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A penalization method to take into account obstacles in incompressible viscous flows

Philippe Angot[1], **Charles-Henri Bruneau**[2], **Pierre Fabrie**[2]

[1] IRPHE, CNRS UMR 6594, Equipe Mathématiques Numériques pour la Modélisation, La Jetée – Technopôle de Château-Gombert, F-13451 Marseille, France
[2] Mathématiques Appliquées de Bordeaux, CNRS et Université Bordeaux 1, 351, cours de la Libération, F-33405 Talence, France

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Summary. From the Navier-Stokes/Brinkman model, a penalization method has been derived by several authors to compute incompressible Navier-Stokes equations around obstacles. In this paper, convergence theorems and error estimates are derived for two kinds of penalization. The first one corresponds to a L^2 penalization inducing a Darcy equation in the solid body, the second one corresponds to a H^1 penalization and induces a Brinkman equation in the body. Numerical tests are performed to confirm the efficiency and accuracy of the method.

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1. Introduction and setting of the problem

About fifteen years ago, there were several attempts to penalize the no-slip boundary condition on the surface of an obstacle surrounded by a fluid. The aim was to avoid body-fitted unstructured meshes in order to use fast and efficient spectral, finite differences or finite volumes approximations on cartesian meshes. A way to do that is to add a penalized velocity term in the momentum equation of the incompressible Navier-Stokes equations. Following the former work of Peskin [12], [13], several authors (for instance [9]) add both a time integral of the velocity and a velocity penalization term only at the points defining the surface of the obstacle. It appears that the penalization has to be extended to the volume of the body to give correct physical solutions at high Reynolds numbers [14]. In independent works, Arquis and Caltagirone [4] add a penalization term on the velocity defined on the volume of a porous body. This corresponds to a Brinkman type model with variable permeability where the fluid domain has a very large permeability in front of the one of the porous medium. This model was generalized later in [1] [2] to deal with fluid-porous-solid systems. In a more recent work [8], it is suggested that this model allows to compute the drag and lift coefficients by integrating the penalization term inside the solid body. Various works use the same methodology to compute incompressible flows around a cylinder or behind a step [3], [5], [6], [10].

The aim of this paper is to establish rigourous estimates of the error induced by such penalizations and to show numerically the efficiency of the method. In Sect. 3, we give the formal asymptotic expansion of the approximate solution with respect to the penalization coefficient. Then Sects. 4 and 5 are devoted to the analysis of the proposed L^2 and H^1 penalization respectively. The L^2 penalization consists in adding a damping term on the velocity in the momentum equation whereas the H^1 penalization includes in addition a perturbation of both the time derivative and the viscous term. The numerical tests to validate the mathematical modelling are presented in the last section.

2. Preliminaries and notations

Let Ω be a regular bounded connected open set in $\mathbb{R}^2$, we assume that Ω contains I regular obstacles $\Omega_{\rm s}^i\ ,\ 1 \leq i \leq I$ (see Fig. 1).

We set

$$\Omega_{\rm s} = \bigcup_{i=1}^{I} \Omega_{\rm s}^i\ , \quad \Omega_{\rm f} = \Omega \backslash \overline{\Omega}_{\rm s}$$
$$\Sigma_{\rm s}^i = \partial \Omega_{\rm s}^i\ , \quad \Gamma = \partial \Omega$$

where $\Omega_{\rm f}$ is the incompressible fluid domain in which the Navier-Stokes equations are prescribed. In the physical case, the motion is given by imposing a non homogeneous Dirichlet boundary condition on Γ, whereas, for sake of simplicity in the mathematical study we assume that the motion, is imposed by an external source term. Consequently, we take an homogeneous Dirichlet boundary condition for the velocity on Γ.

So we are looking for the solution of the following initial boundary value problem:

$$(2.1)\qquad \begin{array}{ll} \partial_t u_{\rm f} - \dfrac{1}{Re} \Delta u_{\rm f} + u_{\rm f} \cdot \nabla u_{\rm f} + \nabla p_{\rm f} = f & \text{in } \mathbb{R}^+ \times \Omega_{\rm f} \\ \text{div } u_{\rm f} = 0 & \text{in } \mathbb{R}^+ \times \Omega_{\rm f} \\ u_{\rm f}(0, \cdot) = u_{f\,0} & \text{in } \Omega_{\rm f} \\ u_{\rm f} = 0 & \text{on } \partial \Omega_{\rm f}. \end{array}$$

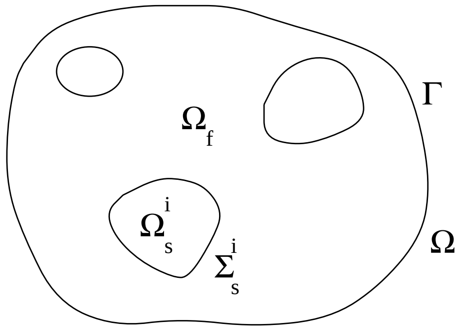

Fig. 1. Domain

The first equation can be written in terms of the stress tensor $\sigma(u,p) = \frac{1}{2\,Re}(\nabla u + \nabla u^t) - pI)$ as:

$$
(2.2)\qquad
\begin{array}{ll}
\partial_t u_{\rm f} + u_{\rm f}\cdot\nabla u_{\rm f} - \mathrm{div}\ \sigma(u_{\rm f},p_{\rm f}) = f & \text{in } \mathbb{R}^+ \times \Omega_{\rm f}\\
\mathrm{div}\ u_{\rm f} = 0 & \text{in } \mathbb{R}^+ \times \Omega_{\rm f}\\
u_{\rm f}(0,\cdot) = u_{f\,0} & \text{in } \Omega_{\rm f}\\
u_{\rm f} = 0 & \text{on } \partial\Omega_{\rm f}.
\end{array}
$$

This form is used in Sect. 3 for the H^1 penalization.
Now, let us introduce the following functional spaces:

$$
\begin{array}{l}
\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^2\\
\boldsymbol{H}^1(\Omega) = \{u \in \boldsymbol{L}^2(\Omega)\ ;\ \nabla u \in (\boldsymbol{L}^2(\Omega))^4\}\\
\boldsymbol{H}^2(\Omega) = \{u \in \boldsymbol{L}^2(\Omega)\ ;\dfrac{\partial^2 u}{\partial x_i\,\partial x_j} \in \boldsymbol{L}^2(\Omega)\ ,\ 1 \le i \le j \le 2\}\\
\boldsymbol{H}^1_0(\Omega) = \{u \in \boldsymbol{H}^1(\Omega)\ ;\ u = 0 \text{ on } \partial\Omega\}\\
\boldsymbol{H} = \{u \in \boldsymbol{L}^2\ ;\ \mathrm{div}\ u = 0\ ;\ u.n = 0 \text{ on } \partial\Omega\}\\
\boldsymbol{V} = \{u \in \boldsymbol{H}^1_0(\Omega)\ ;\ \mathrm{div}\ u = 0\}.
\end{array}
$$

at last, we denote $\mathbf{1}_{\Omega_{\rm s}}$ the function:

$$
\begin{array}{l}
\mathbf{1}_{\Omega_{\rm s}}(x) = 1\ \text{ if } x \in \overline{\Omega}_{\rm s}\\
\mathbf{1}_{\Omega_{\rm s}}(x) = 0\ \text{ if } x \in \Omega_{\rm f}
\end{array}
$$

3. Two models to penalize the Navier-Stokes equations

Instead of solving the problem (2.1) on $\Omega_{\rm f}$, we solve an equivalent problem on the whole domain Ω by penalizing the obstacles. The linear case is studied in [3]

3.1. The $\boldsymbol{L^2}$ penalization

The first idea is to force the velocity to be small in Ω_{s} by solving:

$$\text{(3.1)}\quad \begin{array}{ll} \partial_t u_\eta - \dfrac{1}{Re}\Delta u_\eta + u_\eta \cdot \nabla u_\eta + \dfrac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}} u_\eta + \nabla p_\eta = f & \text{in } \mathbb{R}^+ \times \Omega \\ \operatorname{div}\, u_\eta = 0 & \text{in } \mathbb{R}^+ \times \Omega \\ u_\eta(0,\cdot) = u_0 & \text{in } \Omega \\ u_\eta = 0 & \text{on } \Gamma. \end{array}$$

We set $u_\eta = u + \eta\tilde{u}$ and $p_\eta = p + \eta\tilde{p}$ to derive formally the equations satisfied by $u,\ p$ and $\tilde{u},\ \tilde{p}$. Then we get from the first equation in (3.1) by identifying the terms of same order:

$$\text{(3.2)}\quad \mathbf{1}_{\Omega_{\mathrm{s}}} u = 0$$

$$\text{(3.3)}\quad \partial_t u - \tfrac{1}{Re}\Delta u + u \cdot \nabla u + \mathbf{1}_{\Omega_{\mathrm{s}}}\tilde{u} + \nabla p = f$$

$$\text{(3.4)}\quad \partial_t \tilde{u} - \tfrac{1}{Re}\Delta \tilde{u} + u \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u + \nabla \tilde{p} = 0.$$

Thus by (3.2) u vanishes in Ω_{s}. Consequently u satisfies equation (2.1) in Ω_{f} and $\tilde{u}$ checks:

$$\text{(3.5)}\quad \begin{array}{ll} \tilde{u}_{\mathrm{s}} + \nabla p_{\mathrm{s}} = 0 & \text{in } \Omega_{\mathrm{s}} \\ \operatorname{div}\, \tilde{u} = 0 & \text{in } \Omega_{\mathrm{s}} \end{array}$$

and

$$\text{(3.6)}\quad \begin{array}{ll} \partial_t \tilde{u}_{\mathrm{f}} - \tfrac{1}{Re}\Delta \tilde{u}_{\mathrm{f}} + u_{\mathrm{f}} \cdot \nabla \tilde{u}_{\mathrm{f}} + \tilde{u}_{\mathrm{f}} \cdot \nabla u_{\mathrm{f}} + \nabla \tilde{p}_{\mathrm{f}} = 0 & \text{in } \Omega_{\mathrm{f}} \\ \operatorname{div}\, \tilde{u}_{\mathrm{f}} = 0 & \text{in } \Omega_{\mathrm{f}}. \end{array}$$

Let us remark that $u,\ \tilde{u}$ and $p,\ \tilde{p}$ are continuous on each Σ_{s}^i. Hence p_{s} is given by:

$$\begin{array}{ll} \Delta p_{\mathrm{s}} = 0 & \text{in } \Omega_{\mathrm{s}}^i\ 1 \le i \le I \\ p_{\mathrm{s}} = p_{\mathrm{f}} & \text{on } \Sigma_{\mathrm{s}}^i \end{array}$$

which yields $\tilde{u}_{\mathrm{s}}$ in Ω_{s} by (3.5).
Then $\tilde{u}_{\mathrm{f}}$ is completly determined by adding:

$$\begin{array}{ll} \tilde{u}_{\mathrm{f}} = \tilde{u}_{\mathrm{s}} & \text{on } \Sigma_{\mathrm{s}}^i\ 1 \le i \le I \\ \text{and} & \\ \tilde{u}_{\mathrm{f}}(0,\cdot) = 0 & \end{array}$$

to (3.6).

Remark 3.1 In Ω_{s}, $\tilde{u}$ verifies a Darcy type law associated to a Neumann boundary condition on the pressure. Thus the obstacles are associated to porous media.

3.2. The $\boldsymbol{H^1}$ penalization

In this section we choose to penalize in addition the whole linear part of the Navier-Stokes equations. So according to (2.2) we set:

$$
\begin{aligned}
&\Big(1+\frac{1}{\eta}\mathbf{1}_{\Omega_\mathrm{s}}\Big)\partial_t u_\eta + u_\eta\cdot\nabla u_\eta \\
&-\mathrm{div}\ \Big((1+\tfrac{1}{\eta}\mathbf{1}_{\Omega_\mathrm{s}})\sigma(u_\eta,p_\eta)\Big)+\tfrac{1}{\eta}\mathbf{1}_{\Omega_\mathrm{s}}u_\eta = f \quad \text{in } \mathbb{R}^+\times\Omega \\
&\mathrm{div}\ u_\eta = 0 \quad \text{in } \mathbb{R}^+\times\Omega \\
&u_\eta(0,\cdot)=u_0 \quad \text{in } \Omega \\
&u_\eta = 0 \quad \text{on } \Gamma
\end{aligned} \tag{3.7}
$$

From the expansion form of u_η and p_η, $u_\eta = u+\eta\tilde{u}$ and $p_\eta = p+\eta\tilde{p}$, we get by identification of the terms of same order:

$$
\mathbf{1}_{\Omega_\mathrm{s}}\partial_t u - \mathrm{div}\ (\mathbf{1}_{\Omega_\mathrm{s}}\sigma(u,p)) + \mathbf{1}_{\Omega_\mathrm{s}}u = 0, \tag{3.8}
$$

$$
\begin{aligned}
&\partial_t u + u\nabla u - \mathrm{div}\ \sigma(u,p) + \mathbf{1}_{\Omega_\mathrm{s}}\partial_t\tilde{u} \\
&\quad -\mathrm{div}\ (\mathbf{1}_{\Omega_\mathrm{s}}\sigma(\tilde{u},\tilde{p})) + \mathbf{1}_{\Omega_\mathrm{s}}\tilde{u} = f,
\end{aligned} \tag{3.9}
$$

$$
\partial_t\tilde{u} + u\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla u - \mathrm{div}\ \sigma(\tilde{u},\tilde{p}) = 0. \tag{3.10}
$$

Closing equation (3.8) in Ω_s with the natural boundary condition

$$
\sigma(u_\mathrm{s},p_\mathrm{s})\cdot n_\mathrm{s}^i = 0 \text{ on } \Sigma_\mathrm{s}^i \ 1\le i\le I
$$

where n_s^i denotes the unit normal vector pointing inside of Ω_s^i, we find that $u_\mathrm{s}\equiv 0$ on Ω_s thanks to the damping term. Then by (3.9) u satisfies equation (2.2) in Ω_f and by (3.9) and (3.10) $\tilde{u}$ cheks:

$$
\begin{aligned}
&\partial_t\tilde{u}_\mathrm{s} - \mathrm{div}\ \sigma(\tilde{u}_\mathrm{s},\tilde{p}_\mathrm{s}) + \tilde{u}_\mathrm{s} = 0 \quad \text{in } \mathbb{R}^+\times\Omega_\mathrm{s} \\
&\mathrm{div}\ \tilde{u}_\mathrm{s} = 0 \quad \text{in } \mathbb{R}^+\times\Omega_\mathrm{s}
\end{aligned} \tag{3.11}
$$

and

$$
\begin{aligned}
&\partial_t\tilde{u}_\mathrm{f} + u_\mathrm{f}\cdot\nabla\tilde{u}_\mathrm{f} + \tilde{u}_\mathrm{f}\cdot\nabla u_\mathrm{f} - \mathrm{div}\ \sigma(\tilde{u}_\mathrm{f},\tilde{p}_\mathrm{f}) = 0 \quad \text{in } \mathbb{R}^+\times\Omega_\mathrm{f} \\
&\mathrm{div}\ \tilde{u}_\mathrm{f} = 0 \quad \text{in } \mathbb{R}^+\times\Omega_\mathrm{f}
\end{aligned} \tag{3.12}
$$

Now, as in the previous subsection, the continuity of $(u,\tilde{u})$ and $(p,\tilde{p})$ on each Σ_s^i allows to close the problem by adding:

$$
\begin{aligned}
&\sigma(\tilde{u}_\mathrm{s},\tilde{p}_\mathrm{s})\cdot n_\mathrm{s}^i = \sigma(u_\mathrm{f},p_\mathrm{f})\cdot n_\mathrm{s}^i \quad \text{on } \mathbb{R}^+\times\Sigma_\mathrm{s}^i\ ,\ 1\le i\le I \\
&\tilde{u}_\mathrm{s}(0,\cdot) = 0 \quad \text{in } \Omega_\mathrm{s}
\end{aligned}
$$

to (3.11) and

$$\tilde{u}_{\mathrm{f}} = \tilde{u}_{\mathrm{s}} \qquad \text{on } \mathbb{R}^+ \times \Sigma_{\mathrm{s}}^i\ ,\ 1 \le i \le I$$

$$\tilde{u}_{\mathrm{f}}(0,\cdot) = 0 \ \text{ in } \Omega_{\mathrm{f}}$$

to (3.12).

Remark 3.2 In Ω_{s}, $\tilde{u}$ satisfies a Brinkman type equation associated to a Neumann type condition for the stress tensor. Once again the obstacles can be viewed as sparse porous media.

Remark 3.3 Instead of (3.7) we study a similar problem given by :

$$(3.13)\quad \begin{aligned}
&\left(1+\frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}\right)\partial_t u_\eta - \frac{1}{Re}\mathrm{div}\ \left((1+\frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}})\nabla u_\eta\right) \\
&+u_\eta\cdot\nabla u_\eta + \frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}u_\eta + \nabla p_\eta = f && \text{in } \mathbb{R}^+\times\Omega \\
&\mathrm{div}\ u_\eta = 0 && \text{in } \mathbb{R}^+\times\Omega \\
&u_\eta(0,\cdot) = u_0 && \text{in } \Omega \\
&u_\eta = 0 && \text{on } \Gamma.
\end{aligned}$$

The mathematical analysis of (3.7) can be achieved through a mixed formulation as in [7]

4. The L^2 penalization

In this section we study the behaviour of the solution u_η of problem (3.1) when η goes to zero.

$$(4.1)\quad \begin{aligned}
&\partial_t u_\eta - \frac{1}{Re}\Delta u_\eta + u_\eta\cdot\nabla u_\eta + \frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}u_\eta + \nabla p_\eta = f && \text{in } \mathbb{R}^+\times\Omega \\
&\mathrm{div}\ u_\eta = 0 && \text{in } \mathbb{R}^+\times\Omega \\
&u_\eta(0,\cdot) = u_0 && \text{in } \Omega \\
&u_\eta = 0 && \text{on } \Gamma
\end{aligned}$$

where f is a given function in $L^\infty(\mathbb{R}^+; L^2(\Omega))$ which support is included in Ω_{f}. As it is well known, for η given there exists an unique solution u_η of (4.1) satisfying:

$$u_\eta \in C^0(\mathbb{R}^+; \boldsymbol{H}) \cap L^2_{loc}(\mathbb{R}^+; \boldsymbol{V})$$

$$\partial_t u_\eta \in L^2_{loc}(\mathbb{R}^+; \boldsymbol{V}')$$

Remark 4.1 Note that this solution cannot be more regular in space as $\mathbf{1}_{\Omega_s}$ is a discontinuous function.

As the results of this section are derived from energy estimates, we recall the weak formulation of (4.1), for any $\varphi \in \boldsymbol{V}$

$$\begin{aligned} &\langle \partial_t u_\eta \,,\, \varphi \rangle_{\boldsymbol{V}',\boldsymbol{V}} + \frac{1}{Re}\int_\Omega \nabla u_\eta \cdot \nabla\varphi dx + \int_\Omega u_\eta \cdot \nabla u_\eta \varphi dx \\ &+\frac{1}{\eta}\int_\Omega \mathbf{1}_{\Omega_s} u_\eta \varphi dx = \int_\Omega f\varphi dx. \\ &u_\eta(0,\cdot) = 0 \end{aligned} \tag{4.2}$$

Remark 4.2 It is allowed to extend (4.1) to test functions φ in $\boldsymbol{W}$ where:

$$\boldsymbol{W} = \left\{\varphi \in L^2(0,T;\boldsymbol{V})\ ;\ \partial_t\varphi \in L^2(0,T;\boldsymbol{V}')\ ;\ \varphi(T) = 0\right\}$$

4.1. Convergence result

In order to get a convergence result we need the following *a priori* estimates:

Lemma 4.1

$$\sup_{t\in\mathbb{R}^+} |u_\eta|^2 \le |u_0|^2 + \frac{Re^2}{\lambda_1}\sup_{t\in\mathbb{R}^+} \|f\|_{-1}^2 \tag{4.3}$$

$$\int_0^t |\nabla u_\eta(\tau)|^2 d\tau \le |u_0|^2 + Re^2 \sup_{t\in\mathbb{R}^+} \|f\|_{-1}^2\, t \tag{4.4}$$

where $\|f\|_{-1}$ denotes the norm in $\boldsymbol{H}^{-1}(\Omega)$.

Proof. From (4.2) with $\varphi = u_\eta$ we get:

$$\frac{1}{2}\frac{d}{dt}|u_\eta(t)|^2 + \frac{1}{Re}|\nabla u_\eta(t)|^2 + \frac{1}{\eta}\int_\Omega \mathbf{1}_{\Omega_s}(x)u_\eta^2(t,x)dx \le \|f\|_{-1}|\nabla u_\eta(t)|,$$

so,

$$\begin{aligned} &\frac{d}{dt}|u_\eta(t)|^2 + \frac{1}{Re}|\nabla u_\eta(t)|^2 + \frac{2}{\eta}\int_\Omega \mathbf{1}_{\Omega_s}(x)u_\eta^2(t,x)dx \\ &\le \frac{1}{Re}\|f\|_{-1}^2, \end{aligned} \tag{4.5}$$

that gives after elementary computations (4.3), (4.4). □

From (4.5) we deduce directly:

Corollary 4.1

$$\text{(4.6)}\quad \frac{2}{\eta}\int_0^t\int_\Omega \mathbf{1}_{\Omega_s}(x)u_\eta^2(\tau,x)dxd\tau \le |u_0|^2 + Re\sup_{t\in\mathbb{R}^+}\|f(t)\|_{-1}^2\, t.$$

As we are not able to derive an estimate on $\partial_t u_\eta$ in $\boldsymbol{V}'$, we choose to evaluate a fractional derivative in time of u_η to obtain compactness.

Lemma 4.2 *There exists a generic function g in $C^0(\mathbb{R}^+)$ independent of η such that:*

$$\text{(4.7)}\qquad |\partial_t^\gamma u_\eta(t)|_{L^2(0,t;\Omega)} \le g(t),\quad \forall\gamma\le\frac{1}{4}$$

Sketch of proof: As the term $\frac{1}{\eta}\mathbf{1}_{\Omega_s}u_\eta$ is a damping term, the ideas developed in [11] , [16] can be applied (see also [7]).

Following the idea of [7], we introduce the Hilbert space:

$$\boldsymbol{W} = \left\{\varphi\in L^2(0,T;\boldsymbol{V})\ ;\ \partial_t\varphi\in L^2(0,T;\boldsymbol{V}')\ ;\ \varphi(T)=0\right\}$$

to obtain:

Lemma 4.3 *There exists a generic function g in $C^0(\mathbb{R}^+)$ depending on the data such that:*

$$\text{(4.8)}\qquad \frac{1}{\eta}\left|\int_0^t\int_\Omega \mathbf{1}_{\Omega_s}(x)u_\eta(\tau,x)\varphi(\tau,x)dxd\tau\right| \le g(t)\,\|\varphi\|_{\boldsymbol{W}}$$

Proof. Integrating by part in time the first term of (4.2) we get for $\varphi\in\boldsymbol{W}$:

$$\begin{aligned}\frac{1}{\eta}\int_0^T \mathbf{1}_{\Omega_s}(x)u_\eta(t,x)\varphi(t,x)dxdt &= \int_0^T\langle u_\eta(t),\partial_t\varphi\rangle_{\boldsymbol{V},\mathbf{V}'}dt\\ &-\frac{1}{Re}\int_0^T\int_\Omega\nabla u_\eta(t,x)\cdot\nabla\varphi(t,x)dxdt\\ &-\int_0^T\int_\Omega u_\eta(t,x)\cdot\nabla u_\eta(t,x)\varphi(t,x)dxdt\\ &+\int_0^T\int_\Omega f(t,x)\varphi(t,x)dxdt+\int_\Omega u_0(x)\varphi(0,x)dx.\end{aligned}$$

The estimate (4.8) comes from straightforward majorations using (4.3) and (4.4). □

Finally, to show that the limit u of the sequence $(u_\eta)_\eta$ is solution of homogeneous Navier-Stokes equations in Ω_f we need the last result:

Lemma 4.4 *There exists a generic function g in $C^0(\mathbb{R}^+)$ depending on the data such that*

$$|u_\eta|_{L^2(0,t;L^2(\Sigma_{\mathrm{s}}^i))} \leq g(t)\eta^{\frac{1}{4}},\ \forall i\ 1 \leq i \leq I \tag{4.9}$$

Proof. As u_η belongs to $L^2(0,T;\boldsymbol{V})$ the trace of u_η is well defined on each boundary Σ_{s}^i and we have the following inequality where c denotes always a generic constant.

$$|u_\eta(t)|_{L^2(\Omega_{\mathrm{s}}^i)} \leq c|u_\eta(t)|^{\frac{1}{2}}_{L^2(\Omega_{\mathrm{s}}^i)}\|u_\eta(t)\|^{\frac{1}{2}}_{H^1(\Omega_{\mathrm{s}}^i)}. \tag{4.10}$$

We first bound the H^1 norm on Ω_{s}^i by the H^1 norm on Ω ; then using (4.3) and (4.4) we get:

$$\int_0^t |u_\eta(\tau)|^2_{L^2(\Sigma_{\mathrm{s}}^i)}d\tau \leq g(t)\int_0^t |u_\eta(\tau)|_{L^2(\Sigma_{\mathrm{s}}^i)}d\tau$$

$$\leq g(t)\left\{\int_0^t |u_\eta(\tau)|^2_{L^2(\Sigma_{\mathrm{s}}^i)}d\tau\right\}^{\frac{1}{2}},$$

which gives with (4.10) and (4.6)

$$\int_0^t |u_\eta(\tau)|^2_{L^2(\Sigma_{\mathrm{s}}^i)}d\tau \leq g(t)\eta^{\frac{1}{2}}\ ,\ \forall t \in \mathbb{R}^+,$$

where g is a generic function. □

Therefore we can show:

Theorem 4.1 *When η goes to zero, the sequence $(u_\eta)_\eta$ converges to a limit u which satisfies:*

$$u_{|\Omega_{\mathrm{s}}} = 0,$$

and $u_{|\Omega_{\mathrm{f}}}$ is the unique weak solution of Navier-Stokes equation in Ω_{f}. Moreover there exists $h \in \mathbf{W}'$ such that:

$$\frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}u_\eta \rightharpoonup h \text{ in } \boldsymbol{W}' \text{ weakly}$$

and

$$\begin{aligned}&-\int_0^t \langle u(\tau), \partial_t\varphi(\tau)\rangle_{\mathbf{V},\mathbf{V}'}d\tau + \int_0^t\int_\Omega \nabla u(\tau,x)\cdot\nabla\varphi(\tau,x)dxd\tau\\ &+\int_0^t\int_\Omega u(\tau,x)\cdot\nabla u(\tau,x)\varphi(\tau,x)dxd\tau + \langle h,\varphi\rangle_{\boldsymbol{W},\boldsymbol{W}'}\\ &= \int_0^t\int_\Omega f(\tau,x)\varphi(\tau,x)dxd\tau + \int_\Omega u_0(x)\varphi(0,x)dx\end{aligned} \tag{4.11}$$

Proof. From (4.3), (4.4) and (4.7) we get by a compactness result [16],

$$\begin{array}{l} u_\eta \to u \text{ in } L^2(0,T;\boldsymbol{H}) \text{ strong} \\ u_\eta \to u \text{ in } L^2(0,T;\boldsymbol{V}) \text{ weak} \\ u_\eta \rightharpoonup u \text{ in } L^\infty(0,T;\boldsymbol{H}) \text{ weak}\star \end{array}$$

for a subsequence still denoted $(u_\eta)_\eta$.
Moreover from (4.8) we have

$$\frac{1}{\eta}\mathbf{1}_{\Omega_{\rm s}} u_\eta \rightharpoonup h \text{ in } \boldsymbol{W}'.$$

At last, from (4.9)

$$u = 0 \text{ on } \mathbb{R}^+ \times \Sigma_{\rm s}^i.$$

As h is the weak limit of $\frac{1}{\eta}\mathbf{1}_{\Omega_{\rm s}} u_\eta$, one has for any φ in $\boldsymbol{W}$ such that supp $\varphi(t,\cdot) \subset \Omega_{\rm f}$

$$\langle h, \varphi \rangle = 0.$$

Then (4.11) is straightforward and from the remark above u is the weak solution of Navier-Stokes equation in $\Omega_{\rm f}$. Finally, from Corollary 4.1 $u = 0$ in $\Omega_{\rm s}$ and by uniqueness of $u_{|\Omega_{\rm f}}$ the whole sequence $(u_{\eta|\Omega_{\rm f}})_\eta$ converges. □

4.2. Error estimate

Now assuming that the solution $u_{\rm f}$ of Navier-Stokes equations in $\Omega_{\rm f}$ is regular enough, we can derive an error estimate.

So, for $u_{\rm f} \in L^\infty(0,T,\boldsymbol{H}_0^1(\Omega_{\rm f})) \cap L^2(0,T;\boldsymbol{H}^2(\Omega_{\rm f}))$ (which is true as soon as $u_{f\,0} \in \boldsymbol{H}_0^1(\Omega_{\rm f})$), let us define u by:

$$\begin{array}{ll} u = u_{\rm f} & \text{in } \Omega_{\rm f} \\ u = 0 & \text{in } \Omega_{\rm s}. \end{array}$$

It is obvious that:

$$u \in L^\infty(0,T;\boldsymbol{H}_0^1(\Omega)) \cap L^2(0,T;\{\boldsymbol{H}^2(\Omega_{\rm f}) \cap \boldsymbol{H}^2(\Omega_{\rm s})\})$$

and that

$$\text{div } u = 0 \text{ in } \mathbb{R}^+ \times \Omega.$$

So, we get for $\varphi \in \boldsymbol{V}$,

$$0 = \int_{\Omega_{\rm f}} \left(\partial_t u - \frac{1}{Re}\Delta u + u\cdot\nabla u + \nabla p - f \right) \varphi dx,$$

and after integration by parts

$$0 = \int_{\Omega_\mathrm{f}} \left(\partial_t u \varphi + \frac{1}{Re} \nabla u \cdot \nabla \varphi + u \cdot \nabla u \varphi - f \varphi \right) dx$$

$$- \sum_{i=1}^{I} \int_{\Sigma_\mathrm{s}^i} \sigma(u,p) n_\mathrm{f}^i \varphi d\gamma \tag{4.12}$$

where n_f^i is the outward unit normal vector to Ω_f on Σ_s^i.

In addition, as the 2D weak solution of Navier-Stokes equations is unique, the weak limit u obtained in the previous theorem is identical to u_f in Ω_f as soon as $u_{|\Omega_\mathrm{f}}$ belongs to $\boldsymbol{H}_0^1(\Omega_\mathrm{f})$ and is divergence free in Ω_f.

Theorem 4.2 *Let*

$$u_\mathrm{f} \in L^\infty(0,T;\boldsymbol{H}_0^1(\Omega_\mathrm{f})) \cap L^2(0,T;\boldsymbol{H}^2(\Omega_\mathrm{f}))$$

be the solution of Navier-Stokes equations in Ω_f and u defined as:

$$u = u_\mathrm{f} \quad \textit{in } \Omega_\mathrm{f}$$
$$u = 0 \quad \textit{in } \Omega_\mathrm{s}.$$

Then there exists v_η bounded in $L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$ such that:

$$u_\eta = u + \eta^{\frac{1}{4}} v_\eta.$$

Moreover there exists a generic function g in $C^0(\mathbb{R}^+)$ such that:

$$|v_\eta|_{L^2(0,t;\boldsymbol{L}^2(\Omega_\mathrm{s}))} \leq g(t) \eta^{\frac{1}{2}}.$$

Proof. To get the error estimate above we seek u_η on the following *a priori* form $u_\eta = u + \eta w_\eta$ and get from (4.2) as $\mathbf{1}_{\Omega_\mathrm{s}} u = 0$, for $\varphi \in \boldsymbol{V}$

$$\int_\Omega \left(\partial_t u \varphi + \frac{1}{Re} \nabla u \cdot \nabla \varphi + u \cdot \nabla u \varphi - f \varphi \right) dx + \int_\Omega \mathbf{1}_{\Omega_\mathrm{s}} w_\eta \varphi dx$$

$$+\eta \int_\Omega \left(\partial_t w_\eta \varphi + \frac{1}{Re} \nabla w_\eta \cdot \nabla \varphi + u \cdot \nabla w_\eta \varphi + w_\eta \cdot \nabla u \varphi \right) dx$$

$$+\eta^2 \int_\Omega w_\eta \cdot \nabla w_\eta \varphi dx = 0.$$

According to (4.12) this reduce, after dividing by η to:

$$\int_{\Omega}\left(\partial_t w_\eta \varphi + \frac{1}{Re}\nabla w_\eta \cdot \nabla\varphi + u\cdot\nabla w_\eta\varphi + w_\eta\cdot\nabla u\varphi\right)dx$$

$$+\frac{1}{\eta}\int_{\Omega}\mathbf{1}_{\Omega_{\mathrm{s}}} w_\eta\varphi dx + \eta\int_{\Omega} w_n\cdot\nabla w_\eta\varphi dx \tag{4.13}$$

$$= \frac{1}{\eta}\sum_{i=1}^{I}\int_{\Sigma_{\mathrm{s}}^i}\sigma(u,p)\cdot n_{\mathrm{f}}^i w_\eta d\gamma.$$

For fixed η , w_η is regular enough to take $\varphi = w_\eta$ in (4.13). So we get the following energy estimate:

$$\frac{1}{2}\frac{d}{dt}|w_\eta|^2 + \frac{1}{Re}|\nabla w_\eta|^2 + \frac{1}{\eta}\int_{\Omega}\mathbf{1}_{\Omega_{\mathrm{s}}} w_\eta^2 dx$$

$$\le \frac{1}{\eta}\sum_{i=1}^{I}\left|\int_{\Sigma_{\mathrm{s}}^i}\sigma(u,p)\cdot n_{\mathrm{f}}^i w_\eta d\gamma\right| + \left|\int_{\Omega} w_\eta\cdot\nabla u w_\eta dx\right|$$
$$\le \frac{c}{\eta}\sum_{i=1}^{I}|w_\eta|_{L^2(\Sigma_{\mathrm{s}}^i)} + c\,|w_\eta|\,|\nabla w_\eta|$$
$$\le \frac{c}{\eta}\sum_{i=1}^{I}|w_\eta|_{L^2(\Omega_{\mathrm{s}}^i)}^{\frac{1}{2}}\,|\nabla w_\eta|_{L^2(\Omega_{\mathrm{s}}^i)}^{\frac{1}{2}} + c\,|w_\eta|\,|\nabla w_\eta|\,,$$

as in the proof of Lemma 4.4.

Then, using Young inequality to absorb the gradient terms, one gets:

$$\frac{d}{dt}|w_\eta|^2 + \frac{1}{Re}|\nabla w_\eta|^2 + \frac{2}{\eta}\int_{\Omega}\mathbf{1}_{\Omega_{\mathrm{s}}} w_\eta^2 dx$$

$$\le c\,|w_\eta|^2 + c\left(\frac{1}{\eta}\,|w_\eta|_{L^2(\Omega_{\mathrm{s}})}^{\frac{1}{2}}\right)^{\frac{4}{3}},$$

and absorbing again $|w_\eta|_{L^2(\Omega_{\mathrm{s}})}$ by the third term of the left hand side, one has:

$$\frac{d}{dt}|w_\eta|^2 + \frac{1}{Re}|\nabla w_\eta|^2 + \frac{2}{\eta}\int_{\Omega}\mathbf{1}_{\Omega_{\mathrm{s}}} w_\eta^2 dx$$

$$\le c\,|w_\eta|^2 + c\eta^{-\frac{3}{2}}$$

Then, by Gronwall lemma, we get for $w_\eta(0) = 0$ as we choose $u_\eta(0) = u(0)$

$$|w_\eta(t)|^2 \leq g(t)\eta^{-\frac{3}{2}},$$

and

$$\frac{1}{Re}\int_0^t |\nabla w_\eta(\tau)|^2\, d\tau + \frac{1}{\eta}\int_0^t \int_\Omega \mathbf{1}_{\Omega_{\mathrm{s}}}(x) w_\eta^2(\tau,x)dxd\tau \leq g(t)\eta^{-\frac{3}{2}},$$

where g is a generic continuous function on $\mathbb{R}^+$. This gives the result for $w_\eta = \frac{u-u_\eta}{\eta}$. □

4.3. *Interpretation of* h

For weak solutions given by (4.11), we are not able to give a rigorous interpretation of the weak limit h of $(\frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}u_\eta)_\eta$. Nevertheless, for regular solution in time, we can derive an explicit formula.
Following [15], we can show, under the compatibility condition:

$$\begin{aligned}-\frac{1}{Re}\Delta u_\eta(0) + u_\eta(0)\cdot\nabla u_\eta(0) + \frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}u_\eta(0)\\ +\nabla p_\eta(0) - f(0) \quad \text{is bounded in } \boldsymbol{L}^2(\Omega)\end{aligned} \tag{4.14}$$

and under the regularity assumption:

$$\partial_t f \in L^\infty(\mathbb{R}^+;\boldsymbol{L}^2(\Omega)) \tag{4.15}$$

that $\partial_t u_\eta$ is bounded in $L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$.
Thus h belongs to $L^\infty(0,T;\boldsymbol{V}')$.

Remark 4.3 The condition (4.14) is fulfilled as soon as $u_\eta(0)_{|\Omega_{\mathrm{s}}} = 0$ and $u_\eta(0) \in \boldsymbol{H}^2(\Omega)$ which is physically relevant.

In this case, instead of (4.11) the limit u satisfies:

$$\begin{aligned}&\int_\Omega \partial_t u\varphi dx + \frac{1}{Re}\int_\Omega \nabla u\cdot\nabla\varphi\\ &+\int_\Omega u\cdot\nabla u\varphi dx + \langle h(t),\varphi(t)\rangle_{\mathbf{V}',\boldsymbol{V}} = \int_\Omega f\varphi dx.\end{aligned} \tag{4.16}$$

Theorem 4.3 *Under compatibility condition (4.14) and regularity condition (4.15) h is uniquely determined by*

$$\langle h(t),\varphi(t)\rangle_{\boldsymbol{V}',\boldsymbol{V}} = -\sum_{i=1}^I \int_{\Sigma_{\mathrm{s}}^i} \sigma(u,p)n_{\mathrm{f}}^i\varphi d\gamma \quad \forall\varphi\in\boldsymbol{V}. \tag{4.17}$$

Proof. The result follows from (4.12) and (4.16). □

Remark 4.4 From (4.17) we can write

$$\sum_{i=1}^{I} \int_{\Sigma_s^i} \sigma(u,p) n_f^i \varphi d\gamma = - \lim_{\eta \to 0} \frac{1}{\eta} \int_{\Omega_s} \mathbf{1}_{\Omega_s} u_\eta \varphi dx.$$

When $\varphi \in \boldsymbol{V}$ is such that $\varphi \equiv 1$ in Ω_s^i and $\varphi \equiv 0$ in Ω_s^j , $j \neq i$, we get

$$\int_{\Sigma_s^i} \sigma(u,p) n_f^i d\gamma = - \lim_{\eta \to 0} \frac{1}{\eta} \int_{\Omega_s} \mathbf{1}_{\Omega_s^i} u_\eta dx.$$

This last equation gives an explicit way to compute the lift and drag coefficients through an integration over the volume of the obstacle [3], [8].

5. The $\boldsymbol{H^1}$ penalization

We study in this section the behaviour of the solution $(u_\eta)_\eta$ of the penalized problem (3.7) presented in Sect. 3 when η goes to zero.

$$(5.1) \quad \begin{aligned} & \left(1 + \frac{1}{\eta}\mathbf{1}_{\Omega_s}\right) \partial_t u_\eta - \frac{1}{Re} \text{div } \left((1 + \frac{1}{\eta}\mathbf{1}_{\Omega_s}) \nabla u_\eta\right) \\ & + u_\eta \cdot \nabla u_\eta + \frac{1}{\eta}\mathbf{1}_{\Omega_s} u_\eta + \nabla p_\eta = f && \text{in } \mathbb{R}^+ \times \Omega \\ & \text{div } u_\eta = 0 && \text{in } \mathbb{R}^+ \times \Omega \\ & u_\eta(0,\cdot) = u_0 && \text{in } \Omega \\ & u_\eta = 0 && \text{on } \Gamma. \end{aligned}$$

As the aim of this section is to derive a better error estimate, we work directly with regular solution in time, and so we take by example

$$(5.2) \qquad f \in L^\infty(\mathbb{R}^+; \boldsymbol{L}^2(\Omega)) \text{ such that } \partial_t f \in L^\infty(\mathbb{R}^+; \boldsymbol{L}^2(\Omega)),$$

and we assume that the following compatibility solution is fulfilled.

$$(5.3) \qquad \begin{aligned} & -\frac{1}{Re} \text{div } \left((1 + \frac{1}{\eta}\mathbf{1}_{\Omega_s}) \nabla u_\eta(0)\right) \\ & + u_\eta(0) \cdot \nabla u_\eta(0) + \frac{1}{\eta}\mathbf{1}_{\Omega_s} u_\eta(0) + \nabla p_\eta(0) - f(0) \end{aligned}$$

is bounded in $L^2(\Omega)$.

Remark 5.1 The condition (5.3) is satisfied as soon as

$$u_\eta(0) \in \boldsymbol{H}^2(\Omega)\ u_\eta(0)_{|\Omega_{\rm s}} = 0\ ,\ \operatorname{div}\ u_\eta(0) = 0$$

By usual techniques, we can show that (5.1) has a unique solution under the compatibility condition (5.3) and under the regularity assumption (5.2) which checks

$(u_\eta)_\eta$ is bounded in $C^0(0,T;\boldsymbol{V})$

$(\partial_t u_\eta)_\eta$ is bouded in $L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$.

The variational formulation of (5.1) is, for any $\varphi \in \boldsymbol{V}$

$$\begin{aligned}
&\int_\Omega \left(1 + \frac{1}{\eta}\mathbf{1}_{\Omega_{\rm s}}\right)\partial_t u_\eta dx + \frac{1}{Re}\int_\Omega \left(1 + \frac{1}{\eta}\mathbf{1}_{\Omega_{\rm s}}\right)\nabla u_\eta \cdot \nabla\varphi dx \\
&\qquad + \int_\Omega u_\eta \cdot \nabla u_\eta \varphi dx + \frac{1}{\eta}\int_\Omega \mathbf{1}_{\Omega_{\rm s}} u_\eta \varphi dx = \int_\Omega f\varphi dx \qquad (5.4)\\
&\qquad\qquad u_\eta(0,\cdot) = u_0.
\end{aligned}$$

As pointed out in Sect. 3, we introduce u and $\tilde{u}$ in $C^0(0,T;\boldsymbol{V})$ solution of:

$$\begin{aligned} u_{|\Omega_{\rm s}} &= 0 \\ u_{|\Omega_{\rm f}} &= u_{\rm f}, \end{aligned} \qquad (5.5)$$

where $u_{\rm f}$ is the solution of Navier-Stokes equations in $\Omega_{\rm f}$ with $u_{\rm f} = 0$ on $\Gamma \cup \Sigma$ and $\tilde{u}$ defined as:

$\tilde{u}_{|\Omega_{\rm s}} = \tilde{u}_{\rm s}$ where

$$\begin{aligned}
&\partial_t \tilde{u}_{\rm s} - \tfrac{1}{Re}\Delta\tilde{u}_{\rm s} + \tilde{u}_{\rm s} + \nabla\tilde{p}_{\rm s} = 0 && \text{in } \mathbb{R}^+ \times \Omega_{\rm s} \\
&\operatorname{div}\ \tilde{u}_{\rm s} = 0 && \text{in } \mathbb{R}^+ \times \Omega_{\rm s} \\
&\tilde{u}_{\rm s}(0) = 0 && \text{in } \Omega_{\rm s} \\
&\sigma(\tilde{u}_{\rm s},\tilde{p}_{\rm s})n_{\rm s}^i = -\sigma(u_{\rm f},p_{\rm f})n_{\rm s}^i && \text{on } \Sigma_{\rm s}^i\ 1 \le i \le I
\end{aligned} \qquad (5.6)$$

$\tilde{u}_{|\Omega_{\rm f}} = \tilde{u}_{\rm f}$ where

$$\begin{aligned}
&\partial_t \tilde{u}_{\rm f} - \tfrac{1}{Re}\Delta\tilde{u}_{\rm f} + \tilde{u}_{\rm f} \cdot \nabla u_{\rm f} \\
&\quad + u_{\rm f} \cdot \nabla\tilde{u}_{\rm f} + \nabla\tilde{p}_{\rm f} = 0 && \text{in } \mathbb{R}^+ \times \Omega_{\rm f} \\
&\operatorname{div} \tilde{u}_{\rm f} = 0 && \text{in } \mathbb{R}^+ \times \Omega_{\rm f} \\
&\tilde{u}_{\rm f}(0) = 0 && \text{in } \Omega_{\rm f} \\
&\tilde{u}_{\rm f} = \tilde{u}_{\rm s} && \text{on } \Sigma_{\rm s}^i\ 1 \le i \le I
\end{aligned} \qquad (5.7)$$

Remark 5.2 If the penalization is suppressed in the time derivative in (5.1), the resulting Brinkman equation in the solid (5.7) becomes a steady equation.

Remark 5.3 The existence and the regularity of $\tilde{u}_{\mathrm{f}}$ is classical as

$$\int_{\Sigma_{\mathrm{s}}^i} \tilde{u}_{\mathrm{s}} n_{\mathrm{f}}^i d\gamma = 0,\ 1 \le i \le I.$$

The existence of $\tilde{u}_{\mathrm{s}}$ can be derived from [7] where such boundary conditions are studied.

The regularity $\tilde{u}_{\mathrm{s}} \in C^0(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{H}^2(\Omega_{\mathrm{s}}))$ can be obtained by the translation method of Nirenberg.

Then we set $u_\eta = u + \eta\tilde{u} + \eta w_\eta$ and we establish energy estimates on w_η.

Theorem 5.1 *Let u and $\tilde{u}$ be defined by (5.5), (5.6) and (5.7), Then there exists $(v_\eta)_\eta$ bounded in $L^\infty(0,t;\boldsymbol{H}) \cap L^2(0,t;\mathbf{V})$ such that:*

$$u_\eta = u + \eta\tilde{u} + \eta^{\frac{3}{2}} v_\eta.$$

Moreover there exists a generics function g in $C^0(\mathbb{R}^+)$ such that:

$$|v_\eta|_{L^\infty(0,T;L^2(\Omega_{\mathrm{s}}))} \le g(t)\eta^{\frac{1}{2}}$$
$$|\nabla v_\eta|_{L^\infty(0,T;L^2(\Omega_{\mathrm{s}}))} \le g(t)\eta^{\frac{1}{2}}$$

Proof. Replacing u_η in (5.4) by its expression, we get as $u_{|\Omega_{\mathrm{s}}} = 0$:

$$\begin{aligned}
&\int_\Omega \mathbf{1}_{\Omega_{\mathrm{s}}} \left(\partial_t u\varphi + \frac{1}{Re}\nabla u \cdot \nabla\varphi + u \cdot \nabla u\varphi - f\varphi\right) dx \\
&+ \int_\Omega \mathbf{1}_{\Omega_{\mathrm{s}}} \left(\partial_t \tilde{u}\varphi + \frac{1}{Re}\nabla\tilde{u} \cdot \nabla\varphi + \tilde{u}\varphi\right) dx \\
&+\eta \int_\Omega \mathbf{1}_{\Omega_{\mathrm{f}}} \left(\partial_t\tilde{u}\varphi + \frac{1}{Re}\nabla\tilde{u} \cdot \nabla\varphi + \tilde{u} \cdot \nabla u\varphi + u \cdot \nabla\tilde{u}\varphi\right) dx \\
&+\eta \int_\Omega \Big((1 + \frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}})(\partial_t w_\eta\varphi + \frac{1}{Re}\nabla w_\eta \cdot \nabla\varphi) + u \cdot \nabla w_\eta\varphi \\
&\qquad + \frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}} w_\eta\varphi + w_\eta \cdot \nabla u\varphi \Big) dx \\
&+\eta^2 \int_\Omega (\tilde{u} \cdot \nabla\tilde{u} + w_\eta \cdot \nabla\tilde{u} + \tilde{u} \cdot \nabla w_\eta + w_\eta \cdot \nabla w_\eta)\,\varphi dx = 0\ .
\end{aligned} \tag{5.8}$$

Using (5.5) , (5.6) and (5.7) the first three integrals reduce, after integrating by parts, to:

$$-\eta \sum_{i=1}^{I} \int_{\Sigma_{\mathrm{s}}^{I}} \sigma(\tilde{u}_{\mathrm{f}}, \tilde{p}_{\mathrm{f}}) n_{\mathrm{f}}^{i} \varphi d\gamma.$$

Then (5.8) becomes when dividing by η:

$$\begin{aligned} &+\int_{\Omega} \Big((1+\frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}})(\partial_t w_\eta \varphi + \frac{1}{Re}\nabla w_\eta \nabla\varphi) + u\cdot\nabla w_\eta \varphi \\ &\qquad + \frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}} w_\eta \varphi + w_\eta \cdot \nabla u \varphi \Big)\, dx \\ &+\eta \int_{\Omega} \left(\tilde{u}\cdot\nabla\tilde{u} + w_\eta\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla w_\eta + w_\eta\cdot\nabla w_\eta\right)\varphi dx \\ &\qquad = \sum_{i=1}^{I} \int_{\Sigma_{\mathrm{s}}^{I}} \sigma(\tilde{u}_{\mathrm{f}}, \tilde{p}_{\mathrm{f}}) n_{\mathrm{f}}^{i} \varphi d\gamma. \end{aligned}$$

Setting $\varphi = w_\eta$ we get by sobolev embeddings the following inequality:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega}\left(1+\frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}\right) w_\eta^2 dx + \frac{1}{Re}\int_{\Omega}\left(1+\frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}\right)\left|\nabla w_\eta\right|^2 dx \\ &\quad + \frac{1}{\eta}\int_{\Omega} \mathbf{1}_{\Omega_{\mathrm{s}}} w_\eta^2 dx \\ &\quad \le c\ \left(|\nabla u| + \eta\,|\nabla\tilde{u}|\right)|w_\eta|\,|\nabla w_\eta| + c\ \eta\,|\tilde{u}|^{\frac{1}{2}}\,|\nabla\tilde{u}|^{\frac{3}{2}}\,|w_\eta|^{\frac{1}{2}}\,|\nabla w_\eta|^{\frac{1}{2}} \\ &\quad + c\ \left(\sum_{i=1}^{I} \left|\sigma(\tilde{u}_{\mathrm{f}}, \tilde{p}_{\mathrm{f}}\right|_{L^2(\Sigma_{\mathrm{s}}^i)}\right)\left(|\mathbf{1}_{\Omega_{\mathrm{s}}} w_\eta| + |\mathbf{1}_{\Omega_{\mathrm{s}}}\nabla w_\eta|\right), \end{aligned}$$

as

$$|w_\eta|_{L^2(\Sigma_{\mathrm{s}}^i)} \le c\ \left(\left|\mathbf{1}_{\Omega_{\mathrm{s}}^i} w_\eta\right| + \left|\mathbf{1}_{\Omega_{\mathrm{s}}^i}\nabla w_\eta\right|\right).$$

where c always denotes a generic constant.
Then, from the regularity of u and $\tilde{u}$, and from Young inequalities we have

for $\eta \leq 1$:

$$\frac{d}{dt}\int_{\Omega}\left(1+\frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}\right)w_{\eta}^{2}dx+\frac{1}{Re}\int_{\Omega}\left(1+\frac{1}{\eta}\mathbf{1}_{\Omega_{\mathrm{s}}}\right)\left|\nabla w_{\eta}\right|^{2}dx$$

$$+\frac{1}{\eta}\int_{\Omega}\mathbf{1}_{\Omega_{\mathrm{s}}}w_{\eta}^{2}dx$$

$$\leq c\ \left(\left|\nabla u\right|^{2}+\eta^{2}\left|\nabla\tilde{u}\right|^{2}\right)\left|w_{\eta}\right|^{2}$$

$$+c\ \eta\left(\left|\tilde{u}\right|\left|\nabla\tilde{u}\right|^{2}+\sum_{i=1}^{I}\left|\sigma(\tilde{u}_{\mathrm{f}},\tilde{p}_{\mathrm{f}}\right|_{L^{2}(\Sigma_{\mathrm{s}}^{i})}^{2}\right)$$

which gives from Gronwall lemma as $w_{\eta}(0)=0$:

$$\left|w_{\eta}(t)\right|^{2}+\frac{1}{\eta}\left|\mathbf{1}_{\Omega_{\mathrm{s}}}w_{\eta}(t)\right|^{2}\leq\eta g(t)$$

and

$$\int_{0}^{t}\left(\left|\nabla w_{\eta}(\tau)\right|^{2}+\frac{1}{\eta}\left|\mathbf{1}_{\Omega_{\mathrm{s}}}w_{\eta}(\tau)\right|^{2}\right)d\tau\leq\eta g(t).$$

where g denotes a generic function in $C^{0}(\mathbb{R}^{+})$. □

6. Numerical validation

To validate the penalization we choose a numerical test easy to implement and for which the geometry is well fitted in order to avoid an additional error. So we compute the two-dimensional flow around a square cylinder in a channel, i.e. with a no-slip boundary condition on the horizontal walls of Γ, as shown on Fig. 2.

We use a cartesian mesh with 320×80 cells on the whole domain $\Omega = \Omega_{\mathrm{f}} \cup \overline{\Omega}_{\mathrm{s}} = (0,4) \times (0,1)$, in such a way that Σ_{s} corresponds to mesh lines. To get the reference solutions u_{ref}, we approximate the Navier-Stokes equations in Ω_{f} associated to homogeneous Dirichlet boundary conditions on Σ_{s}. Then we can compare for various values of η the penalized solutions to these reference solutions.

6.1. The $\boldsymbol{L^2}$ penalization

First, we investigate a steady case at the Reynolds number $Re = 40$ where $Re = \frac{UD}{\nu}$ with $U = 1$ defines the mean velocity of the Poiseuille flow at the entrance section, D the size of the square and ν the cinematic viscosity.

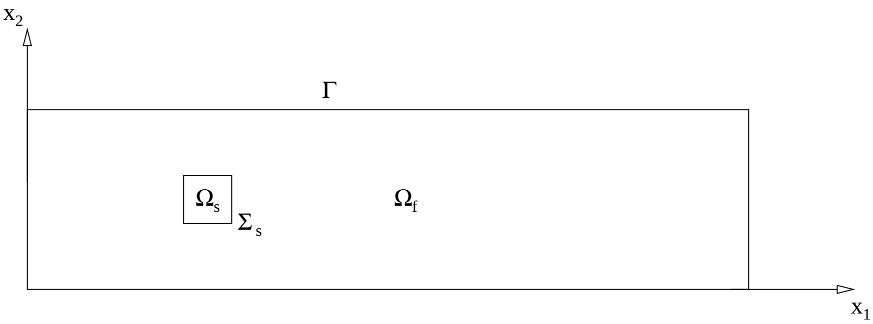

Fig. 2. Computational domain

Table 1. Numerical measurement of the error estimate at $Re = 40$

η	$\|\|u_\eta\|\|_{L^2(\Omega_s)}$	α for $O(\eta^\alpha)$	$\|\|u_\eta - u_{\rm ref}\|\|_{L^2(\Omega_f)}$	α for $O(\eta^\alpha)$
10^{-2}	3.81 10^{-2}		9.69 10^{-2}	
10^{-3}	5.80 10^{-3}	0.82	1.65 10^{-2}	0.77
10^{-4}	6.39 10^{-4}	0.96	1.82 10^{-3}	0.96
10^{-5}	6.46 10^{-5}	0.99	2.01 10^{-4}	0.96
10^{-6}	6.47 10^{-6}	1.00	9.59 10^{-5}	0.32

The isolines of pressure and vorticity are plotted on Fig. 3 for both the exact Dirichlet boundary condition and the L^2 penalization respectively. We can observe that the pressure is continuous through the obstacle with the L^2 penalization.

Table 1 shows that the numerical error estimate varies as $O(\eta)$ in $\Omega_{\rm s}$ and about $O(\eta)$ in $\Omega_{\rm f}$. When the penalty parameter η is going to zero, the error of the solution in the fluid domain is still decreasing as shown on Table 1. However the gain in accuracy, which is in $O(\eta)$ for a range of η values not too small, becomes in fact limited by the discretization error as soon as η is taken below a certain threshold. This is confirmed by the fact that the latter threshold is decreasing when the mesh step is smaller. Hence, the theoretical error estimate given in Theorem 4.2 might not be optimal as it yields $O(\eta^{3/4})$ in $\Omega_{\rm s}$ and $O(\eta^{1/4})$ in $\Omega_{\rm f}$.
Then we compute unsteady solutions at $Re = 80$ to be sure that propagation effects are not spoiled by the penalization method. We see on Fig. 4 and 5 that a large value of η induces a delay of the convection. Consequently, the Strouhal number $S_t = \frac{fD}{U}$, where f is the frequency of the vortex shedding in the wake, is under estimated (Table 2). Let us note that the value of the Strouhal number for this internal flow does not correspond to the well-known value for such a Reynolds number in an open flow.

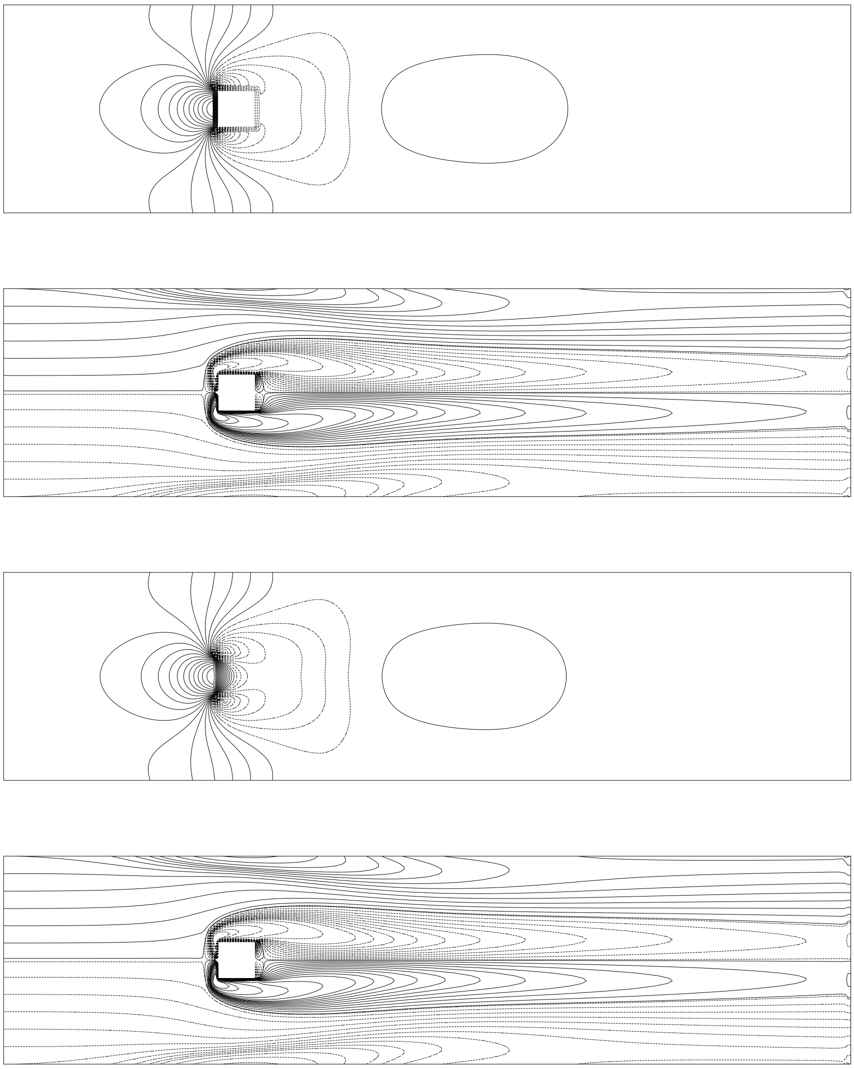

Fig. 3. Comparison of the solution computed with Dirichlet boundary condition and L^2 penalization with $\eta = 10^{-8}$ (pressure and vorticity fields at $Re = 40$)

Table 2. Strouhal number at $Re = 80$

	u_{ref}	$\eta = 10^{-8}$	$\eta = 10^{-6}$	$\eta = 10^{-4}$	$\eta = 10^{-2}$
S_t	0.2390	0.2390	0.2390	0.2387	0.2349

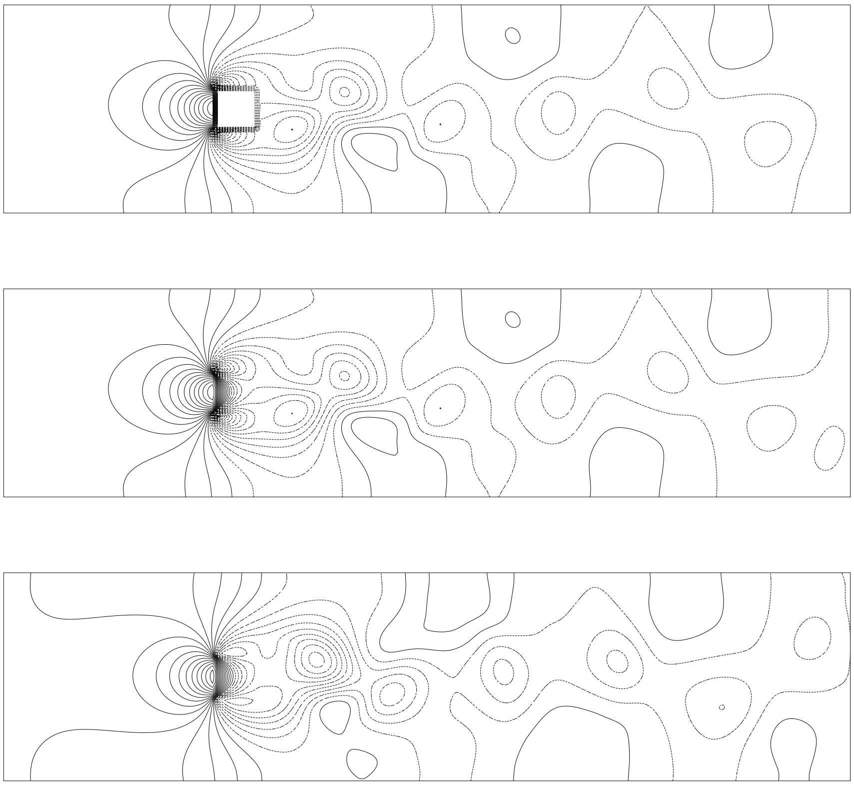

Fig. 4. Comparison of the solutions computed with Dirichlet boundary condition and L^2 penalization with $\eta = 10^{-8}$ and $\eta = 10^{-2}$ at $Re = 80$ (pressure field)

For small values of η, we clearly observe on Figs. 4 and 5 that there is no difference between u_{ref} and u_η when $\eta = 10^{-8}$. This means that the error induced by the penalization method is much lower than the error of approximation.

These numerical tests show the efficiency of the L^2 penalization that gives the same result than Dirichlet boundary condition as soon as the penalty parameter η is small enough. In addition, the pressure field is continuous in the whole domain Ω and the equation

$$\frac{1}{\eta} u_\eta + \nabla p_\eta = 0 \text{ in } \Omega_{\mathrm{s}}$$

is numerically satisfied, which confirms the asymptotic formal expansion given by (3.5). Hence, there is a Darcy flow in the solid at the order η. As it is carried out numerically in [10] and according to (4.17) and Remark 4.4,

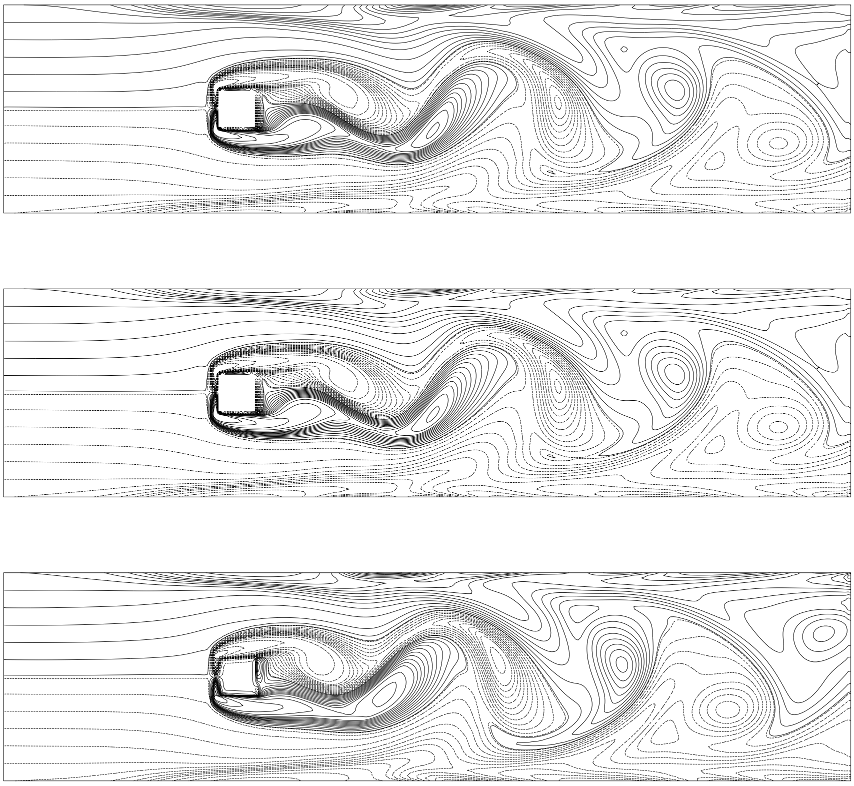

Fig. 5. Comparison of the solutions computed with Dirichlet boundary condition and L^2 penalization with $\eta = 10^{-8}$ and $\eta = 10^{-2}$ at $Re = 80$ (vorticity field)

the lift and drag coefficients can be computed from

$$\lim_{\eta \to 0} \int_{\Omega_\mathrm{s}} \nabla p \, dx = - \lim_{\eta \to 0} \frac{1}{\eta} \int_{\Omega_\mathrm{s}} u_\eta \, dx = \int_{\Sigma_\mathrm{s}} \sigma(u,p) \cdot n_\mathrm{f} \, d\gamma .$$

6.2. The $\boldsymbol{H^1}$ penalization

We perform the same numerical tests than in Sect. 6.1. The results plotted on Fig. 6 at $Re = 40$ show that there is a small discrepancy with the original solution. It is due to the discretization of the penalized viscous term. Indeed, the discretization of this term changes slightly the size of the obstacle and thus the recirculation zone in the wake is a little bit different.

In the unsteady case, we observe also a discrepancy in the determination of the Strouhal number even for $\eta = 10^{-8}$. This is still due to the variation of volume of the body which is effectively taken into account by the

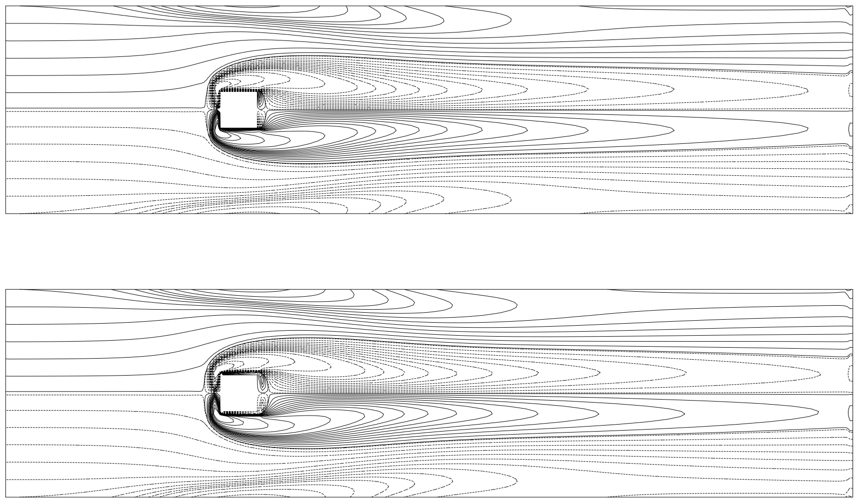

Fig. 6. Comparison of the solutions computed with Dirichlet boundary condition and H^1 penalization with $\eta = 10^{-8}$ at $Re = 40$ (vorticity field)

discretization and induced by the approximation of the penalized viscous term. Hence, the H^1 penalization does not improve in practice the numerical results as it is shown theoretically.

7. Conclusions

This paper gives a rigorous justification of the penalization method to take into account a solid body immersed in an incompressible viscous fluid in motion. This method is a fictitious domain method which has been proved to be very easy to implement, robust and efficient. In addition, since the penalization parameter η can be taken as small as necessary, the penalty error is always negligeable in front of the error of approximation.

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