Anelastic Limits for Euler Type Systems

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Abstract

In this paper, we present rigorous derivations of anelastic limits for compressible Euler type systems when the Mach (or Froude) number tends to zero. The first and main part is to prove local existence and uniqueness of strong solution together with uniform estimates on a time interval independent of the small parameter. The key new remark is that the systems under consideration can be written in a form where ideas from [MeSc1] can be adapted. The second part of the analysis is to pass to the limit as the parameter tends to zero. In this context, the main problem is to study the averaged effect of fast acoustic waves on the slow incompressible motion. In some cases, the averaged system is completely decoupled from acoustic waves. The first example studied in this paper enters this category: it is a shallow-water type system with topography and the limiting system is the inviscid lake equation (rigid lid approximation). This is similar to the low Mach limit analysis for prepared data, following the usual terminology, where the acoustic wave disappears in a pure pressure term for the limit equation. The decoupling also occurs in infinite domains where the fast acoustic waves are rapidly dispersed at infinity and therefore have no time to interact with the slow motion (see [Sc, MeSc1, Al2]).

In other cases, and this should be expected in general for bounded domains or periodic solutions, this phenomenon does not occur and the acoustic waves leave a nontrivial averaged term in the limit fluid equation, which cannot be incorporated in the pressure term. In this case, the limit system involves a fluid equation, coupled to a nontrivial infinite dimensional system of differential equations which models the energy exchange between the fluid and some remanent acoustic energy. This was suspected for the periodic low Mach limit problem for

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nonisentropic Euler equations in [MeSe2] and proved for finite dimensional models. The second example treated in this paper, namely Euler type system with heterogeneous barotropic pressure law, is an example where this scenario is rigorously carried out. To the authors' knowledge, this is the first example in the literature where such a coupling is mathematically justified.

Keywords: Compressible Euler equations, heterogeneous media, shallow-water system, rigid-lid approximation, anelastic limit, lake equation, low mach number, low Froude Number.

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1 Introduction

Anelastic limits starting from diffusive systems have been recently studied from a Mathematical point of view for instance in [BrGiLi], [Ma1] and [FeMaNoSt]. These works concern respectively the degenerate viscous shallow-water equations with bathymetry and the compressible Navier-Stokes equations with high potential and constant viscosities. In all these papers, the authors consider global weak solutions where the time interval is fixed and, in the ill-prepared case, they prove that, from an energetical point of view, acoustic waves do not interact with the mean velocity field. In this paper, we consider anelastic limits starting from two compressible Euler-type systems.

I) The *first model* is the two space dimension shallow-water system with topography, namely:

model1 (1.1)
$$\begin{cases} \partial_t h + \operatorname{div}(hv) = 0, \\ \partial_t(hv) + \operatorname{div}(hv \otimes v) + h\dfrac{\nabla(h-h_b)}{\varepsilon^2} = 0 \end{cases}$$

where v denotes the vertical averaged of the hoziontal velocity field component, h the height of water and the bathymetry h_b is a given function depending on the space variables, see for instance [Br], [BrGiLi]. Note the analysis presented here extends to similar inviscid systems, see for instance [Ma1] for corresponding viscous systems.

II) The *second model* we have in mind is the Euler equation with heterogeneous pressure law

model2 (1.2)
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \dfrac{\nabla(c(x)\rho^\gamma)}{\varepsilon^2} = 0 \end{cases}$$

with c a given function depending on space variables x, ρ the density of the fluid and v its velocity.

To prove existence and uniqueness of local strong solution on a time interval which does not depend on the small parameter ε, the main idea is to rewrite such systems under an appropriate form : in Section 2, we prove that after a suitable change of unknows, both system enter the general framework of systems of the following form:

$$\begin{cases} a(\partial_t q + v \cdot \nabla q) + \dfrac{1}{\varepsilon}\mathrm{div}_{\mathbf{x}} u = 0, \\ b(\partial_t m + v \cdot \nabla m) + \dfrac{1}{\varepsilon}\nabla \psi = 0 \end{cases} \tag{1.3}$$

with $a(t,x)$ and $b(t,x)$ known and positive, and

$$q = \frac{1}{\varepsilon} Q(t,x,\varepsilon\psi), \quad m = \mu(t,x)u, \quad v = V(t,x,u,q), \tag{1.4}$$

where Q, μ and V are smooth functions of their arguments with $Q(t,x,0) = 0$, $\partial_\theta Q > 0$ and $\mu > 0$. Note that q is not singular in ε and satisfies

$$q = Q_1(t,x,\varepsilon\psi)\psi, \qquad \text{with} \quad Q_1 > 0. \tag{1.5}$$

For such systems, the existence and uniqueness of strong solutions on a time interval independent of ε can be obtained following the lines of [MeSc1]: first, since the singular term is skew symmetric there are easy L^2 estimates; next one commutes the equation with appropriate operators so that the commutators are controlled. This is explained in Section 3.

Concerning the asymptotic limit, we show that the two models (1.1) and (1.2) yield different analyses. For system (1.1) the main contribution of acoustic waves can be written as a gradient and therefore behaves as a pressure term. On the contrary, for system (1.2), the acoustic waves are strongly coupled to the mean velocity by a term which is *not* a gradient. The novelty of this model, compared to usual models studied in the literature, is that in the pressure law, $p = c\rho^\gamma$, the heterogeneity of the medium is modeled through a function $c(x)$ which changes from a point to another. The strong coupling between the acoustic part and the mean field is well known in many models. For instance it occurs for ill-prepared data for the non-isentropic Euler equations, where an additional equation for the entropy is added to the classical Euler system. For unbounded domains, the decoupling between the acoustics and the mean field is due to the fast dispersion of the acoustic waves at infinity (see [MeSc1, Al1]). On bounded or periodic domains, acoustic waves still travel very fast but are trapped and their averaged effect remains present in the limit. The analysis of this averaging seems to be very difficult in general due to the crossing eigenvalues phenomena, see for instance [BrDeGr, MeSe2]. In [BrDeGrLi] a semi-formal derivation is given for the nonstationary problem with ill prepared initial data. For System (1.2), the fast acoustic waves are governed by a space dependent wave equation, and this induces a nontrivial coupling in the limit. But, this fast wave equation is independent of the solution, in sharp contrast with the nonisentropic Euler's equation. Using this property, the limit can be carried out rigorously. The nonhomogeneity yields an extra non gradient term depending on the waves in the mean momentum equation. The description of the waves dynamics might be of physical and numerical interest.

The paper is organized as follows : in Section 2, we state the main results. In Section 3, we prove the uniform estimates on (u, ψ) which imply existence and uniqueness of strong solutions on a fixed interval of time for the general system. The last section is devoted to the asymptotic analysis when ε go to zero and the rigorous proof of convergence of solutions to solutions of asymptotic models. This section is splitted in four parts: the first one concerns the general problem, the second part (Theorem 2.3) provides a framework where we get an asymptotic decoupling between fast and slow scales. Note that System (1.1) satisfies the asumptions of this part. The third part concerns dispersion of acoustic waves on $\mathbb{R}^d$ which may lead to strong convergence (Theorem 2.4) and in the last part we consider averaged acoustics on the torus leading to a limit system involving a fluid equations coupled to a nontrivial infinite dimensional system of differential equations which models the energy exchange between the fluid and some remanent acoustic energy (Theorem 2.6). System (1.2) satisfies the assumptions of this part and thus it provides an example where this scenario is rigorously carried out. At the end of the paper, we give the coupled limit system corresponding to the heterogenous isentropic Euler system. To the authors' knowledge, this is the first example in the literature where such a coupling is mathematically justified.

2 Main results.

2.1 Reduction to System (1.3)

a) The shallow-water equations. Consider the system (1.1). Using the mass equation, denoting

$$\psi = (h - h_b)/\varepsilon, \qquad q = \frac{1}{\varepsilon}\ln(1 + \varepsilon\psi/h_b),$$

the system (1.1) may reads

model11

$$\begin{cases} h_b(\partial_t q + v\cdot\nabla q) + \dfrac{div(h_b v)}{\varepsilon} = 0, \\ \partial_t v + v\cdot\nabla v + \dfrac{\nabla\psi}{\varepsilon} = 0. \end{cases} \tag{2.1}$$

This system is of the form (1.3) with

$$a = h_b, \qquad b = 1, \qquad V = m = u/h_b, \qquad u = h_b v.$$

b) The heterogeneous Isentropic Euler equations. Consider the system (1.2). Using the mass equation $(2.3)_1$ and denoting

$$\psi = \frac{\gamma}{(\gamma-1)}\frac{(c^{1/\gamma}\rho)^{\gamma-1} - 1}{\varepsilon}, \qquad q = \frac{1}{\varepsilon(\gamma-1)}\ln(1 + \varepsilon(\gamma-1)\psi/\gamma),$$

the system (1.2) reads

model21

$$\begin{cases} c^{-1/\gamma}(\partial_t q + v\cdot\nabla q) + \dfrac{div(c^{-1/\gamma} v)}{\varepsilon} = 0, \\ c^{-1/\gamma}(\partial_t v + v\cdot\nabla v) + \dfrac{\nabla\psi}{\varepsilon} = 0, \end{cases} \tag{2.2}$$

This system is of the form (1.3) with

$$a = c^{-1/\gamma}, \qquad b = c^{-1/\gamma}, \qquad V = m = c^{1/\gamma}u, \qquad u = c^{-1/\gamma}v.$$

2.2 Uniform existence and uniqueness of smooth solution

Our first theorem concerns the existence of local strong solution on a time intervall which does not depend on the small parameter. For such purpose we consider the general form (1.3). We work on the domain $\mathbb{D}^d$ which is either the entire space $\mathbb{R}^d$, or a torus $\mathbb{T}^d$, or a mixed of these two types $\mathbb{T}^{d'} \times \mathbb{R}^{d-d'}$.

mainth **Theorem 2.1.** *There are $\varepsilon_0 > 0$ and $T_0 > 0$ which depend only on the $H^s(\mathbb{D}^d)$ norm of the initial data, $s > \frac{d}{2} + 1$, such that for all $\varepsilon \leq \varepsilon_0$ the Cauchy problem for (1.3) has a unique solution $(u, \psi) \in C^0([0,T]; H^s(\mathbb{D}^d))$.*

In particular, this gives the existence and uniqueness of local strong solution on a time intervall which does not depend on the small parameter ε for Systems (1.1) and (1.2).

2.3 Asymptotic limits

In a second part, we look at the asymptotic procedure, letting ε go to zero, in the ill-prepared case for (1.3) under some assumptions on a, b, μ, V and Q. We consider different domains, Ω and we investigate three different frameworks.

a) In the first case, assumptions on b and V provide a framework where we get an asymptotic decoupling between fast and slow scales. Note that System (1.1) satisfies the asumptions of this part and that this result applies on any domain Ω.

b) Next we work on $\Omega = \mathbb{R}^d$ with specific decreasing assumptions at infinity for the coefficients (similar to the one given in [MeSc1]) which provide dispersion of acoustic waves on $\mathbb{R}^d$. This lead to strong convergence and the decoupling observed in case a) occurs in this case too, but for a different reason.

c) In the last part, we consider averaged acoustics on the torus $\Omega = \mathbb{T}^d$. Under some assumptions on a, b, μ, V and Q, we mathematically justify a limit system involving a fluid equations coupled to a nontrivial infinite dimensional system of differential equations which models the energy exchange between the fluid and some remanent acoustic energy. System (1.2) satisfies the assumptions of this theorem and thus it provides an example where this scenario is rigorously carried out.

We first prove a result under the following

ass4.1 **Assumption 2.2.** *Suppose that b is constant and the function V has the special form*

$$V(t, x, u, q) = d\mu u,$$

where d is a constant.

Theorem 2.3. *Under Assumption 2.2 and assuming (4.1), the family of solutions u^ε (given by Theorem 2.1) converge weakly (in the sense of distributions) to the unique solution of*

$$\partial_t(\mu u) + d\mu u \cdot \nabla(\mu u) + \nabla \pi = 0, \qquad \operatorname{div} u = 0, \tag{2.3}$$

with initial data (4.11). Moreover, $\tilde{u}^\varepsilon$ converges strongly to u in $C^0([0,T]; H^{s'}_{loc}(\mathbb{D}^d))$ where $\tilde{u}^\varepsilon$ is defined by (4.7).

Remark. This theorem applies for system (1.1) and provides the lake equation (4.16) at the limit.

Next, we split the analysis in two parts:

i) *Dispersion of acoustics waves on $\mathbb{R}^d$.* Introduce the notation $\mathcal{F}$ for functions f on $[0,T]\times\mathbb{R}^d$ which have a limit $\underline{f}(t)$ as x tends to infinity and such that

$$|f(t,x) - \underline{f}(t)| \le C|x|^{-1-\delta}, \quad |\nabla_x f(t,x)| \le C|x|^{-2-\delta} \tag{2.4}$$

for some constant C and some $\delta > 0$. Let us define $\underline{Q}_1(t,x) = Q_1(t,x,0) = \partial_\theta Q(t,x,0)$.

Theorem 2.4. *Suppose that the coefficients a, b, μ and $\underline{Q}_1$ belong to the class $\mathcal{F}$. Then, the family $(\psi^\varepsilon, u^\varepsilon)$ converges strongly to $(0,u)$ in $L^2([0,T]; H^{s'}_{loc}(\mathbb{R}^d))$ and the limit equation for u is*

$$b(\partial_t(\mu u) + v\cdot\nabla(\mu u)) + \nabla\pi = 0, \qquad \operatorname{div} u = 0 \tag{2.5}$$

with $v = V(t,x,u,0)$.

Remark. This theorem applies for both systems (1.1) and (1.2) provided that the coefficients h_b or $c(x)$ respectively belong to the class $\mathcal{F}$.

ii) *Averaged acoustics on the torus.* We make the following assumption.

Assumption 2.5. *The reference domain in a torus $\mathbb{T}^d$, the coefficients a, b, μ do not depend on time and are smooth functions of $x \in \mathbb{T}^d$, as well as the function $Q(x,\theta)$. We further assume that $V = d(t,x)u$ is a linear function of u.*

Theorem 2.6. *Under the Assumption 2.5, the limit u satisfies the equation (4.37) with initial data (4.11). The additional term $\mathcal{I}$ is given by (4.36) and the α_j's satisfy the equations (4.38) with the initial conditions (4.39).*

Remark. This theorem applies for system (1.2) and the limit system is given by (4.40), (4.41), (4.42), (4.43) and (4.44). Note the strong coupling between mean velocity and acoutic waves.

3 Proof of Theorem 2.1.

sec3

The system being symmetric hyperbolic, solutions are known to exist and to be unique on a small interval of time depending on ε. The solutions are continued to a fixed interval, using suitable a-priori estimates for the smooth solutions. Following [MeSc1], we are looking for estimates of

defK

$$K := \sup_{t\in[0,T]} \big\| (u,\psi)(t)\big\|_{H^s}, \qquad s > \frac{d}{2} + 1. \tag{3.1}$$

Recall the following nonlinear estimates:

Lemma 3.1 (**Nonlinear estimates**). *i) for $s > \frac{d}{2}$, $0 \le k_j$ and $k := k_1 + \ldots + k_p \le s$:*

nl1

$$\big\| u_1 \ldots u_p \big\|_{s-k} \le C \big\| u_1 \ldots u_p\big\|_{s-k_1} \ldots \big\| u_p \ldots u_p \big\|_{s-k_p} \tag{3.2}$$

ii) for $s > 1 + \frac{d}{2}$, $1 \le k_j \le s$ and $k := k_1 + \ldots + k_p \le s$:

nl2

$$\big\| u_1 \ldots u_p \big\|_{s-k+p-1} \le C \big\| u_1 \ldots u_p\big\|_{s-k_1} \ldots \big\| u_p \ldots u_p \big\|_{s-k_p}. \tag{3.3}$$

Proof. i) well known. For ii) apply (3.2) with $s - 1 > \frac{d}{2}$ and $k_j - 1 \ge 0$ in place of k_j. Note that $k \ge p$, so that $s - k + p - 1 \le s - 1$. $\square$

The first step is to show that the constant K controls various other derivatives of the unknows which will be present in the analysis of commutator estimates.

Lemma 3.2. *Smooth solutions of (1.3) satisfy the following estimates*

eq1.10

$$\widetilde{K} := \sup_{t\in[0,T]} \sum_{k=0}^{s} \big\| (\varepsilon\partial_t)^k (u,\psi)(t) \big\|_{H^{s-k}} \le C(K), \tag{3.4}$$

and for $s \le k$:

eq1.11

$$\sup_{t\in[0,T]} \big\| (\varepsilon\partial_t)^k (q,m,v)(t) \big\|_{H^{s-k}} \le C(\widetilde{K}). \tag{3.5}$$

Here and below, $C(K)$ denotes a constant independent of ε, depending only K, the coefficients a, b, Q, mv, the dimension d and the index s

Proof. From the equation

$$\varepsilon\partial_t(u,\psi) = \Phi_\varepsilon(t,x,u,\psi)\nabla(u,\psi) + \Psi_\varepsilon(t,x,u,\psi)$$

with Φ_ε and Ψ_ε uniformly bounded with respect to $\varepsilon \geq 0$. By induction on k, the nonlinear estimates imply

$$\left\|(\varepsilon\partial_t)^k(u,\psi)(t)\right\|_{H^{s-k}} \leq C(K).$$

Next, because $Q(t,x,0) = 0$, $q = \tilde{Q}(t,x,\varepsilon\psi)\psi$, so the estimate (3.5) for $k = 0$ follows. For $k \geq 1$, $(\varepsilon\partial_t)^k q$ is the sum of terms

eq1.12

$$\varepsilon^{l+p-1}(\partial_t^l\partial_\theta^p Q)(t,x,\varepsilon\psi)(\varepsilon\partial_t\psi)^{k_1}\dots(\varepsilon\partial_t\psi)^{k_p} \tag{3.6}$$

where

$$l+p \geq 1, \qquad k_j \geq 1, \quad l+k_1+\ldots+k_p = k.$$

Thus (3.5) follows from (3.4) and (3.3).

Moreover,

eq1.13

$$(\varepsilon\partial_t)^k q = \partial_\theta Q(t,x,\varepsilon\psi)(\varepsilon\partial_t)^k\psi + r_r \tag{3.7}$$

where r_k is a sum of terms (3.6) with $l+p \geq 2$. In this case, there is (at least) an extra factor ε in front of the derivatives and the $k_j \leq s-1$ for all j. Thus one can apply another full derivative to r_k and

eq1.14

$$\left\|\partial_{t,x} r_k\right\|_{H^{s-k}} \leq C(K). \tag{3.8}$$

There estimates for m and v are similar and easier . $\square$

Lemma 3.3. *Let $(u_k,\psi_k) = (\varepsilon\partial_t)^k(u,\psi)$. Then for $k \leq s$:*

comm

$$\begin{cases} a(\partial_t q_k + v\cdot\nabla q_k) + \dfrac{1}{\varepsilon}\mathrm{div}_{\mathbf{x}} u_k = f_k, \\ b(\partial_t m_k + v\cdot\nabla m_k) + \dfrac{1}{\varepsilon}\nabla\psi_k = g_k \end{cases} \tag{3.9}$$

with

qk

$$q_k = \partial_\theta Q(t,x,\varepsilon\psi)\psi_k, \qquad m_k = \mu u_k, \tag{3.10}$$

estcom

$$\sup_{t\in[0,T]} \left\|(f_k,g_k)(t)\right\|_{L^2} \leq C(\widetilde{K}). \tag{3.11}$$

Proof. By (3.7)

$$(\varepsilon\partial_t)^k\partial_t q = \partial_t q_k + \partial_t r_k.$$

Thus f_k is equal to $a\partial_t r_k$ plus a sum of commutators terms

$$\varepsilon^{l-1}\partial_t^l a\,(\varepsilon\partial_t)^{k-l+1}q, \qquad \varepsilon^l\partial_t^l(av)\,(\varepsilon\partial_t)^{k-l}\partial_x q$$

with $l \geq 1$ and the estimate (3.11) follows. The proof of the estimate for g_k is similar. $\square$

This leads to consider the linearized system

lineq

$$\begin{cases} a(\partial_t(\rho\dot\psi) + v\cdot\nabla(\rho\dot\psi)) + \dfrac{1}{\varepsilon}\mathrm{div}_{\mathbf{x}}\dot u = \dot f, \\ b(\partial_t(\mu\dot u) + v\cdot\nabla(\mu\dot u)) + \dfrac{1}{\varepsilon}\nabla\dot\psi_k = \dot g \end{cases} \tag{3.12}$$

with $\rho = \partial_\theta Q(t, x, \varepsilon\psi)$.

Lemma 3.4. *There are C_0 and $C = C(K)$ such that the solution of (3.12) satisfies for $t \leq 1$:*

estL2

$$\begin{aligned}\big\|(\dot u, \dot\psi)(t)\big\|_{L^2} \leq C_0(1 + tC(K))\big\|(\dot u, \dot\psi)(0)\big\|_{L^2} \\ + C(K)\int_0^t \big\|(\dot f, \dot g)(t')\big\|_{L^2}dt'.\end{aligned} \tag{3.13}$$

Proof. The energy is

$$\mathcal{E}(t) = \frac{1}{2}\big\|\sqrt{a\rho}\dot\psi(t)\big\|_{L^2}^2 + \frac{1}{2}\big\|\sqrt{b\mu}\dot v(t)\big\|_{L^2}^2.$$

Then

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) \leq \mathrm{Re}\,\big(\dot f, \dot\psi\big)_{L^2} + \mathrm{Re}\,\big(\dot g, \dot u\big)_{L^2} \\ + O(\|\rho\partial_t a - a\partial_t\rho\|_{L^\infty} + \|\rho\partial_x(av) - av\partial_x\rho\|_{L^\infty})\|\dot\psi\|_{L^2}^2 \\ + O(\|\mu\partial_t b - b\partial_t\mu\|_{L^\infty} + \|\mu\partial_x(bv) - bv\partial_x\mu\|_{L^\infty})\|\dot u\|_{L^2}^2.\end{aligned}$$

Thus

$$\frac{d}{dt}\mathcal{E}(t) \leq \mathrm{Re}\,\big(\dot f, \dot\psi\big)_{L^2} + \mathrm{Re}\,\big(\dot g, \dot u\big)_{L^2} + C(K)\mathcal{E}(t)$$

and

$$\mathcal{E}^{\frac12}(t) \leq \mathcal{E}^{\frac12}(0) + C(K)\int_0^t \big\|(\dot f, \dot g)(t')\big\|_{L^2}dt'.$$

Note that

$$\|\varepsilon\psi(t)\|_{L^\infty} \le \|\varepsilon\psi(0)\|_{L^\infty} + t\|\varepsilon\partial_t\psi\|_{L^\infty} \le \|\varepsilon\psi(0)\|_{L^\infty} + tC\widetilde{K}.$$

Therefore,

estrho
$$\|\rho(t)\|_{L^\infty} \le C_0 + tC(K) \tag{3.14}$$

and

$$\|(\dot u, \dot\psi)(t)\|_{L^2} \le C_0(1 + tC(K))\mathcal{E}^{\frac{1}{2}}(t)$$

where C_0 depends only on the L^∞ norm of the initial data for u and ψ. The lemma follows. □

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Corollary 3.5. *There is C_0 which depends only on the H^s norm of the initial data for u and ψ and there is $C(K)$ such that the $(u_k, \psi_k) = (\varepsilon\partial_t)^k(u, \psi)$ for $k \le s$, satisfy*

estepsdtk
$$\|(u_k, \psi_k)(t)\|_{L^2} \le C_0 + tC(K). \tag{3.15}$$

Proof. For $k \ge 1$, apply the lemma to (comm 3.9). When $k = 0$, there is a slightly different equation for (u, ψ): writing $q = \tilde Q(t, x, \varepsilon\psi)\psi$ and considering $\tilde Q(t, x, \varepsilon\psi)$ as a known coefficient, yields an equation which is again of the form (lineq 3.12) for (u, ψ). □

Next, we estimate the vorticity.

Lemma 3.6. *Let $\omega :=$ curl$(b\mu u)$. Then $h := (\partial_t + v\nabla)\omega$ satisfies for $l \le s - 1$:*

$$\sup_{0 \le t \le T} \|(\varepsilon\partial_t)^l h(t)\|_{H^{s-1-l}} \le C(K). \tag{3.16}$$

Proof.

$$h = \operatorname{curl}\big((\partial_t b)\mu u + v[\nabla, b]\mu u\big) + [-\operatorname{curl}, v\nabla]b\mu u.$$

□

Corollary 3.7. *There is C_0 which depends only on the H^s norm of the initial data for u and ψ and there is $C(K)$ such that $\omega :=$ curl$(b\mu u)$ satisfies*

estvort
$$\|(\varepsilon\partial_t)^l\omega(t)\|_{H^{s-1-l}} \le C_0 + tC(K). \tag{3.17}$$

where C_0 depends only on the H^s norm of the initial data.

Lemma 3.8 (**Elliptic estimates**).

estdivcurl

$$\|u\|_{H^k} \leq C_k\Big(\|\operatorname{div} u\|_{H^{k-1}} + \|\operatorname{curl}(b\mu u)\|_{H^{k-1}} + \|u\|_{H^{k-1}}\Big). \tag{3.18}$$

Lemma 3.9. *For $0 \leq l \leq k \leq s$,*

estrec1

$$\|(u_{s-k}, \psi_{s-k})(t)\|_{H^l} \leq C_0 + (t+\varepsilon)C(K) + C_1\|(u_{s-k}, \psi_{s-k})(t)\|_{H^{l-1}} \tag{3.19}$$

where C_0 depends only on the H^s norm of the initial data and C_1 is independent of (u, ψ).

Proof. The equation yields

$$\operatorname{div} u = -a\varepsilon\partial_t q - \varepsilon av\nabla q, \quad \nabla\psi = -b\varepsilon\partial_t m - \varepsilon bv\nabla m.$$

Applying $(\varepsilon\partial_t)^l$ to these equation, and using (3.7), we see that

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$$\operatorname{div} u_l = -a\rho\psi_{l+1} - \varepsilon f_l, \quad \nabla\psi_l = -b\mu u_{l+1} - \varepsilon g_l \tag{3.20}$$

where $\rho = \partial_\theta Q(t, x, \varepsilon\psi)$ and

esterr

$$\|(\varepsilon\partial_t)^j (f_l, g_l)(t)\|_{H^{s-l-j-1}} \leq C(K). \tag{3.21}$$

Moreover, applying $(\varepsilon\partial_t)^l$ to $\operatorname{curl}(b\mu u)$ yields

epsdtcurl

$$\operatorname{curl}\big(b\mu u_l\big) - (\varepsilon\partial_t)^l\omega = \varepsilon h_l \tag{3.22}$$

where h_l satisfies estimates similar to (3.21).

We prove (3.19) by induction on k. For $k = 0$, this is (3.15). Assume that it is proven at the order $k-1$. When $l = 0$, the desired estimate is implied by (3.15). Thus, suppose that $1 \leq l \leq k \leq s$.

Then, from (3.20) and the induction hypothesis we find that

$$\|\operatorname{div} u_{s-k}\|_{H^{l-1}} + \|\nabla\psi_{s-k}\|_{H^{l-1}} \leq C_0 + (t+\varepsilon)C(K)$$

With (3.22) and (3.18), we deduce that

$$\|(u_{s-k}, \psi_{s-k})(t)\|_{H^l} \leq C_0 + (t+\varepsilon)C(K) + \|(u_{s-k}, \psi_{s-k})(t)\|_{H^{l-1}}$$

that is (3.19) $\square$

Corollary 3.10. *There is C_0 which depends only on the H^s norm of initial data and $C(K)$ such that*

$$\|(u(t), \psi(t)\|_{H^s} \leq C_0 + (t + \varepsilon C(K)). \tag{3.23}$$

Corollary 3.11. *There are C_0, $\varepsilon_0 > 0$ and $T_0 > 0$ which depend only on the H^s norm of the initial data, such that for $\varepsilon \le \varepsilon_0$ and $t \le T_0$:*

$$\big\|(u(t), \psi(t)\big\|_{H^s} \le 2C_0 \tag{3.24}$$

Proof. Take $T_0 \le \frac{1}{2C(2C_0)}$ and $\varepsilon_0 \le \frac{1}{2C(2C_0)}$. □

Together with the local existence theorem for symmetric hyperbolic systems, this uniform bound implies Theorem 2.1.

4 Asymptotics

4.1 The general problem

Assume that $(u^\varepsilon, \psi^\varepsilon)$ is a family of solutions of (1.3) on $[0,T]\times\mathbb{D}^s$, which satisfy the uniform estimates (3.1) (3.4) and (3.5) with a fixed K: in particular $(\varepsilon\partial_t)^k(u^\varepsilon, \psi^\varepsilon)$ and $(\varepsilon\partial_t)^k(q^\varepsilon, m^\varepsilon, v^\varepsilon)$ are uniformly bounded in $C^0([0,T]; H^{s-k})$. We further assume that the initial data converge strongly in H^s:

initconv
$$(u^\varepsilon{}_{|t=0}, \psi^\varepsilon{}_{|t=0}) \to (u_0, \psi_0) \quad \text{in } H^s. \tag{4.1}$$

In particular, up to the extraction of a subsequence, we can assume the following weak convergences in the sense of distribution for instance, as ε tends to 0:

weakconv
$$(u^\varepsilon, \psi^\varepsilon, q^\varepsilon, m^\varepsilon, v^\varepsilon) \ \rightharpoonup\ (u, \psi, q, m, v). \tag{4.2}$$

Moreover, the constitutive definition of Q and m show that

$$q = Q_1(t,x,0)\psi, \quad m = \mu u. \tag{4.3}$$

Multiplying the first equation by ε and passing to the weak limit shows that

incomp
$$\operatorname{div} u = 0, \qquad \nabla\psi = 0. \tag{4.4}$$

Consider the curl of the second equation equation of (1.3). It reads

curl
$$\operatorname{curl}\big(b(\partial_t + v^\varepsilon\cdot\nabla)m^\varepsilon\big) = 0. \tag{4.5}$$

The first information we get from it is that $\partial_t \operatorname{curl}(b\mu u^\varepsilon) = \operatorname{curl}(b\partial_t m^\varepsilon) + \operatorname{curl}\big((\partial_t b)m^\varepsilon\big)$ is bounded in $C^0([0,T]; H^{s-1})$. Therefore, the vorticity $\omega^\varepsilon = \operatorname{curl}(b\mu u^\varepsilon)$ converges strongly

convvort
$$\omega^\varepsilon \to \omega = \operatorname{curl}(b\mu u) \qquad \text{in } \ C^0([0,T], H^{s'}_{loc}) \tag{4.6}$$

for all $s' < s$.

To pass to the weak limit in the equation (4.5), we split u^ε into its incompressible and acoustic components, namely we write

decomp
$$u^\varepsilon = \widetilde{u}^\varepsilon + \frac{1}{b(t,x)\mu(t,x)} \nabla G^\varepsilon \tag{4.7}$$

with

decomp2
$$G^\varepsilon = (\Delta_{b\,\mu})^{-1} \mathrm{div} u^\varepsilon, \qquad \Delta_{b\,\mu} = \mathrm{div}(\frac{1}{b\mu}\nabla). \tag{4.8}$$

Here, $\Delta_{b\,\mu}$ and $(\Delta_{b\,\mu})^{-1}$ are seen as acting in the space $\dot{H}^{\pm 1}$ or in the space of functions with zero mean on the torus. In particular,

$$\mathrm{div}\, \tilde{u}^\varepsilon = 0, \quad \mathrm{curl}\,(b\mu)\tilde{u}^\varepsilon = \omega^\varepsilon.$$

Since $\mathrm{div}\, u^\varepsilon \rightharpoonup 0$, $\tilde{u}^\varepsilon$ converges weakly to u and the following weak convergence holds

convG
$$\nabla G^\varepsilon \rightharpoonup 0. \tag{4.9}$$

In addition, the uniform estimate (4.6) implies that the convergence of $\tilde{u}^\varepsilon$ is strong ang

convtildeu
$$\widetilde{u}^\varepsilon \to u \text{ in } \mathcal{C}^0([0,T]; H^{s'}_{loc}(\Omega)). \tag{4.10}$$

In particular, the initial data for u is

liminitdata
$$u_{|t=0} = \tilde{u}_0, \tag{4.11}$$

where $u_0 = \tilde{u}_0 + \frac{1}{b\mu}\nabla G_0$ as in (4.7).

We subsitute the splitting (4.7) in the equation (4.5). The first term to consider is

$$b\partial_t m^\varepsilon = b\partial_t\Big(\mu\widetilde{u}^\varepsilon + \frac{1}{b}\nabla G^\varepsilon\Big) = b\partial_t(\mu\widetilde{u}^\varepsilon) + \nabla\partial_t G^\varepsilon - \frac{\partial_t b}{b}\nabla G^\varepsilon$$

thus

$$\mathrm{curl}\,\big(b\partial_t m^\varepsilon\big) \rightharpoonup \mathrm{curl}\,\big(b\partial_t(\mu u)\big).$$

The next term to consider is

$$\begin{aligned} I^\varepsilon &:= b v^\varepsilon \cdot \nabla m^\varepsilon \\ &= bV\Big(t, x, \widetilde{u}^\varepsilon + \frac{1}{b\mu}\nabla G^\varepsilon, \frac{1}{\varepsilon}Q(t,x,\varepsilon\psi)\Big) \cdot \nabla\big(\mu\widetilde{u}^\varepsilon + \frac{1}{b}\nabla G^\varepsilon\big) \end{aligned} \tag{4.12}$$

Hence, passing to the limit in the sense of distributions in the equation (4.5) yields

lim1 (4.13)
$$\operatorname{curl} \partial_t(b\mu u) + \operatorname{curl} I = 0$$

where I is the weak limit of $I^\varepsilon := v^\varepsilon \cdot \nabla m^\varepsilon$, or

model6 (4.14)
$$b\partial_t(\mu u) + \nabla \pi + I = 0,$$

for some pressure term $\nabla\pi$ which is in accordance with the divergence free condition $\operatorname{div} u = 0$.

Conditions on V and Q are necessary to compute the limit I.

4.2 Asymptotic decoupling between fast and slow scales

The limit of $\operatorname{curl} I^\varepsilon$ is easily computed under the following assumption (2.2).
Proof of Theorem 2.3. Due to the form of V, the expression of I^ε simplifies to

$$I^\varepsilon = bd\mu u^\varepsilon \cdot \nabla(\mu u^\varepsilon) = d\big(b\mu\widetilde{u}^\varepsilon + \nabla G^\varepsilon\big) \cdot \nabla\big(\mu\widetilde{u}^\varepsilon + \frac{1}{b}\nabla G^\varepsilon\big).$$

There is no difficulty in passing to the weak limit in quadratic terms involving at least one strongly convergent factor, namely those with $\tilde{u}^\varepsilon$. The remaining term which involves two weakly convergent factors is

$$d(\nabla G^\varepsilon) \cdot \nabla(\frac{1}{b}\nabla G^\varepsilon) = \frac{d}{2b}\nabla|\nabla G^\varepsilon|^2.$$

It is an exact gradient, so that

$$\operatorname{curl}(I^\varepsilon) \rightharpoonup \operatorname{curl}\big(bd\mu u \cdot \nabla(\mu u)\big).$$

and Theorem 2.3 follows.

Shallow-water equations. Note that the model (1.1) satisfies this assumption, since then

$$V = \frac{u}{h_b}, \qquad \mu = \frac{1}{h_b} \qquad b = 1.$$

Thus analysis above applies to (1.1), and the limit system reads

model12 (4.15)
$$\begin{cases} \partial_t(\frac{u}{h_b}) + (\frac{u}{h_b}) \cdot \nabla(\frac{u}{h_b}) + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases}$$

Recalling that $v = u/h_b$, we get exactly the lake equation, namely

lake (4.16)
$$\partial_t v + v \cdot \nabla v + \nabla p = 0, \qquad \operatorname{div}(h_b v) = 0.$$

This proves the convergence to the inviscid lake equations.

4.3 Dispersion of acoustic waves on $\mathbb{R}^d$

The solutions of (1.3) satisfy

modelfastwaves

$$\begin{cases} \varepsilon a \partial_t \underline{Q}_1 \psi^\varepsilon + \operatorname{div} u^\varepsilon = \varepsilon f^\varepsilon \\ \varepsilon b \partial_t (\mu u^\varepsilon) + \nabla \psi^\varepsilon = \varepsilon g^\varepsilon . \end{cases} \tag{4.17}$$

with $\underline{Q}_1(t,x) = Q_1(t,x,0) = \partial_\theta Q(t,x,0)$. From the estimates for $(u^\varepsilon, \psi^\varepsilon)$, we know that f^ε and g^ε are bounded in $C^0([0,T]; H^{s-1})$. This systems governs the evolution on small scales of time or order ε.

In this paragraph, we consider solutions on $\mathbb{R}^d$. We sketch the analysis of [MeSc1] which proves that the family u^ε converges strongly to u in $L^2([0,T]; H^{s'}_{loc}(\mathbb{R}^d))$ for $s' < s$.

Sketch of proof of Theorem 2.4. The system (4.17) can be written

$$\varepsilon E_1 \partial_t E_2 U^\varepsilon + L(\partial_x) U^\varepsilon = \varepsilon F^\varepsilon \tag{4.18}$$

with

$$U = \begin{pmatrix} \psi \\ u \end{pmatrix}, \quad E_1 \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad E_2 = \begin{pmatrix} \underline{Q}_1 & 0 \\ 0 & \mu \end{pmatrix} \quad L(\partial_x) = \begin{pmatrix} 0 & \operatorname{div}_x \\ \nabla_x & 0 \end{pmatrix}. \tag{4.19}$$

Following P. Gérard ([Gér]), see also the introduction in [Bu], one introduces the microlocal defect measures of subsequences of u^ε. They are measures $\mathcal{M}$, on $\mathbb{R}_t \times \mathbb{R}_\tau$ valued in the space $\mathcal{L}$ of trace class operators on $L^2(\mathbb{R}^d)$. They can be written

$$\mathcal{M}(dt, d\tau) = M(t,\tau) \alpha(dt, d\tau) ,$$

where α is a scalar nonnegative Radon measure and M is an integrable function with respect to α with values in $\mathcal{L}$. The usual feature of defect measures is that they are supported in the characteristic variety of the equation. In our case, this means that for α-almost all (t,τ), $M(t,\tau)$ is valued in $H^1(\mathbb{R}^d)$ and

suppM

$$\big(i E_1(t) \tau E_2(t) \, + \, L(\partial_x) \big) M(t,\tau) \, = \, 0, \tag{4.20}$$

where $E_1(t)$ and $E_2(t)$ are seen as multiplication operators on $L^2(\mathbb{R}^d)$.

When $\tau \neq 0$, the kernel of $i\tau E_1 E_2 + L(\partial_x)$ is non trivial if and only if τ^2 is a positive eigenvalue of $\dfrac{1}{a\underline{Q}_1} \operatorname{div} \big(\dfrac{1}{b\mu} \nabla_x \big)$. When the coefficients belong to the class $\mathcal{F}$ this never occurs (see [RS, Hö] and [MeSc1]), implying that $M(t,\tau)=0$ for α-almost all t and $\tau \neq 0$. Thus, M is supported in $\tau = 0$ so (4.20) implies that

$$L(\partial_x)M(t,\tau) \,=\, 0 \quad \text{or} \quad (I - \Pi(D_x))\, M(t,\tau) \,=\, 0\,, \quad \alpha\ a.e$$

where $\Pi(D_x)$ is the orthogonal projector on $\ker L(\partial_x)$:

$$\Pi(D_x)\begin{pmatrix}\psi\\ u\end{pmatrix} = \begin{pmatrix}0\\ u - \nabla(\Delta^{-1}\operatorname{div} u)\end{pmatrix}.$$

As a corollary, the microlocal defect measure of $(I-\Pi(D_x))u^\varepsilon$ vanishes and, together with the uniform bounds in H^s, this implies that $(I-\Pi(D_x))U^\varepsilon$ converges strongly in $L^2_{loc}([0,T]\times\mathbb{R}^d)$. Together with the strong convergence of $\tilde u^\varepsilon$, this implies the local strong convergence of u^ε and ψ^ε. Since the limit $\psi(t,\cdot) \in H^s(\mathbb{R}^d)$ and $\nabla\psi = 0$, the limit ψ is equal to 0. For further details, we refer the reader to [MeSc1]. $\square$

4.4 Averaged acoustics on the torus

For the heterogeneous isentropic Euler equations (1.2), the coefficients of the fast acoustic operator depend only on x, not on time. In this case, the fast evolution is easily analyzed using a spectral decomposition. We first present the averaging method in the extended framework of Assumption (2.5). We then prove Theorem 2.6.

Under Assumption (2.5), the fast dynamics (4.17) reads:

$$\varepsilon E(x)\partial_t U^\varepsilon + L(\partial_x)U^\varepsilon = \varepsilon F^\varepsilon \tag{4.21}$$

with

$$E = \begin{pmatrix} a\underline{Q}_1 & 0\\ 0 & b\mu\end{pmatrix}, \quad L(\partial_x) = \begin{pmatrix} 0 & \operatorname{div}_x\\ \nabla_x & 0\end{pmatrix} \tag{4.22}$$

and

$$F^\varepsilon = \mathcal{F}_2(t,x,U^\varepsilon,\nabla U^\varepsilon) + \varepsilon\widetilde{F}^\varepsilon. \tag{4.23}$$

with $\mathcal{F}_2$ quadratic in $U^\varepsilon, \nabla U^\varepsilon$:

$$\mathcal{F}_2 = \begin{pmatrix} (\underline{Q}_1)^{-1}\partial_\theta^2 Q(x,0)\psi\operatorname{div} u - ad\, u\cdot\nabla(\underline{Q}_1\psi)\\ -bd\, u\cdot\nabla(\mu u)\end{pmatrix} \tag{4.24}$$

The problem reduces to study interaction of resonant time oscillations linked to the spectral properties of the operator $\tilde{L}(x, \partial_x) := E^{-1}(x) L(\partial_x)$. This operator is self adjoint with respect to the scalar product

$$\big\langle U, V \big\rangle := \int_{\mathbb{T}^d} E(x) U(x) \cdot \overline{V}(x) dx. \tag{4.25}$$

The fast evolution is governed by the group $\mathcal{E}(t) := e^{-tL(x,\partial_x)}$ and the solution of (4.21) is

$$U^\varepsilon(t) = \mathcal{E}(\frac{t}{\varepsilon}) U(0) + \int_0^t \mathcal{E}(\frac{t-s}{\varepsilon}) E^{-1} F^\varepsilon(s) ds. \tag{4.26}$$

The filtering method (see e.g. [Sc1]) consists in studying the limit of

$$V^\varepsilon(t) := \mathcal{E}(-\frac{t}{\varepsilon}) U^\varepsilon = U(0) + \int_0^t \mathcal{E}(-\frac{s}{\varepsilon}) E^{-1} F^\varepsilon(s) ds.$$

Equivalently, V^ε solves

$$\partial_t V^\varepsilon = \mathcal{E}(-\frac{t}{\varepsilon}) E^{-1} F^\varepsilon. \tag{4.27}$$

The group $\mathcal{E}(t)$ is unitary in L^2 for the scalar product (4.25) but it is also bounded in $H^s(\mathbb{T}^d)$, as a consequence of Theorem 2.1 applied to the linear equation (4.21). In particular, this implies that V^ε and $\partial_t V^\varepsilon$ are bounded in $C^0([0,T]; H^s)$ and $C^0([0,T]; H^{s-1})$ respectively. As a consequence

Lemma 4.1. *Extracting further a subsequence if necessary, V^ε converges strongly to a limit V in $C^0([0,T]; H^{s'}(\mathbb{T}^d))$. In particular,*

$$U^\varepsilon(t) \sim \mathcal{E}(\frac{t}{\varepsilon}) V(t) \quad in\ C^0([0,T]; H^{s'}(\mathbb{T}^d)). \tag{4.28}$$

Next, we analyze the evolution group $\mathcal{E}(t)$ using the spectral decomposition of $\widetilde{L}(x, \partial_x)$. Its kernel is

$$\mathbb{K} = \ker \widetilde{L} = \Big\{ U = \begin{pmatrix} \psi \\ u \end{pmatrix} : \ \nabla \psi = 0, \ \operatorname{div} u = 0 \Big\}. \tag{4.29}$$

The orthogonal projector $\Pi_{\mathbb{K}}$ on $\mathbb{K}$ is

$$\Pi_{\mathbb{K}} \begin{pmatrix} \psi \\ u \end{pmatrix} = \begin{pmatrix} \psi_0 \\ u_0 = u - \frac{1}{b\mu} \nabla G \end{pmatrix}, \tag{4.30}$$

where ψ_0 is the average of ψ for the measure $a\underline{Q}_1 dx$ and G solves $\operatorname{div}\left(\frac{1}{b\mu}\nabla G\right) = \operatorname{div} u$ as in (4.8).

The non zero eigenvalues of $\widetilde{L}(x, \partial_x)$ are $\pm i\sqrt{\kappa_j}$ where the κ_j are the positive eigenvalues of the acoustic wave operator

$$W(a\underline{Q}_1, b\mu)(\cdot) = -\frac{1}{a\underline{Q}_1}\operatorname{div}\big(\frac{1}{b\mu}\nabla\cdot\big) \tag{4.31}$$

Note that $W(a\underline{Q}_1, b\mu)$ is self-adjoint in $L^2(\mathbb{T}^d, a\underline{Q}_1\, dx)$. The eigenvectors of $\widetilde{L}$ for the eigenvalue $\pm i\sqrt{\kappa_j}$ are

defPhij
$$\Phi_{\pm}j = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_j \\ \pm\dfrac{1}{ib\mu\sqrt{\kappa_j}}\nabla\psi_j \end{pmatrix} \tag{4.32}$$

where ψ_j is an eigenvector of W associated to the eigenvalue κ_j. From now on, we fix an orthonormal basis in $L^2(\mathbb{T}^d, a\underline{Q}_1 dx)$ made of eigenfunctions $(\psi_0, \psi_1 \ldots, \psi_j \ldots)$ associated to the eigenvalues $\kappa_0 = 0 < \kappa_1 \le \kappa_2 \le \ldots$ of W. Note that κ_0 is simple and that ψ_0 is a constant. For an integer $j \in \mathbb{Z}\backslash\{0\}$ we set $\lambda_j = \sqrt{\kappa_j}$ when $j > 0$ and $\lambda_j = -\sqrt{\kappa_{-j}}$ when $j < 0$. Moreover, Φ_j denotes the function defined in (4.32). The $\{\Phi_j\}$ form an orthonormal basis of $\mathbb{K}^\perp$, the orthogonal of $\mathbb{K}$ for the scalar product (4.25).

Classical properties of the elliptic operator W on the torus imply that

$$\kappa_j \approx j^{2/d}$$

and that v belongs to $H^s(\mathbb{T}^d)$ if and only if

$$v = \sum \alpha_j\psi_j \qquad \text{with} \qquad \sum j^{2s/d}|\alpha_j|^2 < +\infty.$$

Accordingly we expand V^ε using the spectral decomposition of $\widetilde{L}$:

$$V^\varepsilon(t) = \Pi_{\mathbb{K}}V^\varepsilon(t) + \sum_{j\neq 0}\alpha_j^\varepsilon(t)\Phi_j, \qquad \alpha_j(t) = \big\langle V^\varepsilon(t), \Phi_j\big\rangle. \tag{4.33}$$

and the series converges in $C^0([0,T]; H^s)$. Therefore $\Pi_{\mathbb{K}}U^\varepsilon = \Pi_{\mathbb{K}}V^\varepsilon$ and :

$$U^\varepsilon(t) = \Pi_{\mathbb{K}}U^\varepsilon(t) + \sum_{j\neq 0} e^{-it\lambda_j/\varepsilon}\alpha_j^\varepsilon(t)\Phi_j.$$

The strong convergence $V^\varepsilon \to V$ implies the following.

Lemma 4.2. *$\Pi_{\mathbb{K}} U^\varepsilon$ converges strongly to $\Pi_{\mathbb{K}} V$ in $C^0([0,T]; H^{s'}(\mathbb{T}^d))$ and the α_j^ε converge strongly to $\alpha_j := \langle V, \Phi_j \rangle$ in $C^0([0,T])$.*

Thus,

asympprofb
$$U^\varepsilon(t) \sim U(t) + \sum_{j \neq 0} e^{-it\lambda_j/\varepsilon} \alpha_j(t) \Phi_j \quad \text{in } C^0([0,T]; H^{s'}(\mathbb{T}^d)). \tag{4.34}$$

and in particular U^ε converges weakly to $U = \Pi_{\mathbb{K}} U = \Pi_{\mathbb{K}} V$.

The equation (4.27) implies that

neweq
$$\begin{cases} \partial_t \Pi_{\mathbb{K}} U^\varepsilon(t) = \Pi_{\mathbb{K}} E^{-1} F^\varepsilon(t) \\ \partial_t \alpha_j^\varepsilon(t) = e^{it\lambda_j/\varepsilon} \big\langle E^{-1} F^\varepsilon(t), \Phi_j \big\rangle. \end{cases} \tag{4.35}$$

The limiting equations are obtained by taking the weak limits if the right hand sides. To compute them, we substitute the asymptotic form (4.34) of U^ε in (4.23):

Lemma 4.3. *One has $F^\varepsilon \sim F_2^\varepsilon$ in $C^0([0,T]; H^{s'-1}(\mathbb{T}^d))$ with*

$$\begin{aligned} F_2^\varepsilon = \mathcal{F}_2(U, \nabla U) &+ \sum_{j \neq 0} \alpha_j e^{-t\lambda_j/\varepsilon} \widetilde{\mathcal{F}_2}(U, \nabla U; \Phi_j, \nabla \Phi_j) \\ &+ \sum_{j \neq 0, k \neq 0} \alpha_k \alpha_j e^{-t(\lambda_j + \lambda_k)/\varepsilon} \widetilde{\mathcal{F}_2}(\Phi_k, \nabla \Phi_k; \Phi_j, \nabla \Phi_j), \end{aligned}$$

where $\widetilde{\mathcal{F}_2}$ is the bilinear form associated to the quadratic term $\mathcal{F}_2$.

Therefore, the stationary phase Theorem implies the following.

Lemma 4.4. *In the sense of distributions, one has the following weak convergence :*

$$\Pi_{\mathbb{K}}(E^{-1} F^\varepsilon) \rightharpoonup \Pi_{\mathbb{K}}\big(E^{-1}(\mathcal{F}_2(U, \nabla U) + \mathcal{I}\big)$$

where

defcall
$$\mathcal{I} = \sum_{\lambda_k + \lambda_j = 0} \alpha_k \alpha_j \widetilde{\mathcal{F}_2}(\Phi_k, \nabla \Phi_k; \Phi_j, \nabla \Phi_j). \tag{4.36}$$

Moreover,

$$\begin{aligned} e^{is\lambda_l/\varepsilon} \big\langle E^{-1} F^\varepsilon(s), \Phi_l \big\rangle \rightharpoonup &\sum_{\lambda_j = \lambda_l} \alpha_j \big\langle E^{-1}\big(\mathcal{F}_2(U, \nabla U; \Phi_j, \nabla \Phi_j)\big), \Phi_l \big\rangle \\ &+ \sum_{\lambda_j + \lambda_k = \lambda_l} \alpha_k \alpha_j \big\langle E^{-1} \mathcal{F}_2(\Phi_k, \nabla \Phi_k; \Phi_j, \nabla \Phi_j), \Phi_l \big\rangle. \end{aligned}$$

With this lemma, we can pass to the limit in the equations (4.35). In particular, we get that

$$\partial_t U = \Pi_{\mathbb{K}}\big(E^{-1}(\mathcal{F}_2(U, \nabla U) + \mathcal{I}\big).$$

Tracing back the definition, this gives the condition $\operatorname{div} u = 0$ implies that $\partial_t \psi = 0$ and the limit ψ is not present in the other equations. The equation for u is

limequ

$$b\mu\partial_t u + bd\, u \cdot \nabla(\mu u) + \mathcal{I} + \nabla\pi = 0. \tag{4.37}$$

The initial condition for u is still given by (4.11). Summing up, we have proved Theorem 2.6 with

alphaj

$$\begin{aligned} \partial_t \alpha_j = \sum_{\lambda_j = \lambda_l} \alpha_j \big\langle E^{-1}\big(\mathcal{F}_2(U, \nabla U; \Phi_j, \nabla\Phi_j)\big), \Phi_l\big\rangle \\ + \sum_{\lambda_j + \lambda_k = \lambda_l} \alpha_k \alpha_j \big\langle E^{-1}\mathcal{F}_2(\Phi_k, \nabla\Phi_k; \Phi_j, \nabla\Phi_j), \Phi_l\big\rangle. \end{aligned} \tag{4.38}$$

with the initial conditions

alphajinit

$$\alpha_j(0) = \langle U_0, \Phi_j\rangle. \tag{4.39}$$

Remark 4.5. *Generically, that means for general coefficients or for general tori, one expects that the eigenvalues of the wave equations κ_j are simple and that there are no resonances $\pm\sqrt{\kappa}_j \pm \sqrt{\kappa}_k \pm \sqrt{\kappa}_l = 0$, in which case the formulas above become simpler.*

Heterogeneous isentropic Euler equations. The analysis above applies to (1.2). Recall, see Section 2, that for this model

$$a = c^{-1/\gamma}, \qquad b = c^{-1/\gamma}, \qquad \mu = c^{1/\gamma}, \qquad V = du = c^{1/\gamma} u$$

and that

$$\underline{Q}_1(t, x) = 1/\gamma, \qquad \partial_\theta^2 Q(x, 0) = -(\gamma - 1)/\gamma^2.$$

Using Theorem 2.6, the limit system (4.37) reads

model23

$$\begin{cases} \partial_t u + u \cdot \nabla(c^{1/\gamma} u) + \mathcal{I} + \nabla\pi = 0 \\ \operatorname{div} u = 0 \end{cases} \tag{4.40}$$

where $\mathcal{I}$ is given by (4.36). From (4.24), $\mathcal{F}_2$ reads

$$\mathcal{F}_2 = \begin{pmatrix} -\dfrac{\gamma-1}{\gamma}\psi \operatorname{div} u - \dfrac{1}{\gamma}\, u\cdot\nabla\psi \\ -u\cdot\nabla(c^{1/\gamma}u) \end{pmatrix}$$

Thus, using that Φ_j denotes the function defined by

defPhi
$$\Phi_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_j \\ \dfrac{1}{i\lambda_j}\nabla\psi_j \end{pmatrix} \tag{4.41}$$

where ψ_j is an eigenvector of W namely

ell
$$W(\psi_j) = -\gamma c^{1/\gamma}\Delta\,\psi_j = \lambda_j^2\psi_j \tag{4.42}$$

associated to the eigenvalue λ_j^2. We get from (4.36)

233
$$\mathcal{I} = \sum_{j,\ell,\lambda_k+\lambda_j=0} \frac{\alpha_k\alpha_j}{2|\lambda_k|^2}\big(\nabla\psi_k\cdot\nabla(c^{1/\gamma}\nabla\psi_j) + \nabla\psi_j\cdot\nabla(c^{1/\gamma}\nabla\psi_k)\big) \tag{4.43}$$

The coefficients α_j are calculated using (4.38). The first term in right-hand side of (4.38) is given by

$$\sum_{\lambda_j=\lambda_l} \alpha_j\big\langle E^{-1}\big(\mathcal{F}_2(U,\nabla U;\Phi_j,\nabla\Phi_j)\big),\Phi_l\big\rangle$$

$$= -\sum_{\lambda_j=\lambda_\ell}\alpha_j\int_{\mathbb{T}^d}\big(\frac{1}{\sqrt{2}i\lambda_j}\nabla\psi_j\cdot\nabla(c^{1/\gamma}u)+u\cdot\nabla(c^{1/\gamma}\frac{1}{\sqrt{2}i\lambda_j}\nabla\psi_j)\big)\cdot\big(\frac{1}{\sqrt{2}i\lambda_\ell}\nabla\psi_\ell\big)$$

$$= \sum_{\lambda_j=\lambda_\ell}\frac{\alpha_j}{2|\lambda_j|^2}\int_{\mathbb{T}^d}\big(\nabla\psi_j\cdot\nabla(c^{1/\gamma}u) + u\cdot\nabla(c^{1/\gamma}\nabla\psi_j)\big)\cdot\nabla\psi_\ell$$

The second term in the right-hand side of (4.38) is given by

$$\sum_{\lambda_j+\lambda_k=\lambda_l}\alpha_k\alpha_j\big\langle E^{-1}\mathcal{F}_2(\Phi_k,\nabla\Phi_k;\Phi_j,\nabla\Phi_j),\Phi_l\big\rangle$$

$$= -\sum_{\lambda_j+\lambda_k=\lambda_\ell}\alpha_j\alpha_k\int_{\mathbb{T}^d}\Big(\frac{1}{\sqrt{2}i\lambda_j}\nabla\psi_j\cdot\nabla(c^{1/\gamma}\frac{1}{\sqrt{2}i\lambda_k}\nabla\psi_k)$$
$$+\frac{1}{\sqrt{2}i\lambda_k}\nabla\psi_k\cdot\nabla(c^{1/\gamma}\frac{1}{\sqrt{2}i\lambda_j}\nabla\psi_j)\Big)\cdot\frac{1}{\sqrt{2}i\lambda_\ell}\nabla\psi_\ell$$

$$
-\sum_{\lambda_j+\lambda_k=\lambda_\ell}\alpha_j\alpha_k\int_{\mathbb{T}^d}\frac{\gamma-1}{\gamma}\Big(\psi_j\mathrm{div}(\frac{1}{\sqrt{2}i\lambda_k}\nabla\psi_k)+\psi_k\mathrm{div}(\frac{1}{\sqrt{2}i\lambda_j}\nabla\psi_j))\psi_\ell
$$

$$
-\sum_{\lambda_j+\lambda_k=\lambda_\ell}\alpha_j\alpha_k\frac{1}{\gamma}\int_{\mathbb{T}^d}\Big(\frac{1}{\sqrt{2}i\lambda_j}\nabla\psi_j\cdot\nabla\psi_k+\frac{1}{\sqrt{2}i\lambda_k}\nabla\psi_k\cdot\nabla\psi_j\Big)\psi_\ell
$$

Therefore, we conclude that the coefficients α_j are given by the following system of ODE's

model24

$$
\partial_t\alpha_j=\sum_{\lambda_j=\lambda_\ell}\frac{\alpha_j}{2|\lambda_j|^2}\int_{\mathbb{T}^d}(u\cdot\nabla(c^{1/\gamma}\nabla\psi_j)+\nabla\psi_j\cdot\nabla\nabla(c^{1/\gamma}u))\cdot\nabla\psi_\ell \tag{4.44}
$$

$$
-\sum_{\lambda_j+\lambda_k=\lambda_\ell}\frac{i\alpha_j\alpha_k}{2\sqrt{2}}\frac{1}{\lambda_\ell\lambda_k\lambda_j}\int_{\mathbb{T}^d}\big((\nabla\psi_j\cdot\nabla(c^{1/\gamma}\nabla\psi_k))\cdot\nabla\psi_\ell+(\nabla\psi_k\cdot\nabla(c^{1/\gamma}\nabla\psi_j))\cdot\nabla\psi_\ell\big).
$$

$$
-\sum_{\lambda_j+\lambda_k=\lambda_\ell}\Big(\frac{i(\gamma-1)}{\sqrt{2}\gamma^2}\int_{\mathbb{T}^d}c^{-1/\gamma}\alpha_j\alpha_k\lambda_\ell\psi_j\psi_k\psi_\ell-\frac{i}{\sqrt{2}\gamma}\int_{\mathbb{T}^d}\alpha_j\alpha_k\frac{\lambda_\ell}{\lambda_j\lambda_k}\nabla\psi_j\cdot\nabla\psi_k\,\psi_\ell\Big).
$$

The first two terms in the right-hand side of (4.44) correspond to the part of $\mathcal{F}_2$ in the momentum equation. The last two terms correspond to part of $\mathcal{F}_2$ in the mass equation.

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