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CHAPTER II

Finiteness Theorems for Abelian Varieties over Number Fields

GERD FALTINGS

§1. Introduction

Let K be a finite extension of $\mathbb{Q}$, A an abelian variety defined over K, $\pi = \mathrm{Gal}(\bar{K}/K)$ the absolute Galois group of K, and l a prime number. Then π acts on the (so-called) Tate module

$$T_l(A) = \varprojlim_n A[l^n](\bar{K}).$$

The goal of this chapter is to give a proof of the following results:

(a) The representation of π on $T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is semisimple.
(b) The map

$$\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \mathrm{End}_\pi(T_l(A))$$

is an isomorphism.
(c) Let S be a finite set of places of K, and let $d > 0$. Then there are only finitely many isomorphism classes of abelian varieties over K with polarizations of degree d which have good reduction outside of S.

(a) and (b) are known as the Tate conjectures, (c) as the Shafarevich conjecture. Furthermore, one knows [9] that the Mordell conjecture follows from (c). The Tate conjectures for abelian varieties over finite fields have already been proven by Tate himself. Zarhin generalized this to function fields over such fields, [15], [16], and our proof is an adaptation of his method to the case of a number field. Arakelov supplied the dictionary necessary for this translation [2], and the author has built upon his methods [5]. In brief, what is needed is to provide "everything" with a hermitian metric.

The proof of (c) is achieved by first showing finiteness only for isogeny classes. The basic idea for this was communicated to me by a referee for *Inventiones* in connection with the publication of my paper [6], and I then had only to translate it from Hodge theory into étale cohomology. I would therefore like to heartily thank this referee, who is personally unknown to me, for his suggestion.

The rest of the proof of (c) uses a variant of the methods employed in proving (a) and (b).

The paper begins, first of all, with some technical details concerning heights. The complications arise because, at least to my knowledge, no good moduli space for semiabelian varieties over $\mathbb{Z}$ exists yet. (L. Moret-Bailly, who investigated the situation over functions fields, had to struggle with similar problems [7].) After that, we use the very beautiful results of Tate [13] on p-divisible groups. The conclusion is then again somewhat technical.

I have learned much about the subject from L. Szpiro, and I want to thank him here for introducing me to this circle of problems. P. Deligne called my attention to a discrepancy in an earlier version of this work.

§2. Semiabelian Varieties

Definition. Let S be a scheme (or an algebraic stack). A semiabelian variety of relative dimension g over S is a smooth algebraic group $p: G \to S$ whose fibres are connected of dimension g, and are extensions of an abelian variety by a torus.

EXAMPLE. Let $q: C \to S$ be a stable curve of genus g [4]. Then

$$J = \mathrm{Pic}^{\tau}(C/S) \to S$$

is a semiabelian variety of relative dimension g.

We need the following:

Lemma 1. *Let S be normal, $U \subset S$ open and dense, $p_1: A_1 \to S$ and $p_2: A_2 \to S$ two semiabelian varieties, $\phi: A_1/U \to A_2/U$ a homomorphism of algebraic groups defined over U. Then ϕ can be extended uniquely over all of S.*

PROOF. This is well known in case S is the spectrum of a complete discrete valuation ring. In general, one reduces immediately to the case in which S is noetherian and excellent, and writes

$$X \subseteq A_1 \times_S A_2$$

for the closure of the graph of ϕ.

After base change with suitable valuation rings, one sees that the projection $\mathrm{pr}_1: X \to A_1$ is proper, and that its fibres have only one point. Since A_1 is normal, pr_1 must be an isomorphism, and X the graph of the uniquely determined extension of ϕ. (Uniqueness follows, for example, by consideration of torsion points, or in a thousand other ways.) □

Definition. Let $p: A \to S$ be a semiabelian variety of relative dimension g, $s: S \to A$ the zero section.

Let

$$\omega_{A/S} = s^*(\Omega^g_{A/S}),$$

$\omega_{A/S}$ is a line bundle on S.

Remarks. (a) If p is proper, then $\omega_{A/S} \cong p_*(\Omega^g_{A/S})$.

(b) $\omega_{A/S}$ commutes with a change of base.

(c) If $A = \mathrm{Pic}^{\tau}(C/S)$ for a stable curve $q: C \to S$, then $\omega_{A/S} \cong \Lambda^g q_*(\omega_{C/S})$, where $\omega_{C/S}$ denotes the relative dualizing module.

(d) If $S = \mathrm{Spec}(\mathbb{C})$ and p is proper (i.e., $A/\mathbb{C}$ is a complex abelian variety), then $\omega_{A/S} \cong \Gamma(A, \Omega^g_{A/\mathbb{C}})$ admits a canonical hermitian scalar product, namely: If α, β are holomorphic differential forms on A, then

$$\langle \alpha, \beta \rangle = \left(\frac{i}{2}\right)^g \int_A \alpha \wedge \bar{\beta}.$$

We need some facts about the moduli spaces for stable curves and abelian varieties. For this the language of algebraic stacks seems to be the most appropriate. Should this notation appear too abstract to the reader, he might think through the following considerations:

We are really concerned with finiteness statements. If $\mathfrak{G}$ is one of the stacks to be introduced below, and S denotes the corresponding coarse moduli space, there is always an open covering

$$S = \bigcup_{i=1}^{r} U_i$$

and finite surjective maps $V_i \to U_i$, such that over V_i the "universal object for $\mathfrak{G}$" exists. One can then carry out all calculations in the V_i.

Now for the algebraic stacks to be used here.

(1) $\overline{\mathfrak{M}}_g$ classifies stable curves of genus g [4]. $\overline{\mathfrak{M}}_g$ is proper over $\mathrm{Spec}(\mathbb{Z})$, and the coarse moduli variety belonging to it is called $\overline{M}_g$.
(2) $\mathfrak{A}_g$ classifies the principally polarized abelian varieties of relative dimension g, and A_g the corresponding moduli variety.

$\mathfrak{A}_g$ is not proper over $\mathrm{Spec}(\mathbb{Z})$, but the following facts are known:

(a) If

$$p: A \to \mathfrak{A}_g$$

denotes the universal abelian variety over $\mathfrak{A}_g$, then there exists an $r > 0$ for which $(\omega_{A/\mathfrak{A}_g})^{\otimes r}$ defines a very ample line bundle on $A_g/\mathbb{Q}$ [3].

Let $\bar{A}_g/\mathbb{Q}$ be the Zariski closure in the corresponding projective space $\mathbb{P}^N_{\mathbb{Q}}$, $\bar{A}_g/\mathbb{Z}$ the Zariski closure in $\mathbb{P}^N_{\mathbb{Z}}$, and $\mathscr{M}$ the line bundle $\mathcal{O}(1)$ on $\bar{A}_g/\mathbb{Z}$. ($\mathscr{M}$ extends $(\omega_{A/A_g})^{\otimes r}$ on $\bar{A}_g/\mathbb{Q}$.)

(b) Over $\mathbb{C}$ there is a proper dominating morphism

$$\phi\colon \mathfrak{N} \to \bar{A}_g/\mathbb{C},$$

such that there exists a semiabelian variety over $\mathfrak{N}$ which extends the universal variety over $\mathfrak{A}_g$ (see [8, §9]). Moreover, it is known that the $\omega^{\otimes r}$ of this semiabelian variety is isomorphic to $\phi^*(\mathscr{M})$. (This is proven by a direct calculation; see my exposition in [6, §2]).

Lemma 2. *Over* Spec($\mathbb{Z}$), *there exists a proper algebraic stack* $\mathfrak{Z}$, *an open subset* $\mathfrak{U} \subset \mathfrak{Z}$, *and a proper morphism* $\psi\colon \mathfrak{U} \to \mathfrak{A}_g$, *which extends to a* $\bar{\psi}\colon \mathfrak{Z}/\mathbb{Q} \to \bar{A}_g/\mathbb{Q}$, *such that the following objects exist:*

(a) *A stable curve* $q\colon C \to \mathfrak{Z}$.

(b) *A sub-line-bundle* (= *local direct summand*) $\mathscr{L} \subseteq \Lambda^g q_*(\omega_{C/\mathbb{Z}})$.

(c) *A pair of group homomorphisms over* $\mathfrak{U}$

$$\alpha\colon \mathrm{Pic}^\tau(C/\mathfrak{Z}) \to \psi^*(A),$$

$$\beta\colon \psi^*(A) \to \mathrm{Pic}^\tau(C/\mathfrak{Z}),$$

with

$$\alpha \circ \beta = \textit{multiplication by } d \quad d \in \mathbb{N}, d \neq 0.$$

(Here A is again the universal abelian variety over $\mathfrak{A}_g$.)

(d) *There exists an isomorphism* $\mathscr{L}^{\otimes r} = \bar{\psi}^*(\mathscr{M})$ *over* $\mathfrak{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ *and* $\mathscr{L}$ *is the image of*

$$\alpha^*\colon \psi^*(\omega_{A/A_g}) \to \Lambda^g q_*(\omega_{C/\mathbb{Z}})$$

over $\mathfrak{U}/\mathbb{Q}$. *The resulting isomorphism (over* $\mathfrak{U}/\mathbb{Q}$)

$$\psi^*(\omega_{A/A_g})^{\otimes r} \cong \psi^*(\mathscr{M})$$

is the ψ^**-pullback of the isomorphism over* A_g *resulting from the construction of* $\mathscr{M}$.

PROOF. The abelian variety associated to the generic point of $\mathfrak{A}_g$ is the quotient of a Jacobian. The curve thus obtained corresponds to a rational map from $\mathfrak{A}_g$ to $\overline{\mathfrak{M}}_{\tilde{g}}$, for some $\tilde{g}$.

If one considers the graph of this map, one obtains (with the help of some trivial additional considerations) a first candidate $\mathfrak{Z}$, such that conditions (a) and (by Lemma 1) (c) are already fulfilled. $\mathscr{L}$ is then already determined over $\mathfrak{U} \otimes_{\mathbb{Z}} \mathbb{Q}$ by (d), and furnishes a rational map from $\mathfrak{U} \otimes_{\mathbb{Z}} \mathbb{Q}$ to a certain projective bundle over $\mathfrak{Z}$. One replaces $\mathfrak{Z}$ by the normalization of the closure of the graph of this map and then (b) and the second part of (d) are also fulfilled. For the rest of (d), one notes that we have already constructed the required isomorphism over $\mathfrak{U} \otimes_{\mathbb{Z}} \mathbb{Q}$, and one need now only prove the existence of an extension to $\mathfrak{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$. To do this, one can extend the ground field from $\mathbb{Q}$ to $\mathbb{C}$, and it suffices to prove the existence of an extension for a $\tilde{\mathfrak{Z}}/\mathbb{C}$ which is dominant and proper over $\mathfrak{Z}$.

With the help of the map $\phi\colon \mathfrak{N} \to \bar{A}_g/\mathbb{C}$ introduced above, one constructs a normal $\tilde{\mathfrak{Z}}$, such that $\psi^*(A)$ extends to a semiabelian variety on $\tilde{\mathfrak{Z}}$. By Lemma 1 one can also extend α and β, and this furnishes the desired isomorphism over $\tilde{\mathfrak{Z}}$. □

Corollary. *There exists a natural number e with the following property:*

Let K be a number field, R its ring of integers,

$$p\colon A \to \mathrm{Spec}(R)$$

a semiabelian variety such that the generic fibre A/K is proper over K and possesses a principal polarization. Then the corresponding map $\rho\colon \mathrm{Spec}(K) \to A_g/\mathbb{Q}$ *extends to a* $\rho\colon \mathrm{Spec}(R) \to \bar{A}_g/\mathbb{Z}$.

By construction, there exists an isomorphism

$$\rho^*(\mathscr{M}) \otimes_R K \cong (\omega_{A/R})^{\otimes r} \otimes_R K.$$

Using this isomorphism one gets:

$$e \cdot \rho^*(\mathscr{M}) \subseteq (\omega_{A/R})^{\otimes r} \subseteq e^{-1} \cdot \rho^*(\mathscr{M}) \quad (\subseteq \rho^*(\mathscr{M}) \otimes_R K).$$

PROOF. We may assume that $\bar{\psi}\colon \mathfrak{Z}/\mathbb{Q} \to \bar{A}_g/\mathbb{Q}$ can be extended to a proper $\bar{\psi}\colon \mathfrak{Z}/\mathbb{Z} \to \bar{A}_g/\mathbb{Z}$. Then there is a finite field extension $K' \supseteq K$ (with integers $R' \subseteq K'$), such that ρ can be lifted to

$$\tilde{\rho}\colon \mathrm{Spec}(R') \to \mathfrak{Z}.$$

Since $\bar{\psi}^*(\mathscr{M})$ and $\mathscr{L}^{\otimes r}$ are isomorphic over $\mathfrak{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$, there is an $e_1 > 0$ such that over $\mathfrak{Z}$

$$e_1 \cdot \mathscr{L}^{\otimes r} \subseteq \bar{\psi}^*(\mathscr{M}) \subseteq e_1^{-1} \cdot \mathscr{L}^{\otimes r}.$$

It suffices to prove the claim after a change of base to R', and then we need only compare $\omega_{A/R'}$ and $\tilde{\rho}^*(\mathscr{L})$.

By pullback one obtains a stable curve

$$q\colon C \to \mathrm{Spec}(R')$$

and

$$\alpha\colon \mathrm{Pic}^\tau(C/R') \to A/R', \qquad \beta\colon A/R' \to \mathrm{Pic}^\tau(C/R')$$

with $\alpha \circ \beta = d \cdot id$ (use Lemma 2 over R'), such that $\tilde{\rho}^*(R)$ is a subbundle of $\Lambda^g q_*(\omega_{C/R'})$, which is generated by the image of

$$\alpha^*\colon \omega_{A/R'} \to \Lambda^g q_*(\omega_{C/R'}).$$

From this, it follows immediately that $d^g \cdot \tilde{\rho}^*(\mathscr{L}) \subseteq \omega_{A/R'} \subseteq \tilde{\rho}^*(\mathscr{L})$, and we are done. □

§3. Heights

Again let K be a number field, R the ring of integers in K. By analogy to [5], we define a metrized line bundle on Spec(R) to be a projective R-module P of rank 1, together with norms $\| \ \|_v$ on $P \otimes_R K_v$ for all infinite places of K. K_v denotes the completion of K at v, and we define $\varepsilon_v = 1$ or 2 according to whether $K_v \cong \mathbb{R}$ or $K_v \cong \mathbb{C}$. The degree of the metrized line bundle is defined as ("#" = order)

$$\mathrm{Deg}(P, \| \ \|) = \log(\#(P/R \cdot p)) - \sum_v \varepsilon_v \log \|p\|_v,$$

where p is a nonzero element of P, the sum runs over all infinite places of K, and the right-hand side is of course independent of p.

Remark. The idea of metrized line bundle was introduced by Arakelov ([2]). The degree of P is naturally also connected with the volume of P.

We are especially interested in the metrized line bundle $\omega_{A/R}$, where

$$p: A \to \mathrm{Spec}(R)$$

is a semiabelian variety, with proper generic fibre A/K. The metrics at the infinite places come from the scalar product mentioned above

$$\|\alpha\|_v^2 = \left(\frac{i}{2}\right)^g \cdot \int_{A(\bar{K}_v)} \alpha \wedge \bar{\alpha}.$$

Definition. The moduli-theoretic height $h(A)$ is

$$h(A) = \frac{1}{[K:\mathbb{Q}]} \deg(\omega_{A/R}).$$

One sees immediately that $h(A)$ is invariant under extension of the ground field. The name "height" is justified by the following:

In general one defines the height of a point $x \in \mathbb{P}^n(K)$ by associating to x a morphism ρ: Spec(R) $\to \mathbb{P}^n_{\mathbb{Z}}$, providing the bundle $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathbb{C}}$ with a metric, and then defining the height of x to be

$$\frac{1}{[K:\mathbb{Q}]} \cdot \deg(\rho^* \mathcal{O}(1)).$$

Changing the hermitian metric only changes the height function by a bounded amount, and it is known that for every c, there are only finitely many K-rational points of $\mathbb{P}^n$ with height $\leq c$. Corresponding considerations apply to closed subvarieties of $\mathbb{P}^n$. In our situation, one embeds A_g in $\mathbb{P}^n_{\mathbb{Z}}$ as above, by means of $\mathscr{M}$. We have already defined a metric $\| \ \|$ on the line bundle induced by $\mathscr{M}$ on $A_g(\mathbb{C})$. Should this metric admit an extension to $\bar{A}_g(\mathbb{C})$, one could use it to define the height, and then the corollary to Lemma 2 would show for a semiabelian variety A over R (as above), which has principal polarization over K and so defines an $x \in A_g(K)$, that $h(x)$ and $r \cdot h(A)$ differ only by a bounded amount.

Unfortunately, the metric $\| \ \|$ has singularities along $\bar{A}_g(\mathbb{C}) - A_g(\mathbb{C})$; however, these are so mild that the fundamental finiteness property of heights remains true.

Definition. Let $X/\mathbb{C}$ be a compact complex variety, $Y \subseteq X$ a closed subvariety, $\mathscr{M}$ a line bundle of X, $\| \ \|$ a hermitian metric on $\mathscr{M}|X - Y$. The metric has logarithmic singularities along Y if the following holds:

There is a proper dominant map

$$\phi: \tilde{X} \to X,$$

such that $\tilde{X}$ is smooth and $\phi^{-1}(Y)$ is a divisor with normal crossings, and such that for a local generator h of $\phi^*(\mathscr{M})$ and a local equation f of $\phi^{-1}(Y)$,

$$\mathrm{Sup}\{\|h\|, \|h\|^{-1}\} \leq c_1 \cdot |(\log|f|)|^{c_2} \qquad \text{(with constants } c_1, c_2 > 0\text{) holds.}$$

EXAMPLE.

$$X = \bar{A}_g(\mathbb{C}), \qquad Y = \bar{A}_g(\mathbb{C}) - A_g(\mathbb{C}), \qquad \mathscr{M} \text{ and } \| \ \| \text{ as before.}$$

Indeed this was already proven in [6, end of §2], but at the request of the referees we present a short sketch of the proof here:

More generally, it is true that for a smooth X, a divisor Y on X with normal crossings and a semiabelian variety

$$p: A \to X,$$

such that p is proper and A is principally polarized over $X - Y$, the canonical metric on $\omega_{A/X}$ has logarithmic singularities along Y.

To see this one considers $p_*(\Omega^1_{A/X})$ instead of $\omega_{A/X}$ ("logarithmic singularities" can also be defined for vector bundles), and by the methods of Section 2 one reduces the problem to the case in which A is the Jacobian of a semistable curve $q: C \to X$. We will treat briefly the case of a semistable curve over the unit disc $\mathbb{D}$. The general case goes exactly the same way. If

$$q: C \to \mathbb{D} = \{t \,|\, |t| < 1\}$$

is a semistable curve, with good reduction except at 0, then C admits a covering

$$C = \bigcup_{i=1}^{l} U_i$$

such that either

(a) $\qquad U_i = \{(z, t) \,|\, |z| < 1, |t| < 1\}, \qquad q|U_i: U_i \to \mathbb{D}$

is smooth, and z furnishes coordinates on all fibres; or

(b) $\qquad U_i = \{(z_1, z_2, t) \,|\, |z_1| < \varepsilon, |z_2| < \varepsilon, z_1 z_2 = t^m\}, \qquad q(z_1, z_2, t) = t.$

If α is a local section of $q_*(\omega_{C/\mathbb{D}})$, then on the U_i's is of the form

(a) $$\alpha = (\text{holomorph}) \cdot dz \quad \text{resp.}$$

(b) $$\alpha = (\text{holomorph}) \frac{dz_1}{z_1}.$$

An explicit calculation shows that for $t \to 0$

$$\frac{i}{2} \int_{U_i \cap q^{-1}(t)} \alpha \wedge \bar{\alpha}$$

either remains bounded, or grows at most like $|\log|t||$. Further, one sees immediately that

$$\| \; \| \geq (\text{pos. const.}) \; \| \; \|_1,$$

where $\| \; \|_1$ denotes a hermitian metric on $q_*(\omega_{C/X})$ defined on all of X.

Lemma 3. *Let $X \subseteq \mathbb{P}^n_{\mathbb{Z}}$ be Zariski-closed, $Y \subseteq X$ closed, $\| \; \|$ a hermitian metric on $\mathcal{O}(1)|(X(\mathbb{C}) - Y(\mathbb{C}))$, with logarithmic singularities along Y. For a number field K, and $x \in X(K) - Y(K)$ one defines $h(x)$ as before. Then for every c, there are only finitely many $x \in X(K) - Y(K)$ with $h(x) \leq c$,*

Proof. Let $\| \; \|_1$ be a hermitian metric for $\mathcal{O}(1)|X(\mathbb{C})$, h_1 the corresponding height function, and choose an $s > 0$ and

$$f_1, \ldots, f_t \in \Gamma(X/\mathbb{Z}, \mathcal{O}(s)),$$

whose set of common zeros is exactly Y. Then $\| \; \|_1$ defines a metric on $\mathcal{O}(s)$ (which is also called $\| \; \|_1$), and from the hypotheses it follows immediately that there exist constants $c_1, c_2 > 0$ with

$$\log \left| \frac{\| \; \|}{\| \; \|_1}(z) \right| \leq c_1 + c_2 \cdot \inf\{\log(|\log \| f_i(z) \|_1|)\}$$

for $z \in X(\mathbb{C})$.

If $x \in X(K) - Y(K)$, there corresponds a

$$\rho \colon \text{Spec}(R) \to X,$$

and then the f_i define sections $\rho^*(f_i)$ of $\rho^*(\mathcal{O}(s))$, with whose help the height $h_1(x)$ can be calculated. Since $\| f_i(z) \|_1$ is bounded above on $X(\mathbb{C})$, one immediately obtains constants $c_3, c_4 > 0$ with

$$|h(x) - h_1(x)| \leq c_3 + c_4 \log(h_1(x)).$$

The claim follows directly. □

We can now reap the fruits of our labors. The following result is almost already proven.

Theorem 1. *Let c be given. Then there are only finitely many isomorphism classes of pairs of:*

(i) *a semiabelian variety of relative dimension g*

$$p \colon A \to \text{Spec}(R)$$

with proper generic fibre A/K.

(ii) *a principal polarization of A/K for which*

$$h(A) \leq c$$

holds.

Proof. According to the corollary to Lemma 2 the difference between $h(x)$ and $r \cdot h(A)$ is bounded ($x \in A_g(X)$ corresponding to A). According to Lemma 3, the A for which $h(A) \leq c$ provide only finitely many different $x \in A_g(K)$. We must now note that only finitely many K-isomorphism classes can induce the same isomorphism class over the algebraic closure $\bar{K}$. Thus, we fix such a class over $\bar{K}$ and consider the A/K belonging to it. It is known that all of these have bad reduction at the same places of K. It follows immediately from Lemma 4 below that there exists a finite extension $K' \supseteq K$, which for some $n \geq 3$ contains the nth division points of all A/K. It is known that our A's are already isomorphic over K', and the rest follows from basic general theorems of Galois cohomology. □

There remains to be added the

Lemma 4. *Let K be a number field, S a finite set of places of K. Then there are only finitely many field extensions of a given degree which are unramified outside of S.*

Proof. Well known (Hermite–Minkowski). □

§4. Isogenies

We examine the behavior of $h(A)$ under isogeny. As always K is a number field, $R \subset K$ its ring of integers.

Let

$$p_1 \colon A_1 \to \text{Spec}(R)$$

and

$$p_2 \colon A_2 \to \text{Spec}(R)$$

be semiabelian varieties with proper generic fibre,

$$s \colon \text{Spec}(R) \to A_1$$

the zero section, and $\phi \colon A_1 \to A_2$ an isogeny. (Of course, by Lemma 1, it is enough that ϕ be defined over K.)

We set $G = \mathrm{Ker}(\phi) \subseteq A_1$. Since ϕ is automatically flat, G is a quasi-finite flat group scheme over $\mathrm{Spec}(R)$.

ϕ induces an injection

$$\phi^*: \omega_{A_2/R} \to \omega_{A_1/R},$$

and one sees at once that

$$\#(\omega_{A_1/R}/\phi^*(\omega_{A_2/R})) = \# s^*(\Omega^1_{A_1/A_2}) = \# s^*(\Omega^1_{G/R}).$$

Since moreover ϕ^* changes the norms at the infinite places by $(\deg(\phi))^{1/2}$, there follows directly

Lemma 5.

$$h(A_2) = h(A_1) + \frac{1}{2}\log(\deg(\phi)) - \frac{1}{[K:\mathbb{Q}]}\cdot \log(\# s^*(\Omega^1_{G/R})).$$

Remark. If G is annihilated by a number $n \in \mathbb{N}$, then n also annihilates $\Omega^1_{G/R}$. It follows that

$$\exp(2[K:\mathbb{Q}]\cdot(h(A_2) - h(A_1)))$$

is a rational number in whose numerator and denominator only the prime factors of $\deg(\phi)$ appear. The exponents of these primes can be bounded by their exponents in $\deg(\phi)$.

We now investigate the behavior of the $h(A_n)$, in the case $A_n = A/G_n$, where G_n runs through the levels of an l-divisible group $G \subseteq A[l^\infty]$.

Theorem 2. *Let $p: A \to \mathrm{Spec}(R)$ be a semiabelian variety with proper generic fibre, l a prime number, and $G/K \subseteq A[l^\infty]/K$ an l-divisible subgroup.*

Furthermore, let G_n be the kernel of l^n in G, and A_n the semiabelian variety $A_n = A/G_n$. Then

$$h(A_n) = h(A).$$

PROOF. Let $v_1, \ldots, v_n$ be the places of k lying over l, $K_i = K_{v_i}$ the corresponding local fields, $R_i \subseteq K_i$ the valuation rings, $m_i = [K_i : \mathbb{Q}_l]$, so that

$$m = [K:\mathbb{Q}] = \sum_{i=1}^{r} m_i.$$

We fix an i, and consider the formal group scheme $\hat{A}$ over $\mathrm{Spf}(R_i)$, the completion of A/R_i along the fibre A_s over the closed point s of $\mathrm{Spec}(R_i)$.

A_s is an extension $0 \to T_s \to A_s \to B_s \to 0$, with T_s a torus, B_s an abelian variety.

According to general fundamental theorems, one can lift T_s to a torus T over $\mathrm{Spec}(R_i)$, and $\hat{T}$ is a closed formal subscheme of $\hat{A}$. (Morphisms from T_s into smooth group schemes can be lifted.)

Let

$$\hat{H}_i = \hat{A}[l^\infty]$$

be the associated l-divisible group. $\hat{H}_i$ is the formal completion of an l-divisible group H_i over R_i, and H_i/K_i is an l-divisible subgroup of $A[l^\infty]/K_i$. The same holds for $T[l^\infty]$, and to these subgroups, there correspond $\mathbb{Z}_l$-sublattices

$$T_l(T) \subseteq T_l(H_i) \subseteq T_l(A). \qquad \square$$

Lemma 6. *Let $D_i = \mathrm{Gal}(\bar{K}_i/K_i)$ be the absolute Galois group of K_i, $I_i \subseteq D_i$ the ramification group.*

Then I_i acts trivially on $T_l(A)/T_l(H_i)$, and the induced action of $D_i/I_i \cong \hat{\mathbb{Z}}$ factors through a finite quotient of $\hat{\mathbb{Z}}$.

PROOF. Let

$$\langle\ ,\ \rangle: T_l(A) \times T_l(A) \to \mathbb{Z}_l(1) = T_l(\mathbb{G}_m)$$

be the symplectic form induced by a polarization of A/K. $\langle\ ,\ \rangle$ is not degenerate, and it is known [SGA VII, Exp IX, §7] that

$$\langle T_l(T), T_l(H_i)\rangle = 0.$$

By a dimension argument, $T_l(H_i) = T_l(T)^\perp$, and we have an injection

$$T_l(A)/T_l(H_i) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(T_l(T), \mathbb{Z}_l(1)).$$

This injection is D_i linear, and D_i acts on $\mathrm{Hom}_{\mathbb{Z}_l}(T_l(T), \mathbb{Z}_l(1))$ in the required way. $\square$

Now, once again, back to our $G/K \subseteq A[l^\infty]/K$. After base extension $K \subseteq K_i$, we can form the intersection $G_i = G \cap H_i$. This is the maximal l-divisible subgroup of G_i/K_i which can be extended over R_i, and we have

$$\#(s^*\Omega^1_{A/A_n} \otimes_R R_i) = \# s^*(\Omega^1_{(G_i)_n/R_i}).$$

By [13, Prop. 2], one can calculate this immediately: Let d_i be the dimension of the maximal formal subgroup of G_i. Then

$$\# s^*(\Omega^1_{(G_i)_n/R_i}) = l^{n\cdot m_i \cdot d_i}.$$

If C_i denotes the completion of the algebraic closure of K_i, then it is known furthermore [13, Theorem 3, Cor. 2], that as D_i-modules

$$T_l(G_i) \otimes_{\mathbb{Z}_l} C_i \cong C_i^{h_i - d_i} \oplus C_i^{d_i}(+1) \qquad (h_i = \text{height}\ (G_i),\ \text{“}(+1)\text{”} = \text{Tate twist}).$$

Together with Lemma 6, this implies that D_i acts on

$$\Lambda^h T_l(G) \otimes_{\mathbb{Z}_l} C_i \subseteq \Lambda^h T_l(A) \otimes_{\mathbb{Z}_l} C_i \qquad (h = \text{height}\ (G))$$

as on

$$C_i(\chi_0^{d_i}) = C_i(d_i) \qquad (\chi_0 = \text{cyclotomic character}).$$

We now carry this over to the global case.

We have

$$\# s^*(\Omega^1_{A/A_n}) = l^{n\sum_{i=1}^{r} m_i d_i} \qquad (l^n \cdot \Omega^1_{A/A_n} = 0)$$

and

$$h(A_n) - h(A) = n \cdot \log(l) \cdot \left(\frac{h}{2} - \sum_{i=1}^{r} \frac{m_i}{m} \cdot d_i\right).$$

We must therefore show that $\sum_{i=1}^{r} m_i d_i = \frac{1}{2} mh$. For this purpose, we consider the absolute Galois group $\tilde{\pi} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and the $\tilde{\pi}$-module

$$\tilde{V} = \mathrm{Ind}_{\pi}^{\tilde{\pi}}(T_l(A)) \qquad (\pi = \mathrm{Gal}(\bar{K}/K)).$$

This contains the submodule

$$\tilde{W} = \mathrm{Ind}_{\pi}^{\tilde{\pi}}(T_l(G))$$

of rank mh, and $\tilde{\pi}$ acts on the line

$$L = \Lambda^{mh}(\tilde{W}) \subseteq \Lambda^{mh}(\tilde{V})$$

via a character $\chi\colon \tilde{\pi} \to \mathbb{Z}_l^*$.

From class field theory it follows that χ is of the form

$$\chi = (l\text{-adic power of } \chi_0) \cdot (\text{character of finite order}).$$

The above l-adic power of χ_0 is determined as follows:

Let C be the completion of the algebraic closure of $\mathbb{Q}_l$,

$$D \cong \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l) \subseteq \tilde{\pi}$$

the decomposition group of l. Then as D-modules we have

$$L \otimes_{\mathbb{Z}_l} C \cong C\left(+ \sum_{i=1}^{r} m_i d_i\right)$$

(this follows from our previous calculations), and hence by [13, Theorem 2],

$$\chi \cdot \chi_0^{-\sum_{i=1}^{r} m_i d_i}$$

is a character of finite order of D and also of $\tilde{\pi}$. Finally, from the part of the Weil conjectures already proven by Weil, together with some local considerations, it follows that, for almost all p, $\chi(F_p)$ (p = a prime number, F_p = Frobenius) is an algebraic number, all of whose conjugates have absolute value $p^{mh/2}$. Since $\chi_0(F_p) = p$, we have, as desired

$$\sum_{i=1}^{r} m_i d_i = \frac{mh}{2}.$$

§5. Endomorphisms

Let K be a number field, A/K an abelian variety of dimension g, l a prime number, $T_l = T_l(A)$ the Tate module, on which $\pi = \mathrm{Gal}(\bar{K}/K)$ acts.

Theorem 3. *The action of π on $T_l \otimes_{\mathbb{Z}} \mathbb{Q}_l$ is semisimple.*

Theorem 4. *The map*

$$\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \mathrm{End}_{\pi}(T_l)$$

is an isomorphism.

PROOF. The two theorems are proven together. It is well known that it suffices, instead of Theorem 4, to prove the somewhat weaker statement that the map

$$\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \mathrm{End}_{\pi}(T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$$

is bijective.

For the proof one may extend the ground field, or replace A by an isogenous abelian variety. We can also assume that A/K is principally polarized, and that A extends to a semiabelian variety over $\mathrm{Spec}(R)$. Then T_l admits a nondegenerate skew-symmetric bilinear form. Let

$$W \subseteq T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

be a π-invariant maximal isotropic subspace. This corresponds to an l-divisible subgroup $G \subseteq A[l^\infty]$, and the semiabelian varieties $A_n = A/G_n$ again admit principal polarizations.

By Theorem 2, $h(A_n) = h(A)$, and by Theorem 1, infinitely many A_n's are isomorphic.

As in [16], it follows that W is the image of an idempotent in $\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}_l$. The rest of the proof goes exactly as in [16], and it will only be sketched here:

Choose $a, b, c, d \in \mathbb{Q}_l$ with $a^2 + b^2 + c^2 + d^2 = 1$.

Set

$$v = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$$

(corresponding to the quaternion $a + bi + cj + dk$), so $v \cdot {}^t v = -1$. If W is an arbitrary π-invariant subspace of $T_l \otimes_{\mathbb{Z}} \mathbb{Q}_l$, then one applies the above considerations to the maximal isotropic subspace.

$$W_1 = \{(x, vx) \mid x \in W^4\} \oplus \{(y, -vy) \mid y \in (W^{\perp})^4\} \subseteq T_l(A)^8 \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Corollary 1. *Let A_1 and A_2 be abelian varieties over K. Then*

$$\mathrm{Hom}_K(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \mathrm{Hom}_{\pi}(T_l(A_1), T_l(A_2))$$

is an isomorphism. □

PROOF. Theorem 4 applied to $A_1 \times A_2$.

The L-series of A is defined, as is well known, as

$$L(A, s) = \prod_v \frac{1}{\det(1 - (N_v)^{-s} \cdot F_v | T_l(A))} = \prod_v L_v(A, s),$$

where the product runs over almost all places of K. The local L-factors are independent of l.

Corollary 2. *Let A_1, A_2 be as in Corollary 1. The following are equivalent:*

(i) *A_1 and A_2 are isogenous.*
(ii) *$T_l(A_1) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \cong T_l(A_2) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ as π-modules.*
(iii) *$L_v(s, A_1) = L_v(s, A_2)$ for almost all places v of K.*
(iv) *$L_v(s, A_1) = L_v(s, A_2)$ for all v.*

PROOF. The equivalence of (i) and (ii) follow from Theorem 4, that of (ii) and (iii) from Theorem 3 (+ Čebotarev), and that (ii) implies (iv) implies (iii) is trivial. □

Corollary 3. *Let A/K be an abelian variety, $d > 0$. Then there are only finitely many isomorphism classes of abelian varieties B/K, with polarization of degree d, such that, for all l, $T_l(A) \cong T_l(B)$.*

PROOF. The assumptions imply that for every l there exists an isogeny between A and B which is of degree prime to l. Furthermore, for the purpose of the proof, we may extend the ground field, and then assume that A and all B's extend to semiabelian varieties over Spec(R), and that for all B's, there exists an isogeny of degree $\sqrt{d}$ with a principally polarized abelian variety. One then comes easily to the following assumptions:

(a) all B's have semistable reduction;
(b) all B/K are principally polarized;
(c) there exists an N such that for every prime number l and all B, there exist isogenies $\phi: A \to B$, for which the greatest power of l in $\deg(\phi)$ divides N.

The remark after Lemma 5 then shows that $\exp(2[K:\mathbb{Q}](h(B) - h(A)))$ is a rational number, whose numerator and denominator divide a certain power of N.

Thus $h(B)$ is bounded, and one can apply Theorem 1. □

§6. Finiteness Theorems

Theorem 5. *Let S be a finite set of places of K. Then there are only finitely many isogeny classes of abelian varieties over K of a given dimension which have good reduction outside S.*

PROOF. Let A be such an abelian variety. According to the Weil conjectures, for $v \notin S$ there are only finitely many possibilities for the local L-factors $L_v(A, s)$. We will construct finitely many places $v_1, \ldots, v_r$, such that two A's are isogenous if they have the same local L-factor at these places. For this purpose, one chooses a prime number l. By Lemma 4, there exists a finite Galois extension $K' \supseteq K$ that contains all field extensions of K of degree $\leq l^{8g^2}$ which are unramified outside l and S ($g = \dim(A)$).

Let $G = \mathrm{Gal}(K'/K)$; and choose $v_1, \ldots, v_r$ such that every conjugacy class in G contains the image of a Frobenius F_v for $v \in \{v_1, \ldots, v_r\}$ (Čebotarev). Then $v_1, \ldots, v_r$ fulfills our condition: Let A_1 and A_2 be two abelian varieties over K which have the same local L-factors at $v_1, \ldots, v_r$.

Let

$$M \subseteq \mathrm{End}_{\mathbb{Z}_l}(T_l(A_1)) \times \mathrm{End}_{\mathbb{Z}_l}(T_l(A_2))$$

be the $\mathbb{Z}_l$-subalgebra which is generated by the image of π.

Then M is a free $\mathbb{Z}_l$-module of rank $\leq 8g^2$, and M has representations on $T_l(A_1)$ and $T_l(A_2)$.

We must show that for every $m \in M$

$$\mathrm{Tr}(m | T_l(A_1)) = \mathrm{Tr}(m | T_l(A_2)).$$

It naturally suffices to prove this for m in a $\mathbb{Z}_l$-module basis of M, and by assumption the equality already holds if m is the image of an element of the conjugacy class of F_v, for $v \in \{v_1, \ldots, v_r\}$. We show that these images generate M over $\mathbb{Z}_l$. By Nakayama it is enough that they generate M/lM. This holds for the following reason:

We have a representation

$$\rho: \pi \to (M/lM)^* = \text{units of } M/lM,$$

whose image generates M/lM.

Since $\#(M/lM)^* \leq l^{8g^2}$, ρ factors through G, and $\rho(\pi)$ is the union of the images of the conjugacy classes of F_v, $v \in \{v_1, \ldots, v_r\}$. □

Theorem 6 (Shafarevich Conjecture). *Let S be a finite set of places of K, $d > 0$. Then there are only finitely many isomorphism classes of abelian varieties over K of a given dimension, with polarization of degree d, which have good reduction outside S.*

PROOF. By Theorem 5, we may assume that all the abelian varieties under consideration are isogenous to a fixed A/K. As in the proof of Corollary 3 to Theorem 4, we may further assume that all the B's extend to semiabelian varieties over Spec(R), and that $d = 1$. We already know that

$$\exp(2[K:\mathbb{Q}](h(B) - h(A)))$$

is a rational number. We will construct a number N such that the numerator and denominator of this rational number have no prime factor $l > N$, and so that the powers of l dividing them are bounded for the prime numbers $l \leq N$.

The latter is very easy: If, for two abelian varieties, B_1/K and B_2/K, $T_l(B_1)$ and $T_l(B_1)$ are isomorphic as π-modules, then by Theorem 4 there exists an isogeny between B_1 and B_2, of degree prime to l, and l does not occur in

$$\exp(2[K:\mathbb{Q}](h(B_1) - h(B_2))).$$

It thus suffices to show that there are only finitely many isomorphism classes of π invariant lattices in $T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. For this let M_l be the $\mathbb{Z}_l$-subalgebra of $\mathrm{End}_{\mathbb{Z}_l}(T_l(A))$ generated by π. Everything follows then from the fact that $M_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is semisimple (Theorem 3).

We now come to the choice of N. For this, let n be the product of prime numbers l, for which either the extension $K \supseteq \mathbb{Q}$ is ramified at l, or A does not have good reduction at all of the places of characteristic l.

Choose any prime number p which does not divide n. Again let

$$\tilde{\pi} = \mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q}) \supseteq \pi = \mathrm{Gal}(\bar{K}/K),$$

and, for $0 \leq h \leq 2gm$ ($g = \dim(A)$, $m = [K:\mathbb{Q}]$), let

$$P_h(T) = \det[T - F_p | \Lambda^h(\mathrm{Ind}_{\pi}^{\tilde{\pi}}(T_l(A)))].$$

Here l is a prime number, prime to pn, and F_p denotes the Frobenius at the place p.

The $P_h(T)$ are independent of l, have coefficients in $\mathbb{Z}$, and their zeros have absolute value $p^{+h/2}$ (Weil conjecture, or better, theorem).

We now choose $N \geq 2$ so large that no prime number $l > N$ divides $P_h(\pm p^j)$ for

$$0 \leq h \leq 2gm,$$
$$0 \leq j \leq gm,$$
$$j \neq \tfrac{1}{2}h.$$

In addition, choose $N \geq np$.

We will show for every isogeny

$$\phi: B_1 \to B_2$$

of abelian varieties isogenous to A, whose degree is a power of l for a prime number $l > N$, that $h(B_1)$ and $h(B_2)$ are equal. This argument is similar to that in the proof of Theorem 2: We may assume that l annihilates the kernel G of ϕ. Let

$$V_l = T_l(B_1)/l \cdot T_l(B_1) \cong B_1[l](\bar{K}),$$
$$\tilde{V}_l = \mathrm{Ind}_{\pi}^{\tilde{\pi}}(V_l),$$
$$W_l = G(\bar{K}) \subseteq V_l$$
$$\tilde{W}_l = \mathrm{Ind}_{\pi}^{\tilde{\pi}}(W_l) \subseteq \tilde{V}_l.$$

If ϕ has degree l^h, then $\tilde{\pi}$ acts on

$$L = \Lambda^{mh}(\tilde{W}_l) \subseteq \Lambda^{mh}(\tilde{V}_l)$$

via a character $\chi: \tilde{\pi} \to (\mathbb{Z}/l\mathbb{Z})^*$.

If $\varepsilon: \tilde{\pi} \to \{\pm 1\}$ denotes the character through which $\tilde{\pi}$ acts on $\Lambda^m \mathrm{Ind}_{\pi}^{\tilde{\pi}}(\mathbb{Z})$, then $\chi \cdot \varepsilon^h$ is unramified outside l, because the inertia groups of places of K which do not divide l act unipotently on V_l (semistable reduction). By class field theory $\chi \cdot \varepsilon^h$ is a power of the cyclotomic character χ_0. The exponent of this power can be determined with the help of [10, Theorem 4.11] (instead of Tate's theory [13]) as follows:

Let

$$l^d = \# s^*(\Omega^1_{G/R}),$$
$$0 \leq d \leq gm.$$

Then $\chi \cdot \varepsilon^h = \chi_0^{+d}$ (according to Raynaud). Thus $\chi_0^d(F_p) = \pm p^d$ is a zero of $P_{mh}(T)$ modulo l, and by our choice of N, $d = hm/2$ must hold.

Since again

$$h(B_2) - h(B_1) = \log(l)\left(\frac{h}{2} - \frac{d}{m}\right),$$

our claim is proved, and it follows that the $h(B)$'s of the B's under consideration are bounded. Thus Theorem 6 follows from Theorem 1. □

Corollary 1. *There are only finitely many isomorphism classes of smooth curves of genus $g \geq 2$ which have good reduction outside of S.*

PROOF. Torelli. □

Theorem 7 (Mordell Conjecture). *Let X/K be a smooth curve of genus $g \geq 2$. Then $X(K)$ is finite.*

PROOF. This argument is in [9]: After extending the ground field if necessary, there is an unramified covering of degree $m > 2$:

$$\phi: X_1 \to X.$$

Lemma 4 furnishes a finite extension field $K_1 \supseteq K$ such that for every $x \in X(K)$, $\phi^{-1}(x)$ consists of m different K_1-rational points. Choose one of these points, say $y \in p^{-1}(x)$.

Let $D = \phi^{-1}(x) - \{y\}$ and A/K_1 the generalized Jacobian of the pair (X_1, D). With the help of y one constructs a map from $X_1 - D$ to A.

Multiplication by 2 on A then induces a covering $Y(X) \to X_1$, ramified exactly over D, for which the curve $Y(x)$ can have bad reduction only at those places v of K_1, and for which one of the following three conditions hold:

(a) v divides 2.
(b) X_1 has bad reduction at v.
(c) ϕ ramifies in the fibre mod v.

There are only finitely many such places, and thus only finitely many possibilities for $Y(x)$.

The same holds for the map $Y(x) \to X_1 \to X$, which ramifies exactly over x. The claim follows. □

Remarks. (1) In this way one also obtains a proof of the Siegel theorem about integral points, which makes no use of diophantine approximation.

(2) With the help of the methods of [16], one can conclude from Theorem 6, that for almost all prime numbers l, the subalgebra M_l of $\mathrm{End}_{\mathbb{Z}_l}(T_l(A))$ generated by π is the full commutator of $\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Z}_l$.

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ERRATUM

N. Katz has remarked that Theorem 2 in the above work is not completely correct. (O. Gabber constructed a counterexample.) The statement of the theorem should be replaced by the following which suffices for what comes after it:

The sequence $h(A_n)$ becomes stationary.

The mistake was in overlooking two subtle points. However, the original proof works if one replaces from the beginning, $A = A_0$ by A_m, for large enough m.

The problems are as follows:

(a) If $W \subseteq T_l(\hat{A})$ is a D_l-invariant sublattice, then of course there is a corresponding l-divisible subgroup of A/K_i, and by forming the Zariski closures, one obtains a system of finite flat group schemes over $\mathrm{Spf}(R_i)$ or also over $\mathrm{Spec}(R_i)$. However, these form an l-divisible group only when the mappings

$$G_{i,n+1}/G_{i,n} \to G_{i,n}/G_{i,n-1}$$

are isomorphisms for $n \geq 1$. In general, one cannot expect this. Nevertheless, a consideration of the discriminant shows that this is the case for large n. Passing from $A = A_0$ to A_m means that one need only consider these mappings for $n > m$. This argument is already found in Tate [1].

(b) In general, the intersection $G_i = G \cap H_i$ of l-divisible groups over $\mathrm{Spec}(R_i)$ does not define an l-divisible group even over K_i. This problem also disappears if we go to a suitable A_m. One may then continue as in (a).

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