathbb{Z}}) \in CH_0(\mathcal{X}_{\mathbb{F}_p})$$

is a localized Chern class introduced in [4].

The Bloch conductor was introduced in [4] and further studied in [5],[23],[15],[24]. The deepest result about the Bloch conductor is its equality with the Artin conductor, defined in terms of the l-adic cohomology of $\mathcal{X}$, in certain cases. This equality was proven for $d = 2$ in [4] and if $\mathcal{X}$ has everywhere semistable reduction in [15]. For general regular $\mathcal{X}$ it is conjectured but still

open. The equality of the Bloch and the Artin conductor is important for establishing cases of Conjecture 1.3 via the Langlands correspondence but plays no role in this section. Here we only review the (slightly) more elementary results of [23] and [24] about $A(\mathcal{X})$. Also note that our normalization of $A(\mathcal{X})$ is different from these references so that $A(\mathcal{X})$ equals the Artin conductor rather than its inverse.
The following theorem was proven by T. Saito in [24][Cor. 4.9]. The case $d = 2$, $r = 1$ is due to Bloch [5][Thm. 2.3] and the case $r \geq d-1$ can already be found in T. Saito's earlier article [23]. We give some details of Saito's proof since the exposition in [24] is rather short.

Theorem 3.3. *For any $r \in \mathbb{Z}$ let $C^r_{\mathcal{X}/\mathbb{Z}}$ be the mapping cone of (16), a perfect complex of $\mathcal{O}_{\mathcal{X}}$-modules. Then $R\Gamma(\mathcal{X}, C^r_{\mathcal{X}/\mathbb{Z}})$ has finite cohomology and*

$$\chi^\times R\Gamma(\mathcal{X}, C^r_{\mathcal{X}/\mathbb{Z}}) = A(\mathcal{X})^{(-1)^r}$$

where $\chi^\times$ is the multiplicative Euler characteristic (see Definition 2.19).

Proof. First note that over the open subset $\mathcal{X}^{sm} \subseteq \mathcal{X}$ where $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is smooth the complex $L\wedge^r L_{\mathcal{X}/\mathbb{Z}}$ is concentrated in degree 0 with cohomology the locally free sheaf $\Omega^r_{\mathcal{X}^{sm}/\mathbb{Z}} = \wedge^r \Omega_{\mathcal{X}^{sm}/\mathbb{Z}}$. The map (15) is also an isomorphism over $\mathcal{X}^{sm}$. Hence, by linear algebra, the map (16) is an isomorphism over $\mathcal{X}^{sm}$ and $C^r_{\mathcal{X}/\mathbb{Z}}$ is supported in $\mathcal{X} \setminus \mathcal{X}^{sm}$. Since $\mathcal{X}_{\mathbb{Q}} \to \mathrm{Spec}(\mathbb{Q})$ is smooth $\mathcal{X} \setminus \mathcal{X}^{sm}$ is contained in a finite union of closed fibres $\mathcal{X}_{\mathbb{F}_p}$. By [15][Lemma 5.1.1] any point $x \in \mathcal{X} \setminus \mathcal{X}^{sm}$ has a Zariski open neighborhood $U \subseteq \mathcal{X}$ such that there exists a closed embedding

$$U \to P$$

where $P \to \mathrm{Spec}(\mathbb{Z})$ is smooth of relative dimension d. The exact sequence

$$0 \to N_{U/P} \to \Omega_{P/\mathbb{Z}} \otimes_{\mathcal{O}_P} \mathcal{O}_U \to \Omega_{U/\mathbb{Z}} \to 0 \tag{17}$$

then shows that $\Omega_{\mathcal{X}/\mathbb{Z}}$ can be locally generated by d sections and that $\wedge^d \Omega_{\mathcal{X}/\mathbb{Z}}$ is locally monogenic. Following [15][Lemma 5.1.3] let

$$i : Z \to \mathcal{X}$$

be the closed subscheme with support $\mathcal{X} \setminus \mathcal{X}^{sm}$ [15][Lemma 3.1.2] defined by the ideal sheaf

$$\mathrm{Ann} \wedge^d \Omega_{\mathcal{X}/\mathbb{Z}}.$$

Then $i^* \wedge^d \Omega_{\mathcal{X}/\mathbb{Z}}$ is an invertible $\mathcal{O}_Z$-module by definition and hence $i^*\Omega_{\mathcal{X}/\mathbb{Z}}$ is locally free of rank d, as the d generating sections have no relation on Z. It follows that

$$Li^*\Omega_{\mathcal{X}/\mathbb{Z}}|_U = \left(N_{U/P} \otimes_{\mathcal{O}_U} \mathcal{O}_{U\cap Z} \xrightarrow{0} \Omega_{P/\mathbb{Z}} \otimes_{\mathcal{O}_P} \mathcal{O}_{U\cap Z}\right)$$

and hence that

$$\mathcal{L} := L^1 i^* \Omega_{\mathcal{X}/\mathbb{Z}}$$

is an invertible $\mathcal{O}_Z$-module.

Lemma 3.4. *The coherent sheaves $\mathcal{H}^i(C^r_{\mathcal{X}/\mathbb{Z}})$ are $\mathcal{O}_Z$-modules and there are canonical isomorphisms*

$$\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{H}^i(C^r_{\mathcal{X}/\mathbb{Z}}) \cong \mathcal{H}^{i-1}(C^{r+1}_{\mathcal{X}/\mathbb{Z}}) \tag{18}$$

for any $i, r \in \mathbb{Z}$.

Proof. We follow the proof of [23][Prop. 1.7] where the case $r \geq d-1$ is treated, see also [15][Lemma 2.4.2]. Recall that

$$L_{\mathcal{X}/\mathbb{Z}}|_U \cong \Omega_{U/\mathbb{Z}}[0]$$

is represented by the strictly perfect complex (17) where the conormal bundle $N_U := N_{U/P}$ is invertible and $E_U := \Omega_{P/U} \otimes_{\mathcal{O}_P} \mathcal{O}_U$ is a vector bundle of rank d. For $r \geq 0$ we have isomorphisms

$$\begin{aligned} L \wedge^r L_{\mathcal{X}/\mathbb{Z}}|_U \cong & L \wedge^r \left(N_U \xrightarrow{v} E_U \right) \\ \cong & \Gamma^r N_U \to \Gamma^{r-1} N_U \otimes E_U \to \Gamma^{r-2} N_U \otimes \wedge^2 E_U \to \cdots \to \wedge^r E_U \\ \cong & N_U^{\otimes r} \to N_U^{\otimes r-1} \otimes E_U \to N_U^{\otimes r-2} \otimes \wedge^2 E_U \to \cdots \to \wedge^r E_U \end{aligned} \tag{19}$$

where Γ^i denotes the divided power functor and $\Gamma^i N_U \cong N_U^{\otimes i}$ since N_U is invertible. The differential is given by

$$x' \otimes x \otimes y \in N_U^{\otimes i-1} \otimes N_U \otimes \wedge^{r-i} E_U \ \mapsto \ x' \otimes v(x) \wedge y \in N_U^{\otimes i-1} \otimes \wedge^{r-i+1} E_U \tag{20}$$

on local sections. This computation of the derived exterior powers of a strictly perfect two-term complex goes back to Illusie [13][4.3.1.3] and is also recalled in [15][1.2.7.2]. From this description it is clear that there is an identity of complexes

$$N_U \otimes L \wedge^r \left(N_U \xrightarrow{v} E_U \right) = \left(\sigma^{<0} L \wedge^{r+1} \left(N_U \xrightarrow{v} E_U \right) \right) [-1] \tag{21}$$

where $\sigma^{<0}$ refers to the naive truncation. Similarly we find

$$\begin{aligned} & \underline{R\,\mathrm{Hom}}(L \wedge^{d-1-r} L_{\mathcal{X}/\mathbb{Z}}, \omega_{\mathcal{X}/\mathbb{Z}})|_U \cong \underline{\mathrm{Hom}}(L \wedge^{d-1-r} \left(N_U \xrightarrow{v} E_U \right), K_U) \\ \cong & \underline{\mathrm{Hom}}(\wedge^{d-1-r} E_U, K_U) \to \cdots \to \underline{\mathrm{Hom}}(N_U^{\otimes i} \otimes \wedge^{d-1-r-i} E_U, K_U) \to \cdots \end{aligned} \tag{22}$$

where

$$K_U := N_U^{-1} \otimes \wedge^d E_U \cong \omega_{\mathcal{X}/\mathbb{Z}}|_U.$$

Using the canonical isomorphism

$$\begin{aligned} N_U \otimes \underline{\mathrm{Hom}}(& N_U^{\otimes i} \otimes \wedge^{d-1-r-i} E_U, K_U) \\ & \cong \underline{\mathrm{Hom}}(N_U^{\otimes i-1} \otimes \wedge^{d-1-(r+1)-(i-1)} E_U, K_U) \end{aligned} \tag{23}$$

we find a canonical isomorphism of complexes

$$\sigma^{>0}N_U\otimes \underline{\mathrm{Hom}}(L\wedge^{d-1-r}\left(N_U\xrightarrow{v}E_U\right),K_U)$$
$$\cong \underline{\mathrm{Hom}}(L\wedge^{d-1-(r+1)}\left(N_U\xrightarrow{v}E_U\right),K_U)[-1]. \quad (24)$$

The complex $C^r_{\mathcal{X}/\mathbb{Z}}|_U$ is obtained by splicing together (19) placed in degrees ≤ -1 with (22) placed in degrees ≥ 0 via the map

$$\phi_r:\wedge^r E_U\to \underline{\mathrm{Hom}}(\wedge^{d-1-r}E_U,K_U)\cong \underline{\mathrm{Hom}}(N_U\otimes\wedge^{d-1-r}E_U,\wedge^d E_U)$$

dual to

$$\wedge^r E_U\otimes N_U\otimes\wedge^{d-1-r}E_U\to\wedge^d E_U;\quad y\otimes x\otimes y'\mapsto v(x)\wedge y\wedge y'.$$

Denoting by ψ the canonical isomorphism

$$\psi:\wedge^{r+1}E_U\cong \underline{\mathrm{Hom}}(\wedge^{d-1-r}E_U,\wedge^d E_U)\cong N_U\otimes \underline{\mathrm{Hom}}(\wedge^{d-1-r}E_U,K_U)$$

we have a commutative diagram

$$\begin{tikzcd}
N_U\otimes\wedge^r E_U \arrow[r, equal] \arrow[d, "(20)"'] & N_U\otimes\wedge^r E_U \arrow[d, "\mathrm{id}\otimes\phi_r"'] \\
\wedge^{r+1}E_U \arrow[r, "\psi", "\sim"'] \arrow[d, "\phi_{r+1}"'] & N_U\otimes \underline{\mathrm{Hom}}(\wedge^{d-1-r}E_U,K_U) \arrow[d] \\
\underline{\mathrm{Hom}}(\wedge^{d-2-r}E_U,K_U) \arrow[r, "(23)^{-1}", "\sim"'] & N_U\otimes \underline{\mathrm{Hom}}(N_U\otimes\wedge^{d-2-r}E_U,K_U)
\end{tikzcd} \quad (25)$$

as one verifies easily on local sections. Combining (21), (24) and (25) we obtain a canonical isomorphism

$$N_U\otimes C^r_{\mathcal{X}/\mathbb{Z}}|_U\cong C^{r+1}_{\mathcal{X}/\mathbb{Z}}|_U[-1]. \quad (26)$$

As in [23][(1.6.1)] one has an isomorphism

$$C^{d-1}_{\mathcal{X}/\mathbb{Z}}|_U\cong K_U\otimes \mathrm{Kos}(E_U^*\otimes N_U\xrightarrow{v^*\otimes\mathrm{id}}N_U^*\otimes N_U\cong\mathcal{O}_U)$$

where $\mathrm{Kos}(P\to A)$ denotes the Koszul algebra associated to a A-module homomorphism $P\to A$ where P is finitely generated projective over A. Using the fact that $H^i(\mathrm{Kos}(P\to A))$ is a module over the ring $H^0(\mathrm{Kos}(P\to A))$ [26][15.28.6] one deduces that all coherent sheaves $\mathcal{H}^i(C^{d-1}_{\mathcal{X}/\mathbb{Z}}|_U)$ are modules over $H^0(\mathrm{Kos})\cong\mathcal{O}_{U\cap Z}$. Using (26) and the fact that N_U is invertible we deduce that all coherent sheaves $\mathcal{H}^i(C^r_{\mathcal{X}/\mathbb{Z}}|_U)$ are modules over $\mathcal{O}_{U\cap Z}$, and an isomorphism

$$\left(\mathcal{L}\otimes_{\mathcal{O}_Z}\mathcal{H}^i(C^r_{\mathcal{X}/\mathbb{Z}})\right)|_U\cong\mathcal{H}^{i-1}(C^{r+1}_{\mathcal{X}/\mathbb{Z}})|_U \quad (27)$$

whose construction a priori depends on the choice of $U \to P$. However, as in the proof of [23][(1.7.2)] one shows that for a different embedding $U \to P'$, leading to a different strictly perfect resolution $N'_U \to E'_U$ of $L_{\mathcal{X}/\mathbb{Z}}|_U$, one has a quasi-isomorphism

$$g : (N'_U \to E'_U) \to (N_U \to E_U)$$

unique up to homotopy, inducing quasi-isomorphisms

$$g^r : L \wedge^r (N'_U \to E'_U) \to L \wedge^r (N_U \to E_U)$$

for all r, unique up to homotopy, which commute with the isomorphisms (21), (24) and (25). Hence (27) is in fact independent of the choice of $U \to P$ which also implies that the local isomorphisms (27) glue to the global isomorphism (18). $\square$

Since $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is proper and $C^r_{\mathcal{X}/\mathbb{Z}}$ is a perfect complex complex of $\mathcal{O}_{\mathcal{X}}$-modules $R\Gamma(\mathcal{X}, C^r_{\mathcal{X}/\mathbb{Z}})$ is a perfect complex of abelian groups. It has a finite filtration with subquotients

$$R\Gamma(\mathcal{X}, \mathcal{H}^i(C^r_{\mathcal{X}/\mathbb{Z}})[-i]) \cong R\Gamma(Z, \mathcal{H}^i(C^r_{\mathcal{X}/\mathbb{Z}})[-i])$$

which are perfect complexes of abelian groups with torsion cohomology, as Z is supported in a finite union of closed fibres $\mathcal{X}_{\mathbb{F}_p}$. Hence $R\Gamma(\mathcal{X}, C^r_{\mathcal{X}/\mathbb{Z}})$ has finite cohomology. We can view $\chi^\times$ as a homomorphism

$$\chi^\times : G(Z) \to K_0(\mathbb{Z}; \mathbb{Q}) \cong \mathbb{Q}^{\times,>0}; \quad [\mathcal{F}] \mapsto [R\Gamma(Z, \mathcal{F})]$$

where $G(Z)$ is the Grothendieck group of the category of coherent sheaves on Z and $K_0(\mathbb{Z}; \mathbb{Q})$ is the Grothendieck group of the category of finite abelian groups (which is also the relative K_0 for the ring homomorphism $\mathbb{Z} \to \mathbb{Q}$). By [15][Lemma 5.1.3.3] one has $[\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{F}] = [\mathcal{F}]$ in $G(Z)$ for any coherent sheaf $\mathcal{F}$ on Z. Hence (18) implies

$$\chi^\times R\Gamma(Z, \mathcal{H}^i(C^r_{\mathcal{X}/\mathbb{Z}})) = \chi^\times R\Gamma(Z, \mathcal{H}^{i-1}(C^{r+1}_{\mathcal{X}/\mathbb{Z}}))$$

and therefore

$$\chi^\times R\Gamma(\mathcal{X}, C^r_{\mathcal{X}/\mathbb{Z}}) = \chi^\times R\Gamma(\mathcal{X}, C^{r+1}_{\mathcal{X}/\mathbb{Z}})^{-1}$$

for any $r \in \mathbb{Z}$. On the other hand we have

$$\chi^\times R\Gamma(\mathcal{X}, C^d_{\mathcal{X}/\mathbb{Z}}) = \chi^\times R\Gamma(\mathcal{X}, L \wedge^d L_{\mathcal{X}/\mathbb{Z}}[1]) = A(\mathcal{X})^{(-1)^d}$$

by [23][Prop. 2.3]. This finishes the proof of the theorem. $\square$

3.2 Duality for derived de Rham cohomology

Denote by

$$\cdots \to F^{r+1} \to F^r \to \cdots \to R\Gamma_{dR}(\mathcal{X}/\mathbb{Z}) = F^0 = F^{-1} = \cdots$$

the Hodge filtration of (Hodge completed) derived de Rham cohomology and by F^n/F^m the mapping cone of $F^m \to F^n$ for $m \geq n$. Since

$$F^r/F^{r+1} \cong R\Gamma(\mathcal{X}, L\wedge^r L_{\mathcal{X}/\mathbb{Z}}[-r]) \tag{28}$$

is a perfect complex of abelian groups, so are all F^n/F^m for $m \geq n$. Denote by $C^* = R\operatorname{Hom}(C, \mathbb{Z})$ the $\mathbb{Z}$-dual of a perfect complex of abelian groups.

Proposition 3.5. *a) For $n \leq d$ there is a (Poincaré) duality map*

$$\epsilon_n : F^n/F^d \to (R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n})^*[-2d+2] \tag{29}$$

satisfying

$$\chi^\times \operatorname{Cone}(\epsilon_n) = A(\mathcal{X})^{d-n}. \tag{30}$$

b) In particular, the discriminant of the Poincaré duality pairing

$$R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \otimes^L_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \to \mathbb{Z}[-2d+2] \tag{31}$$

has absolute value $A(\mathcal{X})^d$.

Remark 3.6. *For $d = 1$ we have $\mathcal{X} = \operatorname{Spec}(\mathcal{O}_F)$ and $A(\mathcal{X}) = |D_F|$, and b) reduces to the fact that the trace pairing*

$$\mathcal{O}_F \times \mathcal{O}_F \to \mathbb{Z}; \quad (a, b) \mapsto \operatorname{Tr}(ab)$$

has discriminant D_F. For $d = 2$ it was shown by Bloch in [5]*[Thm. 2] that the Poincaré duality pairing on the complex*

$$R\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}} \to \Omega_{\mathcal{X}/\mathbb{Z}}) \cong R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^2$$

has discriminant $\pm A(\mathcal{X})^2$. For $d \geq 3$ it seems harder to describe the complex $R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d$ more explicitly.

Remark 3.7. *If P is a perfect complex of abelian groups and $P \otimes^L_{\mathbb{Z}} P \to \mathbb{Z}[2\delta]$ is a pairing which induces an isogeny $\phi : P \to P^*[2\delta]$ in the sense that* $\operatorname{Cone}(\phi)$ *has finite cohomology groups, we obtain isomorphisms*

$$\det_{\mathbb{Z}} P^* \simeq \det_{\mathbb{Z}} P \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \operatorname{Cone}(\phi)$$

and

$$\det_{\mathbb{Z}} P \otimes \det_{\mathbb{Z}} P \simeq \det^{-1}_{\mathbb{Z}} \operatorname{Cone}(\phi)$$

and hence a duality pairing on determinants

$$\langle \cdot, \cdot \rangle : \det_{\mathbb{Q}} P_{\mathbb{Q}} \otimes \det_{\mathbb{Q}} P_{\mathbb{Q}} \simeq \mathbb{Q}.$$

The discriminant of the pairing is $\langle b,b\rangle\in\mathbb{Q}$ where b is a $\mathbb{Z}$-basis of $\det_{\mathbb{Z}}P$. Since

$$\langle -b,-b\rangle=(-1)^2\langle b,b\rangle=\langle b,b\rangle$$

the discriminant is a well-defined rational number (of absolute value $\chi^\times\operatorname{Cone}(\phi)$).

Proof. Poincaré duality for algebraic de Rham cohomology of $\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}$ is discussed in [26][Prop. 50.20.4]. It turns out that one can lift the construction of the cup product pairing in loc. cit. to the derived de Rham complex on $\mathcal{X}$ since we are truncating by F^d. More precisely, choose a simplicial resolution $P_\bullet\to\mathcal{O}_{\mathcal{X}}$ in $\mathcal{X}_{Zar}$ where P_i is a free $\mathbb{Z}$-algebra in $\mathcal{X}_{Zar}$ and denote by $\Omega^{[n,m]}_{P_\bullet/\mathbb{Z}}$ the complex (of simplicial modules)

$$\Omega^n_{P_\bullet/\mathbb{Z}}\to\cdots\to\Omega^m_{P_\bullet/\mathbb{Z}}$$

in degrees $[n,m]$, zero for $n>m$, where the differential is the de Rham differential. Define a complex of sheaves of abelian groups on $\mathcal{X}_{Zar}$

$$L\Omega^{[n,m]}_{\mathcal{X}/\mathbb{Z}}:=\operatorname{Tot}^*_\bullet\Omega^{[n,m],\sim}_{P_\bullet/\mathbb{Z}}$$

so that

$$L\wedge^n L_{\mathcal{X}/\mathbb{Z}}[-n]=L\Omega^{[n,n]}_{\mathcal{X}/\mathbb{Z}};\quad R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n=R\Gamma(\mathcal{X},L\Omega^{[0,n-1]}_{\mathcal{X}/\mathbb{Z}}).$$

Here and in the following we denote by $M_\bullet^\sim$ the (n-tuple) chain complex associated to a (n-tuple) simplicial module $M_\bullet$ [13][1.1] and we decorate the (partial) totalization of an n-tuple complex with the indices that are contracted into one. We use the convention that totalization of an upper and a lower index leads to an upper index. As in [26][50.4.0.1] the wedge product on differential forms induces a map of bicomplexes

$$\begin{aligned}\operatorname{Tot}^{*,*}\operatorname{Tot}_{\bullet,\bullet}\Omega^{\star,\sim}_{P_\bullet/\mathbb{Z}}\otimes_{\mathbb{Z}}\Omega^{\star,\sim}_{P_\bullet/\mathbb{Z}}&=\operatorname{Tot}^{*,*}\operatorname{Tot}_{\bullet,\bullet}\left(\Omega^\star_{P_\bullet/\mathbb{Z}}\otimes_{\mathbb{Z}}\Omega^\star_{P_\bullet/\mathbb{Z}}\right)^\sim\\&\xrightarrow{\sigma}\operatorname{Tot}^{*,*}\left(\Delta\left(\Omega^\star_{P_\bullet/\mathbb{Z}}\otimes_{\mathbb{Z}}\Omega^\star_{P_\bullet/\mathbb{Z}}\right)\right)^\sim\\&=\left(\operatorname{Tot}^{*,*}\Delta\left(\Omega^\star_{P_\bullet/\mathbb{Z}}\otimes_{\mathbb{Z}}\Omega^\star_{P_\bullet/\mathbb{Z}}\right)\right)^\sim\\&\xrightarrow{\cup}\Omega^{\star,\sim}_{P_\bullet/\mathbb{Z}}\to\Omega^{[0,d-1],\sim}_{P_\bullet/\mathbb{Z}}\end{aligned}$$

where σ is induced by shuffle map $\operatorname{Tot}_{\bullet,\bullet}(M_\bullet\otimes N_\bullet)^\sim\to(\Delta(M_\bullet\otimes N_\bullet))^\sim$ of [13][(1.2.2.1)] and Δ denotes the diagonal simplicial object of a bisimplicial object. Since we have truncated to degrees $\leq d-1$ the above pairing factors through a pairing

$$\operatorname{Tot}^{*,*}\operatorname{Tot}_{\bullet,\bullet}\Omega^{[n,d-1],\sim}_{P_\bullet/\mathbb{Z}}\otimes_{\mathbb{Z}}\Omega^{[0,d-1-n],\sim}_{P_\bullet/\mathbb{Z}}\to\Omega^{[0,d-1],\sim}_{P_\bullet/\mathbb{Z}}$$

and hence we obtain a pairing

$$\begin{aligned}
L\Omega^{[n,d-1]}_{\mathcal{X}/\mathbb{Z}} \otimes^L_{\mathbb{Z}} L\Omega^{[0,d-1-n]}_{\mathcal{X}/\mathbb{Z}} &= \mathrm{Tot}^{*,*}\, \mathrm{Tot}^*_\bullet\, \Omega^{[n,d-1],\sim}_{P_\bullet/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{Tot}^*_\bullet\, \Omega^{[0,d-1-n],\sim}_{P_\bullet/\mathbb{Z}} \\
&\simeq \mathrm{Tot}^{*,*}\, \mathrm{Tot}^*_\bullet\, \mathrm{Tot}^*_\bullet\ \Omega^{[n,d-1],\sim}_{P_\bullet/\mathbb{Z}} \otimes_{\mathbb{Z}} \Omega^{[0,d-1-n],\sim}_{P_\bullet/\mathbb{Z}} \\
&\simeq \mathrm{Tot}^{*,*}_{\bullet,\bullet}\ \Omega^{[n,d-1],\sim}_{P_\bullet/\mathbb{Z}} \otimes_{\mathbb{Z}} \Omega^{[0,d-1-n],\sim}_{P_\bullet/\mathbb{Z}} \\
&\simeq \mathrm{Tot}^*_\bullet\, \mathrm{Tot}^{*,*}\, \mathrm{Tot}_{\bullet,\bullet}\ \Omega^{[n,d-1],\sim}_{P_\bullet/\mathbb{Z}} \otimes_{\mathbb{Z}} \Omega^{[0,d-1-n],\sim}_{P_\bullet/\mathbb{Z}} \\
&\to \mathrm{Tot}^*_\bullet\, \Omega^{[0,d-1],\sim}_{P_\bullet/\mathbb{Z}} = L\Omega^{[0,d-1]}_{\mathcal{X}/\mathbb{Z}}
\end{aligned}$$

and an induced pairing on cohomology

$$\begin{array}{ccc}
R\Gamma(\mathcal{X}, L\Omega^{[n,d-1]}_{\mathcal{X}/\mathbb{Z}}) \otimes^L_{\mathbb{Z}} R\Gamma(\mathcal{X}, L\Omega^{[0,d-1-n]}_{\mathcal{X}/\mathbb{Z}}) & \longrightarrow & R\Gamma(\mathcal{X}, L\Omega^{[0,d-1]}_{X/\mathbb{Z}}) \\
\Vert & & \Vert \\
F^n/F^d \otimes^L_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n} & \longrightarrow & R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d.
\end{array}$$

LEMMA 3.8. *One has* $H^i(\mathcal{X}, L\Omega^{[0,d-1]}_{\mathcal{X}/\mathbb{Z}}) = 0$ *for* $i > 2d-2$. *Moreover, the natural map*

$$H^{2d-2}(\mathcal{X}, L\wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1]) \to H^{2d-2}(\mathcal{X}, L\Omega^{[0,d-1]}_{\mathcal{X}/\mathbb{Z}})$$

induces an isomorphism

$$g : H^{2d-2}(\mathcal{X}, L\wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1])/\mathrm{tor} \cong H^{2d-2}(\mathcal{X}, L\Omega^{[0,d-1]}_{\mathcal{X}/\mathbb{Z}})/\mathrm{tor}$$

and therefore a trace map

$$\begin{aligned}
&R\Gamma(\mathcal{X}, L\Omega^{[0,d-1]}_{X/\mathbb{Z}})[2d-2] \to H^{2d-2}(\mathcal{X}, L\Omega^{[0,d-1]}_{\mathcal{X}/\mathbb{Z}})/\mathrm{tor} \xrightarrow{g^{-1}} \\
&H^{2d-2}(\mathcal{X}, L\wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1])/\mathrm{tor} \xrightarrow{(15)_*} H^{2d-2}(\mathcal{X}, \omega_{\mathcal{X}/\mathbb{Z}}[-d+1])/\mathrm{tor} \xrightarrow{\mathrm{Tr}} \mathbb{Z}
\end{aligned} \tag{32}$$

Proof. We first remark that $H^i(\mathcal{X}, \mathcal{F}) = 0$ for $i \geq d$ and any coherent sheaf $\mathcal{F}$ on $\mathcal{X}$. Indeed, this is clear for $i > d$ since the cohomological dimension of $\mathcal{X}_{Zar}$ is d. Duality for $f : \mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$

$$R\,\mathrm{Hom}_{\mathbb{Z}}(Rf_*\mathcal{F}, \mathbb{Z}) \cong R\,\mathrm{Hom}_{\mathcal{X}}(\mathcal{F}, \omega_{\mathcal{X}/\mathbb{Z}}[d-1])$$

evaluated in degree $-d$

$$\mathrm{Hom}_{\mathbb{Z}}(H^d(\mathcal{X}, \mathcal{F}), \mathbb{Z}) \cong H^{-1}R\,\mathrm{Hom}_{\mathcal{X}}(\mathcal{F}, \omega_{\mathcal{X}/\mathbb{Z}}) = 0$$

shows that $H^d(\mathcal{X}, \mathcal{F})$ is torsion. Evaluation in degree $-d+1$

$$0 \to \mathrm{Ext}^1(H^d(\mathcal{X}, \mathcal{F}), \mathbb{Z}) \to \mathrm{Hom}_{\mathcal{X}}(\mathcal{F}, \omega_{\mathcal{X}/\mathbb{Z}}) \to \mathrm{Hom}_{\mathbb{Z}}(H^{d-1}(\mathcal{X}, \mathcal{F}), \mathbb{Z}) \to 0$$

shows that $H^d(\mathcal{X},\mathcal{F})=0$ since $\omega_{\mathcal{X}/\mathbb{Z}}$ is a line bundle, f is flat, and therefore $\mathrm{Hom}_{\mathcal{X}}(\mathcal{F},\omega_{\mathcal{X}/\mathbb{Z}})$ is torsion free.
Since $L\wedge^r L_{\mathcal{X}/\mathbb{Z}}$ is an object of the derived category of coherent sheaves concentrated in degrees ≤ 0 we also have $H^i(\mathcal{X},L\wedge^r L_{\mathcal{X}/\mathbb{Z}})=0$ for $i\geq d$. The exact triangle

$$R\Gamma(\mathcal{X},L\wedge^r L_{\mathcal{X}/\mathbb{Z}})[-r])\to R\Gamma(\mathcal{X},L\Omega^{[n,r]}_{\mathcal{X}/\mathbb{Z}})\to R\Gamma(\mathcal{X},L\Omega^{[n,r-1]}_{\mathcal{X}/\mathbb{Z}})\to$$

and an easy induction then show that $H^i(\mathcal{X},L\Omega^{[n,m]}_{\mathcal{X}/\mathbb{Z}})=0$ for $i\geq d+m$. In particular, the map

$$H^{2d-2}(\mathcal{X},L\wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1])\to H^{2d-2}(\mathcal{X},L\Omega^{[0,d-1]}_{\mathcal{X}/\mathbb{Z}})$$

is surjective and an isomorphism after tensoring with $\mathbb{Q}$ (see the proof of [26][Prop. 50.20.4]), hence induces an isomorphism

$$g:H^{2d-2}(\mathcal{X},L\wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1])/\mathrm{tor}\cong H^{2d-2}(\mathcal{X},L\Omega^{[0,d-1]}_{\mathcal{X}/\mathbb{Z}})/\mathrm{tor}.$$

□

We now prove (30) by downward induction on n starting with the trivial case $n=d$. The induction step is provided by the diagram with exact rows and columns

$$\begin{array}{ccccc}
F^{n+1}/F^d & \xrightarrow{\epsilon_{n+1}} & (F^0/F^{d-n-1})^*[-2d+2] & \longrightarrow & \mathrm{Cone}(\epsilon_{n+1})\\
\downarrow & & \downarrow & & \downarrow\\
F^n/F^d & \xrightarrow{\epsilon_n} & (F^0/F^{d-n})^*[-2d+2] & \longrightarrow & \mathrm{Cone}(\epsilon_n)\\
\downarrow & & \downarrow & & \downarrow\\
F^n/F^{n+1} & \longrightarrow & (F^{d-n-1}/F^{d-n})^*[-2d+2] & \longrightarrow & R\Gamma(\mathcal{X},C^n_{\mathcal{X}/\mathbb{Z}})[-n]
\end{array}$$

where the bottom exact triangle is $R\Gamma(\mathcal{X},-)[-n]$ applied to (16) in view of (28) and coherent sheaf duality for $f:\mathcal{X}\to\mathrm{Spec}(\mathbb{Z})$:

$$\begin{aligned}
&(F^{d-n-1}/F^{d-n})^*[-2d+2]\\
\cong& R\,\mathrm{Hom}(Rf_*(L\wedge^{d-1-n} L_{\mathcal{X}/\mathbb{Z}})[-d+1+n],\mathbb{Z})[-2d+2]\\
\cong& R\,\mathrm{Hom}_{\mathcal{X}}(L\wedge^{d-1-n} L_{\mathcal{X}/\mathbb{Z}}[-d+1+n],\omega_{\mathcal{X}/\mathbb{Z}})[-d+1]\\
\cong& R\Gamma(\mathcal{X},\underline{R\,\mathrm{Hom}}_{\mathcal{X}}(L\wedge^{d-1-n} L_{\mathcal{X}/\mathbb{Z}},\omega_{\mathcal{X}/\mathbb{Z}}))[-n].
\end{aligned}$$

By Theorem 3.3 we have

$$\chi^\times(\mathrm{Cone}(\epsilon_n))=\chi^\times(\mathrm{Cone}(\epsilon_{n+1}))\cdot A(\mathcal{X})$$

which gives $\chi^\times(\mathrm{Cone}(\epsilon_n))=A(\mathcal{X})^{d-n}$ by induction. □

For any $n \in \mathbb{Z}$ we have an exact triangle on the generic fibre $X = \mathcal{X}_{\mathbb{Q}}$

$$F^n \to R\Gamma_{dR}(X/\mathbb{Q}) \to R\Gamma_{dR}(X/\mathbb{Q})/F^n \tag{33}$$

and we also have a duality isomorphism $(29)_{\mathbb{Q}}$ for any $n \in \mathbb{Z}$ since $F^n = F^d = 0$ on the generic fibre for $n \geq d$.

Corollary 3.9. *Let $n \in \mathbb{Z}$ and denote by λ_{dR} the isomorphism*

$$\begin{aligned} &\left(\det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \otimes \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n}\right)_{\mathbb{Q}} \\ \simeq\ &\det_{\mathbb{Q}}^{-1} R\Gamma_{dR}(X/\mathbb{Q})/F^n \otimes \det_{\mathbb{Q}} R\Gamma_{dR}(X/\mathbb{Q})/F^{d-n} \\ \overset{(29)_{\mathbb{Q}}}{\simeq}\ &\det_{\mathbb{Q}}^{-1} R\Gamma_{dR}(X/\mathbb{Q})/F^n \otimes \det_{\mathbb{Q}}^{-1} F^n \\ \overset{(33)}{\simeq}\ &\det_{\mathbb{Q}}^{-1} R\Gamma_{dR}(X/\mathbb{Q}) \\ \simeq\ &\left(\det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d\right)_{\mathbb{Q}}. \end{aligned}$$

Then

$$\begin{aligned} &\lambda_{dR}\left(\det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \otimes \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n}\right) \\ =&A(\mathcal{X})^{d-n} \cdot \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d. \end{aligned}$$

Proof. For $n \leq d$ this is clear from Prop. 3.5 and the fact that (33) is the scalar extension to $\mathbb{Q}$ of the exact triangle

$$F^n/F^d \to R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \to R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n.$$

For $n > d$ we have $R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n} = 0$ and an exact triangle

$$F^d/F^n \to R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \to R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d$$

where

$$\begin{aligned} \chi^\times(F^d/F^n) &= \prod_{r=d}^{n-1} \chi^\times\left(R\Gamma(\mathcal{X}, L\wedge^r L_{\mathcal{X}/\mathbb{Z}}[-r])\right) \\ &= \prod_{r=d}^{n-1} \chi^\times\left(R\Gamma(\mathcal{X}, C^r_{\mathcal{X}/\mathbb{Z}}[-r-1])\right) \\ &= A(\mathcal{X})^{d-n} \end{aligned}$$

by (28) and Theorem 3.3. Hence

$$\det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n = A(\mathcal{X})^{d-n} \cdot \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d$$

inside $\det_{\mathbb{Q}}^{-1} R\Gamma_{dR}(X/\mathbb{Q})$.

□

4 The archimedean Euler factor

Following [22], for any pure $\mathbb{R}$-Hodge structure M over $\mathbb{R}$ of weight $w(M)$ we define

$$\begin{aligned} h_j(M) &= \dim F^j/F^{j+1} = h^{j,w(M)-j}(M) \\ d_\pm(M) &= \dim_{\mathbb{R}} M^{F_\infty=\pm 1} \\ t_H(M) &= \sum_j jh_j(M) = \frac{w(M)\cdot\dim_{\mathbb{R}} M}{2} = \frac{w(\det(M))}{2} \end{aligned}$$

$$L_\infty(M,s) = \prod_{p<q:=w(M)-p} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \cdot \prod_{p=\frac{w(M)}{2}} \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}}\Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}}$$

where

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2); \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Note that the factorization of $L_\infty(M,s)$ corresponds to the decomposition of M into simple $\mathbb{R}$-Hodge structures over $\mathbb{R}$. Also recall the leading coefficient of the Γ-function at $j \in \mathbb{Z}$

$$\Gamma^*(j) = \begin{cases} (j-1)! & j \geq 1 \\ (-1)^j/(-j)! & j \leq 0 \end{cases} \tag{34}$$

Lemma 4.1. *(see also [22][4.3.2, Lemme C.3.7]) For any pure $\mathbb{R}$-Hodge structure M over $\mathbb{R}$ one has*

$$\frac{L_\infty^*(M,0)}{L_\infty^*(M^*(1),0)} = \pm 2^{d_+(M)-d_-(M)}(2\pi)^{d_-(M)+t_H(M)}\prod_j \Gamma^*(-j)^{h_j(M)}$$

Proof. The functional equation of the Γ-function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{35}$$

implies

$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(1-s) = \frac{2}{\sin(\pi s)}; \qquad \Gamma_{\mathbb{R}}(1+s)\Gamma_{\mathbb{R}}(1-s) = \cos(\frac{\pi s}{2})^{-1}.$$

Hence

$$\frac{\Gamma_{\mathbb{C}}(s-p)}{\Gamma_{\mathbb{C}}(-s-(-q-1))} = \frac{\Gamma_{\mathbb{C}}(s-p)}{\Gamma_{\mathbb{C}}(1-(s-q))} = \Gamma_{\mathbb{C}}(s-p)\Gamma_{\mathbb{C}}(s-q)\frac{\sin(\pi(s-q))}{2}. \tag{36}$$

Using in addition the identity $\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s)$ we find

$$\begin{aligned} \frac{\Gamma_{\mathbb{R}}(s-p)}{\Gamma_{\mathbb{R}}(-s-(-p-1))} &= \frac{\Gamma_{\mathbb{R}}(s-p)\Gamma_{\mathbb{R}}(s-p+1)}{\Gamma_{\mathbb{R}}(1-(s-p))\Gamma_{\mathbb{R}}(s-p+1)} \\ &= \Gamma_{\mathbb{C}}(s-p)\cos(\frac{\pi(s-p)}{2}) \end{aligned} \tag{37}$$

and similarly

$$\frac{\Gamma_{\mathbb{R}}(s-p+1)}{\Gamma_{\mathbb{R}}(-s-(-p-1)+1)} = \frac{\Gamma_{\mathbb{R}}(s-p+1)\Gamma_{\mathbb{R}}(s-p)}{\Gamma_{\mathbb{R}}(2-(s-p))\Gamma_{\mathbb{R}}(s-p)}$$
$$=\Gamma_{\mathbb{C}}(s-p)\cos(\frac{\pi(s-p-1)}{2}). \tag{38}$$

Every pure $\mathbb{R}$-Hodge structure M over $\mathbb{R}$ is the direct sum of simple $\mathbb{R}$-Hodge structures. The simple $\mathbb{R}$-Hodge structures are $M_{p,q}$ of dimension 2 for integers $p<q$ and $M_{p,\pm}$ of dimension 1 for integers p (where F_∞ operates via $\pm(-1)^p$). From the above definition of $L_\infty(M,s)$ and (36), (37), (38) we obtain the following table

M	$M^*(1)$	$L_\infty(M,s)$	$\frac{L_\infty(M,s)}{L_\infty(M^*(1),-s)}$
$M_{p,q}$	$M_{-p-1,-q-1}$	$\Gamma_{\mathbb{C}}(s-p)$	$\Gamma_{\mathbb{C}}(s-p)\Gamma_{\mathbb{C}}(s-q)\cdot\frac{\sin(\pi(s-q))}{2}$
$M_{p,+}$	$M_{-p-1,+}$	$\Gamma_{\mathbb{R}}(s-p)$	$\Gamma_{\mathbb{C}}(s-p)\cdot\cos(\frac{\pi(s-p)}{2})$
$M_{p,-}$	$M_{-p-1,-}$	$\Gamma_{\mathbb{R}}(s-p+1)$	$\Gamma_{\mathbb{C}}(s-p)\cdot\cos(\frac{\pi(s-p-1)}{2})$

We have

$$\sin(\pi(s-q))\big|^*_{s=0} = (-1)^q\pi$$

and

$$\cos(\frac{\pi(s-p)}{2})\Big|^*_{s=0} = \begin{cases}(-1)^{p/2} & p \text{ even}\\ (-1)^{(p-1)/2}\frac{\pi}{2} & p \text{ odd}\end{cases}$$

It is now straightforward to verify the entries of the following table which confirm Lemma 4.1 for simple $\mathbb{R}$-Hodge structures. Since all quantities are additive in M the general case follows by writing M as a sum of simple $\mathbb{R}$-Hodge structures.

M	$d_+(M)$	$d_-(M)$	$h_j(M)$	$t_H(M)$	$\pm\frac{L^*_\infty(M,0)}{L^*_\infty(M^*(1),0)}$
$M_{p,q}$	1	1	1 for $j=p,q$ 0 else	$p+q$	$(2\pi)^{p+q+1}\Gamma^*(-p)\Gamma^*(-q)$
$M_{p,+}$					
p even	1	0	1 for $j=p$	p	$2(2\pi)^p\Gamma^*(-p)$
p odd	0	1	1 for $j=p$	p	$2(2\pi)^p\Gamma^*(-p)\cdot\frac{\pi}{2}$
$M_{p,-}$					
p even	0	1	1 for $j=p$	p	$2(2\pi)^p\Gamma^*(-p)\cdot\frac{\pi}{2}$
p odd	1	0	1 for $j=p$	p	$2(2\pi)^p\Gamma^*(-p)$

□

Suppose now $\mathcal{X}$ is a regular scheme, proper and flat over $\mathrm{Spec}(\mathbb{Z})$ with generic fibre $X:=\mathcal{X}_{\mathbb{Q}}$. The archimedean Euler factor of $\mathcal{X}$ is defined as

$$\zeta(\mathcal{X}_\infty,s)=\prod_{i\in\mathbb{Z}}L_\infty(h^i(X),s)^{(-1)^i} \tag{39}$$

where $h^i(X)$ is the $\mathbb{R}$-Hodge structure on $H^i(\mathcal{X}(\mathbb{C}),\mathbb{R})$.

Corollary 4.2. *One has*

$$\frac{\zeta^*(\mathcal{X}_\infty,n)}{\zeta^*(\mathcal{X}_\infty,d-n)} = \pm\, 2^{d_+(\mathcal{X},n)-d_-(\mathcal{X},n)}\cdot(2\pi)^{d_-(\mathcal{X},n)+t_H(\mathcal{X},n)} \cdot \prod_{p,q}\Gamma^*(n-p)^{h^{p,q}\cdot(-1)^{p+q}}$$

where

$$d_\pm(\mathcal{X},n)=\sum_i(-1)^i d_\pm(h^i(X)(n)),\quad t_H(\mathcal{X},n)=\sum_i(-1)^i t_H(h^i(X)(n))$$

and $h^i(X)(n)$ denotes the $\mathbb{R}$-Hodge structure on $H^i(\mathcal{X}(\mathbb{C}),(2\pi i)^n\mathbb{R})$.

Proof. For $M=h^i(X)(n)$ one has $M^*(1)\cong h^{2d-2-i}(X)(d-n)$ and

$$h_j(M)=h^{j,i-2n-j}(M)=h^{p-n,i-p-n}(M)=h^{p,i-p}=h^{p,q}$$

with $p+q=i$, $j=p-n$. Therefore Lemma 4.1 implies

$$\begin{aligned}\frac{\zeta^*(\mathcal{X}_\infty,n)}{\zeta^*(\mathcal{X}_\infty,d-n)} =&\prod_i\frac{L^*_\infty(h^i(X)(n),0)^{(-1)^i}}{L^*_\infty(h^{2d-2-i}(X)(d-n),0)^{(-1)^{2d-2-i}}}\\ =&2^{d_+(\mathcal{X},n)-d_-(\mathcal{X},n)}(2\pi)^{d_-(\mathcal{X},n)+t_H(\mathcal{X},n)}\prod_{p,q}\Gamma^*(n-p)^{h^{p,q}\cdot(-1)^{p+q}}.\end{aligned}$$

□

Lemma 4.3. *One has*

$$\frac{C_\infty(\mathcal{X},n)}{C_\infty(\mathcal{X},d-n)}=\pm\prod_{p,q}\Gamma^*(n-p)^{h^{p,q}\cdot(-1)^{p+q}}$$

Proof. Since $\mathcal{X}(\mathbb{C})$ is smooth proper of dimension $d-1$ the Hodge numbers $h^{p,q}$ are nonzero only for $0\le p\le d-1$. By definition (2)

$$\begin{aligned}C_\infty(\mathcal{X},n)=&\prod_{0\le p\le n-1,q}(n-p-1)!^{h^{p,q}\cdot(-1)^{p+q}}\\ =&\prod_{0\le p\le n-1,q}\Gamma^*(n-p)^{h^{p,q}\cdot(-1)^{p+q}}.\end{aligned}\tag{40}$$

On the other hand (35) implies

$$\Gamma^*(j)\Gamma^*(1-j)=\pm1$$

and therefore we have

$$\begin{aligned}\prod_{n\le p\le d-1,q}\Gamma^*(n-p)^{h^{p,q}\cdot(-1)^{p+q}} =&\pm\prod_{n\le p\le d-1,q}\Gamma^*(1-(n-p))^{-h^{p,q}\cdot(-1)^{p+q}}\\ =&\pm\prod_{0\le p\le d-n-1,q}\Gamma^*(d-n-p)^{-h^{p,q}\cdot(-1)^{p+q}}\\ =&\pm\, C_\infty(\mathcal{X},d-n)^{-1}.\end{aligned}\tag{41}$$

Combining (40) and (41) gives the Lemma. □

5 The main result

Recall the definition of the completed Zeta-function of $\mathcal{X}$

$$\zeta(\overline{\mathcal{X}},s):=\zeta(\mathcal{X},s)\cdot\zeta(\mathcal{X}_\infty,s)$$

where $\zeta(\mathcal{X}_\infty,s)$ was defined in (39). We repeat Conjecture 1.3 from the introduction.

Conjecture 1.3. *(Functional Equation) Let $\mathcal{X}$ be a regular scheme of dimension d, proper and flat over* $\mathrm{Spec}(\mathbb{Z})$*. Then $\zeta(\mathcal{X},s)$ has a meromorphic continuation to all $s\in\mathbb{C}$ and*

$$A(\mathcal{X})^{(d-s)/2}\cdot\zeta(\overline{\mathcal{X}},d-s)=\pm A(\mathcal{X})^{s/2}\cdot\zeta(\overline{\mathcal{X}},s).$$

This conjecture is true for $d=1$ where it reduces to the functional equation of the Dedekind Zeta function. It is true for $d=2$ by [5][Prop. 1.1] provided that the L-function $L(h^1(\mathcal{X}_{\mathbb{Q}}),s)$ satisfies the expected functional equation involving the Artin conductor of the l-adic representation $H^1(\mathcal{X}_{\bar{\mathbb{Q}}},\mathbb{Q}_l)$. This is the case if $\mathcal{X}$ is a regular model of a potentially modular elliptic curve over a number field F in view of the compatibility of the (local) Langlands correspondence for GL_2 with ϵ-factors and hence conductors. Potential modularity of elliptic curves is known if F is totally real or quadratic over a totally real field. We refer to [7][1.1] for a discussion of these results and for the original references. In [7] potential modularity is also shown for abelian surfaces over totally real fields F and hence Conjecture 1.3 should hold for regular models of genus 2 curves over totally real fields F (since this involves the local Langlands correspondence for GSp_4/F we are unsure whether the conductor in the functional equation is indeed the Artin conductor).

Remark 5.1. *We do not actually know a published reference for Conjecture 1.3. Serre's article* [25] *deals with what one might call the Hasse-Weil Zeta function of $\mathcal{X}$ which only depends on the generic fibre $\mathcal{X}_{\mathbb{Q}}$ and he conjectures its functional equation with $A(\mathcal{X})$ replaced by the Artin conductor of $\mathcal{X}_{\mathbb{Q}}$, defined in terms of the l-adic representations $H^i(\mathcal{X}_{\bar{\mathbb{Q}}},\mathbb{Q}_l)$. In the case $d=2$ Bloch* [5] *discusses Conjecture 1.3 with $A(\mathcal{X})$ replaced by a modified Artin conductor $\tilde{A}(\mathcal{X})$ which also depends on the l-adic representations $H^i(\mathcal{X}_{\bar{\mathbb{F}}_p},\mathbb{Q}_l)$ for bad primes $p\neq l$. In* [4] *Bloch proves that $A(\mathcal{X})=\tilde{A}(\mathcal{X})$ for $d=2$ and conjectures that $A(\mathcal{X})=\tilde{A}(\mathcal{X})$ for general regular $\mathcal{X}$. This identity has since become known as "Bloch's conductor formula" but it is still conjectural for general regular $\mathcal{X}$. For $\mathcal{X}$ with semistable reduction it was proven by Kato and Saito in* [15]*, and we have followed their terminology in calling $\tilde{A}(\mathcal{X})$ the Artin conductor of $\mathcal{X}$. We prefer to state Conjecture 1.3 with $A(\mathcal{X})$ rather than $\tilde{A}(\mathcal{X})$ in order to avoid Bloch's conductor formula which is a very deep result.*

We repeat Theorem 1.4 from the introduction which is the main result of this paper.

Theorem 1.4. *Assume $\mathcal{X}$ is a regular scheme of dimension d, proper and flat over* $\mathrm{Spec}(\mathbb{Z})$ *which satisfies Conjectures* $\mathbf{L}(\overline{\mathcal{X}}_{et},n)$*,* $\mathbf{L}(\overline{\mathcal{X}}_{et},d-n)$*,* $\mathbf{AV}(\overline{\mathcal{X}}_{et},n)$*,* $\mathbf{B}(\mathcal{X},n)$ *and* $\mathbf{B}(\mathcal{X},d-n)$ *in* [8]*. Assume that $\zeta(\overline{\mathcal{X}},s)$ satisfies Conjecture 1.3. Then Conjecture 1.1 for $(\mathcal{X},n)$ is equivalent to Conjecture 1.1 for $(\mathcal{X},d-n)$.*

Proof. The reduction of Theorem 1.4 to Theorem 1.2 was already made in [8][Cor. 5.31]. We repeat the argument here with the assumptions of Theorem 1.4 in effect. From [8][Prop. 5.29] recall the invertible $\mathbb{Z}$-module

$$\begin{aligned}\Xi_\infty(\mathcal{X}/\mathbb{Z},n) \quad := \quad & \det{}_{\mathbb{Z}}R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\otimes\det{}^{-1}_{\mathbb{Z}}R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n\\ & \otimes\det{}^{-1}_{\mathbb{Z}}R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n))\otimes\det{}_{\mathbb{Z}}R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n},\end{aligned}$$

the canonical isomorphism

$$\Delta(\mathcal{X}/\mathbb{Z},n)\otimes\Xi_\infty(\mathcal{X}/\mathbb{Z},n)\xrightarrow{\sim}\Delta(\mathcal{X}/\mathbb{Z},d-n) \tag{42}$$

and the canonical trivialization

$$\xi_\infty:\mathbb{R}\xrightarrow{\sim}\Xi_\infty(\mathcal{X}/\mathbb{Z},n)\otimes\mathbb{R}$$

such that the diagram

$$\begin{tikzcd}\Delta(\mathcal{X}/\mathbb{Z},n)\otimes\Xi_\infty(\mathcal{X}/\mathbb{Z},n)\otimes\mathbb{R} \arrow[r] & \Delta(\mathcal{X}/\mathbb{Z},d-n)\otimes\mathbb{R}\\ \mathbb{R}\otimes\mathbb{R} \arrow[u, "\lambda_\infty\otimes\xi_\infty"] \arrow[r, "="] & \mathbb{R} \arrow[u, "\lambda_\infty^*"']\end{tikzcd} \tag{43}$$

commutes. Here λ^*_∞ is the isomorphism (3) associated to $(\mathcal{X},d-n)$. Taking leading terms at $s=n$ in Conjecture 1.3 we find

$$\zeta^*(\mathcal{X},d-n)=\pm A(\mathcal{X})^{n-d/2}\cdot\frac{\zeta^*(\mathcal{X}_\infty,n)}{\zeta^*(\mathcal{X}_\infty,d-n)}\cdot\zeta^*(\mathcal{X},n)$$

or equivalently

$$\begin{aligned}C_\infty(\mathcal{X},d-n)\zeta^*(\mathcal{X},d-n)=\pm\, A(\mathcal{X})^{n-d/2}\cdot\frac{\zeta^*(\mathcal{X}_\infty,n)}{\zeta^*(\mathcal{X}_\infty,d-n)}\cdot\frac{C_\infty(\mathcal{X},d-n)}{C_\infty(\mathcal{X},n)}\\ \cdot C_\infty(\mathcal{X},n)\zeta^*(\mathcal{X},n).\end{aligned}$$

It is then clear from (43) and (42) that any two of the identities

$$\begin{aligned}\lambda^*_\infty(\mathbb{Z}) &= C_\infty(\mathcal{X},d-n)\zeta^*(\mathcal{X},d-n))\cdot\Delta(\mathcal{X}/\mathbb{Z},d-n)\\ \xi_\infty(\mathbb{Z}) &= \left(A(\mathcal{X})^{n-d/2}\cdot\frac{\zeta^*(\mathcal{X}_\infty,n)}{\zeta^*(\mathcal{X}_\infty,d-n)}\cdot\frac{C_\infty(\mathcal{X},d-n)}{C_\infty(\mathcal{X},n)}\right)\cdot\Xi_\infty(\mathcal{X}/\mathbb{Z},n)\\ \lambda_\infty(\mathbb{Z}) &= C_\infty(\mathcal{X},n)\zeta^*(\mathcal{X},n)\cdot\Delta(\mathcal{X}/\mathbb{Z},n)\end{aligned}$$

imply the third. Recall that

$$x_\infty(\mathcal{X},n)^2\in\mathbb{R}_{>0}$$

is the strictly positive real number such that

$$\xi_\infty(\mathbb{Z}) = x_\infty(\mathcal{X},n)^2 \cdot \Xi_\infty(\mathcal{X}/\mathbb{Z},n).$$

Theorem 1.2 shows that the second identity holds, and we therefore obtain Theorem 1.4 (equivalence of the first and third identity). □

Remark 5.2. *The reason for writing $x_\infty(\mathcal{X},n)^2$ as a square is the following. The canonical isomorphism*

$$\Xi_\infty(\mathcal{X}/\mathbb{Z},n) \otimes \Xi_\infty(\mathcal{X}/\mathbb{Z},d-n) \cong \mathbb{Z}$$

implies that

$$x_\infty(\mathcal{X},n)^2 \cdot x_\infty(\mathcal{X},d-n)^2 = 1$$

and hence

$$x_\infty(\mathcal{X},n)^2 = \pm x_\infty(\mathcal{X},n) \cdot x_\infty(\mathcal{X},d-n)^{-1}.$$

The identity of Theorem 1.2 can therefore be written in the following more symmetric form

$$\begin{aligned} &A(\mathcal{X})^{n/2} \cdot \zeta^*(\mathcal{X}_\infty,n) \cdot x_\infty(\mathcal{X},n)^{-1} \cdot C_\infty(\mathcal{X},n)^{-1} \\ =&\pm A(\mathcal{X})^{(d-n)/2} \cdot \zeta^*(\mathcal{X}_\infty,d-n) \cdot x_\infty(\mathcal{X},d-n)^{-1} \cdot C_\infty(\mathcal{X},d-n)^{-1} \end{aligned}$$

and this is how it was presented in [8]*[Cor. 5.31] (note that there is a typo in the statement of* [8]*[Cor. 5.31] and $C(\mathcal{X},n)$ and $C(\mathcal{X},d-n)$ have to be replaced by their inverses). However, we do not know any deeper significance of this symmetric rewritting.*

It remains to prove Theorem 1.2 which we repeat here for the convenience of the reader.

Theorem 1.2. *Let $\mathcal{X}$ be a regular scheme of dimension d, proper and flat over* $\mathrm{Spec}(\mathbb{Z})$*. Then we have*

$$x_\infty(\mathcal{X},n)^2 = \pm A(\mathcal{X})^{n-d/2} \cdot \frac{\zeta^*(\mathcal{X}_\infty,n)}{\zeta^*(\mathcal{X}_\infty,d-n)} \cdot \frac{C_\infty(\mathcal{X},d-n)}{C_\infty(\mathcal{X},n)}.$$

Proof. By Corollary 4.2 and Lemma 4.3 this identity is equivalent to

$$x_\infty(\mathcal{X},n)^2 = \pm A(\mathcal{X})^{n-d/2} \cdot 2^{d_+(\mathcal{X},n)-d_-(\mathcal{X},n)} \cdot (2\pi)^{d_-(\mathcal{X},n)+t_H(\mathcal{X},n)}. \tag{44}$$

Lemma 5.2. *The isomorphism ξ_∞ is induced by the sequence of isomorphisms*

$$\begin{aligned} &\left(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n))\right)_{\mathbb{R}} \qquad (45)\\ \xrightarrow{(9)_{\mathbb{R}}}\ & \det_{\mathbb{R}}\left(R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))^+ \oplus R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n-1))^+\right) \\ \xrightarrow{(47)}\ & \det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C})^+ \\ \xrightarrow{(48)^+}\ & \det_{\mathbb{R}} R\Gamma_{dR}(\mathcal{X}_{\mathbb{C}}/\mathbb{C})^+ \\ \simeq\ & \det_{\mathbb{R}} R\Gamma_{dR}(\mathcal{X}_{\mathbb{R}}/\mathbb{R}) \simeq \left(\det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d\right)_{\mathbb{R}} \\ \xrightarrow{\lambda_{dR}^{-1}}\ & \left(\det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n}\right)_{\mathbb{R}} \end{aligned}$$

where λ_{dR} was defined in Cor. 3.9.

Proof. The isomorphism ξ_∞ was defined in [8][Prop. 5.29]

$$\begin{aligned}
&\big(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\det_{\mathbb{Z}}^{-1}R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n\big)_{\mathbb{R}}\\
\simeq\;& \det_{\mathbb{R}} R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{R},\mathbb{R}(n))\\
\simeq\;& \det_{\mathbb{R}} R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{R},\mathbb{R}(d-n))^*[-2d+1]\\
\simeq\;& \det_{\mathbb{R}} R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{R},\mathbb{R}(d-n))\\
\simeq\;& \big(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n))\otimes_{\mathbb{Z}}\det_{\mathbb{Z}}^{-1}R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n}\big)_{\mathbb{R}}
\end{aligned}$$

using the defining exact triangle

$$(R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n)_{\mathbb{R}}[-1]\to R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}},\mathbb{R}(n))\to R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))^+$$

and duality

$$R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}},\mathbb{R}(n))\simeq R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}},\mathbb{R}(d-n))^*[-2d+1]$$

for Deligne cohomology. This duality is constructed in [8][Lemma 2.3] by taking $G_{\mathbb{R}}$-invariants in

$$R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}},\mathbb{R}(n))\simeq R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}},\mathbb{R}(d-n))^*[-2d+1] \tag{46}$$

which is obtained as follows. Dualizing the defining exact triangle

$$(R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n})_{\mathbb{C}}[-1]\to R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}},\mathbb{R}(d-n))\to R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(d-n))$$

and using Poincaré duality (7) and $(29)_{\mathbb{C}}$ on $\mathcal{X}(\mathbb{C})$ we obtain the bottom exact triangle in the diagram

$$\begin{tikzcd}
 & R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))[-1] \arrow[r, equal] \arrow[d] & R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))[-1] \arrow[d] \\
F^n_{\mathbb{C}} & (R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n)_{\mathbb{C}}[-1] \arrow[l] & R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C})[-1] \arrow[l, "\beta"'] \arrow[d] \\
F^n_{\mathbb{C}} \arrow[u, equal] & R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}},\mathbb{R}(d-n))^*[-2d+1] \arrow[l] \arrow[from=u] & R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n-1))[-1] \arrow[l]
\end{tikzcd}$$

The right hand column is induced by the decomposition

$$\mathbb{C}\cong\mathbb{R}(n)\oplus\mathbb{R}(n-1) \tag{47}$$

on coefficients, and the map β is the comparison isomorphism

$$R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C})\simeq R\Gamma_{dR}(\mathcal{X}_{\mathbb{C}}/\mathbb{C}) \tag{48}$$

composed with the natural projection. It is then clear that all rows and columns in the diagram are exact, and the middle column is the defining exact triangle of $R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}},\mathbb{R}(n))$, giving (46). Recalling that (9) was also defined using Poincaré duality (7) we find that the isomorphisms used in (45) are precisely those used in the construction of $(46)^+$. □

We call the real line $\det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C})^+$ the de Rham real structure of $\det_{\mathbb{C}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C})$ and the real line $\det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))$ the Betti real structure of $\det_{\mathbb{C}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C})$. By (8) we have

$$\det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C})^+ \cdot (2\pi i)^{d_-(\mathcal{X},n)} = \det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n)). \tag{49}$$

In the remaining computations of the proof of Theorem 1.2 all identities should be understood up to sign. We choose bases of the various $\mathbb{Z}$-structures of the de Rham real line appearing in (45)

$$\begin{aligned}
\mathbb{Z}\cdot \tilde{b}_B &= \det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n))\\
\mathbb{Z}\cdot b_{dR} &= \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d\\
\mathbb{Z}\cdot \tilde{b}_{dR} &= \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n}
\end{aligned}$$

and we also choose a basis

$$\mathbb{Z}\cdot b_B = \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n))$$

of the natural $\mathbb{Z}$-structure in the Betti real structure. Let $P\in\mathbb{C}^\times$ be the Betti-de Rham period, i.e. we have

$$b_{dR} = P\cdot b_B$$

under the comparison isomorphism (48). Note that P depends on n which is fixed in this proof.

Lemma 5.3. *Let $\varepsilon_B\in\{\pm 1\}$ be the discriminant (see Remark 3.7) of the Poincaré duality pairing*

$$R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n))\otimes R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n)) \xrightarrow{\mathrm{Tr}\circ\cup} \mathbb{Z}(2n-d+1)[-2d+2]$$

and $\varepsilon_{dR}\cdot A(\mathcal{X})^d$ the discriminant of the deRham duality pairing (31). Then

$$P = \sqrt{\varepsilon_B\varepsilon_{dR}}\cdot (2\pi i)^{t_H(\mathcal{X},n)}\cdot A(\mathcal{X})^{\frac{d}{2}}.$$

Moreover $P\cdot(2\pi i)^{d_-(\mathcal{X},n)}$ is real and hence we have

$$P\cdot(2\pi i)^{d_-(\mathcal{X},n)} = (2\pi)^{d_-(\mathcal{X},n)+t_H(\mathcal{X},n)}\cdot A(\mathcal{X})^{\frac{d}{2}}.$$

Proof. We have a commutative diagram

$$\begin{array}{ccc}
R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n))\otimes R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n)) & \xrightarrow{\mathrm{Tr}\circ\cup} & \mathbb{Z}(2n-d+1)[-2d+2]\\
\downarrow & & \downarrow\\
R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C})\otimes R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C}) & \xrightarrow{\mathrm{Tr}\circ\cup} & \mathbb{C}[-2d+2]\\
\uparrow & & \uparrow\\
R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d\otimes^L_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d & \xrightarrow{(31)} & \mathbb{Z}[-2d+2]
\end{array}$$

where the bottom square commutes since the comparison isomorphism

$$R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C}) \simeq R\Gamma_{dR}(\mathcal{X}_{\mathbb{C}}/\mathbb{C}) \simeq R\Gamma_{dR}(\mathcal{X}_{\mathbb{Q}}/\mathbb{Q})_{\mathbb{C}}$$

is compatible with cup product and cycle classes, and the trace map sends the cycle class of a closed point to its degree over the base field. We also use the fact that the trace map in algebraic de Rham cohomology

$$H^{2d-2}_{dR}(\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}) \xleftarrow{\sim} H^{d-1}(\mathcal{X}_{\mathbb{Q}},\Omega^{d-1}_{\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}}) \xrightarrow{\mathrm{Tr}} \mathbb{Q}$$

is the base change of the Trace map (32) under $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$ by [27]. Applying the construction of Remark 3.7 we then obtain a pairing

$$\langle\cdot,\cdot\rangle : \det_{\mathbb{C}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C}) \otimes_{\mathbb{C}} \det_{\mathbb{C}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{C}) \simeq \mathbb{C}$$

which restricts to the corresponding $\mathbb{Q}$-valued pairing on $\det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d$ and a $\mathbb{Q}\cdot(2\pi i)^{(2n-d+1)\chi}$-valued pairing on $\det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n))$. Here

$$\begin{aligned}\chi :=& \operatorname{rank}_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n)) = \dim_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))\\ =&\sum_i (-1)^i \dim_{\mathbb{R}} H^i(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))\end{aligned}$$

is the Euler characteristic of the manifold $\mathcal{X}(\mathbb{C})$. We then have

$$\varepsilon_{dR}\cdot A(\mathcal{X})^d = \langle b_{dR}, b_{dR}\rangle = P^2\langle b_B, b_B\rangle = P^2\varepsilon_B(2\pi i)^{(2n-d+1)\chi}$$

and moreover

$$\begin{aligned}-(2n-d+1)\chi =& \sum_{i<d-1}(-1)^i(i-2n+2d-2-i-2n)\dim_{\mathbb{R}} H^i(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))\\ &+(-1)^{d-1}(d-1-2n)\dim_{\mathbb{R}} H^{d-1}(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))\\ =&2\,t_H(\mathcal{X},n).\end{aligned}$$

Hence

$$P^2 = \varepsilon_{dR}\varepsilon_B\cdot(2\pi i)^{2\,t_H(\mathcal{X},n)}\cdot A(\mathcal{X})^d$$

which shows the first statement. Since both b_B and $b_{dR}\cdot(2\pi i)^{d_-(\mathcal{X},n)} = P\cdot(2\pi i)^{d_-(\mathcal{X},n)}\cdot b_B$ lie in the Betti real structure, the factor $P\cdot(2\pi i)^{d_-(\mathcal{X},n)}$ is real. This proves the second statement. □

COROLLARY 5.4. *With notation as in Lemma 5.3 we have*

$$\varepsilon_B\varepsilon_{dR} = (-1)^{d_-(\mathcal{X},n)+t_H(\mathcal{X},n)} = (-1)^{d_-(\mathcal{X},0)+\frac{d-1}{2}\chi}.$$

Proof. This follows from tracking powers of i in Lemma 5.3 and the fact that ε_B and ε_{dR} are independent of n (hence so is the right hand side). □

REMARK 5.5. *Corollary 5.4 generalizes the classical fact that the sign of the discriminant of a number field F is $(-1)^{r_2}$ where $r_2 = d_-(\mathrm{Spec}(\mathcal{O}_F), 0)$ is the number of complex places. In this case $\varepsilon_B = 1$. For an example with $\varepsilon_B = -1$ consider $\mathcal{X} = \mathbb{P}^1_{\mathrm{Spec}(\mathbb{Z})}$. The intersection pairing on*

$$R\Gamma(\mathbb{P}^1(\mathbb{C}), \mathbb{Z}) \simeq R\Gamma(S^2, \mathbb{Z}) \simeq \mathbb{Z}[0] \oplus \mathbb{Z}[-2]$$

has Gram matrix

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and hence $\varepsilon_B = \det(A) = -1$. The intersection pairing on

$$R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^2 \simeq H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})[0] \oplus H^1(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathbb{Z}})[-2] \simeq \mathbb{Z}[0] \oplus \mathbb{Z}[-2]$$

has the same Gram matrix, hence $\varepsilon_{dR} = -1$. Since $d_-(\mathcal{X}, 0) = d_-(h^2(\mathcal{X})) = 1$ and $d = \chi = 2$ we indeed have

$$\varepsilon_B \varepsilon_{dR} = 1 = (-1)^{d_-(\mathcal{X},0)+\frac{d-1}{2}\chi}.$$

We can now finish the proof of Theorem 1.2 by verifying (44). By Prop. 2.23 we have

$$\tilde{b}_B \cdot (2\pi i)^{d_-(\mathcal{X},n)} = b_B \cdot 2^{d_-(\mathcal{X},n)-d_+(\mathcal{X},n)}$$

and by Corollary 3.9

$$\tilde{b}_{dR}^{-1} = A(\mathcal{X})^{d-n} \cdot b_{dR}^{-1}.$$

Therefore

$$\begin{aligned} x_\infty(\mathcal{X}, n)^{-2} =& \tilde{b}_B \cdot \tilde{b}_{dR}^{-1} \\ =& (2\pi i)^{-d_-(\mathcal{X},n)} \cdot 2^{d_-(\mathcal{X},n)-d_+(\mathcal{X},n)} \cdot b_B \cdot A(\mathcal{X})^{d-n} \cdot b_{dR}^{-1} \\ =& 2^{d_-(\mathcal{X},n)-d_+(\mathcal{X},n)} \cdot A(\mathcal{X})^{d-n} \cdot (2\pi i)^{-d_-(\mathcal{X},n)} \cdot P^{-1} \\ =& 2^{d_-(\mathcal{X},n)-d_+(\mathcal{X},n)} \cdot A(\mathcal{X})^{d-n} \cdot (2\pi)^{-d_-(\mathcal{X},n)-t_H(\mathcal{X},n)} \cdot A(\mathcal{X})^{-\frac{d}{2}} \\ =& A(\mathcal{X})^{\frac{d}{2}-n} \cdot 2^{d_-(\mathcal{X},n)-d_+(\mathcal{X},n)} \cdot (2\pi)^{-d_-(\mathcal{X},n)-t_H(\mathcal{X},n)} \end{aligned}$$

which is (44). □

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