ECE 285 – Assignment #5
Fourier transform

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In this assignment we will study the Fourier transform, and next implement convolutions in the Fourier domain as part of our image manipulation library `imagetools`.

First, start a Jupyter Notebook, go into the subdirectory `ece285_IVR_assignments` (or whatever you named it), and create a new notebook `assignment5_fourier.ipynb` with

```
%load_ext autoreload
%autoreload 2
import numpy as np
import numpy.fft as npf
import matplotlib
import matplotlib.pyplot as plt
import time
import imagetools.assignment5 as im

%matplotlib notebook
```

We will use

- `assets/house.png`
- `assets/map.png`
- `assets/motionblur.npy`
- `assets/montreuil.png`
- `assets/lady.png`

For the following questions, please write your code and answers directly in your notebook. Organize your notebook with headings, markdown and code cells (following the numbering of the questions).

1 Spectral analysis

1. In your notebook, create an image with a horizontal sinusoidal `a`:

   ```
   n = 256
   i = np.arange(n)
   j = np.pi / 16 * np.ones(n)
   x = np.sin(np.outer(i, j))
   ```

   Visualize this image and the amplitude of its spectrum in one single figure as

   ```
   fig, axes = plt.subplots(ncols=2, figsize=(7,3))
   im.show(x, ax=axes[0])
   im.showfft(x, ax=axes[1], apply_fft=True)
   fig.show()
   ```

Explain what you observe: number of bright points, their locations and their amplitude.

2. Repeat the same analysis for

   - a vertical sinusoidal:

     ```
     i2 = 2 * np.pi * 3 / 64 * np.ones(n)
     j2 = np.arange(n)
     x = 4 * np.sin(np.outer(i2, j2))
     ```

   - a variant of the vertical sinusoidal:

     ```
     i3 = 2 * np.pi * (3 + 1/8) / 64 * np.ones(256)
     x = np.sin(np.outer(i3, j2))
     ```

   - a diagonal sinusoidal

     ```
     x = np.sin(np.outer(i, j) + np.outer(i2, j2))
     ```

   - a square

     ```
     x = np.zeros((256, 256))
     x[62:190, 62:190] = 1
     ```

   - a sinusoidal in a window

     ```
     x = x * np.sin(np.outer(i, j))
     ```

3. Load the image `x = house`. Display this image and the logarithm of its spectrum as

   ```
   im.showfft(x, ax=axes[1], apply_fft=True, apply_log=True)
   ```

   What does the horizontal, vertical and oblique structures that you observe represent?

4. Repeat with the image `map`, `montreuil` and `lady`, and interpret what you observe.

5. For the three images, perform a spatial sub-sampling by a factor 2 in each direction

   ```
   x = x[::2, ::2]
   ```

   Inspect their spectrum, repeat with a factor 4, and explain how aliasing impacts these images.

2 Spectral convolution

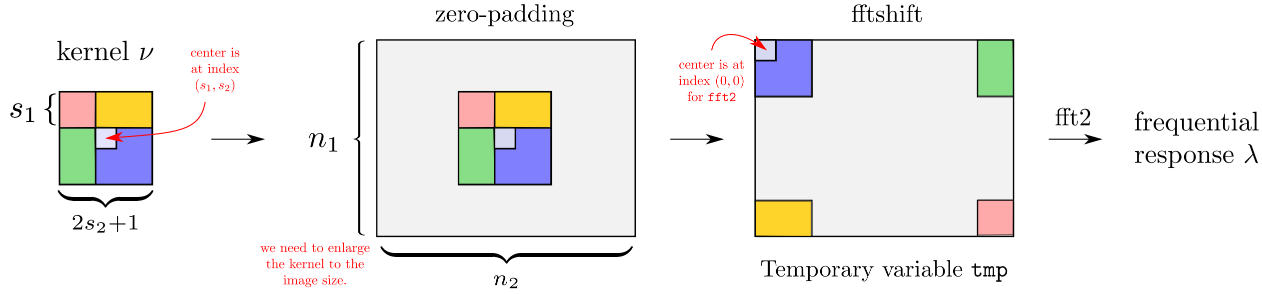
6. Run `convolve` on `x = lady` with a Gaussian convolution of bandwidth $\tau = 2$ using periodical boundary conditions. Display the image and its spectrum (in logarithm) before and after convolution. Explain what you observe.

7. Repeat with a box kernel $\tau = 5$. Explain what you observe.

8. Create a function

```python
def kernel2fft(nu, n1, n2, separable=None):
    ...
    tmp = np.zeros((n1, n2))
    tmp[:s1+1, :s2+1] = nu[s1:2*s1+1, s2:2*s2+1]
    ...
    return lbd
```

that creates a $n_1 \times n_2$ complex valued array `lambda` corresponding to the frequential response of the convolution kernel function with impulse response `nu` that is limited to window of support $[-s_1, s_1] \times [-s_2, s_2]$. The format of `nu` is the same as the one for `convolve` and is determined by the optional argument `separable` (default: `None`). Refer to the following figure for more details:



9. Create in `imagetools/assignment5.py`, the function

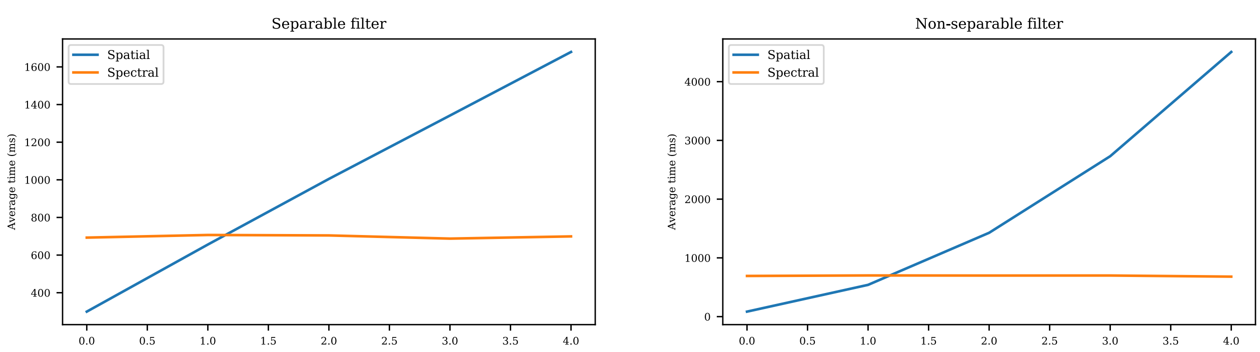
```python
def convolvefft(x, lbd)
```

that implements the convolution (in the Fourier domain) of x with a convolution kernel whose frequential response is given by λ.

Hint: Don't forget to take the real part.

10. Test your new convolution function on `x = lady`. Compare the results with `convolve` for different kernel functions using periodical boundary conditions. Check that the error between the two is zero (up to machine precision).

11. Compare the computation time of `convolve` and `convolvefft` for a box kernel with $\tau = 0$ to 4. Make the comparison when the spatial convolution is separable and non-separable (with the box kernel). Average the computation time on 100 runs. Draw the curves of average computation time as a function of τ (do not forget about title, axis names, and legend). Check that your curves are consistent with the ones below.



What are the complexities of the two methods with respect to τ and the separability? When is the spectral convolution favorable against the spatial convolution?

3 Adjoint

The convolution $\mathbf{H} : \mathbf{x} \mapsto \nu * \mathbf{x}$ where ν is the convolution kernel, is a linear operator with respect to x. As any linear operators, it has a unique adjoint $\mathbf{H}^*$ such that for all images x and y, $\langle \mathbf{H}x, y \rangle = \langle x, \mathbf{H}^* y \rangle$.

12. Copy paste the function `kernel` from `imagetools/assignment3.py` into `assignment4.py`. Modify the function in order to implement `name='motion'`, in which case it loads the convolution kernel stored in `assets/motionblur.npy`. All optional arguments are ignored in this case.

13. In your notebook, load the motion kernel, display it and display its application to `x = house` using `convolve`.

14. The adjoint $\mathbf{H}^*$ is also a convolution, $\mathbf{H}^* : \mathbf{x} \mapsto \mu * \mathbf{x}$. What is μ compared to ν? In your notebook, modify ν into μ. Using `convolve`, check that $\langle \mathbf{H}x, y \rangle = \langle x, \mathbf{H}^* y \rangle$ for `x = house` and `y = map`.

15. What is the frequential response of μ compared to the one of ν? In your notebook, using `kernel2fft`, create λ the frequential response of ν, and modify it into the one of μ. Using `convolvefft`, check that $\langle \mathbf{H}x, y \rangle = \langle x, \mathbf{H}^* y \rangle$ for `x = house` and `y = map`.