Stability of Small Viscosity Noncharacteristic Boundary Layers

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Introduction

These notes correspond to lectures and graduate courses given in Brescia, Bordeaux and Toulouse. They are intended to serve as an introduction to the *stability analysis* of small viscosity boundary layers which is developed in [MZ1]. There is a huge literature concerning formal asymptotic analysis or WKB expansions for boundary layers using multiple scale analyses, including sophisticated multiple layers expansions and matched asymptotics. Here we study the basic problem of a single layer, but focus on the stability analysis, aiming to give rigorous justification of the approximation of the exact solutions by the formal or asymptotic expansions. These notes are not intended to give an hypothetic complete treatment of the problem. Instead they aim to point out a few important features:

- existence of exact solutions of hyperbolic and parabolic problem, including *stability estimates* allowing to estimate the difference between exact and approximate solutions;
- analysis of the nonlinear inner layer o.d.e. which arises for quasilinear systems, making a link between its stability and a geometric transversality condition;
- formulate almost necessary and sufficient conditions for the multidimensional plane wave stability of the inner layer, in terms of an *Evans function*;
- extend the Kreiss construction of multidimensional symmetrizers to hyperbolic-parabolic problem, proving the basic L^2 estimate for linear stability.

To be more specific, consider a $N \times N$ system

$$\partial_t u + \sum_{j=1}^{d} A_j(u)\partial_j u - \varepsilon \sum_{j,k=1}^{d} \partial_j\big(B_{j,k}(u)\partial_k u\big) = F(u)\,,$$

The variables are $(t, x) \in \mathbb{R} \times \mathbb{R}^d$. The equation is considered for x in a smooth domain $\Omega \subset \mathbb{R}^d$ and supplemented with boundary conditions. Here we consider homogeneous Dirichlet boundary conditions:

$$u_{|\partial\Omega} = 0\,.$$

We assume that the first order part is sufficiently hyperbolic, for states $u \in \mathcal{U} \subset \mathbb{R}^N$ and that the second order singular perturbation is elliptic. A major assumption in the analysis below is that $\partial\Omega$ *is noncharacteristic* for the hyperbolic part:

$$\forall u \in \mathcal{U}, \forall x \in \partial\Omega : \quad \det\big(\sum n_j A_j(u)\big) \neq 0\,,$$

where $n = (n_1, \ldots, n_d)$ is the normal to the boundary. ε is a positive parameter which, in applications, measures the strength of the viscous or dissipative phenomena. The goal is to understand the behavior of solutions when ε tends to zero. The limit is expected to satisfy the hyperbolic equation

$$\partial_t u + \sum_{j=1}^{d} A_j(u)\partial_j u = F(u)\,.$$

The main questions are:

1 - existence of solutions for t in an interval of time independent of ε. This is *not* an easy consequence of know results for parabolic systems, since the parabolic estimates break down as $\varepsilon \to 0$;

2 - identification of the boundary conditions for the limiting hyperbolic problem; well-posedness of this boundary value problem;

3 - convergence of the solutions of the viscous equation to the solution of the inviscid problem.

A general approach to answer these questions is to proceed in two steps:

- first, one constructs approximate solutions by multiple scale expansions. This construction provides the good candidate for the limiting boundary conditions of question 2; it relies on the well-posedness both of the inner-layer equation and of the limiting hyperbolic boundary value problem;

- second, one studies the linear and nonlinear stability of the approximate solutions. The stability conditions are in general much stronger than the conditions necessary for the construction of approximate solutions. There are examples of strongly unstable approximate (or even exact) solutions see for instance [Gr-Gu]. Indeed, the existence of unstable layers is a well known occurrence in fluid dynamics. In the favorable cases when the layers are linearly stable under multidimensional perturbations, they also are nonlinearly stable and one can exhibit exact solutions which are small perturbations of the approximate solutions, answering questions 1 and 3 above.

These notes can be divided in two parts: the first one, Chapters one to four is devoted to linear and semilinear systems. We present most of the analysis in [Gu1] about noncharacteristic semilinear layers, including a detailed exposition of the existence and regularity theory for symmetric hyperbolic boundary value problems. The second part, from Chapter five to the end concerns quasilinear systems. We focus on two important aspects of the analysis: the study of the nonlinear inner layer equation, and the plane wave stability analysis which yields to the stability estimates in [MZ1].

Part 1: semilinear layers.

For linear equations the analysis of boundary layers in a general setting as above, is studied in [BBB], [Ba-Ra], [Lio]. Next we mention the important work of O.Guès ([Gu1]) where the semilinear case is solved in great details. He constructs and rigorously justifies high order approximations. Moreover, his analysis includes the completely different case of characteristic equations.

In Chapter one we study the elementary example of viscous perturbations of a transport equation:

$$\partial_t u + a\partial_x u - \varepsilon \partial_x^2 u = f\,.$$

In this example, explicit computations are available and most of the phenomena can be easily observed.

In Chapter two, we give a detailed analysis of the hyperbolic mixed Cauchy problem for symmetric operators (see e.g. [Fr1], [Fr2], [Fr-La], [Tar]). For simplicity, we restrict attention to constant coefficients equations, but, with the additional use of Friedrichs Lemma our presentation is immediately adaptable to variable coefficients. We follow closely the presentation in [Ch-Pi] to derive existence, uniqueness, smoothness, the causality principle for linear equations, from weighted L^2 estimates. Next, we carry out the analysis of semi-linear equations, using Picard's iterations. With one more degree of smoothness, this method extends to quasilinear systems, as shown, for instance, in [Maj],[Mok],[Mé2]. We also give a detailed account of the compatibility conditions at the corner edge, which are necessary for the existence of smooth solutions, both in the linear and nonlinear mixed Cauchy problem (see [Ra-Ma], [Ch-Pi]).

In Chapter three, we first review the classical construction of smooth solutions for parabolic systems, including the analysis of compatibility conditions. The most important part is devoted to the proof of estimates which are uniform with respect to the viscosity ε. Because of the layers, there are no uniform estimates in the usual Sobolev spaces. At most, one can expect *tangential or conormal* smoothness and we prove uniform estimates in spaces H^s_{tg} and H^s_{co} of functions possessing this tangential or conormal regularity. Moreover, although these spaces are not imbedded in L^∞, one can prove L^∞ bounds for the solutions of the equations in H^s_{tg} or in H^s_{co}. This allows to use iterative scheme to solve nonlinear equations.

The analysis of semilinear boundary layers is done Chapter four, following [Gu1]. First, one uses BK W multiple scale expansions, to find approximate solutions

$$u^\varepsilon_{app}(t,x) = \sum_{n=0}^{m} \varepsilon^n U_n(t,x,x_d/\varepsilon)\,, \quad U_n(t,x,z) = U^*_n(t,x,z) + u_n(t,x)\,.$$

Each *profile* U_n is the sum of an inner term U^*_n which is rapidly decreasing in z and carries the rapid variations of u^ε_{app} in the layer $0 \le x_d = 0(\varepsilon)$. The outer part, u_n, describes the behavior of u^ε_{app} in the interior of the domain, that is for $x_d \ge \delta$, for all $\delta > 0$. The U_n are determined inductively. The u_n are solutions of hyperbolic boundary value problems. There we use the results of Chapter two. The U^*_n are given by explicit integration. Next we construct the exact solutions

$$u^\varepsilon = u^\varepsilon_{app} + \varepsilon^m v^\varepsilon_m\,.$$

We use the uniform estimates of Chapter three to solve the equation for v^ε, proving in the same time the existence of u^ε on a uniform interval of time $[0,T]$ and the estimate

$$u^\varepsilon - u^\varepsilon_{app} = O(\varepsilon^m)\,.$$

Note that the estimates of Chapter three are stronger than the estimates used in [Gu1], so that the result given in Chapter four apply as soon as $m \ge 1$.

Part 2: quasilinear layers.

For quasilinear equations the analysis is much more delicate. A first result is given in [Gi-Se] in one space dimension and in [Gr-Gu] for multidimensional problems. Indeed, the analysis in [Gr-Gu] has two parts. In the first part, approximate solutions are obtained using formal expansions in power series of the the viscosity ε. In the second part, the authors prove the stability of this approximate solution, proving that the exact solution is actually close to the approximate one, *using a smallness condition* (as in [Gi-Se]). By an example, they also show that some condition is needed. However, the smallness condition is not natural and does not allow large boundary layers.

In the model case of *planar layers* (in analogy with planar shocks) that is boundary layers created by viscous perturbations of constant state solutions $\underline{u}$ of hyperbolic equations on a half space $\{x_d > 0\}$:

$$u^\varepsilon(t,x) = w(x_d/\varepsilon) \quad \text{with } w(0) = 0 \quad \text{and } \lim_{z \to +\infty} = \underline{u}\,,$$

rescaling the variables transforms the problem into a long time stability analysis for parabolic systems. In the study of reaction–diffusion equations, is has been shown that the accurate long time stability conditions are based on the analysis of an *Evans function*, see, e.g. [Ev1]-[Ev4], [Jon], [AGJ], [Pe-We], [Kap]. Evans functions have also been introduced in the study of the stability of planar viscous shock and boundary layers (see, e.g., [Ga-Zu], [ZH], [ZS], [Zum], [Ser], [Ro1], and references therein. They play the role of the Lopatinski determinant for constant coefficient boundary value problems. When they vanish in the open left half plane, the problem is strongly unstable and when they do not vanish in the closed half space, the problem is expected to be strongly stable. This indicates that assumptions on the Evans function should be the correct approach in the study of the stability of boundary layers. This has been proved to be correct in space dimension one [Gr-Ro] and in [MZ1] for multidimensional problems.

The one space dimensional analysis in [Gr-Ro] is based on integrations along characteristics for the hyperbolic equations and on pointwise estimates of the Green's function for the parabolic part, which are then combined to yield L^1 bounds on the Green's function for the linearized equations about the full boundary layer expansion. In multi-dimensions, both ingredients break down, due to more complicated geometry of characteristic surfaces. In particular, the known estimates of the parabolic Green's function [Zum] consist of L^p bounds, $p \geq 2$, and do not include pointwise behavior. Moreover, it is known from study of the constant-coefficient case [Ho-Zu] that the L^1 norm of the Green's function is *not* necessarily bounded in multi-dimensions, but in general may grow time-algebraically. This is a consequence of focusing and spreading in the underlying hyperbolic propagation, the effects of which are even more dramatic without parabolic regularization; indeed, as pointed out by Rauch [Rau1], there is good reason to believe that L^2 is the only norm in which we can expect that multi-dimensional hyperbolic problems be well-posed.

Thus, in multi-dimensions, the hyperbolic (or "outer") part of the solution forces to seek $L^2 \to L^2$ bounds. For hyperbolic equations satisfying a "uniform Lopatinski condition", this has been done by H.O. Kreiss ([Kre], see also [Ch-Pi]) using *symmetrizers*. In [MZ1], it is shown that this method can be extended to the parabolic regularizations of the systems, under the analogous "uniform Evans condition". It provides us with maximal estimates which are sharp *for both hyperbolic and parabolic parts of the equations*, as seen by comparison with explicit representations of the resolvant in the planar case (see [Agm] and [Zum]).

In Chapter five, we study the inner-layer equation, that is the equation satisfied by $w(z)$ if $w(x_d/\varepsilon)$ is a stationary solution of the viscous equations. Moreover, this analysis also gives the natural boundary conditions for the limiting hyperbolic equation: the boundary value of the hyperbolic solution must be a limit at $z = +\infty$ of a solution of the inner-layer o.d.e..

Chapter six is mainly devoted to the definition of the *Evans function*, starting from general considerations about the plane wave stability analysis of boundary value problems. In particular, we also recall the definition of the *Lopatinski determinant*. A key point of the analysis, due to [Ro1] (see also [ZS] for an analogous result about viscous shocks), is that the uniform Evans stability condition for the viscous problem implies, first, that the natural boundary conditions are smooth and second, that the limiting inviscid hyperbolic problem satisfies the uniform Lopatinski conditions. These are the two ingredients which are necessary to construct BKW solutions in [Gr-Gu].

The basic L^2 estimates for the linearized equations around a planar layer, are given in Chapter seven. The method of symmetrizers is recalled, as well as their construction for "large" and "medium" frequencies, which means here that the space-time wave numbers ζ satisfy $\varepsilon|\zeta| \gg 1$ and $0 < r \leq \varepsilon|\zeta| \leq R < +\infty$ respectively.

The analysis of low frequencies, that is $\varepsilon|\zeta| \ll 1$, is performed in Chapter eight, where Kreiss' construction of symmetrizers is extended to the viscous equations.

Finally, in Chapter nine, we state (without proof) the main result of of [MZ1], indicating where the key results of the previous chapters are used.

Further remarks

We end this introduction with a few remarks about applications and further developments. There are many motivations for the analysis of small viscosity perturbation of hyperbolic problems. For instance, for scalar conservation laws, there is the basic Krushkov analysis ([Kru], see also [Ole]). In space dimension one, for systems there are analogous results in particular cases ([Go-Xi], [DiP]) and now in a general context by S.Bianchini and A.Bressan [Bi-Br]. Boundary layers occur in many circumstances and applications, it is impossible to make a complete list here. Many examples come from fluid mechanics, starting with the analysis of Navier-Stokes equations with small viscosity. However, it is important to note here that the framework developed in these notes rules out the specific case of compressible Navier-Stokes equations. For two reasons. First, for Navier-Stokes equations, the viscous part is only partially parabolic: there are no viscous term in the equation of mass conservation. This is probably not fundamental, and one can reasonably expect that the analysis of [MZ1] extends to partial viscosity, with suitable assumptions on the kernel of the parabolic symbol. The second reason, is much more serious: the boundary condition for the limiting Euler equations usually include that the velocity is tangent to the boundary, so that the boundary *is characteristic*. This is a dramatic change. In the semi-linear case, the analysis of O.Guès shows that characteristic boundary layers are of order $\sqrt{\varepsilon}$ instead of ε in the noncharacteristic case and are not governed by an o.d.e. but by a parabolic partial differential equation. When applied to Navier-Stokes equations, this analysis yields Prandtl equations, which have been shown to be *strongly unstable* at least in some cases, but this is probably a general phenomenon, see the works by E.Grenier [Gr1] [Gr2]. This reflects the well known fact that many layers in fluid dynamics are unstable.

An important example of noncharacteristic boundary value problem, for general system of conservation laws, is the equations of sock waves. It is a transmission problem, with transmission conditions given by Rankine-Hugoniot conditions. The main new difficulty is that it is a *free boundary* value problem, but the analysis of classical multidimensional hyperbolic boundary value problem has been successfully extended to the shock problem by A.Majda (see [Maj]). The analysis of viscous perturbations of shocks has been done in dimension one, see [Go-Xi] for sufficiently weak shocks, [Ro2] under an Evans function hypothesis. In higher dimension, O.Guès and M.Williams, have constructed BKW approximate solutions to any order [Gu-Wi]. Extending the stability analysis of [MZ1] which do not directly apply due to a singularity of the Evans function at the origin, their stability is studied in [GMWZ1] [GMWZ2], first in the context of long time stability for fixed viscosity, and next in the small viscosity framework, using additional technical assumptions which are expected to be removed soon. This work should extend to partial viscosity and thus should apply to the analysis of multidimensional small viscosity shock waves for the real Navier-Stokes equations.

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Chapter 1

An Example

This chapter is an introduction to the main topics developed in this course. We consider a very simple example: the viscous perturbation of a transport equation. The advantage is that we can perform explicit computations and show when and how boundary layers occur.

1.1 The equation

We consider the equation

$$\partial_t u^\varepsilon + a\partial_x u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon = f \tag{1.1.1}$$

The variables are $t \in \mathbb{R}$ (time) and $x \in \mathbb{R}$ (space). The unknown u is a real (or complex) valued function. The parameter ε (viscosity) is "small" and the main goal is to understand the behavior of u^ε as ε goes to zero. The limit u is expected to be a solution of

$$\partial_t u + a\partial_x u = f\,. \tag{1.1.2}$$

We consider the equation (1.1.1) for x in $\mathbb{R}_+ = \{x > 0\}$. Then a boundary condition must be added to (1.1.1):

$$u^\varepsilon(t, 0) = 0\,. \tag{1.1.3}$$

However, depending on a, this boundary condition may be incompatible with the limit equation (1.1.2). Thus if u^ε converges to a solution u of (1.1.2) which does not satisfy (1.1.3), u^ε must experience a rapid transition between something close to u in the interior $x > 0$ and 0 on the boundary: this transition is called a boundary layer.

We may also consider the Cauchy problem, solving (1.1.1) for $t > 0$ with an initial condition

$$u^\varepsilon(0, x) = h(x) . \tag{1.1.4}$$

The goal of this chapter is to introduce on the toy model (1.1.1) different methods and tools which will be useful in the more general analysis developed in the next chapters.

We first investigate the wellposedness of boundary value problems for the transport equation (1.1.2). Next we study the viscous equation (1.1.1) and, by explicit computations, we study the convergence of viscous solutions to solutions of the inviscid equation, revealing when $a < 0$ the phenomenon of *boundary layers*. Then, we will introduce on the example two methods developed later: the BKW asymptotic analysis and the use of Fourier-Laplace transform in the analysis of boundary value problems.

1.2 Transport equation

In this section we consider the transport equation (1.1.2). We consider here only *classical* solutions, that is C^1 solutions, for which the equation has a clear sense. The case of *weak* solutions could be considered too, see Chapter two.

1.2.1 The Cauchy problem

Using the change of variables $y = x - at$ and setting

$$v(t, y) = u(t, x) = u(t, y + at) , \qquad g(t, y) = f(t, y + at) \tag{1.2.1}$$

the equation (1.1.2) is equivalent, for C^1 functions, to $\partial_t v = g$. Thus, for C^1 source term f and initial data u_0, the Cauchy problem has a unique solution:

$$v(t, y) = v(0, y) + \int_0^t g(s, y) ds$$

that is

$$u(t, x) = h(x - at) + \int_0^t f(s, x - a(t - s)) ds . \tag{1.2.2}$$

1.2.2 The mixed Cauchy-boundary value problems

Consider the problem (1.1.2) on $\Omega = \{(t,x) : t > 0, x > 0\}$ with $f \in C^1(\overline{\Omega})$ and initial data $h \in C^1(\overline{\mathbb{R}}_+)$.

1) When $a \leq 0$, the formula (1.2.2) defines a solution of (1.1.2) satisfying the initial condition $u_{|t=0} = u_0$. (since $x - a(t-s) \geq 0$ when $x \geq 0$, $t \geq s \geq 0$).

Proposition 1.2.1. *When $a \leq 0$, for $f \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\mathbb{R}}_+)$, the Cauchy problem* (1.1.2)(1.1.4) *on Ω, without any boundary condition, has a unique solution $u \in C^1(\overline{\Omega})$ given by* (1.2.2). *If the data are C^k, with $k \geq 1$, the solution is C^k.*

2) On the other hand, when $a > 0$ the formula (1.2.2) defines a solution only for $x \geq at$. For $x < at$, one has

$$u(t,x) = u(t - x/a, 0) + \frac{1}{a}\int_0^x f(t - y/a, x - y)dy\,. \tag{1.2.3}$$

Thus to determine the solution, one boundary condition

$$u(t,0) = \ell(t) \tag{1.2.4}$$

must be added. The solution is (uniquely) determined by f, u_0 and ℓ.

Remark 1.2.2. For the function u defined by (1.2.2) for $x > at$ and by (1.2.3) when $x < at$ to be continuous on the half line $\{x = at \geq 0\}$, the Cauchy data h and the boundary data ℓ must satisfy

$$h(0) = \ell(0) \quad (= u(0,0))\,. \tag{1.2.5}$$

For u to be C^1, one has to impose the condition

$$\partial_t \ell(0) = f(0,0) - a\partial_x h(0)\,, \tag{1.2.6}$$

which is clearly necessary from the equation evaluated at the origin.

Exercise : show that these conditions are necessary and sufficient to define a solution $u \in C^1(\overline{\Omega})$.

This is an example of *compatibility conditions* to be discussed in Chapter 2.

Proposition 1.2.3. *When $a > 0$, for $f \in C^1(\overline{\Omega})$, $h \in C^1(\overline{\mathbb{R}}_+)$ and $\ell \in C^1(\overline{\mathbb{R}}_+)$ satisfying* (1.2.5)(1.2.6)*, the mixed Cauchy problem* (1.1.2)(1.1.4)(1.2.4) *has a unique solution $u \in C^1(\overline{\Omega})$ given by* (1.2.2) *for $x \geq at$ and by* (1.2.3) *for $x \leq at$.*

If the data are C^k and vanish near the origin, then the solution is C^k.

1.2.3 The boundary value problem

To avoid the problem of compatibility conditions, one can consider the problem (1.1.2) on the half space $\mathbb{R}^2_+ = \{t \in \mathbb{R}, x > 0\}$, assuming that f vanishes far in the past (say for $t < T_0$). In this case there is a unique solution vanishing for $t \le T_0$, given as follows.

1) When $a \le 0$,

$$u(t,x) = \int_{-\infty}^{t} f(s, x - a(t-s))ds\,. \tag{1.2.7}$$

In this case, no boundary condition is required.

2) When $a > 0$, one boundary condition (1.2.4) must be added and

$$u(t,x) = \ell(t - x/a) + \frac{1}{a}\int_0^x f(t - (x-y)/a, y)dy\,. \tag{1.2.8}$$

Remark 1.2.4. In both cases, note that $u = 0$ for $t < T_0$ if f (and ℓ) vanish for $t < T_0$. More generally, the solutions defined by (1.2.7) and (1.2.8) satisfy the *causality principle*: their value at time t only depend on f (and ℓ) for times less than or equal to t.

Remark 1.2.5. Above, we considered the cases $a > 0$ and $a \le 0$. Note that the case $a = 0$ is very particular. When $a < 0$, the formula (1.2.7) shows that classical solutions satisfy

$$u(t,0) = \int_{-\infty}^{t} f(s, -a(t-s))ds$$

thus

$$\|u(\cdot,0)\|_{L^1(]-\infty,T])} \le \frac{1}{|a|}\|f\|_{L^1(]-\infty,T]\times\mathbb{R}_+)}\,. \tag{1.2.9}$$

This extends to weak solutions : one can show using (1.2.7) that they are continuous in x with values in $L^1(]-\infty,T])$ for all T. In particular their trace on $x = 0$ is well defined.

In sharp contrast, when $a = 0$ the formula

$$u(t,x) = \int_{-\infty}^{t} f(s,x)ds$$

shows that the value of u on $x = 0$ is not well defined when $f \in L^1$.

1.3 Viscous perturbation: existence of layers

We come back to the viscous equation (1.1.1)

1.3.1 The Cauchy problem

Perform the change of variables

$$y = x - at\,, \quad u(t,x) = v(t, x - at)\,, \quad f(t,x) = g(t, x-at)\,.$$

In these variables, the Cauchy problem for (1.1.1) reads

$$\partial_t v - \varepsilon \partial_y^2 v = g\,, \quad v_{|t=0} = v_0. \tag{1.3.1}$$

The solution is given by the heat kernel:

$$\begin{aligned} v(t,y) = & \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{\mathbb{R}} e^{-(y-y')^2/4\varepsilon t} v_0(y') dy' \\ & + \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} e^{-(y-y')^2/4\varepsilon(t-s)} f(s,y') dy'\,. \end{aligned} \tag{1.3.2}$$

Exercise : show that (1.3.2) provides a solution v^ε to (1.3.1), for v_0 and f in various spaces. Moreover, when $\varepsilon \to 0$, v^ε converges to the solution of $\partial_t v = g$ with the same initial condition.

1.3.2 The mixed Cauchy-problem

We consider here the mixed Cauchy problem on $\Omega = \{t > 0, x > 0\}$ and for simplicity we assume here that there is no source term:

$$(\partial_t + a\partial_x - \varepsilon\partial_x^2) u^\varepsilon = 0\,, \quad {u^\varepsilon}_{|x=0} = 0\,, \quad {u^\varepsilon}_{|t=0} = h\,. \tag{1.3.3}$$

When $a = 0$, the mixed Cauchy problem with initial data $h \in C_0^\infty(\mathbb{R}_+)$ is transformed into a Cauchy problem by considering odd extensions of h and u^ε for negative x. This yields to the following formula for the solution:

$$u^\varepsilon(t,x) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_0^\infty \big(e^{-(x-y)^2/4\varepsilon t} - e^{-(x+y)^2/4\varepsilon t} \big) h(y) dy$$

The case $a \neq 0$ is reduced to the case $a = 0$ using the change of unknown

$$u^\varepsilon(t,x) = e^{-a^2 t/4\varepsilon + ax/2\varepsilon} \tilde{u}^\varepsilon(t,x)\,.$$

Thus the solution is given explicitly by

$$u^\varepsilon(t,x) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_0^\infty \big(e^{-\Phi/4\varepsilon t} - e^{-\Psi^2/4\varepsilon t}\big) h(y) dy \tag{1.3.4}$$

with

$$\Phi = (x - at - y)^2\,, \quad \Psi = (x + y - at)^2 + 4ayt = (x + y + at)^2 - 4axt\,.$$

We can pass to the limit in (1.3.4) as ε tends to zero. Assuming for simplicity that $h = 0$ near the origin, the stationary phase theorem implies that

$$\frac{1}{\sqrt{4\pi\varepsilon t}} \int_0^\infty e^{-\Phi/4\varepsilon t} h(y) dy \ \to\ \tilde{h}(x - at) := u_0(t,x)$$

where $\tilde{h}(y)$ is the extension of h by zero for $y \le 0$. The convergence is uniform on compact sets, and since h is C^∞, the stationary phase theorem also gives complete expansion of the integral in powers series of ε.

Note that u^0 is a solution of the limit equation $\partial_t u_0 + a\partial_x u_0 = 0$, with $u_0(0,x) = h(x)$. When $a < 0$, $u_0(t,x) = h(x - at)$ for $t \ge 0$ and $x \ge 0$ since then $x - at \ge 0$. In this case, u_0 is the unique solution of the (mixed) Cauchy problem on Ω without boundary condition, see Proposition 1.2.1:

$$(\partial_t + a\partial_x) u_0 = 0\,, \quad u_{0|t=0} = h\,. \tag{1.3.5}$$

When $a > 0$, we find that $u_0(t,x) = h(x - at)$ for $x > at$ and $u_0(t,x) = 0$ for $x \le at$. Thus u_0 is the unique solution of the mixed Cauchy problem with homogeneous boundary condition, see Proposition 1.2.3:

$$(\partial_t + a\partial_x) u_0 = 0\,, \quad u_{0|x=0} = 0\,, \quad u_{0|t=0} = h\,. \tag{1.3.6}$$

The behavior of the second term depends on the sign of a. When $a > 0$, the phase Ψ is strictly positive for $y > 0$ in the support of h and $(t,x) \in \Omega$. In this case the second integral is $O(e^{-\delta/\varepsilon})$ for some $\delta > 0$, proving that

$$u^\varepsilon(t,x) = u_0(t,x) + O(\varepsilon)\,. \tag{1.3.7}$$

When $a < 0$, we write the second term

$$-e^{ax/\varepsilon} v^\varepsilon(t,x) \quad \text{with} \quad v^\varepsilon(t,x) := \frac{1}{\sqrt{4\pi\varepsilon t}} \int_0^\infty e^{-(x+y+at)^2/4\varepsilon t} h(y) dy$$

By the stationary phase theorem

$$v^\varepsilon(t,x) \to \tilde{h}(-x - at) := v_0(t,x)\,.$$

Note that the limit is different from zero when $-(x+at) > 0$ belongs to the support of h. Since $a < 0$, this occurs for points $(t,x) \in \Omega$. In this case, we have

$$u^\varepsilon(t,x) = u_0(t,x) - e^{ax/\varepsilon} v_0(t,x) + O(\varepsilon)\,.$$

Note that $e^{-ax/\varepsilon} v_0(t,x) = e^{-ax/\varepsilon} v_0(t,0) + O(\varepsilon)$, and $v_0(t,0) = u_0(t,0)$. Thus

$$u^\varepsilon(t,x) = u_0(t,x) - e^{ax/\varepsilon} u_0(t,0) + O(\varepsilon)\,. \tag{1.3.8}$$

Summing up we have proved that for $h \in C_0^\infty(]0,+\infty[)$:

Conclusion 1.3.1. *i) When $x > 0$, the solution $u^\varepsilon(t,x)$ of* (1.3.3) *converges to the solution u_0 of the limit problem, which is* (1.3.5) *when $a < 0$ and* (1.3.6) *when $a > 0$.*

ii) When $a > 0$, the convergence is uniform on Ω and (1.3.7) *holds.*

iii) When $a < 0$, the convergence is not uniform on Ω. Near the boundary, a corrector must be added, and the uniform behavior is given by (1.3.8).

When $a > 0$, the limit u_0 satisfies the boundary condition, and therefore it is natural to get that u_0 is a good approximation of u^ε. When $a < 0$, the solution u_0 of (1.3.6) has no reason to satisfy the boundary condition, and thus there must be a corrector of order $O(1)$ near the boundary. The computation above shows that this corrector is $-e^{ax/\varepsilon} u_0(t,0)$, revealing the scale x/ε and the exponential decay in the fast variable $z = x/\varepsilon$. The *boundary layer* is this rapid variation of u^ε near the boundary.

1.4 BKW expansions

This section is an introduction to the general method developed in Chapter four. We construct *formal asymptotic expansions* in power series of ε for the solutions of (1.1.1). They provide *approximate solutions*. We always assume that $a \neq 0$.

Inspired by (1.3.8), we look for solutions of the initial boundary value problem (1.1.1) (1.1.3) (1.1.4) such that

$$u^\varepsilon(t,x) \sim \sum_{n \geq 0} \varepsilon^n U_n(t,x,\frac{x}{\varepsilon}) \tag{1.4.1}$$

with $U_n(t,x,z)$ having limits $u_n(t,x)$ at $z = +\infty$ such that $U_n \to u_n$ rapidly as $z \to \infty$. At this stage, the expansion is to be understood in the sense of *formal* series in powers of ε.

Introduce the space $\mathcal{P}(T)$ of functions of the form

$$V(t,x,z) = v(t,x) + V^*(t,z) \tag{1.4.2}$$

with $v \in C^\infty([0,T] \times \overline{\mathbb{R}}_+)$, $V* \in C_b^\infty([0,T] \times \overline{\mathbb{R}}_+)$[1] and such that for all indices (j,k), there are C and $\delta > 0$ such that

$$|\partial_t^j \partial_z^k V^*(t,z)| \le C e^{-\delta z}\,.$$

The splitting (1.4.2) is unique since

$$v(t,x) = \lim_{z\to\infty} Vt,x,z)\,.$$

For $V \in \mathcal{P}$, we also use the notation $\underline{V}(t,x)$ for this limit. We denote by $\mathcal{P}^*$ the class of $V \in \mathcal{P}$ such that $\underline{V} = 0$. Note that for all $V \in \mathcal{P}$, $\partial_z V \in \mathcal{P}^*$.

The profiles U_n are sought in the class $\mathcal{P}$. The boundary condition for u^ε leads to impose that for all n,

$$U_n(t,0,0) = 0 \tag{1.4.3}$$

Similarly, the initial condition (1.1.4) reads:

$$U_0(0,x,z) = h(x)\,, \quad U_n(0,x,z) = 0 \quad \text{for } n \ge 1\,. \tag{1.4.4}$$

In the sense of formal series, (1.4.1) implies that

$$(\partial_t + a\partial_x - \varepsilon\partial_x^2)u^\varepsilon - f \sim \sum_{n\ge -1} \varepsilon^n F_n(t,x,\frac{x}{\varepsilon})\,,$$

with

$$\begin{aligned}
F_{-1} &= -\mathcal{L}U_0\,,\\
F_0 &= -\mathcal{L}U_1 + L_0U_0 - f\\
F_n &= -\mathcal{L}U_{n+1} + L_0U_n - \partial_x^2 U_{n-1}\,, \quad n \ge 1\,,
\end{aligned}$$

with

$$\mathcal{L} := \partial_z^2 - a\partial_z\,, \qquad L_0 = \partial_t + a\partial_x\,.$$

There is an unessential simplification here: there are no cross term $\partial_x\partial_z U_n$, since the decomposition (1.4.2) of profiles implies that they all vanish.

Definition 1.4.1. *We say that $\sum \varepsilon^n U_n$ is a formal solution of* (1.1.1) *if all the F_n, $n \ge -1$, vanish. It is a formal solution of the boundary value problem* (1.1.1) (1.1.3) *if in addition the conditions* (1.4.3) *hold for $n \ge 0$. It is a formal solution of the mixed Cauchy problem* (1.1.1) (1.1.3) (1.1.4) *if in addition* (1.4.4) *holds.*

[1] $C_b^\infty(\Omega)$ denotes the space of C^∞ functions on Ω which are bounded as well as all their derivative

Theorem 1.4.2. *Given $f \in C_b^\infty([0,T] \times \overline{\mathbb{R}}_+)$ and $h \in C_b^\infty(\overline{\mathbb{R}}_+)$ which both vanish near the origin, there is a unique formal solution of the mixed Cauchy problem* (1.1.1) (1.1.3) (1.1.4).

Proof. **a)** *Analysis of* $F_{-1} = 0$. The equation $\mathcal{L}U_0 = 0$ reads $a\partial_z U_0^* = \partial_z^2 U_0^*$. Thus the solutions are

$$U_0(t,x,z) = \beta_0(t,x) + \alpha_0(t,x)e^{az} .$$

1) When $a > 0$, the bounded solutions in $\mathcal{P}$ are

$$U_0(t,x,z) = u_0(t,x) . \tag{1.4.5}$$

In particular, the boundary condition (1.4.3) for $n = 0$ and the initial condition (1.4.4) imply that necessarily

$$u_0(t,0) = 0 , \quad u_0(0,x) = h(x) . \tag{1.4.6}$$

2) When $a < 0$, the solutions in $\mathcal{P}$ are

$$U_0(t,x,z) = u_0(t,x) + \alpha_0(t)e^{az} \tag{1.4.7}$$

The boundary condition (1.4.3) for $n = 0$ reads

$$\alpha_0(t) = -u_0(t,0) . \tag{1.4.8}$$

while the initial condition is

$$u_0(0,x) = h(x) , \quad \alpha_0(0) = 0 . \tag{1.4.9}$$

Next, we split the equations $F_n = 0$ into $\underline{F}_n = 0$ and $F_n^* = 0$, where $\underline{F}_n = \lim_{z\to\infty} F_n$ and $F_n^* = F_n - \underline{F}_n$.

b) *Analysis of* $\underline{F}_0 = 0$. The equation reduces to

$$L_0 u_0 = (\partial_t + a\partial_x)u_0 = f$$

When $a < 0$, this equation coupled with the initial condition has a unique solution by Proposition 1.2.1. Therefore, $\alpha_0 \in C_b^\infty([0,T])$ is determined by (1.4.8) . With (1.4.5) and (1.4.7) this determines U_0. Note that, because the data vanish near the origin, the solution u_0 also vanishes near the origin, so that α_0 also vanishes near $t = 0$ and the second equation in (1.4.9) is also satisfied.

When $a > 0$, by Proposition 1.2.3, there is a unique solution of the mixed Cauchy problem defined by the equation coupled with (1.4.6),

Remark 1.4.3. In both cases, $u_0 \in C_b^\infty([0,T] \times \overline{\mathbb{R}}_+)$ is determined as the solution of the inviscid problem (1.1.2) (1.1.4) augmented with the boundary condition $u_0(t,0) = 0$ when $a > 0$. In both case, the leading term U_0 given by this formal computation is equal to the leading term rigorously derived in section 3.

c) *Analysis of* $F_0^* = 0$. The equation reduces to

$$\mathcal{L}U_1 = \partial_z^2 U_1 - a\partial_z U_1 = \partial_t U_0^* \,.$$

When $a > 0$, the right has side vanishes and the only bounded solution in $\mathcal{P}$ are

$$U_1(t,x,z) = u_1(t,x)\,, \qquad \text{with} \quad u_1(t,0) = 0\,.$$

The initial-boundary conditions imply that

$$u_1(0,x) = 0\,, \quad u_1(t,0) = 0\,.$$

When $a < 0$, the right hand side is $\partial_t \alpha_0(t) e^{az}$. The solutions in $\mathcal{P}$ are

$$U_1(t,x,z) = u_1(t,x) + \alpha_1(t)e^{az} + \frac{1}{a}\partial_t\alpha_0(t) z e^{az}\,.$$

The boundary condition (1.4.3) for $n = 1$ reads

$$\alpha_1(t) = -u_1(t,0)\,. \tag{1.4.10}$$

and, because α_0 vanishes for t near the origin, the initial condition reduces to

$$u_1(0,x) = 0\,, \quad \alpha_1(0) = 0\,. \tag{1.4.11}$$

d) *Analysis of* $\underline{F}_1 = f$. The equation reads

$$(\partial_t + a\partial_x)u_1 = \partial_x^2 u_0\,,$$

Together with the Cauchy data $u_1(0,x) = 0$ and the boundary condition $u_1(t,0) = 0$ when $a > 0$, there is a unique solution u_1. It vanishes near the origin. When $a < 0$, we determine α_1 by (1.4.10) and the second equation in (1.4.11) is satisfied.

e) The analysis of the other terms is similar. The equations $F_{-1} = 0, \ldots, F_{n-1} = 0$ and $\underline{F}_n = 0$ determine $U_0, \ldots, U_n$. Then, $F_n^* = 0$ determines U_{n+1}^* in terms of u_{n+1}, plus a boundary condition on u_{n+1} when $a > 0$ and a choice of a function $\alpha_{n+1}(t)$ such that $u_{n+1} = -\alpha_{n+1}$ on the boundary when $a < 0$. Next, $\underline{F}_{n+1}$ is a transport equation for u_{n+1} which, together with the Cauchy conditions and the boundary condition when $a > 0$, determines u_{n+1}. The details are left as an exercise. $\square$

Remark 1.4.4. The leading term U_0 is equal to the leading term rigorously derived in section 3. More generally, using complete stationary phase expansions in section 3 provides rigorous expansions (1.4.1) for the exact solutions. We leave as an exercise to check that both method provide the same result.

Remark 1.4.5. If $\sum \varepsilon^n U_n$ is a formal solution, then the partial sums

$$u^\varepsilon_{app}(t,x) = \sum_{n\leq N} \varepsilon^n U_n(t,x,x/\varepsilon)$$

provide approximate solutions: they satisfy

$$\begin{cases} (\partial_t + a\partial_x - \varepsilon\partial_x^2) u^\varepsilon_{app} = f + \varepsilon^N r^\varepsilon\,, \\ u^\varepsilon_{app}(t,0) = 0\,, \quad u^\varepsilon_{app}(0,x) = h\,, \end{cases}$$

with r^ε uniformly bounded.

1.5 Laplace Fourier transform

The Laplace transform is a classical tool in the analysis of evolution equations, starting with ordinary differential equations. In this section, as an introduction to Chapters 6 and 7, we sketch an application of this transformation to the analysis of the boundary value problem (1.1.1) (1.1.3) on $\mathbb{R}^2_+ = \{(t,x) : x > 0\}$.

Suppose that the source term f vanishes for $t \leq 0$ and has a controlled exponential growth when $t \to +\infty$: assume for instance that $e^{-\gamma_0 t} f \in L^2(\mathbb{R}^2_+)$. Then the time-Laplace transform $\tilde f$ of f is defined and holomorphic for $\operatorname{Re}\lambda \geq \gamma_0$. The Laplace transform of the equation is

$$(\lambda + a\partial_x - \varepsilon\partial_x^2)\tilde u = \tilde f \tag{1.5.1}$$

If one can solve this equation, by inverse Laplace transform one can get a solution of the original equation (1.1.1). Moreover, with good control of $\tilde u$ for $\operatorname{Re}\lambda \geq \gamma_0$, one can expect that the solution u vanishes for $t \leq 0$. More generally, this construction agrees with the causality principle.

1.5.1 Definitions

We first recall several definitions and known properties. Consider first functions of one variable t. The Fourier transform $\hat u = \mathcal{F} u$ is defined for $u \in L^1(\mathbb{R})$ by

$$\hat u(\tau) = \int e^{-i\tau t} u(t) dt\,.$$

and then extended to temperate distributions $u \in \mathcal{S}'$. $\mathcal{F}$ is an isomorphism from $\mathcal{S}'$ to $\mathcal{S}'$ and for $\hat{u} \in L^1$,

$$u(t) = \frac{1}{2\pi} \int e^{i\tau t} \hat{u}(\tau) d\tau$$

If $e^{-\gamma t} u \in L^1$, the Laplace transform

$$\widetilde{u}(\lambda) = \int e^{-\lambda t} u(t) dt$$

is defined for $\operatorname{Re} \lambda = \gamma$, where we use the notation $\lambda = \gamma + i\tau$. This extends to $u \in e^{\gamma t} \mathcal{S}'$ as

$$\widetilde{u}(\gamma + i\tau) = \mathcal{F}(e^{-\gamma t} u)(\tau) .$$

In this case, $u = e^{\gamma t} \mathcal{F}^{-1} \big\{ \widetilde{u}(\gamma + i \cdot) \big\}$. When $\widetilde{u}(\gamma + i \cdot) \in L^1$:

$$u(t) = \frac{1}{2i\pi} \int_{\operatorname{Re} \lambda = \gamma} e^{\lambda t} \widetilde{u}(\lambda) d\lambda .$$

Lemma 1.5.1. *i)* (Plancherel) *$u \in e^{\gamma t} L^2$ if and only if $\widetilde{u}(\gamma + i \cdot) \in L^2$ and*

$$\| e^{-\gamma t} u \|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \| \tilde{u} \|_{L^2(\{\operatorname{Re} \lambda = \gamma\})}$$

ii) If $u \in e^{\gamma t} \mathcal{S}'$ and $u = 0$ for $t \le T_0$, then $\widetilde{u}$ is defined and holomorphic for $\operatorname{Re} \lambda \ge \gamma$.

For $u(t, x)$ defined on $\mathbb{R} \times \mathbb{R}_+$, we define (when possible) the Laplace transform in t for fixed x. For instance, for $u \in e^{\gamma t} L^1(\mathbb{R} \times \mathbb{R}_+)$, we can define

$$\widetilde{u}(\lambda, x) = \int_{\mathbb{R}} e^{-\lambda t} u(t, x) \, dt$$

when $\operatorname{Re} \lambda = \gamma$, and $\widetilde{u}(\lambda, \cdot) \in L^1(\mathbb{R}_+)$ for all $\lambda = \gamma + i\tau$.

Similarly, using Plancherel's theorem, one obtains the next result:

Proposition 1.5.2. *The time Laplace transform is an isomorphism from $e^{\gamma t} L^2(\mathbb{R} \times \mathbb{R}_+)$ to $L^2(\{\operatorname{Re} \lambda = \gamma\} \times \mathbb{R}_+)$ and*

$$\| e^{-\gamma t} u \|_{L^2(\mathbb{R} \times \mathbb{R}_+)} = \frac{1}{\sqrt{2\pi}} \| \tilde{u} \|_{L^2(\{\operatorname{Re} \lambda = \gamma\} \times \mathbb{R}_+)}$$

Introduce the space W of functions $u \in L^2(\mathbb{R} \times \mathbb{R}_+)$ such that $\partial_t u$, $\partial_x u$ and $\partial_x^2 u$ belong to $L^2(\mathbb{R} \times \mathbb{R}_+)$. This space, and the weighted spaces $e^{\gamma t} W$, are natural spaces for solutions of (1.1.1).

Proposition 1.5.3. *For $u \in e^{\gamma t}W$, the Laplace transform is defined for* $\operatorname{Re}\lambda = \gamma$ *and belong to the space $\widetilde{W}_\gamma$ of functions $v \in L^2(\{\operatorname{Re}\lambda = \gamma\} \times \mathbb{R}_+)$ such that λv, $\partial_x v$ and $\partial_x^2 v$ belong to $L^2(\{\operatorname{Re}\lambda = \gamma\} \times \mathbb{R}_+)$.*

Conversely, if $v \in \widetilde{W}_\gamma$, its inverse Laplace transform belongs to $e^{\gamma t}W$.

Moreover, for $u \in e^{\gamma t}W$ the trace $u_{|x=0}$ belongs to $e^{\gamma t}L^2$ and its Laplace transform is the restriction of $\tilde{u}$ to $x = 0$.

We can now perform the *Laplace transform of the equation* (1.1.1):

Lemma 1.5.4. *Consider $f \in e^{\gamma t}L^2$. Then $u \in W$ is a solution of* (1.1.1) *if and only if its Laplace transform $\widetilde{u} \in \widetilde{W}_\gamma$ satisfies*

$$-\varepsilon\partial_x^2\widetilde{u} + a\partial_x\widetilde{u} + \lambda\widetilde{u} = \widetilde{f}\,. \tag{1.5.2}$$

In addition, u satisfies the homogeneous Dirichlet boundary condition (1.1.3)*, if and only if the Laplace transform satisfies*

$$\widetilde{u}(\lambda, 0) = 0\,. \tag{1.5.3}$$

1.5.2 Green's functions

We consider here the Laplace transformed equations (1.5.2) (1.5.3), for a fixed λ. Dropping the tildes, the o.d.e. on $\mathbb{R}_+$ reads:

$$-\varepsilon\partial_x^2 u + a\partial_x u + \lambda u = f\,, \qquad u(0) = 0\,, \qquad u \in L^2(\mathbb{R}_+)\,. \tag{1.5.4}$$

We assume that $\varepsilon > 0$, $a \in \mathbb{R}$, $a \neq 0$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and $\gamma = \operatorname{Re}\lambda \geq 0$.

This is a constant coefficient second order equation. The indicial equation

$$-\varepsilon r^2 + ar + \lambda = 0$$

has two solutions

$$r_1 = \frac{a}{\varepsilon} + \mu\,, \quad r_2 = -\mu \tag{1.5.5}$$

where

$$\mu = \frac{a}{2\varepsilon}\Big(\sqrt{1 + \frac{4\varepsilon\lambda}{a^2}} - 1\Big) = \frac{\lambda}{a} - \frac{\varepsilon\lambda^2}{2a^3} + \lambda O\big((\varepsilon\lambda)^2\big) \quad \text{as}\, \varepsilon\lambda \to 0\,.$$

Here, $\sqrt{z}$ is the principal determination of the square root in $\mathbb{C}\backslash] - \infty, 0]$ $(\operatorname{Re}\sqrt{z} > 0)$.

Lemma 1.5.5. *The indicial equation has one root r_- such that* $\operatorname{Re} r_- < 0$ *and one root r_+ such that* $\operatorname{Re} r_+ > 0$. *Namely:*

$$\begin{aligned} &\text{if } a > 0 : \qquad r_- = r_2 \quad \text{and} \quad r_+ = r_1 \,, \\ &\text{if } a < 0 : \qquad r_- = r_1 \quad \text{and} \quad r_+ = r_2 \,. \end{aligned} \tag{1.5.6}$$

Proof. Consider

$$\Lambda = \frac{\varepsilon \lambda}{a^2} \,.$$

Then

$$\begin{aligned} \big(\operatorname{Re} \sqrt{1 + 4\Lambda}\big)^2 &= \frac{1}{2}\Big\{1 + 4\operatorname{Re}\Lambda + \Big((1 + 4\operatorname{Re}\Lambda)^2 + (4\operatorname{Im}\Lambda)^2\Big)^{\frac{1}{2}}\Big\} \\ &\geq 1 + 4\operatorname{Re}\Lambda + \frac{|\operatorname{Im}\Lambda|^2}{1 + |\Lambda|} \,. \end{aligned}$$

Thus

$$\operatorname{Re}\sqrt{1 + 4\Lambda} \geq 1 + c\Big\{\frac{\operatorname{Re}\Lambda}{\sqrt{1 + |\Lambda|}} + \frac{|\operatorname{Im}\Lambda|^2}{(1 + |\Lambda|)^{3/2}}\Big\} \,. \tag{1.5.7}$$

for some $c > 0$. In particular, $\operatorname{Re}\big(\sqrt{1 + \Lambda} - 1\big) > 0$ for $\Lambda \neq 0$, $\operatorname{Re}\Lambda \geq 0$ and $\operatorname{Re}\mu \neq 0$ and has the sign of a. With (1.5.5) the lemma follows. $\square$

Definition 1.5.6 (Green's functions). *For* $\lambda \neq 0$, $\operatorname{Re}\lambda \geq 0$, *define*

$$G^\varepsilon_\lambda(x, y) = \begin{cases} \dfrac{1}{\varepsilon(r_+ - r_-)} \, \varphi(x)\psi(y) e^{-ay/\varepsilon} & \text{for } x < y \,, \\ \dfrac{1}{\varepsilon(r_+ - r_-)} \, \psi(x)\varphi(y) e^{-ay/\varepsilon} & \text{for } y < x \,, \end{cases}$$

with

$$\varphi(x) = e^{r_+ x} - e^{r_- x} \,, \qquad \psi(x) = e^{r_- x} \,,$$

where $r_\pm$ are the solutions of the indicial equations given by (1.5.6).

Proposition 1.5.7. *For* $\lambda \neq 0$, $\operatorname{Re}\lambda \geq 0$ *and* $f \in L^2(\mathbb{R}_+)$ *the equation* (1.5.4) *has a unique solution* $u \in H^2(\mathbb{R}_+)$ *given by*

$$u(x) = \int_0^\infty G^\varepsilon_\lambda(x, y) f(y) dy \tag{1.5.8}$$

Sketch of proof. First check that (1.5.8) gives a solution when f is continuous with compact support, and that u is exponentially decaying at $+\infty$. The Lemma 1.5.5 implies that the homogeneous equation has no nontrivial solution in L^2. Thus the solution is unique in L^2. The explicit form of the definition reads

$$
\begin{aligned}
u(x) = \frac{1}{\varepsilon(r_+ - r_-)} \Big(\int_0^x & e^{r_-(x-y)} f(y) dy \\
& + \int_x^\infty e^{r_+(x-y)} f(y) dy - e^{r_- x} \int_0^\infty e^{-r_+ y} f(y) dy \Big) .
\end{aligned}
\tag{1.5.9}
$$

Next use the explicit form of G^ε_λ to check that

$$
\sup_x \| G^\varepsilon_\lambda(x, \cdot\,) \|_{L^1} < +\infty , \qquad \sup_y \| G^\varepsilon_\lambda(\,\cdot\,, y) \|_{L^1} < +\infty .
$$

Using Schur's lemma, this implies that (1.5.8) defines a bounded mapping G^ε_λ from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R}_+)$. Similarly, $\partial_x G^\varepsilon_\lambda$ and $\partial_x^2 G^\varepsilon_\lambda$ are shown to map L^2 to L^2. Thus G^ε_λ maps L^2 to H^2. Because $G^\varepsilon_\lambda f$ is a classical solution of (1.5.4) when f is continuous with compact support, by density, $G^\varepsilon_\lambda f$ is a weak solution of (1.5.4) when $f \in L^2$. $\square$

1.5.3 The inviscid limit: layers

We now pass to the limit in (1.5.4) as ε tends to zero. More precisely, we consider the limits of the solutions u^ε given by (1.5.8). Assume that $f \in L^1 \cap L^\infty$. In this section, we also assume that

$$
\gamma = \operatorname{Re} \lambda > 0 . \tag{1.5.10}
$$

Proposition 1.5.8 (Case 1 : $a > 0$). *One has*

$$
\| u^\varepsilon(x) - u_0(x) \|_{L^\infty} = O(\varepsilon) \tag{1.5.11}
$$

where u_0 is the unique bounded solution of

$$
a \partial_x u_0 + \lambda u_0 = f , \qquad u(0) = 0 . \tag{1.5.12}
$$

Proof. There holds $\varepsilon(r_+ - r_-) \to a$, $r_- \to -\lambda/a$ and $r_+ \sim a/\varepsilon$. Thus

$$
\int_0^\infty e^{-r_+ y} |f(y)| dy \lesssim \varepsilon \|f\|_{L^\infty} , \qquad \int_x^\infty e^{r_+(x-y)} |f(y)| dy \lesssim \varepsilon \|f\|_{L^\infty}
$$

$$
\int_0^x e^{r_-(x-y)} f(y) dy \to \int_0^x e^{-\lambda(x-y)/a} f(y) dy .
$$

Therefore,

$$\|u^\varepsilon - u_0\|_{L^\infty} = O(\varepsilon)$$

with

$$u_0(x) = \frac{1}{a}\int_0^x e^{-\lambda(x-y)/a} f(y)dy$$

which is the unique solution of (1.5.12). □

Proposition 1.5.9 (Case 2 : $a < 0$). *There holds*

$$u^\varepsilon(x) = u_0(x) - e^{ax/\varepsilon}u_0(0) + O(\varepsilon) \quad \text{in } L^\infty([0,\infty[) \tag{1.5.13}$$

where u_0 is the unique bounded solution of

$$a\partial_x u_0 + \lambda u_0 = f\,, \qquad u \in L^\infty\,. \tag{1.5.14}$$

Proof. When $a < 0$, $\varepsilon(r_+ - r_-) \to -a$, $r_+ \to -\lambda/a$, $r_- \sim a/\varepsilon$ and thus

$$\int_0^x e^{r_-(x-y)}|f(y)|dy \lesssim \varepsilon\|f\|_{L^\infty}\,,$$
$$\int_x^\infty e^{r_+(x-y)}f(y)dy \to \int_x^\infty e^{\lambda(y-x)/a}f(y)dy\,.$$

Hence

$$\|u^\varepsilon - u_0 - e^{ax/\varepsilon}u_0(0)\|_{L^\infty} = O(\varepsilon)$$

where

$$u_0(x) = -\frac{1}{a}\int_x^\infty e^{\lambda(y-x)/a}f(y)dy\,.$$

Note that u_0 is the unique bounded solution of (1.5.14). □

Comment. The analysis above is quite similar to the time dependent analysis of section 3. More precisely, (1.5.11) (1.5.13) can be seen as the Laplace transform of (1.3.7)(1.3.8)

When $a < 0$, u_0 does not satisfy in general the boundary condition $u_0(0) = 0$. This is why a *boundary layer* appears: the solution of the singular perturbation (1.5.4) is u_0 plus a corrector $-e^{ax/\varepsilon}\alpha$.

1.5.4 Estimates

The Propositions above provide solutions $\widetilde{u}(\lambda, \cdot\,)$ to the Laplace transformed equations (1.5.2). To construct solutions of (1.1.1) we need to apply the inverse Laplace transform to $\widetilde{u}$. In order to apply Proposition 1.5.3 and Lemma 1.5.4, we want to show that $\widetilde{u}$ belongs to a space $\widetilde{W}_\gamma$. This means that we need estimates for the solutions of (1.5.4).

One can use the explicit formulas (1.5.8) to find the suitable estimates. To prepare the multidimensional analysis, we will prove them using the method of symmetrizers.

Theorem 1.5.10. *For $\lambda = \gamma + i\tau \neq 0$ with $\gamma \geq 0$ and $f \in L^2(\mathbb{R}_+)$, the solution $u \in H^2(\mathbb{R}_+)$ of* (1.5.4) *satisfies*

$$\rho\|u\|_{L^2(\mathbb{R}_+)} + \sqrt{\rho}\sqrt{\varepsilon}\|\partial_x u\|_{L^2(\mathbb{R}_+)} \leq C\|f\|_{L^2(\mathbb{R}_+)} . \tag{1.5.15}$$

with C independent of λ, ε and f and

$$\rho \approx \begin{cases} \gamma + \varepsilon\tau^2 & when \quad \varepsilon|\lambda| \leq 1 \\ |\lambda| & when \quad \varepsilon|\lambda| \geq 1 \end{cases} \tag{1.5.16}$$

According to Proposition 1.5.7, for $f \in L^2(\mathbb{R}_+)$, $\lambda \neq 0$ with $\operatorname{Re}\lambda \geq 0$, the boundary value problem (1.5.4) has a unique solution $u \in H^2(\mathbb{R}_+)$. We first give the *easy* estimates.

Lemma 1.5.11. *For $f \in L^2(\mathbb{R}_+)$, $\lambda \neq 0$ with $\operatorname{Re}\lambda \geq 0$, the solution $u \in H^2(\mathbb{R}_+)$ of* (1.5.4) *satisfies*

$$\gamma\|u\|^2_{L^2(\mathbb{R}_+)} + 2\varepsilon\|\partial_x u\|^2_{L^2(\mathbb{R}_+)} \leq \frac{1}{\gamma}\|f\|^2_{L^2(\mathbb{R}_+)} . \tag{1.5.17}$$

Proof. Multiply (1.5.4) by $\overline{u}$, integrate over $\mathbb{R}_+$ and take the real part. This yields

$$\gamma\|u\|^2_{L^2(\mathbb{R}_+)} + \varepsilon\|\partial_x u\|^2_{L^2(\mathbb{R}_+)} = \operatorname{Re}\int_0^\infty f(x)\overline{u}(x)dx \leq \|f\|_{L^2(\mathbb{R}_+)}\|u\|_{L^2(\mathbb{R}_+)} .$$

□

Proof of Theorem 1.5.10. To get the sharp estimates (1.5.15) we consider (1.5.4) as a first order system: with

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad v = \varepsilon\partial_x u \,, \qquad F = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

the equation (1.5.4) reads

$$\partial_x U = \frac{1}{\varepsilon} GU + F\,, \qquad \Gamma U(0) = 0 \tag{1.5.18}$$

with

$$G = \begin{pmatrix} 0 & 1 \\ \varepsilon\lambda & a \end{pmatrix}, \qquad \Gamma \begin{pmatrix} u \\ v \end{pmatrix} = u\,.$$

The eigenvalues of G are $\varepsilon r_\pm$ thus $a + \varepsilon\mu$ and $-\varepsilon\mu$. Thus G can be diagonalized. Consider

$$\Omega = \begin{pmatrix} 1 & 1 \\ \varepsilon r_+ & \varepsilon r_- \end{pmatrix}, \qquad \Omega^{-1} = \frac{1}{\varepsilon(r_- - r_+)} \begin{pmatrix} \varepsilon r_- & -1 \\ -\varepsilon r_+ & 1 \end{pmatrix}.$$

Thus U is solution to (1.5.18) if and only if

$$V = \begin{pmatrix} v_+ \\ v_- \end{pmatrix} := \Omega^{-1} U$$

satisfies

$$\partial_x v_+ = r_+ v_+ + f_+ \qquad \text{on } \mathbb{R}_+\,, \tag{1.5.19}$$

$$\partial_x v_- = r_- v_- + f_- \qquad \text{on } \mathbb{R}_+\,, \tag{1.5.20}$$

$$v_+(0) + v_-(0) = 0\,, \tag{1.5.21}$$

with

$$f_+ = -f_- = \frac{1}{\varepsilon(r_+ - r_-)} f\,. \tag{1.5.22}$$

Multiplying (1.5.19) by $\overline{v}_+$ and integrating over $\mathbb{R}_+$ yields:

$$\frac{1}{2}|v_+(0)|^2 + \mathrm{Re}\, r_+ \|v_+\|^2 \leq \|f_+\|\,\|v_+\|$$

where the norms are taken in $L^2(\mathbb{R}_+)$. Thus

$$|v_+(0)|^2 + \mathrm{Re}\, r_+ \|v_+\|^2 \leq \frac{1}{\mathrm{Re}\, r_+} \|f_+\|^2\,. \tag{1.5.23}$$

Similarly, multiplying (1.5.20) by $-\overline{v}_-$ and integrating by parts yields

$$-|v_-(0)|^2 + |\mathrm{Re}\, r_-| \|v_-\|^2 \leq \frac{1}{|\mathrm{Re}\, r_-|} \|f_-\|^2\,. \tag{1.5.24}$$

Using the boundary condition (1.5.21), the definition (1.5.22) of $f_\pm$, adding $2\times$(1.5.23) and (1.5.24) yields

$$\frac{1}{2}|V(0)|^2 + \operatorname{Re} r_+ \|v_+\|^2 + |\operatorname{Re} r_-|\,\|v_-\|^2 \leq \frac{2}{\varepsilon^2|r_+ - r_-|^2}\Big(\frac{1}{\operatorname{Re} r_+} + \frac{1}{|\operatorname{Re} r_-|}\Big)\|f\|^2 .$$

Denoting by r_1 and r_2 the roots as in subsection 1.3.1 and labeling accordingly (v_1, v_2) the components of v, we also get

$$\frac{1}{2}|V(0)|^2 + |\operatorname{Re} r_1|\|v_1\|^2 + |\operatorname{Re} r_2|\|v_2\|^2 \leq \frac{2}{\varepsilon^2|r_1 - r_2|^2}\Big(\frac{1}{|\operatorname{Re} r_1|} + \frac{1}{|\operatorname{Re} r_2|}\Big)\|f\|^2 .$$

Forgetting the traces, we get the following estimates for $u = v_1 + v_2$ and $\partial_x u = r_1 v_1 + r_2 v_2$:

$$\|u\| \leq \frac{2}{\varepsilon|r_1 - r_2|}\Big(\frac{1}{|\operatorname{Re} r_1|} + \frac{1}{|\operatorname{Re} r_2|}\Big)\|f\| , \tag{1.5.25}$$

$$\|\partial_x u\| \leq \frac{2}{\varepsilon|r_1 - r_2|}\Big(\frac{1}{|\operatorname{Re} r_1|} + \frac{1}{|\operatorname{Re} r_2|}\Big)^{\frac{1}{2}}\Big(\frac{|r_1|^2}{|\operatorname{Re} r_1|} + \frac{|r_2|^2}{|\operatorname{Re} r_2|}\Big)^{\frac{1}{2}}\|f\| \tag{1.5.26}$$

Introduce the weight ρ such that $\rho \approx \gamma + \varepsilon\tau^2$ when $\varepsilon|\lambda| \leq 1$ and $\rho \approx |\lambda|$ when $\varepsilon|\lambda| \geq 1$ as indicated in (1.5.16). Note that both definitions agree when $\varepsilon|\lambda| \approx 1$, in which case $\rho \approx \varepsilon^{-1}$. The goal is to prove that there is a constant C such that for all $\lambda \neq 0$ with $\operatorname{Re}\lambda \geq 0$ and all $u \in H^2(\mathbb{R}_+)$ solution of (1.5.4)

$$\rho\|u\| \leq C\|f\| , \tag{1.5.27}$$

$$\sqrt{\varepsilon}\sqrt{\rho}\|\partial_x u\| \leq C\|f\| . \tag{1.5.28}$$

a) *The LF regime.* Suppose that $\varepsilon|\lambda| \ll 1$.

In this case, $\varepsilon r_1 \sim a$, $r_2 = -\lambda/a + \varepsilon\lambda^2/(2a^3) + O(\varepsilon^2|\lambda|^3$. Thus, with $\lambda = \gamma - i\tau$:

$$|\operatorname{Re} r_1| \sim \frac{|a|}{\varepsilon} , \quad |\operatorname{Re} r_2| = \frac{1}{|a|}\big(\gamma + \varepsilon(\tau^2 - \gamma^2) + O(\varepsilon^2|\lambda^3|\big) \geq c\rho ,$$

and

$$\varepsilon|r_1 - r_2| \sim |a| , \quad \frac{1}{|\operatorname{Re} r_1|} + \frac{1}{|\operatorname{Re} r_2|} \leq \frac{1}{c\rho} .$$

This yields (1.5.27). Moreover,

$$\frac{|r_1|^2}{|\mathrm{Re}\, r_1|} \le \frac{C}{\varepsilon}\,, \qquad \frac{|r_2|^2}{|\mathrm{Re}\, r_2|} \le \frac{C|\lambda|^2}{\rho} \le \frac{C'}{\varepsilon}\,.$$

Thus (1.5.28) follows.

b) *The HF regime* : $|\varepsilon\lambda| \gg 1$.
In this case,

$$r_\pm \sim \pm\frac{\sqrt{\varepsilon\lambda}}{\varepsilon}\,, \qquad \mathrm{Re}\, r_\pm \approx \pm\frac{\sqrt{\varepsilon\lambda}}{\varepsilon}$$

thus $|r_1|$, $|r_2$, $\mathrm{Re}\, r_1$, $\mathrm{Re}\, r_2$ and $|r_1 - r_2|$ are of order $\sqrt{\rho}/\varepsilon$. Hence (1.5.25) and (1.5.26) imply the estimates (1.5.27) and (1.5.28)

c) *The MF regime* : $0 < c \le |\varepsilon\lambda| \le C < +\infty$.
In this case, $|r_\pm|$, $r_+ - r_-$, $\mathrm{Re}\, r_\pm$ and ρ are of order $1/\varepsilon$ and (1.5.27) (1.5.28) follow from (1.5.25) and (1.5.26). □

1.5.5 Solutions of the BVP

With the estimates, one can perform the inverse Laplace transform to get a solution of (1.1.1) on $\mathbb{R} \times \mathbb{R}_+$. We do not give the details here, most of them can be found in the next chapter.

Chapter 2

Hyperbolic Mixed Problems

In this chapter, we discuss the classical theory of mixed Cauchy boundary value problem for symmetric hyperbolic systems see [Fr1], [Fr2], [Fr-La] and also [Tar], [Ra-Ma]. We follow closely the presentation in [Ch-Pi]. For simplicity, we consider here only constant coefficients equations, and flat boundaries, but all the technics can be adapted to variable coefficients and general smooth domains.

2.1 The equations

Consider a $N \times N$ system

$$Lu := \partial_t u + \sum_{j=1}^{d} A_j \partial_j u = F(u) + f \tag{2.1.1}$$

For simplicity, we assume that the coefficients A_j are constant. F is a C^∞ mapping from $\mathbb{R}^N$ to $\mathbb{R}^N$. The variables are $t \in \mathbb{R}$, $y = (y_1, \ldots, y_{d-1}) \in \mathbb{R}^{d-1}$ and $x \in \mathbb{R}$. The derivations are $\partial_j = \partial_{y_j}$ for $j \in \{1, \ldots, d-1\}$ and $\partial_d = \partial_x$.

For simplicity, we work in the class of symmetric hyperbolic operators:

Assumption 2.1.1.
(H1) *There is a positive definite symmetric matrix* $S = {}^tS \gg 0$ *such that for all* j, SA_j *is symmetric.*
(H2) $\det A_d \neq 0$

The assumption (H2) means that the boundary is *not characteristic* for L. The eigenvalues of A_d are real and different from zero. We denote by N_+ [resp. N_-] the number of positive [resp. negative] eigenvalues of A_d. Then $N = N_+ + N_-$.

Lemma 2.1.2. *The matrix SA_d has only real eigenvalues. Counted with their multiplicities, N_+ are positive and N_- are negative.*

Proof. Dropping the subscript d, $SA = S^{1/2}\big(S^{1/2}AS^{1/2}\big)S^{-1/2}$ is conjugated to the symmetric matrix $A' := S^{1/2}AS^{1/2}$. Therefore the eigenvalues of SA are those of A', thus are real. In addition, A' has the same signature (N_+, N_-) as A. $\square$

We consider the equations (2.1.1) on the half space $\{x \geq 0\}$ together with boundary conditions:

$$Mu_{|x=0} = Mg \,. \tag{2.1.2}$$

where M is a $N' \times N$ matrix.

In the theory of hyperbolic boundary problems, the simplest case occurs when the boundary conditions are *maximal dissipative*:

Definition 2.1.3. *The boundary condition* (2.1.2) *is maximal dissipative for L if and only if dim* $\ker M = N_-$ *and the symmetric matrix SA_d is definite negative on* $\ker M$.

In this Chapter we study the well-posedness of the hyperbolic boundary value problem (2.1.1) (2.1.2). We always assume that Assumption 2.1.1 holds and that the boundary condition is maximal dissipative. Restricting attention to the image of M, there is no loss of generality in assuming that $N' = N_+$, so that M is a $N_+ \times N$ matrix.

Remark 2.1.4. The number of boundary conditions is $N' = N_+$, and there is an easy way to see that is the correct number of conditions. In space dimension one, consider a diagonal system $\partial_t + A\partial_x$ with $A = \mathrm{diag}(a_1, \ldots, a_N)$. The diagonal entries are real, and $N+$ are positive, N_- are negative. We have seen in the first chapter, that a boundary condition is needed for $\partial_t + a_j\partial_x$ for positive a_j. So, the total number of boundary conditions must be N_+.

Remark 2.1.5. The dissipativity condition is satisfied in many physical examples (wave equations with Dirichlet boundary conditions, Maxwell equations with usual boundary conditions, etc). However, it is far from being necessary (see the discussion in Chapter 6 for an approach to necessary conditions). In the analysis below, it appears as a trick to warranty good energy estimates, but in applications these computations mean dissipation of a physical energy.

2.2 Hyperbolic boundary value problems

In this section we consider the problem

$$\left\{\begin{array}{ll} Lu = f & \textit{on } \mathbb{R}\times\mathbb{R}^d_+ \\ Mu_{|x=0} = g & \textit{on } \mathbb{R}\times\mathbb{R}^{d-1} \end{array}\right. \tag{2.2.1}$$

We use the notation $\mathbb{R}^d_+ = \{(y,x)\in\mathbb{R}^d : x>0\}$. We assume that the Assumptions (H1) and (H2) are satisfied, that M is a $N_+\times N$ matrix and that the boundary condition is maximal dissipative.

We first solve this equation in weighted spaces: we look for solutions $u = e^{\gamma t}\widetilde{u}$, assuming that $f = e^{\gamma t}\widetilde{f}$ and $g = e^{\gamma t}\widetilde{g}$, with $\widetilde{u}$, $\widetilde{f}$ and $\widetilde{g}$ at least in L^2. This yields the equations

$$\left\{\begin{array}{ll} (L+\gamma)\widetilde{u} = \widetilde{f} & \textit{on } \mathbb{R}\times\mathbb{R}^d_+ \\ M\widetilde{u}_{|x=0} = \widetilde{g} & \textit{on } \mathbb{R}\times\mathbb{R}^{d-1}\,. \end{array}\right. \tag{2.2.2}$$

The choice $\gamma>0$ corresponds to the idea that the functions u, f and g vanish at $t=-\infty$ and thus to an orientation of time.

We first study (2.2.2), dropping the tildes. We denote by H^s the usual Sobolev spaces. We also use the notation $\mathbb{R}^{1+d}_+ = \mathbb{R}\times\mathbb{R}^d_+$.

2.2.1 The adjoint problem

The adjoint of L (in the sense of distributions) is $L^* := -\partial_t - \sum A_j^*\partial_j$. Thus $-L^*$ has the same form as L.

Lemma 2.2.1. *S^{-1} is a symmetrizer for $-L^*$.*

Proof. Since S is symmetric definite positive, S^{-1} is also definite positive. Moreover, $S^{-1}A_j^* = S^{-1}A_j^*SS^{-1} = S^{-1}SA_jS^{-1} = A_jS^{-1}$ is symmetric. □

For C^1 functions with compact support in $\overline{\mathbb{R}}^{1+d}_+$, one has

$$\big(Lu, v\big)_{L^2} = \big(u, L^*v\big)_{L^2} - \big(A_d u_{|x=0}, v_{|x=0}\big)_{L^2} \tag{2.2.3}$$

where $(\cdot,\cdot)_{L^2}$ denotes the scalar product in L^2. Consider a space of dimension N_+ on which SA_d is definite positive. There is a $N_-\times N$ matrix M_1 such that this space is $\ker M_1$. Since M is maximal dissipative, SA_d is definite negative on $\ker M$ and therefore

$$\mathbb{R}^N = \ker M \oplus \ker M_1 \tag{2.2.4}$$

Lemma 2.2.2. *There are matrices R and R_1 of size $N_- \times N$ and $N_+ \times N$ respectively, such that for all vectors u and v in $\mathbb{R}^N$:*

$$(2.2.5) \qquad \big(A_d u, v\big) = (Mu, R_1 v) + (M_1 u, Rv)\,.$$

Moreover, $\ker R = (A_d \ker M)^\perp$ *has dimension N_+ and $S^{-1}A_d^*$ is definite positive on* $\ker R$.

Proof. The identity (2.2.5) is equivalent to

$$\begin{aligned}(A_d u, v) &= (Mu, R_1 v), \quad \forall u \in \ker M_1\\ (A_d u, v) &= (M_1 u, Rv), \quad \forall u \in \ker M\,.\end{aligned}$$

Since M is an isomorphism from $\ker M_1$ to $\mathbb{R}^{N_+}$, the first equation determines $R_1 v \in \mathbb{R}^{N_+}$. Similarly, the second equation determines $Rv \in \mathbb{R}^{N_-}$.

The identity (2.2.5) implies that $(A_d u, v) = 0$ when $u \in \ker M$ and $v \in \ker R$, thus $\ker R \subset (A_d \ker M)^\perp$. Because the two spaces have the same dimension, they are equal.

Suppose that $(S^{-1}A_d^* v, v) \le 0$ for some $v \in \ker R$. Then for all $u \in \ker M$, $(SA_d u, S^{-1}v) = 0$ by (2.2.5) and for all $\alpha \in \mathbb{R}$

$$\big(SA_d(u + \alpha S^{-1}v), u + \alpha S^{-1}v\big) = (SA_d u, u) + \alpha^2 (A_d S^{-1} v, v) \le 0\,.$$

Since $\ker M$ has maximal dimension among spaces on which SA_d is non positive, this implies that $S^{-1}v \in \ker M$. Because $\ker R$ and $A_d \ker M$ are orthogonal, one has $(A_d S^{-1} v, v) = (SA_d S^{-1}v, S^{-1}v) = 0$. Since SA_d is definite negative on $\ker M$, this shows that $S^{-1}v = 0$, hence $v = 0$. □

Definition 2.2.3. *The system L^* with boundary condition R is the adjoint problem of* (L, M).

Note that R is not unique, but the key object $\ker R = (A_d \ker M)^\perp$ is uniquely determined from L and M.

With (2.2.3), the lemma implies that for all u and v in $C_0^1(\overline{\mathbb{R}}_+^{1+d})$

$$\begin{aligned}\big((L+\gamma)u, v\big)_{L^2} = \big(u, (L^* + \gamma)v\big)_{L^2} &- \big(Mu_{|x=0}, R_1 v_{|x=0}\big)_{L^2}\\ &- \big(M_1 u_{|x=0}, Rv_{|x=0}\big)_{L^2}\,.\end{aligned}$$

In particular, if u is a solution of (2.2.2) and $Rv = 0$ on $\{x = 0\}$, one has

$$\big(f, v\big)_{L^2} = \big(u, (L^* + \gamma)v\big)_{L^2} - \big(g, R_1 v_{|x=0}\big)_{L^2}$$

This motivates the following definition of weak solutions.

Definition 2.2.4. *Given* $f \in L^2(\mathbb{R}^{1+d}_+)$ *and* $g \in L^2(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^{1+d}_+)$ *is a weak solution of* (2.2.2), *if and only if for all* $\Phi \in C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ *such that* $R\Phi_{|x=0} = 0$ *one has*

$$\big(u, (L^* + \gamma)\Phi\big)_{L^2} = \big(f, \Phi\big)_{L^2} + \big(g, R_1\Phi_{|x=0}\big)_{L^2} . \tag{2.2.6}$$

We now discuss in which sense weak solutions are indeed solutions of (2.2.2) Introduce the spaces $H^{0,s}(\mathbb{R}^{1+d})$ of temperate distributions such that their Fourier transform satisfy

$$\int \big(1 + \tau^2 + |\eta|^2\big)^s |\hat{u}(\tau, \eta, \xi)|^2 d\tau d\eta d\xi < +\infty . \tag{2.2.7}$$

For $s \in \mathbb{N}$, this is the space of functions $u \in L^2$ such that the tangential derivatives $D^\alpha_{t,y}$ of order $|\alpha| \le s$ belong to L^2. When s is a negative integer, this is the space of

$$u = \sum_{|\alpha| \le -s} \partial^\alpha_{t,y} u_\alpha , \qquad u_\alpha \in L^2 .$$

The space $H^{0,s}(\mathbb{R}^{1+d}_+)$ is the set of restrictions to $\{x > 0\}$ of functions in $H^{0,s}(\mathbb{R}^{1+d})$. When s is a positive or negative integer, there are equivalent definitions analogous to those given on the whole space.

Lemma 2.2.5. *For all* $s \in \mathbb{R}$*:*

i) the space $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ *is dense in the space* $H^{1,s}(\mathbb{R}^{1+d}_+)$ *of functions* $u \in H^{0,s+1}(\mathbb{R}^{1+d}_+)$ *such that* $D_x u \in H^{0,s}(\mathbb{R}^{1+d}_+)$*;*

ii) the mapping $u \mapsto u_{|x=0}$ *extends continuously from* $H^{1,s}(\mathbb{R}^{1+d}_+)$ *to* $H^{s+\frac{1}{2}}(\mathbb{R}^d)$.

Proof. The first part is proved by usual smoothing arguments. The details are left as an exercise.

Consider next $u \in C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ and denote by $\hat{u}(\tau, \eta, x)$ its partial Fourier transform with respect to the tangential variables (t, y). Integrating $\partial_x |\hat{u}|^2$ on $\mathbb{R}_+$, yields

$$|\hat{u}(\tau, \eta, 0)|^2 \le 2 \int_0^\infty |\partial_x \hat{u}(\tau, \eta, x)||\hat{u}(\tau, \eta, x|dx,$$

Thus, with $\Lambda = (1 + \tau^2 + |\eta|^2)^{1/2}$,

$$\Lambda^{2s+1} |\hat{u}(\cdot, 0)|^2 \le \Lambda^{2s} \int_0^\infty |\partial_x \hat{u}(\cdot, x)|^2 dx + \Lambda^{2s+2} \int_0^\infty |\hat{u}(\cdot, x)|^2 dx$$

Integrating in (τ, η) implies

$$\|u(\cdot, 0)\|^2_{H^{s+1/2}(\mathbb{R}^d)} \le \|\partial_x u\|^2_{H^{0,s}(\mathbb{R}^{1+d}_+)} + \|u\|^2_{H^{0,s+1}(\mathbb{R}^{1+d}_+)} = \|u\|^2_{H^{1,s}(\mathbb{R}^{1+d}_+)} \,.$$

Thus the mapping $u \mapsto u_{|x=0}$ extends by density continuity to $H^{1,s}(\mathbb{R}^{1+d}_+)$ with values in $H^{s+1/2}(\mathbb{R}^d)$. $\square$

We apply this lemma to the space

$$\mathcal{D}(L) = \left\{ u \in L^2(\mathbb{R}^{1+d}_+) \,:\, Lu \in L^2(\mathbb{R}^{1+d}_+) \right\} \,. \tag{2.2.8}$$

Here Lu is computed in the sense of distributions on $\{x > 0\}$. This space is equipped with the norm $\|u\|_{L^2} + \|Lu\|_{L^2}$. Because A_d is invertible, for $u \in \mathcal{D}(L)$ one has

$$\partial_x u = A_d^{-1} L u - A_d^{-1} \partial_t u - \sum_{j=1}^{d-1} A_d^{-1} A_j \partial_{y_j} u \tag{2.2.9}$$

and therefore $\mathcal{D}(L) \subset H^{1,-1}(\mathbb{R}^{1+d}_+)$. This shows that all $u \in \mathcal{D}(L)$ has a trace in $H^{-\frac{1}{2}}$.

Proposition 2.2.6. *i)* $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ *is dense in* $\mathcal{D}(L)$

ii) For all $u \in \mathcal{D}(L)$ *and* $v \in H^1(\mathbb{R}^{1+d}_+)$*, there holds*

$$\big(Lu, v\big)_{L^2} = \big(u, L^* v\big)_{L^2} - \big\langle A_d u_{|x=0}, v_{|x=0} \big\rangle_{H^{-1/2} \times H^{1/2}} \tag{2.2.10}$$

Proof. Consider a tangential mollifier $\jmath \in C_0^\infty(\mathbb{R} \times \mathbb{R}^{d-1})$, with $\jmath \ge 0$ and such that $\int \jmath(t, y) dt dy = 1$. For $\varepsilon > 0$, let

$$\jmath_\varepsilon(t, y) = \frac{1}{\varepsilon^d} \jmath\Big(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\Big) \,, \qquad t \in \mathbb{R} \,, \ y \in \mathbb{R}^{d-1} \,. \tag{2.2.11}$$

Denote by J_ε the convolution operator $\jmath_\varepsilon *$.

If $u \in \mathcal{D}(L)$ and $\Phi \in C_0^\infty(\mathbb{R}^{1+d})$ then $J_\varepsilon \Phi \in C_0^\infty(\mathbb{R}^{1+d})$ and in the sense of distributions

$$\big(u, L^* J_\varepsilon \Phi\big)_{L^2} = \big(Lu, J_\varepsilon \Phi\big)_{L^2} \,.$$

Note that we assume here that the support of Φ is contained in the open half space $\{x > 0\}$. Because J_ε commutes with differentiation and with multiplication by constants, $L^* J_\varepsilon \Phi = J_\varepsilon L^* \Phi$. Moreover, for all u and v in $L^2(\mathbb{R}^{1+d}_+)$, one has

$$\big(u, J_\varepsilon v\big)_{L^2} = \big(J_\varepsilon u, v\big)_{L^2}$$

Thus, there holds in the sense of distributions on $\{x > 0\}$:

$$LJ_\varepsilon u = J_\varepsilon Lu \ .$$

In particular $u_\varepsilon = J_\varepsilon u \in \mathcal{D}(L)$. Moreover, for all v in $L^2(\mathbb{R}^{1+d}_+)$, $J_\varepsilon v$ converges to v in L^2 when ε tends to zero. Thus, for $u \in \mathcal{D}(L)$, u_ε converges to u in $\mathcal{D}(L)$.

Next we note that for all v in L^2, $J_\varepsilon v \in H^{0,s}$ for all $s \in \mathbb{N}$, since for all α $\partial^\alpha_{t,y}(J_\varepsilon v) = (\partial^\alpha_{t,y} \jmath_\varepsilon) * v \in L^2$. Thus, $u_\varepsilon \in H^{0,s}$ for all s. Using (2.2.9) we see that $u_\varepsilon \in H^{1,s}$ for all s. In particular, $u_\varepsilon \in H^1(\mathbb{R}^{1+d}_+)$ and this shows that $H^1(\mathbb{R}^{1+d}_+)$ is dense in $\mathcal{D}(L)$. Since $C^\infty_0(\overline{\mathbb{R}}^{1+d}_+)$ is dense in $H^1(\mathbb{R}^{1+d}_+)$ this implies $i)$.

By (2.2.9), we see that $\mathcal{D}(L) \subset H^{1,-1}$ and

$$\|u\|_{H^{1,-1}} \lesssim \|u\|_{L^2} + \|Lu\|_{L^2} \ .$$

Thus by the trace lemma, the trace $u_{|x=0}$ is well defined on $\mathcal{D}(L)$ and

$$\|u_{|x=0}\|_{H^{-1/2}} \lesssim \|u\|_{L^2} + \|Lu\|_{L^2} \ .$$

The identity (2.2.10) holds when u and v belong to $C^\infty_0(\overline{\mathbb{R}}^{1+d}_+)$. Both side are continuous for the norms of u in $\mathcal{D}(L)$ and v in H^1. Thus, the identity extends by density to $\mathcal{D}(L) \times H^1$. □

Corollary 2.2.7. *Given $f \in L^2(\mathbb{R}^{1+d}_+)$ and $g \in L^2(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^{1+d}_+)$ is a weak solution of* (2.2.2) *if and only if*

i) $u \in \mathcal{D}(L)$ and $Lu = f - \gamma u$ in the sense of distributions on $\{x > 0\}$,

ii) the trace $u_{|x=0}$ which is defined in $H^{-1/2}$ by i) satisfies $Mu_{|x=0} = g$.

Proof. If u is a weak solution, taking Φ with compact support in the open half space implies that $Lu + \gamma u = f$ in the sense of distributions. Thus $u \in \mathcal{D}(L)$.

Comparing (2.2.10) and (2.2.6) we see that for all $\Phi \in C^\infty_0(\overline{\mathbb{R}}^{1+d}_+)$ such that $R\Phi = 0$ on the boundary, there holds

$$\big(g, R_1\Phi_{|x=0}\big)_{L^2} = \big\langle A_d u_{|x=0}, \Phi_{|x=0}\big\rangle_{H^{-1/2}\times H^{1/2}}$$

Next we use Lemma 2.2.2, which means that $A_d = (R_1)^* M + R^* M_1$ to see that the right hand side is equal to

$$\big\langle M u_{|x=0}, R_1\Phi_{|x=0}\big\rangle_{H^{-1/2}\times H^{1/2}} \ .$$

For all $\phi \in C_0^\infty(\mathbb{R}^d)$ there is $\Phi \in C_0^\infty(\overline{\mathbb{R}}_+^{1+d})$ such that $\Phi_{|x=0} = \phi$. Thus, for all $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $R\phi = 0$,

$$\big(g, R_1\phi\big)_0 = \big\langle Mu_{|x=0}, R_1\phi\big\rangle_{H^{-1/2}\times H^{1/2}}\,.$$

Similar to (2.2.4), there is a splitting

$$\mathbb{R}^N = \ker R \oplus \ker R_1$$

Therefore, for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ with values in $\mathbb{R}^{N_+}$, there is $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $R\phi = 0$ and $R_1\phi = \varphi$. Thus for all $\varphi \in C_0^\infty(\mathbb{R}^d)$:

$$\big(g, \varphi\big)_0 = \big\langle Mu_{|x=0}, \varphi\big\rangle_{H^{-1/2}\times H^{1/2}}\,.$$

This means that $Mu_{|x=0} = g$.

Conversely, if $u \in \mathcal{D}(L)$ and $Lu + \gamma u = f$, for all test function Φ, one has

$$\begin{aligned}\big(u, (L^* + \gamma)\Phi\big)_0 - \big(f, \Phi\big)_0 =& \big\langle Mu_{|x=0}, R_1\phi\big\rangle_{H^{-1/2}\times H^{1/2}}\\ &+ \big\langle M_1u_{|x=0}, R\phi\big\rangle_{H^{-1/2}\times H^{1/2}}\,.\end{aligned}$$

with $\phi = \Phi_{|x=0}$. Taking Φ such that $R\phi = 0$, we see that if $Mu_{|x=0} = g$ then u is a weak solution of (2.2.2). □

2.2.2 Energy estimates. Existence of weak solutions

Lemma 2.2.8. *The symmetric matrix SA_d is definite negative on* $\ker M$ *if and only if there are constants $c > 0$ and C such that for all vector $h \in \mathbb{C}^N$:*

$$-(SA_dh, h) \geq c|h|^2 - C|Mh|^2\,.$$

Proof. Since SA_d is definite negative on $\ker M$, there is $c > 0$ such that

$$\forall h \in \ker M\ :\quad -(SA_dh, h) \geq c|h|^2\,.$$

Since SA_d is invertible, $\dim(SA_d \ker M) = \dim\ker M = N_-$, thus $K = (SA_d \ker M)^\perp$ has dimension $N - N_- = N_+$. In addition since SA_d is definite negative on $\ker M$, $K \cap \ker M = \{0\}$ and $\mathbb{R}^N = K \oplus \ker M$. In particular, there is C_0 such that for all $v \in K$, $|v| \leq C_0|Mv|$. By definition of K, if $h = v + w$ with $v \in K$ and $w \in \ker M$, there holds

$$\begin{aligned}-(SA_dh, h) = -(SA_dv, v) - (SA_dw, w) &\geq c|w|^2 - C|v|^2\\ &\geq c(|w|^2 + |v|^2) - (C + c)C_0^2|Mv|^2\,.\\ &\geq \frac{c}{2}|h|^2 - (C + c)C_0^2|Mh|^2\,.\end{aligned}$$

The converse statement is clear. □

Proposition 2.2.9 (Energy estimates). *There is C such that for all $\gamma > 0$ and all test function $u \in H^1(\mathbb{R} \times \mathbb{R}^d_+)$, one has*

$$\gamma \|u\|^2_{L^2} + \|u_{|x=0}\|^2_{L^2} \leq C\big(\frac{1}{\gamma}\|(L+\gamma)u\|^2_{L^2} + \|Mu_{|x=0}\|^2_{L^2}\big) \tag{2.2.12}$$

$$\gamma \|v\|^2_{L^2} + \|v_{|x=0}\|^2_{L^2} \leq C\big(\frac{1}{\gamma}\|(L^*+\gamma)v\|^2_{L^2} + \|Ru_{|x=0}\|^2_{L^2}\big) \tag{2.2.13}$$

Proof. Both side of the estimates are continuous for the H^1 norm. Since $C^\infty_0(\overline{\mathbb{R}}^{1+d}_+)$ is dense in $H^1(\mathbb{R}^{1+d}_+)$ it is sufficient to make the proof when $u \in C^\infty_0$. Then, using that the SA_j are self adjoint and integrating by parts yields

$$2\mathrm{Re}\,\big(S(L+\gamma)u, u\big)_{L^2} = \gamma\big(Su, u\big)_{L^2} - \big(SA_d u_{|x=0}, u_{|x=0}\big)_{L^2}$$

By Lemma 2.2.8, there are $c > 0$ and $C \geq 0$ such that

$$-\big(SA_d u_{|x=0}, u_{|x=0}\big)_{L^2} \geq c\|u_{|x=0}\|^2_{L^2} - C\|Mu_{|x=0}\|^2_{L^2}\,.$$

Because S is definite positive, there is $c_1 > 0$ such that

$$\big(Su, u\big)_{L^2} \geq c_1\|u\|^2_{L^2}\,.$$

Therefore

$$c_1\gamma\|u\|^2_{L^2} + c\|u_{|x=0}\|^2_{L^2} \leq 2|S|\|(L+\gamma)u\|_{L^2}\|u\|_{L^2} + C\|Mu_{|x=0}\|^2_{L^2}\,.$$

This implies (2.2.12). The proof of (2.2.13) is similar. $\square$

Proposition 2.2.10. *For all $\gamma > 0$, f and g in L^2, the problem* (2.2.2) *has a weak solution in L^2.*

Proof. Consider the space $\mathcal{H}$ of $\Phi \in H^1(\mathbb{R}\times\mathbb{R}^d_+)$ such that $R\Phi_{|x=0} = 0$. Let $\mathcal{H}_1 = (L^* + \gamma)\mathcal{H} \subset L^2$. By (2.2.13), the mapping $L^* + \gamma$ is one to one from $\mathcal{H}$ to $\mathcal{H}_1$ and the reciprocal mapping $\mathcal{F}$ satisfies

$$\gamma\|\mathcal{F}\varphi\|_{L^2} + \sqrt{\gamma}\|R_1\mathcal{F}\varphi_{|x=0}\|_{L^2} \leq C\|\varphi\|_{L^2}\,.$$

Thus the linear form

$$\varphi \quad\mapsto\quad \ell(\varphi) := \big(f, \mathcal{F}\varphi\big)_{L^2} + \big(g, R_1\mathcal{F}\varphi_{|x=0}\big)_{L^2}$$

is continuous on $\mathcal{H}_1$ equipped with the norm $\|\cdot\|_{L^2}$. Therefore it extends as a continuous linear form on L^2 and there is $u \in L^2$ such that $\ell(\varphi) = (u,\varphi)_0$. The definition of ℓ implies that u is a weak solution. $\square$

2.2.3 Strong solutions

Definition 2.2.11. *Given f and g in L^2, $u \in L^2$ is a strong solution of* (2.2.2) *if there exists sequences (u_n, f_n) in $H^1(\mathbb{R}^{1+d}_+)$, and g_n in $H^1(\mathbb{R}^d)$ solutions of* (2.2.2) *and converging to (u, f) in $L^2(\mathbb{R}^{1+d}_+)$. and to g in $L^2(\mathbb{R}^d)$ respectively.*

By the density of $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ in $H^1(\mathbb{R}^{1+d}_+)$ and continuity from H^1 to L^2 of L and the traces, one obtains an equivalent definition if one requires that there is a sequence (u_n, f_n, g_n) in $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ solutions of (2.2.2) and converging to (u, f, g) in L^2.

Proposition 2.2.12 (Weak= strong). *For all $\gamma > 0$, f and g in L^2, any weak solution of* (2.2.2) *in L^2 is a strong solution and*

$$\gamma \|u\|^2_{L^2} + \|u_{|x=0}\|^2_{L^2} \leq C\big(\frac{1}{\gamma}\|f\|^2_{L^2} + \|g\|^2_{L^2}\big) \tag{2.2.14}$$

In particular the weak=strong solution is unique.

Proof. Consider again the mollifiers $\jmath$ (2.2.11) and the convolution operator $J_\varepsilon u = \jmath_\varepsilon * u$.

Suppose that $u \in L^2$ is a weak solution of (2.2.2). For all test function Φ, $J_\varepsilon \Phi$ is also a test function and $RJ_\varepsilon\Phi = 0$. Therefore,

$$\big(u, (L^* + \gamma)J_\varepsilon\Phi\big)_{L^2} = \big(f, J_\varepsilon\Phi\big)_{L^2} + \big(g, R_1 J_\varepsilon \Phi_{|x=0}\big)_{L^2} .$$

As in the proof of Proposition 2.2.6 this implies that

$$\big(J_\varepsilon u, (L^* + \gamma)\Phi\big)_{L^2} = \big(J_\varepsilon f, \Phi\big)_{L^2} + \big(J_\varepsilon g, R_1 \Phi_{|x=0}\big)_{L^2} .$$

This means that $u_\varepsilon = J_\varepsilon u$ is a weak solution of

$$\begin{cases} (L + \gamma)u_\varepsilon = f_\varepsilon \, , \\ \; Mu_{\varepsilon|x=0} = g_\varepsilon \, . \end{cases} \tag{2.2.15}$$

with $f_\varepsilon = J_\varepsilon f$ and $g_\varepsilon = J_\varepsilon g$.

The proof of Proposition 2.2.6 shows that for all $\varepsilon > 0$, $u_\varepsilon \in H^1(\mathbb{R}^{1+d})$ and by Corollary 2.2.7 the equations (2.2.15) hold in L^2.

Since u_ε, f_ε and g_ε converge in L^2 to (u, f, g) respectively, this shows that u is a strong solution.

In addition, the energy estimates (2.2.12) hold for u_ε. Passing to the limit, we obtain that u satisfies (2.2.14). □

2.2.4 Regularity of solutions

We prove that if the data are regular, then the solution is regular. It is convenient to equip the spaces $H^s(\mathbb{R}^{1+d}_+)$ with a family of parameter dependent norms:

$$\|u\|_{s,\gamma} = \sum_{|\alpha|\le s} \gamma^{s-|\alpha|} \|\partial^\alpha_{t,y,x} u\|_{L^2} . \tag{2.2.16}$$

We define similar norms on the spaces $H^s(\mathbb{R}^d)$, using only tangential derivatives $\partial^\alpha_{t,y}$.

Proposition 2.2.13. *Let s be a non negative integer. For $\gamma > 0$, $f \in H^s$ and $g \in H^s$ the solution of* (2.2.2) *belongs to H^s and*

$$\gamma \|u\|^2_{s,\gamma} + \|u_{|x=0}\|^2_{s,\gamma} \le C\big(\frac{1}{\gamma}\|f\|^2_{s,\gamma} + \|g\|^2_{s,\gamma}\big) \tag{2.2.17}$$

Proof. First prove the tangential regularity. We use the mollified equation (2.2.15). Since $u_\varepsilon \in H^{1,s}$ for all s, we can differentiate this equation as many times as we want in (t,y) and $\partial^\alpha_{t,y} u_\varepsilon \in H^1(\mathbb{R}^{1+d}_+)$ satisfies

$$\begin{cases} (L+\gamma)\partial^\alpha_{t,y} u_\varepsilon = \partial^\alpha_{t,y} f_\varepsilon , \\ M\partial^\alpha_{t,y} u_{\varepsilon|x=0} = \partial^\alpha_{t,y} g_\varepsilon . \end{cases}$$

Proposition 2.2.12 implies that

$$\gamma \|u_\varepsilon\|^2_{H^{0,s}} + \|u_{\varepsilon|x=0}\|^2_{H^{0,s}} \le C\big(\frac{1}{\gamma}\|f_\varepsilon\|^2_{H^{0,s}} + \|g_\varepsilon\|^2_{H^{0,s}}\big)$$

with C independent of ε.

Next we use the equation to recover the normal derivatives. We start from (2.2.9) which implies that

$$\|\partial_x u_\varepsilon\|_{H^{0,s-1}} \lesssim \|f_\varepsilon\|_{H^{0,s-1}} + \|u_\varepsilon\|_{H^{0,s}} .$$

In addition, since f_ε can be differentiated s times in x, we see by induction on $k \le s$ that $\partial^k_x u_\varepsilon \in H^{0,s'}$ for all s' with

$$\partial^k_x u_\varepsilon = A_d^{-1}\partial_x^{k-1} f_\varepsilon - A_d^{-1}\partial_x^{k-1}\partial_t u_\varepsilon - \sum_{j=1}^{d-1} A_d^{-1} A_j \partial_x^{k-1}\partial_j u_\varepsilon .$$

Thus

$$\|\partial^k_x u_\varepsilon\|_{H^{0,s-k}} \lesssim \|\partial_x^{k-1} f_\varepsilon\|_{H^{0,s-k}} + \|\partial_x^{k-1} u_\varepsilon\|_{H^{0,s-k+1}}$$

Adding up, we see that $u_\varepsilon \in H^{s+1}$ and that there is C independent of ε and γ such that

$$\gamma \|u_\varepsilon\|^2_{s,\gamma} + \|u_{\varepsilon|x=0}\|^2_{s,\gamma} \leq C\big(\frac{1}{\gamma}\|f_\varepsilon\|^2_{s,\gamma} + \|g_\varepsilon\|^2_{s,\gamma}\big)$$

This means that the u_ε satisfy (2.2.17). Similarly, the differences $u_\varepsilon - u_{\varepsilon'}$ satisfy (2.2.17). Hence the family u_ε is a Cauchy sequence in H^s, so that the limit u belongs to H^s and satisfy (2.2.17). □

2.2.5 Solutions of the boundary value problem (2.2.1)

We now turn to the original equation (2.2.1). Propositions 2.2.10, 2.2.12 and 2.2.13 imply the next result.

Theorem 2.2.14. *Suppose that $\gamma > 0$, $s \in \mathbb{N}$, $f \in e^{\gamma t} H^s$ and $g \in e^{\gamma t} H^s$. Then the problem* (2.2.1) *has a unique strong solution $u \in e^{\gamma t} H^s$ and*

$$\gamma \|e^{-\gamma t} u\|^2_{s,\gamma} + \|e^{-\gamma t} u_{|x=0}\|^2_{s,\gamma} \leq C\big(\frac{1}{\gamma}\|e^{-\gamma t} f\|^2_{s,\gamma} + \|e^{-\gamma t} g\|^2_{s,\gamma}\big) \tag{2.2.18}$$

where C is independent of γ and u, f, g.

2.3 Solutions on $]-\infty, T]$ and the causality principle

In this section, we show that if the data of (2.2.1) vanish in the past, then the solution also does, and we solve the boundary value problem on $\{t \leq T\}$.

First we note that we have a strong uniqueness result:

Lemma 2.3.1. *Assume that $f \in e^{\gamma_0 t} L^2 \cap e^{\gamma_1 t} L^2$ and $g \in e^{\gamma_0 t} L^2 \cap e^{\gamma_1 t} L^2$ with $0 < \gamma_0 < \gamma_1$. Then the solutions u_{γ_0} and u_{γ_1} given by Proposition* 2.2.13 *applied to $\gamma = \gamma_0$ and $\gamma = \gamma_1$ are equal.*

Proof. Note that $f \in e^{\gamma t} L^2$ for all $\gamma \in [\gamma_0, \gamma_1]$. Therefore, for such γ (2.2.1) has a unique strong solution $u_\gamma \in e^{\gamma t} L^2$.

Introduce a function $\theta \in C^\infty(\mathbb{R})$ such that $\theta(t) = 1$ for $t \leq 0$ and $\theta(t) = e^{-t}$ for $t \geq 1$. Thus $\partial_t \theta = h\theta$ with $h \in L^\infty$. With $\delta = \gamma - \gamma_0$, introduce

$$v = \theta(\delta t)\big(u_\gamma - u_{\gamma_0}\big) .$$

The properties of θ imply that $v \in e^{\gamma_0 t} L^2$ and

$$Lv = \delta \partial_t \theta(\delta t)\big(u_\gamma - u_{\gamma_0}\big) = \delta h(\delta t) v\,, \quad M v_{|x=0} = 0\,.$$

Thus, by uniqueness in $e^{\gamma_0 t} L^2$, Theorem 2.2.14 applied to $\gamma = \gamma_0$, implies that there is a constant C, independent of the γ's, such that

$$\gamma_0 \| e^{-\gamma_0 t} v \|_{L^2} \le C \delta \| e^{-\gamma_0 t} v \|_{L^2}$$

If $C\delta < \gamma_0$, this implies that $v = 0$. Summing up, we have proved that for $\gamma \le (1 + 1/2C)\gamma_0$ and $\gamma_0 \le \gamma \le \gamma_1$, one has $u_\gamma = u_{\gamma_0}$. By induction, this implies that for all integer $k \ge 1$, $u_\gamma = u_{\gamma_0}$ for $\gamma \in [\gamma_0, \gamma_1]$ with $\gamma \le (1 + 1/2C)^k \gamma_0$. Hence, $u_\gamma = u_{\gamma_0}$ for $\gamma \in [\gamma_0, \gamma_1]$. $\square$

This implies *local* uniqueness:

Proposition 2.3.2. *If $f \in e^{\gamma t} L^2(\mathbb{R}^{1+d}_+)$ and $g \in e^{\gamma t} L^2(\mathbb{R}^d)$ with $\gamma > 0$ vanish for $t < T$, then the solution $u \in e^{\gamma t} L^2(\mathbb{R}^{1+d}_+)$ of* (2.2.1) *vanishes for $t < T$.*

Proof. Since f and g vanish for $t < T$, f and g belong to $e^{\gamma' t} L^2$ for all $\gamma' \ge \gamma$. Thus, by the lemma above, $u \in e^{\gamma' t} L^2$ for all γ' large and by Theorem 2.2.14 there is C such that for all $\gamma' \ge \gamma$:

$$\gamma' \| e^{-\gamma' t} u \|^2_{L^2} \le C \frac{1}{\gamma'} \| e^{-\gamma' t} f \|^2_{L^2} + C \| e^{-\gamma' t} g \|^2_{L^2}\,.$$

Thus

$$\begin{aligned} \gamma' \| u \|^2_{L^2(\{t \le T\})} \le \gamma' \| e^{\gamma'(T-t)} u \|^2_{L^2} &\le \frac{C}{\gamma'} \| e^{\gamma'(T-t)} f \|^2_{L^2} + C \| e^{\gamma'(T-t)} f \|^2_{L^2} \\ &\le \frac{C}{\gamma'} \| e^{\gamma(T-t)} f \|^2_{L^2} + C \| e^{\gamma(T-t)} f \|^2_{L^2}\,. \end{aligned}$$

The right hand side is bounded as γ' tends to infinity, thus $u_{|\{t \le T\}} = 0$. $\square$

We now consider solutions of (2.2.1) on $]-\infty, T] \times \mathbb{R}^d_+$. First, we note that the trace makes sense.

Lemma 2.3.3. *Suppose that $u \in L^2(]T_1, T_2[\times \mathbb{R}^d_+)$ satisfies $Lu \in L^2(]T_1, T_2[\times \mathbb{R}^d_+)$. Then the trace $u_{|x=0}$ is well defined in $H^{-1/2}_{loc}(]T_1, T_2[\times \mathbb{R}^{d-1})$.*

Proof. Consider $\chi \in C^\infty_0(]T_1, T_2[)$. Then χu, extended by 0 belongs to $L^2(\mathbb{R}^{1+d}_+)$ and $L(\chi u)$, which is the extension by 0 of $\chi L u + \partial_t \chi u$, also belongs to $L^2(\mathbb{R}^{1+d}_+)$. Thus, by Lemma 2.2.5, χu has a trace in $H^{-1/2}$ and the lemma follows. $\square$

Therefore, for $u \in L^2(]T_1, T_2[\times \mathbb{R}^d_+)$ such that $Lu = f \in L^2(]T_1, T_2[\times \mathbb{R}^d_+)$ the equation $Mu_{|x=0} = g \in L^2$ makes sense.

Corollary 2.3.4. *Suppose that $\gamma > 0$ and $u \in e^{\gamma t} L^2(]-\infty, T] \times \mathbb{R}^d_+)$ satisfies*

$$\begin{cases} Lu = 0 & \text{on }]-\infty, T] \times \mathbb{R}_+) \\ Mu_{|x=0} = 0 & \text{on }]-\infty, T]\,. \end{cases}$$

Then $u = 0$.

Proof. For $\delta > 0$ choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = 1 \quad \text{for } t < T - \delta \qquad \text{and} \qquad \chi(t) = 0 \quad \text{for } t \geq T - \delta/2\,.$$

Extend $v = \chi(t)u$ by 0 for $t \geq T$. Then $v \in e^{\gamma t} L^2$, Mv vanishes on the boundary, and $f := Lv$ which is the extension of $(\partial_t \chi)u$ by 0 for $t \geq T$ vanishes for $t \leq T - \delta$ and belongs to $e^{\gamma t} L^2$. Thus, by Proposition 2.3.2 v and hence u vanish for $t \leq T - \delta$. Since δ is arbitrary, this implies that $u = 0$. □

Remark 2.3.5. If u and u_1 are two solutions in $e^{\gamma t} L^2$ of (2.2.1) on $]-\infty, T_1] \times \mathbb{R}^d_+$ associated to L^2 data (f, g) and (f_1, g_1) respectively, and if $f = f_1$ and $g = g_1$ for $t \leq T$, then $u = u_1$ for $t \leq T$. Thus the values of u for times $t \leq T$ only depend on the values of the data f and g for $t \leq T$. This means that the solutions constructed above satisfy the *causality principle*.

Theorem 2.3.6. *Suppose that $f \in e^{\gamma t} H^s(]-\infty, T] \times \mathbb{R}^d_+)$ and $g \in e^{\gamma t} H^s(]-\infty, T] \times \mathbb{R}^{d-1})$, for some $\gamma > 0$ and $s \in \mathbb{N}$. Then the problem* (2.2.1) *has a unique solution $u \in e^{\gamma t} H^s(]-\infty, T] \times \mathbb{R}^d_+)$.*

If f and g vanish for $t \leq T_1$, then the solution u also vanishes for $t \leq T_1$.

Moreover, estimates similar to (2.2.18) *are satisfied.*

Proof. Extend f and g for $t \geq T$ as $\widetilde{f} \in H^s(\mathbb{R} \times \mathbb{R}^d_+)$ and $\widetilde{g} \in H^s(\mathbb{R} \times \mathbb{R}^{d-1})$. We can choose the extension such that they vanish for $t \geq T + 1$. For instance, when $s = 0$, we can extend them by 0. Because $\widetilde{f} = f$ and $\widetilde{g} = g$ for $t \leq T$ and vanish for $t \geq T + 1$, $\widetilde{f}$ and $\widetilde{g}$ belong to $e^{\gamma t} H^s$. Therefore, by Theorem 2.2.14 the problem

$$L\widetilde{u} = f\,, \qquad M\widetilde{u}_{|x=0} = \widetilde{g} \tag{2.3.1}$$

has a unique solution $\widetilde{u} \in e^{\gamma t} H^s$. Its restriction to $\{t \leq T\}$ satisfies (2.2.1). This proves the existence part of the statement.

The uniqueness follows from Corollary 2.3.4. □

2.4 The mixed Cauchy problem

We now consider the mixed Cauchy-boundary value problem:

$$\left\{\begin{aligned} Lu &= f && \textit{on } [0,T]\times\mathbb{R}^d_+ \\ Mu_{|x=0} &= g && \textit{on } [0,T]\times\mathbb{R}^{d-1} \\ u_{|t=0} &= u_0 && \textit{on } \mathbb{R}^d_+ \end{aligned}\right. \tag{2.4.1}$$

We first solve the problem in L^2 and next study the existence of smooth solutions.

When $u \in L^2([0,T]\times\mathbb{R}^d_+)$ and $Lu \in L^2([0,T]\times\mathbb{R}^d_+)$, the trace $u_{|x=0}$ is defined in $H^{-1/2}_{loc}(]0,T[\times\mathbb{R}^{d-1})$ thus the boundary condition makes sense. We will construct solution in the space $C^0([0,T];L^2(\mathbb{R}^d_+))$ identified with a subspace of $L^2([0,T]\times\mathbb{R}^d_+)$ and for such u the initial condition is meaningful.

2.4.1 L^2 solutions

The starting point is an energy estimate. Note that, by standard trace theorems (see also Lemma 2.2.5) all $u \in H^1([0,T]\times\mathbb{R}^d_+)$ belongs to $C^0([0,T];H^{1/2}(\mathbb{R}^d_+)) \subset C^0([0,T];L^2(\mathbb{R}^d_+))$. In particular, for such u, the value of u at time $t\in[0,T]$, denoted by $u(t)$, is well defined in $L^2(\mathbb{R}^d_+)$.

Proposition 2.4.1. *There is a constant C such that for all $T>0$, all $u\in H^1([0,T]\times\mathbb{R}^d_+)$ and all $t\in[0,T]$, the following inequality holds:*

$$\begin{aligned} \|u(t)\|_{L^2(\mathbb{R}^d_+)} + \|u_{|x=0}\|_{L^2([0,t]\times\mathbb{R}^{d-1})} \le C\Big(& \|u_0\|_{L^2(\mathbb{R}^d_+)} \\ & + \int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)} ds + \|g\|_{L^2([0,t]\times\mathbb{R}^{d-1})}\Big). \end{aligned} \tag{2.4.2}$$

where $u_0 = u(0)$, $f := Lu$ and $g := Mu_{|x=0}$.

Since $u\in H^1$, $f = Lu$ belongs to L^2, thus

$$\|f(t)\|_{L^2(\mathbb{R}^d_+)} = \Big(\int |f(t,y,x)|^2 dydx\Big)^{1/2}$$

is well defined in $L^2([0,T])$, thus in $L^1([0,T])$.

Proof. By integration by parts, as in Proposition 2.2.9, there holds:

$$\begin{aligned} 2\mathrm{Re}\,\big(Sf,u\big)_{L^2([0,t]\times\mathbb{R}^d_+)} = \big(Su(t),u(t)\big)_{L^2(\mathbb{R}^d_+)} &- \big(Su(0),u(0)\big)_{L^2(\mathbb{R}^d_+)} \\ &- \big(SA_d u_{|x=0}, u_{|x=0}\big)_{L^2([0,T]\times\mathbb{R}^{d-1})}. \end{aligned}$$

Since S is definite positive and using Lemma 2.2.8, this implies

$$\|u(t)\|^2_{L^2(\mathbb{R}^d_+)} + \|u_{|x=0}\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} \leq C\Big(\|u(0)\|^2_{L^2(\mathbb{R}^d_+)} + \|g\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} + \int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)}\|u(s)\|_{L^2(\mathbb{R}^d_+)}ds\Big)\,.$$

Taking the supremum of these estimates for $t' \in [0,t]$, we can replace in the left hand side $\|u\|^2_{L^2(\mathbb{R}^d_+)}$ by $n^2(t)$ where $n(t) := \sup_{t'\in[0,t]} \|u(t')\|_{L^2(\mathbb{R}^d_+)}$. Moreover, the integral in the right hand side is smaller than

$$n(t)\int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)}ds \leq \varepsilon n^2(t) + \varepsilon^{-1}\Big(\int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)}ds\Big)^2\,.$$

Choosing ε small enough to absorb $C\varepsilon n^2$ from the right to the left, yields, with a new constant C:

$$n^2(t) + \|u_{|x=0}\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} \leq C\Big(\|u(0)\|^2_{L^2(\mathbb{R}^d_+)} + \|g\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} + \big(\int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)}\big)^2 ds\Big)$$

and (2.4.2) follows. □

This estimate has consequences for *strong* solutions of (2.4.1).

Definition 2.4.2. *Given $f \in L^2([0,T]\times\mathbb{R}^d_+)$, $g \in L^2([0,T]\times\mathbb{R}^{d-1})$ and $u_0 \in L^2(\mathbb{R}^d_+)$, we say that $u \in L^2([0,T]\times\mathbb{R}^d_+)$ is a strong L^2-solution of (2.4.1) if there is a sequence $u^n \in L^2([0,T]\times\mathbb{R}^d_+)$ such that $u^n \to u$, $Lu^n \to f$, $Mu^n{}_{|x=0} \to g$ and $u^n(0) \to u_0$ in L^2.*

Proposition 2.4.3. *If $u \in L^2([0,T]\times\mathbb{R}^d_+)$ is a strong L^2-solution of (2.4.1), then u satisfies the equations (2.4.1), $u \in C^0([0,T];L^2(\mathbb{R}^d_+))$, its trace $u_{|x=0}$ belongs to $L^2([0,T]\times\mathbb{R}^{d-1})$ and the energy inequalities (2.4.2) are satisfied.*

Proof. Suppose that u^n is a sequence in H^1 such that $u^n \to u$, $Lu^n \to f$, $Mu^n{}_{|x=0} \to g$ and $u^n(0) \to u_0$ in L^2.

Applying the estimate (2.4.2) to differences $u^n - u^m$, we conclude that u^n is a Cauchy sequence in $C^0([0,T];L^2(\mathbb{R}^d_+))$ and that the traces $u^n{}_{|x=0}$ form a Cauchy sequence in $L^2([0,T]\times\mathbb{R}^{d-1})$. Hence u^n converges to a limit $v \in C^0([0,T];L^2(\mathbb{R}^d_+))$ and the traces $u^n{}_{|x=0}$ converge to a limit $h \in L^2([0,T]\times\mathbb{R}^{d-1})$. Since $u^n \to u$ in L^2, by uniqueness of the limit in the sense of distributions, $v = u$. Moreover, $Lu^n \to Lu$ in the sense of distributions, thus $Lu = f$. Using Lemmas 2.3.3 and 2.2.5, we get that the traces ${u^n}_{|x=0}$ converge to $u_{|x=0}$ in $H^{-1/2}_{loc}(]0,T[\times\mathbb{R}^{d-1})$, and since the traces converge to h in L^2, this implies that $u_{|x=0} = h \in L^2([0,T]\times\mathbb{R}^{d-1})$. In particular, $Mu_{|x=0} = \lim M{u^n}_{|x=0} = g$. Since $u^n \to u$ in $C^0([0,T];L^2)$, there holds $u^n(0) \to u(0)$ and thus $u(0) = u_0$ in L^2. This shows that u is a solution of (2.4.1) and that the trace on $\{x = 0\}$ is in L^2.

Knowing the convergences $u^n \to u$ in $C^0([0,T];L^2)$, $Lu^n \to f$, $u^n_{|x=0} \to u_{|x=0}$ in L^2, we can pass to the limit in the energy estimates for u^n, and so obtain that u satisfies (2.4.2). $\square$

Remark 2.4.4. This statement applies to solutions of (2.2.1). Suppose that $f \in L^2(]-\infty,T]\times\mathbb{R}^d_+)$ and $g \in L^2(]-\infty,T]\times\mathbb{R}^{d-1})$ vanish for $t < 0$. The unique solution $u \in L^2(]-\infty,T]\times\mathbb{R}^d_+)$ of (2.2.1) which vanishes when $t < 0$, given by Theorem 2.3.6 is a strong solution by Proposition 2.2.12, or as seen by writing $f = \lim f^n$, $g = \lim g^n$ with $f^n \in H^1$, $g^n \in H^1$ vanishing when $t \le 0$. Then, by Theorem 2.3.6, the solution u^n of (2.2.1) with data (f^n, g^n) belongs to H^1 and converge in L^2 to u. Since u^n vanishes for $t < 0$ and $u^n \in H^1$, the trace of u^n on $\{t = T_0\}$ vanishes, i.e. $u^n(T_0) = 0$ for all $T_0 \le 0$. This shows that u, restricted to $\{t \ge T_0\}$ is a strong solution of (2.4.1) with vanishing initial data at time T_0. Thus, $u \in C^0(]-\infty,T];L^2(\mathbb{R}^d_+))$ and the estimates (2.4.2) hold.

We can now state the main theorem.

Theorem 2.4.5. *For all $u_0 \in L^2(\mathbb{R}^d_+)$, $f \in L^2([0,T]\times\mathbb{R}^d_+)$ and $g \in L^2([0,T]\times\mathbb{R}^{d-1})$, there is a unique solution $u \in C^0([0,T],L^2(\mathbb{R}^d_+))$ of* (2.4.1). *It is a strong solution, its trace on $\{x = 0\}$ belongs to $L^2([0,T]\times\mathbb{R}^{d-1})$ and the energy estimate* (2.4.2) *is satisfied.*

Proof. **a)** *Existence.*

Denote by $H^1_0(\mathbb{R}^d_+)$ the space of functions in $v \in H^1(\mathbb{R}^d_+)$ such that $v_{|x=0} = 0$. Since $H^1_0(\mathbb{R}^d_+)$ is dense in $L^2(\mathbb{R}^d_+)$, there is a sequence u^n_0 such that:

$$u^n_0 \in H^1_0(\mathbb{R}^d_+)\,, \quad \|u^n_0 - u_0\|_{L^2} \to 0\,.$$

Considered as a function independent of t, u^n_0 belongs to $H^1([0,T]\times\mathbb{R}^d_+)$, its trace on $x = 0$ vanishes and $Lu^n_0 \in L^2([0,T]\times\mathbb{R}^{d+})$. By density of smooth functions with compact support in L^2, there is a function f^n such that

$$f^n \in H^1(]-\infty,T]\times\mathbb{R}^d_+)\,, \quad f^n_{|t<0} = 0\,, \quad \|f^n - (f - Lu^n_0)\|_{L^2([0,T]\times\mathbb{R}^d_+)} \le \frac{1}{n}\,.$$

Similarly, there is g^n such that

$$g^n \in H^1(]-\infty, T] \times \mathbb{R}^{d-1}), \quad g^n_{|t<0} = 0, \quad \|g^n - g\|_{L^2([0,T]\times\mathbb{R}^{d-1})} \leq \frac{1}{n}.$$

By Theorem 2.2.14, there is a unique function v^n, such that

$$v^n \in H^1(]-\infty, T] \times \mathbb{R}^d_+), \quad Lv^n = f^n, \quad v^n{}_{|t<0} = 0, \quad Mv^n{}_{|x=0} = g^n.$$

In particular, since $v^n \in H^1$, $v^n \in C^0(]-\infty, T]; L^2(\mathbb{R}^d_+))$, and since $v^n = 0$ when $t < 0$, this implies that $v^n(0) = 0$.

Consider u^n the restriction on $[0,T] \times \mathbb{R}^d_+$ of $v^n + u^n_0$. It belongs to $H^1(]-\infty, T] \times \mathbb{R}^d_+)$, its trace on $\{x = 0\}$ is equal to the trace of v^n, thus $Mu^n{}_{|x=0} = g^n \to g$ in L^2. Moreover, $u^n(0) = u^n_0 \to u_0$ in L^2 and $Lu^n = f^n + Lu^n_0 \to f$. Thus, applying the estimate (2.4.2) to differences $u^n - u^m$, we conclude that u^n is a Cauchy sequence in $C^0([0,T]; L^2(\mathbb{R}^d_+))$. Thus u^n converges to a limit u in $C^0([0,T]; L^2(\mathbb{R}^d_+))$, thus in $L^2([0,T] \times \mathbb{R}^d_+)$. The properties listed above show that u is a strong solution of (2.4.1), thus a solution which satisfies the estimates (2.4.2).

b) *Uniqueness.*

Suppose that $u \in C^0([0,T]; L^2(\mathbb{R}^d_+))$ satisfies $Lu = 0$, $Mu_{|x=0} = 0$ and $u(0) = 0$. Consider a C^∞ non decreasing function $\chi(t)$ such that $\chi = 0$ for $t < 1$ and $\chi(t) = 1$ for $t > 2$. For $\delta > 0$, let $\chi_\delta(t) = \chi(t/\delta)$. Consider u_δ the extension by 0 for $t \leq 0$ of $\chi_\delta u$. Thus Lu_δ is the extension by 0 of $(\partial_t \chi_\delta) u$ and thus belongs to L^2. Moreover, the trace of u_δ is the extension of $\chi_\delta u_{|x=0}$. Thus $Mu_{\delta|x=0} = 0$. Therefore, u_δ is a solution of (2.2.1) which vanishes in the past. By Remark 2.4.4, it is a strong solution and the energy estimates (2.4.2) are satisfied. Hence, for $t \geq 2\delta$

$$\|u(t)\|_{L^2} \leq C \int_0^t (\partial_t \chi_\delta)(s) \|u(s)\|_{L^2} ds = C \int_1^2 (\partial_t \chi(s) \|u(\delta s)\|_{L^2} ds.$$

Since $u \in C^0([0,T]; L^2)$ and $u(0) = 0$, the right hand side converges to zero as δ tends to zero, implying that $u = 0$. $\square$

2.4.2 Compatibility conditions

In order to solve the mixed Cauchy problem in Sobolev spaces, compatibility conditions are needed. For instance, the initial and boundary conditions imply that necessarily

$$Mu_{0|x=0} = g_{|t=0} = Mu_{|t=0,x=0}, \tag{2.4.3}$$

provided that the traces are defined. Next, denote by A the operator

$$Au = \sum_{j=1}^{d} A_j \partial_j \,.$$

Thus, if $Lu = f$, $\partial_t u = f - Au$ and therefore

$$u_1 := \partial_t u_{|t=0} = -Au_0 + f_0$$

if $f_0 = f_{|t=0}$. Thus, provided that the traces are defined,

$$\text{(2.4.4)} \qquad Mu_{1|x=0} = M(f_0 - Au_0)_{|x=0} = g_1 := \partial_t g_{|t=0} = M\partial_t u_{|t=0,x=0} \,.$$

These conditions are *necessary* for the existence of a smooth solution. Continuing the Taylor expansions to higher order yields higher order condition as we now explain.

For u smooth enough denote by $u_j = \partial_t^j u_{|t=0}$ the traces at $t = 0$ of the derivatives of u. For instance, if $u \in H^s$, $s \geq 1$, they are defined for $j \leq s - 1$. Similarly, we note $f_j = \partial_t^j f_{|t=0}$ and $g_j = \partial_t^j g_{|t=0}$ when they are defined. If u is a solution of $Lu = f$, then for $j \geq 1$:

$$u_j = f_{j-1} - Au_{j-1}$$

By induction, this implies that

$$\text{(2.4.5)} \qquad u_j = (-A)^j u_0 + \sum_{l=0}^{j-1} (-A)^{j-l-1} f_l \,.$$

The boundary condition $Mu_{|x=0} = g$ implies that

$$Mu_{j\,|x=0} = g_j$$

Thus necessarily, for smooth enough functions, solutions of (2.4.1) must satisfy on the edge $\{t = 0, x = 0\}$:

$$\text{(2.4.6)} \qquad M\Big((-A)^j u_0 + \sum_{l=0}^{j-1} M(-A)^{j-l-1} f_l \Big)_{|x=0} = g_j \,.$$

Lemma 2.4.6. *For $s \geq 1$, $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$, the left and right hand sides of (2.4.6) are defined for $j \in \{0, \dots, s-1\}$ and belong to $H^{s-j-1/2}(\mathbb{R}^{d-1})$.*

Proof. For $u_0 \in H^s$, $A^j u_0 \in H^{s-j}$ and the trace $(A^j u_0)_{|x=0}$ is defined for $j < s$ and belongs to $H^{s-j-1/2}(\mathbb{R}^{d-1})$. For $f \in H^s$, the traces f_l are defined for $l \leq s-1$ and belong to $H^{s-l-1/2}$. Thus, $A^{j-l-1} f_l \in H^{s-j+1/2}$ and the traces $(A^{j-l-1} f_l)_{|x=0}$ are defined for $j < s$ and belong to $H^{s-j}(\mathbb{R}^{d-1})$. For $g \in H^s$, the traces g_j are defined for $j < s$ and belong to $H^{s-j-1/2}(\mathbb{R}^{d-1})$. $\square$

The lemma shows that the following definition makes sense.

Definition 2.4.7. *The data $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ satisfy the compatibility conditions to order $\sigma \leq s-1$ if the equations* (2.4.6) *hold for all $j \in \{0, \ldots, \sigma\}$.*

For instance, the first two conditions, given by (2.4.3) and (2.4.4) are

$$Mu_{0|x=0} = g_{|t=0}\,, \tag{2.4.7}$$

$$(MAu_0)_{|x=0} = f_{0|x=0} - g_1\,. \tag{2.4.8}$$

When $s = 0$, there are no compatibility condition. When $s = 1$, there is only one, (2.4.7). When $s = 2$, there are two conditions, (2.4.7) and (2.4.8), etc.

Remark 2.4.8. Suppose that $f = 0$ and $g = 0$. In this case, the compatibility conditions read $M(A^j u_0)_{|x=0} = 0$. Considering the operator A with domain $D(A) = \{u \in L^2(\mathbb{R}^d_+); Au \in L^2(\mathbb{R}^d_+) \,\text{and}\, Mu_{|x=0} = 0\}$, the compatibility conditions of order s reads $u_0 \in D(A^s)$.

The next result is useful in the construction of smooth solutions.

Proposition 2.4.9. *Suppose that $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ are compatible to order $s-1$. Then there are sequences $u_0^n \in H^{s+1}(\mathbb{R}^d_+)$, $f^n \in H^{s+1}([0,T] \times \mathbb{R}^d_+)$ and $g^n \in H^{s+1}([0,T] \times \mathbb{R}^{d-1})$, compatible to order s, such that $u_0^n \to u_0$, $f^n \to f$ and $g^n \to g$ in H^s.*

Proof. **a)** Consider first the case $s = 0$. Then u_0, f and g are arbitrary data in L^2. One easily construct approximating sequences u_0^n, f^n, g^n arbitrarily smooth and compatible to any order, by approximating the data by C^∞ functions which vanish near $t = 0, x = 0$.

b) Suppose now that $s = 1$, u_0, f and g are data in H^1 which satisfy the first compatibility condition (2.4.7). Consider sequences u_0^n, f^n, g^n in H^2, which converge in H^1 to u_0, f and g respectively. By (2.4.7) and the continuity of the traces, $r_0^n := g^n_{|t=0} - Mu^n_{0\,|x=0}$ satisfies

$$r_0^n \in H^{3/2}(\mathbb{R}^{d-1})\,, \quad \|r_0^n\|_{H^{1/2}(\mathbb{R}^{d-1})} \to 0\,.$$

To construct H^2 data $(u_0^n + v^n, f^n, g^n)$ which are compatible to first order, it is sufficient to construct v^n such that:

$$v^n \in H^2(\mathbb{R}^d_+)\,, \quad \|v^n\|_{H^1} \to 0\,, \quad Mv^n_{|x=0} = r_0^n\,, \quad M(Av^n)_{|x=0} = r_1^n\,,$$

with $r_1^n = M(Au_0^n)_{|x=0} - f^n_{|x=t=0} - \partial_t g^n_{|t=0} \in H^{1/2}(\mathbb{R}^{d-1})$. Since M is onto, there is a $N \times N_+$ matrix, M', such that $MM' = \mathrm{Id}$. Thus is is sufficient to find v^n such that

$$v^n \in H^2(\mathbb{R}^d_+)\,, \quad \|v^n\|_{H^1} \to 0\,, \quad v^n_{|x=0} = h_0^n\,, \quad (Av^n)_{|x=0} = h_1^n\,, \tag{2.4.9}$$

with $h_0^n = M'r^n \in H^{3/2}$, $h_1^n = M'r_1^n \in H^{1/2}$. Moreover, $h_0^n \to 0$ in $H^{1/2}$.

Note that (2.4.9) concerns only functions of $(y, x) \in \mathbb{R}^d$ and their traces on $\{x = 0\}$. We recall the classical construction of Poisson operators. Consider $\phi \in C_0^\infty(\mathbb{R})$, $\phi \geq 0$, such that $\phi(x) = 1$ for $|x| \leq 1$ Denoting here by $\hat{v}$ the Fourier transform with respect to y, consider the operator

$$K : h \mapsto Kh\,, \quad \widehat{Kh}(\eta, x) = \phi(x\langle\eta\rangle)\hat{h}(\eta)$$

with $\langle\eta\rangle = (1 + |\eta|^2)^{1/2}$. Then, K is bounded from $H^{1/2}(\mathbb{R}^{d-1})$ to $H^1(\mathbb{R}^d_+)$ and from $H^{3/2}(\mathbb{R}^{d-1})$ to $H^2(\mathbb{R}^d_+)$. Moreover, $(Kh)_{|x=0} = h$. Consider $v_0^n = Kh^n$. Then, $v_0^n \in H^2$, $v^n_{0\,|x=0} = h_0^n$ and $v_0^n \to 0$ in H^1. Therefore, to find a solution $v^n = v_0^n + w^n$ of (2.4.9), it is sufficient to find w^n which satisfy the same properties with $h_0^n = 0$ and h_1^n replaced by $k_1^n = h_1^n - (Av_0^n)_{|x=0} \in H^{1/2}$. In addition, $A = A_d\partial_x + A'$ where $A' = \sum_{j<d} A_j\partial_j$. Thus, is is sufficient to find w^n such that

$$w^n \in H^2(\mathbb{R}^d_+)\,, \quad \|w^n\|_{H^1} \to 0\,, \quad w^n_{|x=0} = 0\,, \quad \partial_x w^n{}_{|x=0} = k^n\,, \tag{2.4.10}$$

with $k^n = A_d^{-1}k_1^n \in H^{1/2}$.

We use a Poisson operator P_n defined by

$$\widehat{P_n h}(\eta, x) = x\phi(\lambda_n x\langle\eta\rangle)\hat{h}(\eta)$$

where $\lambda_n \geq 1$ is to be chosen. We note that P_n maps $H^{1/2}(\mathbb{R}^{d-1})$ to $H^2(\mathbb{R}^d_+)$, that $(P_n h)_{|x=0} = 0$ and $(\partial_x P_n h)_{|x=0} = h$. Thus, $w^n = P_n k^n$ satisfies the first, third and fourth property in (2.4.10). It remains to show that one can choose the sequence λ_n such that $w^n \to 0$ in H^1.

Elementary computations using Plancherel's theorem, show that

$$\|P_n h\|^2_{H^1(\mathbb{R}^d_+)} \leq C \int \psi_n(\eta)|\hat{h}(\eta)|^2 d\eta$$

with C independent of n and h and

$$\psi_n(\eta) = \int_0^\infty \Big((x^2 \langle \eta \rangle^2 + 1) |\phi(\lambda_n x \langle \eta \rangle)|^2 + \lambda_n^2 x^4 \langle \eta \rangle^2 |\phi'(\lambda_n x \langle \eta \rangle)|^2 \Big) dx \,.$$

For $\lambda_n \geq 1$, there holds

$$\psi_n(\eta) \leq \frac{C}{\lambda_n \langle \eta \rangle}$$

with C independent of n. Therefore

$$\| w^n \|_{H^1(\mathbb{R}^d_+)} \leq \frac{C}{\sqrt{\lambda_n}} \| k^n \|_{H^{-1/2}(\mathbb{R}^{d-1})} \,.$$

One can now choose λ_n such that the right hand side converges to zero, showing that w^n satisfies (2.4.10). This finishes the proof of the proposition when $s = 1$.

c) When $s \geq 2$, the proof is similar. One is reduced to find $v^n \in H^{s+1}(\mathbb{R}^d_+)$ such that $v^n \to 0$ in H^n and $(A^j v^n)_{|x=0} = h^n_j$ where the h^n_j are given in $H^{s-j+1/2}(\mathbb{R}^{d-1})$ for $j \leq s$ and converge to zero in $H^{s-j-1/2}(\mathbb{R}^{d-1})$ for $j \leq s-1$. We first lift up the $s-1$ first traces by a fixed Poisson operator, and reduce the problem to find $w^n \in H^{s+1}(\mathbb{R}^d_+)$ such that $w^n \to 0$ in H^n and $(\partial_x^j w^n)_{|x=0} = 0$ when $j \leq s-1$ and $(\partial_x^s w^n)_{|x=0} = k^n \in H^{1/2}(\mathbb{R}^{d-1})$. We lift up the traces using a Poisson operator

$$\widehat{P_n h}(\eta, x) = \frac{x^j}{j!} \phi(\lambda_n x \langle \eta \rangle) \hat{h}(\eta) \,, \tag{2.4.11}$$

and show that if the sequence λ_n is properly chosen $w^n = P_n k^n$ has the desired properties. The details are left as an exercise. □

2.4.3 Smooth solutions

Definition 2.4.10. *$W^s(T)$ denotes the space of functions $u \in C^0([0,T], H^s(\mathbb{R}^d_+))$ such that for all $j \leq s$, $\partial_t^j u \in C^0([0,T], H^{s-j}(\mathbb{R}^d_+))$.*

$W^s(T)$ is considered as a subspace of $H^s([0,T] \times \mathbb{R}^d_+)$ and $H^{s+1}([0,T] \times \mathbb{R}^d_+) \subset W^s(T)$. We also use the notation

$$\| u(t) \|_s = \sum_{j=0}^{s} \| \partial_t^j u(t) \|_{H^{s-j}(\mathbb{R}^d_+)} \,. \tag{2.4.12}$$

This function is bounded (and continuous) in time when $u \in W^s$ and in L^2 when $u \in H^s$.

We first state an a-priori estimate for smooth solutions.

Proposition 2.4.11. *There is a constant C such that for all $T > 0$, all $u \in H^{s+1}([0,T] \times \mathbb{R}^d_+)$ and all $t \in [0,T]$, the following inequality holds:*

$$\begin{aligned} \|\!|u(t)|\!\|_s + \|u_{|x=0}\|_{H^s([0,t]\times\mathbb{R}^{d-1})} \le C\Big(&\|\!|u(0)|\!\|_s + \\ &+ \int_0^t \|\!|f(t')|\!\|_s dt' + \|g\|_{H^s([0,t]\times\mathbb{R}^{d-1})}\Big) . \end{aligned} \tag{2.4.13}$$

where $f := Lu$ and $g := Mu_{|x=0}$.

Proof. Consider the tangential derivatives $u_\alpha := \partial^\alpha_{t,y} u$ for $\alpha \in \mathbb{N}^d$, $|\alpha| \le s$. Since $u \in H^{s+1}$, they satisfy

$$Lu_\alpha = f_\alpha := \partial^\alpha_{t,y} f\,, \quad Mu_{\alpha|x=0} = g_\alpha := \partial^\alpha_{t,y} g\,.$$

Introduce the tangential norm

$$\|\!|u(t)|\!\|'_s := \sum_{|\alpha|\le s} \|\partial^\alpha_{t,y} u(t)\|_{L^2}\,.$$

The L^2 estimates (2.4.2) imply that

$$\begin{aligned} \|\!|u(t)|\!\|'_s + \|u_{|x=0}\|_{H^s([0,t]\times\mathbb{R}^{d-1})} \le C\Big(&\|\!|u(0)|\!\|'_s \\ &+ \int_0^t \|\!|f(t')|\!\|'_s dt' + \|g\|_{H^s([0,t]\times\mathbb{R}^{d-1})}\Big) \end{aligned}$$

which is dominated by the right hand side of (2.4.13). It remains to estimate the normal derivatives by tangential ones, using the equation (2.2.9). By induction, one proves that

$$\|\!|u(t)|\!\|_s \le C\big(\|\!|u(t)|\!\|'_s + \|\!|f(t)|\!\|_{s-1}\big)\,.$$

Since

$$\|\!|f(t)|\!\|_{s-1} \le \|\!|f(0)|\!\|_{s-1} + \int_0^t \|\!|\partial_t f(t')|\!\|_{s-1} dt'$$

and

$$\|\!|f(0)|\!\|_{s-1} \le \|\!|u(0)|\!\|_s\,, \quad \|\!|\partial_t f(t')|\!\|_{s-1} \le \|\!|f(t')|\!\|_s\,,$$

the estimate (2.4.13) follows. □

We can now prove the main theorem of this chapter.

Theorem 2.4.12. *For all $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ satisfying the compatibility conditions up to order $s-1$, there is a unique solution $u \in W^s(T)$ of* (2.4.1)*. Moreover, the trace of the solution u on $\{x = 0\}$ is in $H^s([0,T] \times \mathbb{R}^{d-1})$ and u satisfies the estimates* (2.4.13)*.*

Proof. When $s = 0$, this is Theorem 2.4.5. We suppose now that $s \geq 1$.
Step 1. Solve the equation with a loss of smoothness.

We prove that when u_0, f and g belong to H^{s+2} and satisfy the compatibility condition up to order s, there is a solution in $H^{s+1}([0,T] \times \mathbb{R}^{d_+}) \subset W^{s+1}(T)$.

With $f_l = \partial_t^l f_{|t=0} \in H^{s+1-l}(\mathbb{R}^d_+)$, consider the functions $u_j \in H^{s+2-j}(\mathbb{R}^d_+)$ defined by (2.4.5) for $j \leq s+2$. Then, there is $u^a \in H^{s+2+1/2}(\mathbb{R} \times \mathbb{R}^d_+)$ such that

$$\partial_t^j u^a{}_{|t=0} = u_j\,, \qquad \text{for } j \leq s+2\,. \tag{2.4.14}$$

We look for a solution as $u = u' + u^a$. The equation for u' reads

$$Lu' = f' := f - Lu^a\,, \quad Mu'{}_{|x=0} = g' := g - Mu^a{}_{|x=0}\,, \quad u'{}_{|t=0} = 0\,.$$

We have $f' \in H^{s+2} - H^{s+3/2} \subset H^{s+1}$ and comparing (2.4.14) and (2.4.5) we see that

$$\partial_t^j f'{}_{|t=0} = 0 \qquad \text{for } j \leq s\,. \tag{2.4.15}$$

Moreover, $g' \in H^{s+2}$ and the compatibility conditions imply that

$$\partial_t^j g'{}_{|t=0} = 0 \qquad \text{for } j \leq s\,. \tag{2.4.16}$$

Denote by $\tilde{f}'$ and $\tilde{g}'$ the extensions of f' and g' by 0 for $t < 0$. Then, the trace conditions (2.4.15) and (2.4.16) imply that $\tilde{f}' \in H^{s+1}(]-\infty, T] \times \mathbb{R}^d_+)$ and $\tilde{g}' \in H^{s+1}(]-\infty, T] \times \mathbb{R}^{d-1})$. Thus, by Theorem 2.3.6, the boundary value problem

$$L\tilde{u}' = \tilde{f}'\,, \quad M\tilde{u}'{}_{|x=0} = \tilde{g}'$$

has a unique solution $\tilde{u}' \in H^{s+1}(]-\infty, T] \times \mathbb{R}^d_+)$ which vanishes when $t \leq 0$. Thus $\tilde{u}'(0) = 0$ and denoting by u' the restriction of $\tilde{u}'$ to $t \geq 0$, $u = u' + u^a$ is a solution of (2.4.1).

Step 2. H^s data.

Given $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ satisfying the compatibility conditions up to order $s-1$, by repeated applications of Proposition 2.4.9, there is a sequence $u_0^\nu \in H^{s+2}(\mathbb{R}^d_+)$, $f^\nu \in H^{s+2}([0,T] \times \mathbb{R}^d_+)$ and $g^\nu \in H^{s+2}([0,T] \times \mathbb{R}^{d-1})$ satisfying the compatibility conditions up to order $s+1$ and converging in H^s to u_0, f and g respectively.

We note that for solutions of (2.4.1),

$$\|\!|u(0)|\!\|_s = \sum_{j \leq s} \|u_j\|_{H^{s-j}}$$

where the u_j are defined at (2.4.5). Thus $\|\!|u^\nu(0) - u^\mu(0)|\!\|_s$ tends to zero as μ and ν tend to infinity. Therefore, the energy estimates (2.4.13) imply that the sequence u^ν is a Cauchy sequence in $W^s(T)$ and therefore converges to $u \in W^s(T)$. Since $s \geq 1$, the limit u is clearly a solution of (2.4.1). The uniqueness follows from the L^2 uniqueness of Theorem 2.4.5. passing to the limit in the energy estimates for the u^ν implies that u also satisfies (2.4.13). $\square$

2.5 Nonlinear mixed problems

Consider the equation

$$\text{(2.5.1)} \qquad \begin{cases} Lu = F(u) + f & \textit{on } [0,T] \times \mathbb{R}^d_+ \\ Mu_{|x_d=0} = g & \textit{on } [0,T] \times \mathbb{R}^{d-1} \\ u_{|t=0} = u_0 & \textit{on } \mathbb{R}^d_+ \end{cases}$$

We assume that $F(0) = 0$, so that it makes sense to look for solutions vanishing at infinity and in Sobolev spaces H^s.

Theorem 2.5.1. *Let s be and integer $s > d/2$.*

i) Suppose that $f \in H^s([0,T_0] \times \mathbb{R}^d_+)$, $g \in H^s([0,T_0] \times \mathbb{R}^{d-1})$ and $u_0 \in H^s(\mathbb{R}^d_+)$. Suppose that the compatibility conditions of section 2.5.2 below are satisfied up to the order $s-1$. Then there is $T \in]0,T_0]$ such that the problem (2.5.1) has a unique solution $u \in W^s(T)$.

ii) If $\sigma > s$ and the data (f, g, u_0) belong to $H^\sigma([0,T] \times \mathbb{R}^d_+)$, $H^\sigma([0,T] \times \mathbb{R}^{d-1})$ and $H^\sigma(\mathbb{R}^d_+)$ respectively and satisfy the compatibility conditions to order $\sigma - 1$, then the solution u given by i) belongs to $W^\sigma(T)$.

2.5.1 Nonlinear estimates

Recall the following multiplicative properties of Sobolev spaces.

Proposition 2.5.2. *For non negative integers $s > d/2$ and j, k such that $j + k \le s$ there is C such that for $u \in H^{s-j}(\mathbb{R}^d_+)$ and $v \in H^{s-k}(\mathbb{R}^d_+)$ the product $uv \in H^{s-j-k}(\mathbb{R}^d_+)$ and*

$$\|uv\|_{H^{s-j-k}} \le C\|u\|_{H^{s-j}}\|v\|_{H^{s-k}}\,. \tag{2.5.2}$$

Corollary 2.5.3. *Let F be a C^∞ function such that $F(0) = 0$. For all $s > d/2$, there is a nondecreasing function $C(\cdot)$ on $[0, +\infty[$ such that for all $T > 0$ and $u \in W^s(T)$, $F(u) \in W^s(T)$ and for all $t \in [0, T]$:*

$$|\!|\!| u(t) |\!|\!|_s \le R \quad \Rightarrow \quad |\!|\!| F(u)(t) |\!|\!|_s \le C(R)\,. \tag{2.5.3}$$

Moreover, for all $u \in W^s(T)$ and $v \in W^s(T)$ with $|\!|\!| u(t) |\!|\!|_s \le R$ and $|\!|\!| v(t) |\!|\!|_s \le R$:

$$|\!|\!| \{F(u) - F(v)\}(t) |\!|\!|_s \le C(R) |\!|\!| \{u - v\}(t) |\!|\!|_s\,. \tag{2.5.4}$$

Proof. Since $F(0) = 0$, there holds

$$\|F(u)(t)\|_{L^2} \le \|\nabla_u F\|_{L^\infty(B_R)}\|u(t)\|_{L^2}\,, \quad \text{with } R = \|u(t)\|_{L^\infty} \le |\!|\!| u(t) |\!|\!|_s\,,$$

where B_R denotes the ball of radius R in the space of states u. The last inequality follows from Sobolev embedding $H^s(\mathbb{R}^d_+) \subset L^\infty(\mathbb{R}^d_+)$.

Next we estimate derivatives. For smooth functions u, there holds

$$\partial^\alpha F(u) = \sum_{k=1}^{|\alpha|} \sum_{\alpha^1+\ldots+\alpha^k=\alpha} c(k, \alpha^1, \ldots, \alpha^k) F^k(u)\big(\partial^{\alpha^1}u, \ldots, \partial^{\alpha^k}u\big) \tag{2.5.5}$$

where the $c(k, \alpha^1, \ldots, \alpha^k)$ are numerical coefficients. Since $\partial^{\alpha^j}u(t) \in H^{s-|\alpha^j|}(\mathbb{R}^d_+)$ with

$$\|\partial^{\alpha^j}u(t)\|_{H^{s-|\alpha^j|}} \le |\!|\!| u(t) |\!|\!|_s\,,$$

by Proposition 2.5.2 we see that each term in the right hand side of (2.5.5) belongs to $C^0(L^2)$ and the estimate (2.5.3) follows.

The estimate of differences is similar. □

Recall next the Gagliardo-Nirenberg-Moser's inequalities, which hold with Ω equal to an Euclidian space $\mathbb{R}^n$ or a half space of $\mathbb{R}^n$, or a quadrant :

Proposition 2.5.4. *For all $s \in \mathbb{N}$, there is C such that for all α of length $|\alpha| \le s$, all $p \in [2, 2s/|\alpha|]$ and all $u \in L^\infty(\Omega) \cap H^s(\Omega)$, the derivative $\partial^\alpha u$ belongs to $L^p(\Omega)$, and*

$$\|\partial^\alpha u\|_{L^p} \le C \|u\|_{L^\infty}^{1-2/p} \|u\|_{H^s}^{2/p} . \tag{2.5.6}$$

The condition on p reads $\frac{|\alpha|}{s} \le \frac{2}{p} \le 1$. Recall that the the proof when $\Omega = \mathbb{R}^n$ relies on the identity

$$0 = \int \partial_j (u |\partial_j u|^{p-2} \partial_j u) = \int |\partial_j u|^p + (p-1) \int u \partial_j^2 u |\partial_j u|^{p-2} .$$

With Hölder inequality, this implies that

$$\|\partial_j u\|_{L^p} \lesssim \|u\|_{L^{p'}}^{1/2} \|\partial_j^2 u\|_{L^{p''}}^{1/2} , \quad \frac{2}{p} = \frac{1}{p'} + \frac{1}{p''} .$$

The estimate (2.5.6) follows by induction on s. Note that the proof applies not only to the ∂_j but also to any vector field.

Using extension operators, the estimate holds on any smooth domain Ω, but the constant depends on the domain. For instance, if $\Omega = [0,T] \times \mathbb{R}^d_+$, the constant are unbounded as $T \to 0$. However, splitting $u = \chi(t) u + (1 - \chi(t)) u$ with $\chi \in C^\infty$, $\chi = 0$ for $t \ge 2T/3$ and $\chi = 1$ for $t \le T/3$, reduces the problem to functions $\chi(t) u$ and $(1 - \chi(t)) u$ which can be extended in H^s by 0 for $t \ge T$ and $t \le 0$ respectively, hence reducing the problem on quadrants $[0, +\infty[\times \mathbb{R}^d_+$ or $] - \infty, T] \times \mathbb{R}^d_+$. Therefore:

Lemma 2.5.5. *Given $T_0 > 0$, there is C such that for all $T \ge T_0$ the estimates* (2.5.6) *are satisfied on $\Omega = [0,T] \times \mathbb{R}^d_+$*

Corollary 2.5.6. *Let F be a C^∞ function such that $F(0) = 0$. For all $s \in \mathbb{N}$, and $T_0 > 0$, there is a non decreasing function $C_F(\cdot)$ on $[0, \infty[$ such that for all $T \ge T_0$, for all $u \in L^\infty(\Omega) \cap H^s(\Omega)$ where $\Omega = [0,T] \times \mathbb{R}^d_+$, one has $F(u) \in H^s(\Omega)$ and*

$$\|F(u)\|_{H^s} \le C_F(\|u\|_{L^\infty}) \|u\|_{H^s} . \tag{2.5.7}$$

Proof. We estimate the L^2 norm as above :

$$\|F(u)\|_{L^2} \le \|\nabla_u F\|_{L^\infty(B_R)} \|u\|_{L^2} , \quad \text{with } R = \|u\|_{L^\infty} ,$$

where B_R denotes the ball of radius R in the space of states u. Next we estimate derivatives using (2.5.5) which is valid at least for smooth u. Using the estimate (2.5.6) for $\partial^{\alpha^j} u$ with $2/p_j = |\alpha^j|/s$, we see that each term in the right hand side of (2.5.5) has an L^2 norm bounded by the right hand side of (2.5.7). The formula and the estimates extend to $u \in L^\infty \cap H^s$ by density (Exercise). □

2.5.2 Compatibility conditions

For (2.5.1), the definition of traces u_j is modified as follows. First, with $u_j = \partial_t^j u_{|t=0}$, there holds

$$\partial_t^j F(u)_{|t=0} = \mathcal{F}_j(u_0, \ldots, u_j) \tag{2.5.8}$$

with $\mathcal{F}_j$ of the form

$$\mathcal{F}_j(u_0, \ldots, u_j) = \sum_{k=1}^{j} \sum_{j^1+\ldots+j^k=j} c(k, j^1, \ldots, j^k) F^k(u_0)\big(u_{j^1}, \ldots, u_{j^k}\big)$$

The definition (2.4.5) is modified as follows: by induction let

$$u_j = -Au_{j-1} + f_{j-1} + \mathcal{F}_{j-1}(u_0, \ldots, u_{j-1}) \,. \tag{2.5.9}$$

Then, for $u_0 \in H^s$ and $f \in H^s$ with $s > d/2$, using Proposition 2.5.2, we see that $u_j \in H^{s-j}(\mathbb{R}^d_+)$ for $j \leq s$.

Definition 2.5.7. *The data $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ satisfy the compatibility conditions to order $\sigma \leq s-1$ if the u_j given by* (2.5.9) *satisfy*

$$Mu_{j|x=0} = \partial_t^j g_{|t=0} \,, \quad j \in \{0, \ldots, \sigma\} \,.$$

2.5.3 Existence and uniqueness

We prove here the first part of Theorem 2.5.1. Below, it is always assumed that $s > d/2$, $f \in H^s([0,T_0] \times \mathbb{R}^d_+)$, $g \in H^s([0,T_0] \times \mathbb{R}^{d-1})$ and $u_0 \in H^s(\mathbb{R}^d_+)$.

Proposition 2.5.8. *Suppose that the compatibility conditions are satisfied up to the order $s-1$. Then there is $T \in]0, T_0]$ such that the problem* (2.5.1) *has a solution $u \in W^s(T)$.*

Proof. **a)** *The iterative scheme.*

Let $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$. Define the $u_j \in H^{s-j}(\mathbb{R}^d_+)$ by (2.5.9). Let $u^0 \in H^{s+1/2}(\mathbb{R} \times \mathbb{R}^d_+)$ such that

$$\partial_t^j u^0{}_{|t=0} = u_j \,, \quad 0 \leq j \leq s \,. \tag{2.5.10}$$

We can assume that u^0 vanishes for $|t| \geq 1$ and thus $u^0 \in W^s(T)$ for all T. There is C_0 depending only on the data such that

$$\sum_{j \leq s} \|u_j\|_{H^{s-j}} \leq C_0 \,, \quad \|\!|u^0(t)|\!\|_s \leq C_0 \,.$$

For future use, we note that C_0 depends only on the data: there is a uniform constant C such that

$$C_0 \leq C\|u_0\|_{H^s} + \|\!| f(0) |\!\|_{s-1}\,. \tag{2.5.11}$$

For $n \geq 1$, we solve by induction the linear mixed problems

$$Lu^n = f + F(u^{n-1})\,, \quad M{u^n}_{|x=0} = g\,, \quad {u^n}_{|t=0} = u_0\,. \tag{2.5.12}$$

Suppose that u^{n-1} is constructed in $W^s(T_0)$ and satisfies

$$\partial_t^j {u^{n-1}}_{|t=0} = u_j\,, \quad j \leq s\,. \tag{2.5.13}$$

This is true for $n = 1$. Then, by definition of the $\mathcal{F}_j$ and by (2.5.13), $\partial_t^j F(u^{n-1})_{|t=0} = \mathcal{F}_j(u_0, \ldots, u_j)$. Next, for the linear problem (2.5.12) we compute the u_j^n by (2.4.5). Comparing with the definition (2.5.9), we see that $u_j^n = u_j$. Thus, the compatibility conditions $Mu_{j|x=0} = g_j$ imply that the data $(f + F(u^{n-1}), g, u_0)$ are compatible for the linear problem. Therefore, Theorem 2.4.12 implies that (2.5.12) has a unique solution $u^n \in W^s(T_0)$ and that

$$\partial_t^j {u^n}_{|t=0} = u_j^n = u_j\,.$$

This shows that the construction can be carried on and thus defines a sequence $u^n \in W^s(T_0)$ satisfying (2.5.12)

b) *Uniform bounds*

We show that we can choose R and $T \in]0, T_0]$ such that for all n:

$$\forall t \in [0, T]\ :\quad \|\!| u^n(t) |\!\|_s \leq R\,. \tag{2.5.14}$$

By (2.5.10), this estimate is satisfied for $n = 0$ if $R \geq C_0$.

Assume that (2.5.14) is satisfied at order $n-1$. Next, the energy estimate (2.4.13) and Corollary 2.5.3 imply that there is a constant C and a function $C_F(\cdot)$ such that for $t \leq T$

$$\|\!| u^n(t) |\!\|_s \leq C\big(\|\!| u^n(0) |\!\|_s + TC_F(R) + C_1\big)$$

with

$$C_1 = \|g\|_{H^s([0,T_0]\times\mathbb{R}^{d-1})} + \int_0^{T_0} \|\!| f(t') |\!\|_s dt'\,. \tag{2.5.15}$$

By (2.5.13) at order n and (2.5.10):

$$\|\!| u^n(0) |\!\| = \sum_{j\leq s} \|u_j\|_{H^{s-j}} \leq C_0\,.$$

Thus, (2.5.14) holds provided that

$$R \geq C_0\,, \quad R \geq C(C_0 + C_1 + 1) \quad \text{and} \quad TC(R) \leq 1\,. \tag{2.5.16}$$

This can be achieved, choosing R first and next T. For such a choice, by induction, (2.5.14) is satisfied for all n.

c) *Convergence*

Write the equation satisfied by $w^n = u^{n+1} - u^n$ for $n \geq 1$. By (2.5.13), there holds $\|w^n(0)\|_s = 0$. Knowing the uniform bounds (2.5.14), estimating the nonlinear terms by Corollary 2.5.3 and using the energy estimate (2.4.13) one obtains that for $n \geq 2$ and $t \leq T$:

$$\|w^n(t)\|_s \leq CC_F(R) \int_0^t \|w^{n-1}(t')\|_s dt'$$

Thus there is K such that for all $n \geq 1$ and $t \in [0, T]$:

$$\|w^n(t)\|_s \leq K^{n+1} t^{n-1}/(n-1)!\,.$$

This implies that the sequence u^n converges in $W^s(T)$, thus in the uniform norm and the limit is clearly a solution of (2.5.1). $\square$

Next we prove uniqueness.

Proposition 2.5.9. *If $T \in]0, T_0]$ and u^1 and u^2 are two solutions of* (2.5.1) *in $W^s(T)$, then $u^1 = u^2$.*

Proof. The traces at $\{t = 0\}$ necessarily satisfy

$$\partial_t^j u^1{}_{|t=0} = \partial_t^j u^2{}_{|t=0} = u_j\,.$$

Thus $w = u^2 - u^1$ satisfies $\|w(0)\|_s = 0$. Write the equation for w. Using bounds for the norms of u^1 and u^2 in W^s, the energy estimates and Corollary 2.5.3 to estimate the nonlinear terms, imply that there is C such that for all $t \in [0, T]$:

$$\|w(t)\|_s \leq C \int_0^t \|w(t')\|_s dt'\,.$$

Thus $w = 0$. $\square$

2.5.4 A criterion for blow-up

Suppose that $f \in H^s([0,T_0]\times\mathbb{R}^d_+)$, $g \in H^s([0,T_0]\times\mathbb{R}^{d-1})$ and $u_0 \in H^s(\mathbb{R}^d_+)$, with $s > d/2$. Suppose that the compatibility conditions are satisfied at order $s-1$. We have proved that there is a local solution in $W^s(T)$. The question is how long can the solution be extended. Let T_* denote the supremum of the set of $T \in]0,T_0]$ such that the problem (2.5.1) has a solution in $W^s(T)$. By uniqueness, there is a unique maximal solution u on $[0,T_*[$. The proof of Proposition 2.5.8 above gives an estimate from below of T^*: since by (2.5.11), (2.5.15) and (2.5.16), there is a function $C(\cdot)$ such that the solution is $W^s(T)$ for

$$T = \min\{T_0, C(K)\} \tag{2.5.17}$$

with

$$K = \|u_0\|_{H^s} + \|\!| f(0) \|\!|_{s-1} + \|g\|_{H^s([0,T_0]\times\mathbb{R}^{d-1})} + \int_0^{T_0} \|\!| f(t') \|\!|_s dt' . \tag{2.5.18}$$

Proposition 2.5.10. *If $T^* < T_0$ or if $T^* = T_0$ but $u \notin W^s(T_0)$, then*

$$\limsup_{t\to T^*} \|u(t)\|_{L^\infty} = +\infty . \tag{2.5.19}$$

Proof. Suppose that (2.5.19) is not true. This means that $u \in L^\infty([0,T^*[\times\mathbb{R}^d_+)$. From Proposition 2.5.8 we know that $T^* > T_1$ for some T_1 depending only on the data. Thus, by Corollary 2.5.6 there is a constant C_1, depending only on the L^∞ norm of u such that for all $T \in [T_1, T^*[$:

$$\|F(u)\|_{H^s([0,T]\times\mathbb{R}^d_+)} \le C_1 \|u\|_{H^s([0,T]\times\mathbb{R}^d_+)} .$$

The energy estimate (2.4.13) implies that

$$\|\!| u(t) \|\!|_s^2 \le C_0 + C\Big(\int_0^t \|\!| F(u)(t') \|\!|_s dt'\Big)^2 ,$$

where C_0 only depends on the data and C depends only on the operator L. Thus, using Cauchy-Schwarz inequality, we get that there is C such that for all $t \in [T_1, T^*[$

$$\begin{aligned}\|\!| u(t) \|\!|_s^2 \le C_0 + C\|F(u)\|^2_{H^s([0,t]\times\mathbb{R}^d_+)} &\le C_0 + CC_1\|u\|^2_{H^s([0,t]\times\mathbb{R}^d_+)} \\ &\le C_0 + CC_1 \int_0^t \|\!| u(t') \|\!|_s^2 dt' .\end{aligned}$$

This implies that there is a constant C_3, depending only on C_0, C, C_1 and the norm of u in $W^s(T_1)$, such that

$$\sup_{t<T^*} \|u(t)\|_s \leq C_3 \,. \tag{2.5.20}$$

Next we consider the Cauchy problem for (2.5.1) with initial data $u(T^* - \delta)$ at time $T^* - \delta$. Because $u \in W^s(T^* - \delta/2)$ is a solution, computing the traces from the equation we see that the compatibility conditions are satisfied up to order $s - 1$. Therefore, by Proposition 2.5.8 there is a solution $\tilde{u}$ in W^s on the interval $[T^* - \delta, T_2]$. By (2.5.17), we have an estimate from below for T_2:

$$T_2 = \min\{T_0, T^* - \delta + C(K)\}$$

with

$$K = \|u(T^* - \delta)\|_{H^s} + \|f(T^* - \delta)\|_{s-1} + \|g\|_{H^s([0,T_0]\times\mathbb{R}^{d-1})} + \int_0^{T_0} \|f(t')\|_s dt' \,.$$

Since f and g are given in H^s, the last three terms are bounded independently of $T^* - \delta$. By (2.5.20), the first term is bounded independently of T^* and δ. This shows that the increment $C(K)$ is bounded from below independently of T^* and δ.

If T^* were strictly smaller that T_0, we could choose $\delta = C(K)/2$ so that $T_2 > T^*$. By uniqueness, $\tilde{u}$ would be an extension of u, contradicting the definition of T^*. If $T_* = T_0$, choosing again $\delta = C(K)/2$, we see that $T_2 = T_0$ and thus $u \in W^(T_0)$. □

2.5.5 Regularity of solutions

Suppose that $T > 0$ is given, $f \in H^s([0,T] \times \mathbb{R}^d_+)$, $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ and $u_0 \in H^s(\mathbb{R}^d_+)$, with $s > d/2$. Suppose that the compatibility conditions are satisfied at order $s - 1$ and $u \in W^s(T)$ is a solution of (2.5.1). The next result finishes the proof of Theorem 2.5.1.

Proposition 2.5.11. *Suppose that $\sigma > s$ and (f, g, u_0) belong to $H^\sigma([0,T]\times \mathbb{R}^d_+)$, $H^\sigma([0,T]\times\mathbb{R}^{d-1})$ and $H^\sigma(\mathbb{R}^d_+)$ respectively and satisfy the compatibility conditions to order $\sigma - 1$, then the solution u belongs to $W^\sigma(T)$.*

Proof. By Proposition 2.5.8 there is $T_1 \in]0, T]$ such that the problem has a solution $\tilde{u} \in W^\sigma(T_1)$. Denote by T^* the maximal time of existence of solutions in W^σ. By uniqueness in $W^s(T')$ for $T' < T^*$, $u = \tilde{u}$ for $t < T^*$. Since $u \in W^s(T)$ and $s > d/2$, $u \in L^\infty([0,T] \times \mathbb{R}^d_+)$ and thus $\tilde{u} \in L^\infty([0,T^*[\times\mathbb{R}^d_+)$. Therefore Proposition 2.5.10 implies that $T^* = T$ and $u = \tilde{u} \in W^\sigma(T)$. □

Chapter 3

Hyperbolic-Parabolic Problems

In this Chapter, we first recall the classical existence and uniqueness results for parabolic systems. Next we look for uniform estimates, independent of the viscosity, in spaces with tangential or conormal smoothness.

3.1 The equations

With notations as in Chapter 2, consider an "hyperbolic" $N \times N$ system

$$Lu := \partial_t u + \sum_{j=1}^{d} A_j \partial_j u = F(u) + f \tag{3.1.1}$$

and a "parabolic" viscous perturbation

$$(L - \varepsilon P)u := Lu - \varepsilon \sum_{j,k=1}^{d} B_{j,k} \partial_j \partial_k u = F(u) + f\,. \tag{3.1.2}$$

For simplicity, we assume that the coefficients A_j and $B_{j,k}$ are constant. F is a C^∞ mapping from $\mathbb{R}^N$ to $\mathbb{R}^N$.

We consider the equation (3.1.2) on the half space $\{x \geq 0\}$ together with homogeneous Dirichlet boundary conditions:

$$u_{|x=0} = 0\,. \tag{3.1.3}$$

For simplicity, we work in the class of symmetric operators. The next assumption implies that L is hyperbolic and that P is elliptic.

Assumption 3.1.1. *There is a positive definite symmetric matrix $S = {}^tS \gg 0$ such that for all j the matrix SA_j is symmetric. Moreover, for all $\xi \neq 0$ the matrix $\sum \xi_j\xi_k \mathrm{Re}\, SB_{j,k}$ is symmetric definite positive.*

Recall that for a matrix M, $\mathrm{Re}\, M = \frac{1}{2}(M + M^*)$ is always symmetric. .

In this chapter, we fist solve the equations for fixed ε. This is much easier than the hyperbolic theory, since we can rely on classical variational methods. The results are classical, but we sketch proofs as they will serve for the ε dependent analysis. Because the equation is nonlinear and the viscous regularization depends on ε, the life span is expected to depend strongly on ε. The main question is to prove the existence of solutions on domains independent of ε. This relies on uniform estimates, in suitable spaces. This analysis is performed in the second part of the chapter and will be used in the next chapter to prove the convergence of asymptotic expansions.

3.2 Linear existence theory

In this section we prove the existence of solutions to the linear equations for a fixed ε. Thus, changing P into εP, we assume in this section that $\varepsilon = 1$. In this section, we do not use the full strength of Assumption 3.1.1, we only assume that there is $c > 0$ such that

$$\forall \xi \in \mathbb{R}^d : \quad \sum_{j,k} \xi_j\xi_k \mathrm{Re}\,(SB_{j,k}) \geq c|\xi|^2 \mathrm{Id} \tag{3.2.1}$$

With $G = P - A$, $A = \sum A_j\partial_j$, consider the problem

$$\begin{cases} \partial_t u - Gu = f & \text{on } [0,T] \times \mathbb{R}^d_+ \\ u_{|x=0} = 0 & \text{on } [0,T] \times \mathbb{R}^{d-1} \\ u_{|t=0} = u_0 & \text{on } \mathbb{R}^d_+ \end{cases} \tag{3.2.2}$$

Recall that $\mathbb{R}^d_+ = \{(y,x) \in \mathbb{R}^d : x > 0\}$.

3.2.1 Variational methods

As usual, $C_0^\infty(U)$ denotes the space of C^∞ functions u on $U \subset \mathbb{R}^m$ with compact support contained in U: this means that there is compact set $K \subset \mathbb{R}^m$, such that K is contained in U and u vanishes on $U \setminus K$. We use this notation when U is open, but also when U is closed (typically a closed half space, or o closed strip) or for instance when $U = [0, T[\times\mathbb{R}^d$.

Definition 3.2.1. *i)* $\mathcal{V} := L^2([0,T]; H^1(\mathbb{R}^d_+))$ *denotes the space of functions* $u \in L^2([0,T] \times \mathbb{R}^d_+)$ *such that the spaces derivatives* $\partial_j u$ *for* $j = 1, \dots, d$*, belong to* $L^2([0,T] \times \mathbb{R}^d_+)$ *.*

ii) $\mathcal{V}_0 := L^2([0,T]; H^1_0(\mathbb{R}^d_+))$ *is the closure in* $L^2([0,T]; H^1(\mathbb{R}^d_+)$ *of* $C^\infty_0(]0,T[\times\mathbb{R}^d_+)$*.*

iii) $L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$ *is the set of distributions* f *on* $]0,T[\times\mathbb{R}^d_+$ *such that there exist functions* $g_j \in L^2([0,T] \times \mathbb{R}^d_+)$ *for* $j = 0, \dots, d$ *such that*

$$f = g_0 + \sum_{j=1}^{d} \partial_j g_j \, . \tag{3.2.3}$$

We collect here a few properties satisfied by theses spaces.

• They are Hilbert spaces. $L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$ is equipped with the norm

$$\min \Big(\sum \|g_j\|^2_{L^2} \Big)^{1/2}$$

where the minimum is taken over all the decompositions (3.2.3) of f. They are the spaces of restrictions to $\mathbb{R}^d_+$ of analogous spaces on $\mathbb{R}^d$, which can be characterized using the spatial Fourier transform. $C^\infty_0([0;T] \times \overline{\mathbb{R}}^d_+)$ is dense in $\mathcal{V}$.

• The functions in $\mathcal{V} = L^2([0,T]; H^1(\mathbb{R}^d_+))$ satisfy $v \in L^2$ and $\partial_x v \in L^2$. Therefore they have a trace on $\{x = 0\}$ which belongs to $L^2([0,T] \times \mathbb{R}^{d-1})$. $\mathcal{V}_0$ is the set of $v \in \mathcal{V}$ such that $v_{|x=0} = 0$. Below, the trace condition in equation (3.2.2) is encoded in the condition $u \in \mathcal{V}_0$.

• Because H^{-1} is the dual space of H^1_0, the spaces $L^2([0,T]; H^1_0(\mathbb{R}^d_+))$ and $L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$ are in duality: if f is given by (3.2.3) and $u \in C^\infty_0(]0,T[\times\mathbb{R}^d_+)$ then, in the distribution sense:

$$\langle f, u \rangle = (g_0, u)_{L^2} - \sum_{j=1}^{d} (g_j, \partial_j u)_{L^2} \tag{3.2.4}$$

where $(\cdot,\cdot)_{L^2}$ denotes the scalar product in L^2. By density continuity, this formula extends to $u \in L^2([0,T]; H^1_0(\mathbb{R}^d_+))$ and uniquely defines $\widetilde{f}$ in the dual space of $L^2([0,T]; H^1_0(\mathbb{R}^d_+))$. In particular, the right hand side of (3.2.4) does not depend on the particular decomposition (3.2.3) of f.

Conversely, suppose that $\widetilde{f}$ is a linear form on $L^2([0,T]; H^1_0(\mathbb{R}^d_+))$. Consider the space K_1 of the $U = (u_0, u_1, \dots, u_d)$ with $u_j = \partial_j u_0$ and $u_0 \in L^2([0,T]; H^1_0(\mathbb{R}^d_+))$. This is a closed subspace of $K = \big(L^2([0,T] \times \mathbb{R}^d_+)\big)^{1+d}$, and the linear form $U \mapsto \langle \widetilde{f}, u_0 \rangle$ is continuous on K_1 for the norm of K.

Thus it extends to K and by Riesz theorem, there are $(g_0, g_1, \dots, g_d)$ such that the extension f^* satisfies

$$\langle f^*, U\rangle = (g_0, u_0)_{L^2} - \sum_{j=1}^{d} (g_j, u_j)_{L^2}$$

Comparing with (3.2.4), this shows that $\widetilde{f}$ coincide with the linear form associated to $f = g_0 + \sum \partial_j g_j \in L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$.

Thus we identify $L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$ with the dual space $\mathcal{V}_0'$ of $\mathcal{V}_0 = L^2([0,T]; H^1_0(\mathbb{R}^d_+))$ and the duality $(u,v)_{L^2}$ for smooth functions extends to $(u,v) \in \mathcal{V}_0' \times \mathcal{V}_0$ or to $(u,v) \in \mathcal{V}_0 \times \mathcal{V}_0'$. We denote it by $\langle u, v\rangle$.

Proposition 3.2.2 (Coerciveness). *G is bounded from $L^2([0,T]; H^1(\mathbb{R}^d_+))$ into $L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$. Moreover, there are $\gamma_0 > 0$ and $c > 0$ such that for all $u \in L^2([0,T]; H^1_0(\mathbb{R}^d_+))$ and $\gamma \geq \gamma_0$:*

$$\operatorname{Re} \langle S(\gamma - G)u, u\rangle \geq c \|u\|^2_{L^2([0,T];H^1(\mathbb{R}^d_+))} . \tag{3.2.5}$$

Proof. By definition, ∂_j maps $L^2([0,T]; H^1(\mathbb{R}^d_+))$ into $L^2([0,T] \times \mathbb{R}^d_+)$ and $L^2([0,T] \times \mathbb{R}^d_+)$ to $L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$. Thus G maps $L^2([0,T]; H^1(\mathbb{R}^d_+))$ into $L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$.

Moreover, (3.2.4) implies that for $u \in L^2([0,T]; H^1_0(\mathbb{R}^d_+))$

$$\begin{aligned}\operatorname{Re} \langle S(\gamma - G)u, u\rangle =& \gamma (Su, u)_{L^2} \\ &+ \operatorname{Re} \sum_j (SA_j \partial_j u, u)_{L^2} + \sum_{j,k} \operatorname{Re} (SB_{j,k} \partial_k u, \partial_j u)_{L^2}\end{aligned}$$

By density continuity, it is sufficient to prove (3.2.5) for $u \in C^\infty_0([0,T] \times \mathbb{R}^d_+)$. In this case, extending u by 0 for negative x and denoting by $\hat{u}(t,\xi)$ its spatial Fourier transform, (3.2.1) implies that

$$\operatorname{Re} \sum_{j,k} (\xi_j \xi_k SB_{j,k} \hat{u}, \hat{u})_{L^2} \geq c \sum_{j=1}^{d} \|\xi_j \hat{u}\|^2_{L^2} .$$

where the L^2 norm is now taken for $(t,\xi) \in]0,T[\times \mathbb{R}^d$. By Plancherel's theorem, this implies that for all $u \in C^\infty_0([0,T] \times \mathbb{R}^d)$:

$$\operatorname{Re} \sum_{j,k} (SB_{j,k} \partial_k u, \partial_j u)_{L^2} \geq c \sum_{j=1}^{d} \|\partial_j u\|^2_{L^2} \tag{3.2.6}$$

Since

$$|(SA_j\partial_j u, u)_{L^2}| \leq C\|\partial_j u\|_{L^2}\|u\|_{L^2}$$

Thus (3.2.5) follows for γ large enough. $\square$

Definition 3.2.3. *Let $\mathcal{H}$ denote the space of functions $u \in L^2([0,T]; H^1(\mathbb{R}^d_+))$ such that the time derivative in the sense of distribution, $\partial_t u$, belongs to $L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$.*

$\mathcal{H}_0$ denotes the subspace of functions $u \in L^2([0,T]; H^1_0(\mathbb{R}^d_+))$ such that $\partial_t u \in L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$.

Proposition 3.2.4. *i) $\mathcal{H}$ is an Hilbert space and $C_0^\infty([0,T] \times \overline{\mathbb{R}}^d_+)$ is dense in $\mathcal{H}$.*

ii) $\mathcal{H}_0 = \{v \in \mathcal{H}; v_{|x=0} = 0\}$ is a closed subspace of $\mathcal{H}$.

iii) $\mathcal{H}$ is continuously imbedded in $C^0([0,T]; L^2(\mathbb{R}^d_+))$. Moreover, for all $u \in \mathcal{H}_0$

$$\text{(3.2.7)} \qquad 2\mathrm{Re}\,\langle S\partial_t u, u\rangle = \big(Su(T), u(T)\big)_{L^2(\mathbb{R}^d_+)} - \big(Su(0), u(0)\big)_{L^2(\mathbb{R}^d_+)}\,.$$

Proof. Using cut-off function in time it is sufficient to prove the results for functions on $[0, +\infty[\times\mathbb{R}^d_+$ and functions on $]-\infty, T] \times \mathbb{R}^d_+$. We consider the former case. We leave the proof of the density as as exercise, which can be solved by standard cut-off and mollification. We also leave the characterization of $\mathcal{H}_0$ to the reader.

There is an extension operator E continuous from $H^1(\mathbb{R}^d_+)$ to $H^1(\mathbb{R}^d)$ and from $H^{-1}(\mathbb{R}^d_+)$ to $H^{-1}(\mathbb{R}^d)$. Extending trivially this operator to functions of (t,x) reduces the analysis of analogous spaces on $[0,\infty[\times\mathbb{R}^d$. Using Lemma 2.2.5 with t and x interchanged, yields:

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq \|u\|_{L^2(H^1)}\|\partial_t u\|_{L^2(H^{-1})}$$

Since H^1 is a subspace of $C^0(L^2)$, this implies by density and continuity, that $\mathcal{H} \subset C^0(L^2)$.

For $u \in C_0^\infty([0,T] \times \mathbb{R}^d_+)$, $\partial_t u$ is smooth and the duality $\langle S\partial_t u, u\rangle$ is simply $(S\partial_t u, u)_{L^2}$ and the identity (3.2.7) holds. It extends to $u \in \mathcal{H}_0$, using in the left hand side the duality $\mathcal{V}_0' \times \mathcal{V}_0$. $\square$

Theorem 3.2.5. *For all $f \in L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$ and $u_0 \in L^2(\mathbb{R}^d_+)$, the equation* (3.2.2) *has a unique solution $u \in \mathcal{H}_0$.*

Proof. **a)** *Introducing weights.* The mapping $v \mapsto e^{\gamma t} v$ is an isomorphism both in $\mathcal{V}_0$ and in $\mathcal{V}_0'$. Moreover, $\partial_t e^{\gamma t} v = e^{\gamma t}(\partial_t + \gamma) v$. Thus, solving (3.2.2) in $\mathcal{H}_0$ is equivalent to solving for γ large enough

$$u \in \mathcal{H}_0 \,, \quad (\partial_t - G + \gamma) u = f \,, \quad u_\gamma(0) = u_0 \,. \tag{3.2.8}$$

From now on, we fix $\gamma \geq \gamma_0$, where γ_0 is as in Proposition 3.2.2.

b) *Uniqueness.*

By Propositions 3.2.2 and 3.2.4, for $u \in \mathcal{H}_0$ the following estimate holds

$$\begin{aligned} \big(Su(T), u(T)\big)_{L^2} &+ c \|\nabla_{y,x} u\|^2_{L^2(L^2)} \\ &\leq \big(Su(0), u(0)\big)_{L^2} + 2\mathrm{Re}\, \langle S(\partial_t - G + \gamma)u, u\rangle \,, \end{aligned}$$

thus

$$\|u(T)\|^2_{L^2} + c\|u\|^2_{L^2(H^1)} \leq C\big(\|u(0)\|^2_{L^2} + 2\mathrm{Re}\, \langle S(\partial_t - G + \gamma)u, u\rangle\big) \,. \tag{3.2.9}$$

This immediately implies uniqueness for (3.2.8).

b) *The adjoint equation.*

The adjoint operator of $J := \partial_t - G + \gamma$ is $J^* = -\partial_t - G^* + \gamma$ with $G^* = \sum B^*_{k,j} \partial_k \partial_j + \sum A_j^* \partial_j$. It maps $\mathcal{H}$ to $L^2([0,T]; H^{-1}(\mathbb{R}^d_+))$. S^{-1} is a symmetrizer for the $B^*_{k,j}$ since $S^{-1}B^* = S^{-1}(SB)^* S^{-1}$ and $\mathrm{Re}\, S^{-1}B^* = S^{-1}\mathrm{Re}\,(SB) S^{-1}$. Thus G^* satisfies the assumption (3.2.1). Due to the change of ∂_t into $-\partial_t$, the analogue of (3.2.9) is

$$\|v(0)\|^2_{L^2} + c\|v\|^2_{L^2(H^1)} \leq \|v(T)\|^2_{L^2} + 2\mathrm{Re}\, \langle S^{-1} J^* v, v\rangle \tag{3.2.10}$$

which holds for all $v \in \mathcal{H}_0$.

Moreover, by density continuity, for all u and v in $\mathcal{H}_0$:

$$\begin{aligned} \mathrm{Re}\, \langle Ju, v\rangle - \mathrm{Re}\, \langle u, J^* v\rangle& \\ &= \big(u(T), v(T)\big)_{L^2} - \big(u(0), v(0)\big)_{L^2} \,. \end{aligned} \tag{3.2.11}$$

c) *Existence*

Let $\mathcal{E}$ denote the subspace of $v \in \mathcal{H}_0$ such that $v(T) = 0$. By (3.2.10), there is C such that for all $v \in \mathcal{E}$:

$$\|v(0)\|_{L^2} + \|v\|_{\mathcal{V}_0} \leq C\|J^* v\|_{\mathcal{V}_0'} \,.$$

Consider the space $\mathcal{F} = J^* \mathcal{E} \subset \mathcal{V}_0'$. The estimate above implies that there is a linear operator R from $\mathcal{F}$ to $\mathcal{E}$ such that

$$\forall g \in \mathcal{F} : \quad \|Rg(0)\|_{L^2} + \|Rg\|_{\mathcal{V}_0} \leq C\|g\|_{\mathcal{V}_0'} \,. \tag{3.2.12}$$

Given $f \in \mathcal{V}_0'$ and $u_0 \in L^2$, consider the anti-linear form on $\mathcal{F}$

$$\ell(g) = \langle f, Rg \rangle + \big(u_0, Rg(0)\big)_{L^2} .$$

By (3.2.12), ℓ is continuous on $\mathcal{F}$ for the $\mathcal{V}_0'$-norm. Thus ℓ extends to $\mathcal{V}_0'$ and there is u in $\mathcal{V}_0$ such that

$$\ell(g) = \langle u, g \rangle .$$

Tracing back the definition, this means that for all $v \in \mathcal{H}_0$,

$$\langle u, (-\partial_t - G^* + \gamma)v \rangle = \langle f, v \rangle + \big(u_0, v(0)\big)_{L^2} .$$

Comparing with (3.2.11) yields

$$\langle (\partial_t - G + \gamma)u, v \rangle + \big(u(0), v(0)\big)_{L^2} = \langle f, v \rangle + \big(u_0, v(0)\big)_{L^2} .$$

Choosing test functions with compact support in $]0, T[\times \mathbb{R}^d_+$ implies that $(\partial_t - G + \gamma)u = f$. Thus for all $v \in \mathcal{E}$ we are left with $\big(u(0) - u_0, v(0)\big)_{L^2} = 0$. Since $v(0)$ can be chosen arbitrarily in $C_0^\infty(\mathbb{R}^d_+)$, this implies that $u(0) = u_0$. Therefore we have solved the equation (3.2.8). □

3.2.2 Regularity estimates

We show that for smooth data satisfying compatibility conditions the solution given by Theorem 3.2.5 is smooth. The form of the equation shows that the time derivative and the spatial derivatives have not the same weight. This leads to introduce nonisotropic spaces.

Definition 3.2.6. *For $s \in \mathbb{Z}$, $s \geq -1$, denote by $\mathcal{H}^s$ the space of functions $u \in L^2([0,T]; H^s(\mathbb{R}^d_+))$ such that for all nonnegative integer $j \leq (s+1)/2$, $\partial_t^j u \in L^2([0,T]; H^{s-2j}(\mathbb{R}^d_+))$*

$\mathcal{H}^{-1}$ is simply $L^2([0,T]; H^{-1}(\mathbb{R}^d_+)$ and $\mathcal{H}^0 = L^2([0,T] \times \mathbb{R}^d_+)$. $\mathcal{H}^1$ is the space introduced in Definition 3.2.3. Similarly we note

$$\mathcal{H}_0^1 = \{u \in \mathcal{H}^1 \, ; \, u_{|x=0} = 0\} . \tag{3.2.13}$$

For further use, we note the following result which extends Proposition 3.2.4:

Proposition 3.2.7. *i) For $s \geq -1$, $C_0^\infty([0,T] \times \overline{\mathbb{R}}^d_+)$ is dense in $\mathcal{H}^s$.*

ii) For $s \geq 0$, the spatial derivatives ∂_{y_j} and ∂_x map $\mathcal{H}^s$ into $\mathcal{H}^{s-1}$. For $s \geq 1$, ∂_t maps $\mathcal{H}^s$ into $\mathcal{H}^{s-2}$.

iii) For $s \geq 2$, the tangential spatial derivatives ∂_{y_j} map $\mathcal{H}^s \cap \mathcal{H}^1_0$ into $\mathcal{H}^{s-1} \cap \mathcal{H}^1_0$. For $s \geq 3$, ∂_t maps $\mathcal{H}^s \cap \mathcal{H}^1_0$ into $\mathcal{H}^{s-2} \cap \mathcal{H}^1_0$.

iv) For $s \geq 1$, $\mathcal{H}^s$ is embedded in the space of $u \in C^0([0,T]; H^{s-1}(\mathbb{R}^d_+))$ such that $\partial_t^j u \in C^0([0,T]; H^{s-2j-1}(\mathbb{R}^d_+))$ for $j \leq s/2$.

Before proving smoothness of solution, we prove a-priori estimates. As in the previous subsection, we consider the equation (3.2.8).

Theorem 3.2.8. *For all $s \geq 0$ and $\gamma > 0$ large enough, there is a constant C such that all $u \in \mathcal{H}^{s+1} \cap \mathcal{H}^1_0$ satisfies the following estimate*

$$\|u\|_{\mathcal{H}^{s+1}} \leq C\|u(0)\|_{H^s(\mathbb{R}^d_+)} + C\|(\partial_t - G + \gamma)u\|_{\mathcal{H}^{s-1}} . \tag{3.2.14}$$

By (3.2.9), there is C such that for γ large enough and $u \in \mathcal{H}^1_0$:

$$\|u\|_{L^2(H^1)} \leq C\|u(0)\|_{L^2(\mathbb{R}^d_+)} + C\|(\partial_t - G + \gamma)u\|_{L^2(H^{-1})} .$$

Since

$$\|(G - \gamma)u\|_{L^2(H^{-1})} \leq C\|u\|_{L^2(H^1)},$$

this implies

$$\begin{aligned} \|u\|_{L^2(H^1)} + \|\partial_t u\|_{L^2(H^{-1})} \leq & C\|u(0)\|_{L^2(\mathbb{R}^d_+)} \\ & + C\|(\partial_t - G + \gamma)u\|_{L^2(H^{-1})} . \end{aligned} \tag{3.2.15}$$

Thus the estimate (3.2.14) is proved when $s = 0$. We prove it for $s = 1$ and next conclude by induction on s.

Lemma 3.2.9. *The estimate* (3.2.14) *holds for $s = 1$.*

Proof. **a)** By (3.2.15), we already have an L^2 estimate of u and of the first (y, x) derivatives.

When $u \in \mathcal{H}^2 \cap \mathcal{H}^1_0$, the tangential derivatives ∂_{y_j} belong to $\mathcal{H}^1_0$ and $(\partial_t - G + \gamma)\partial_{y_j} u = \partial_{y_j}(\partial_t - G + \gamma)u$. Thus

$$\|(\partial_t - G + \gamma)\partial_{y_j} u\|_{L^2(H^{-1})} \leq \|(\partial_t - G + \gamma)u\|_{L^2} .$$

Therefore the estimate (3.2.15) implies that

$$\|\partial_y \partial_{y,x} u\|_{L^2} \leq C\|\partial_y u(0)\|_{L^2(\mathbb{R}^d_+)} + C\|(\partial_t - G + \gamma)u\|_{L^2} . \tag{3.2.16}$$

b) Hence, to prove the lemma it remains to estimate the second normal derivative of u. We cannot use the differentiated equation for $\partial_x u$ since $\partial_x u$ does not satisfy the boundary condition. Instead, we deduce from the equation and (3.2.15) (3.2.16) that

$$\partial_t u - B_{d,d}\partial_x^2 u = f$$

where

$$\|f\|_{L^2} \le C\|u(0)\|_{H_0^1(\mathbb{R}_+^d)} + C\|(\partial_t - G + \gamma)u\|_{L^2([0,T]\times\mathbb{R}_+^d)}\,.$$

We multiply this equation by $S\partial_x^2 u$ and integrate by parts in the term $(S\partial_t u, \partial_x^2 u)_{L^2}$. For smooth functions v vanishing on the boundary, there holds

$$2(S\partial_t v, \partial_x^2 v)_{L^2} = -\big(S\partial_x v(T), \partial_x v(T)\big)_{L^2} + \big(S\partial_x v(0), \partial_x v(0)\big)_{L^2}$$

By density continuity, this identity extends to $u \in \mathcal{H}^2 \cap \mathcal{H}_0^1$ and thus

$$\begin{aligned}-2\mathrm{Re}\,\big(Sf, \partial_x^2 u\big)_{L^2} =& 2\big(SB_{d,d}\partial_x^2 u, \partial_x^2 u\big)_{L^2} \\ &+ \big(S\partial_x u(T), \partial_x u(T)\big)_{L^2} - \big(S\partial_x u(0), \partial_x u(0)\big)_{L^2}\end{aligned}$$

Since $SB_{d,d}$ is definite positive, using the Cauchy Schwarz inequality in the left hand side yields:

$$\|\partial_x^2 u\|_{L^2} \le C\|\partial_x u(0)\|_{L^2} + C\|f\|_{L^2}\,.$$

Together with the estimates above, this finishes the proof of the lemma. $\square$

Proof of Theorem 3.2.8. We proceed by induction on s. The estimate is proved for $s = 0$ and $s = 1$. So, consider $s \ge 2$ and assume that the theorem is proved up to order $s - 1$.

Since $s \ge 2$ and $u \in \mathcal{H}^{s+1} \cap \mathcal{H}_0^1$, the tangential spatial derivatives $\partial_{y_j} u$ belong to $\mathcal{H}^s \cap \mathcal{H}_0^1$ and $(\partial_t - G + \gamma)\partial_{y_j} u = \partial_{y_j} f$ with $f := (\partial_t - G + \gamma)u$. Similarly, $\partial_t u \in \mathcal{H}^{s-1} \cap \mathcal{H}_0^1$ and $(\partial_t - G + \gamma)\partial_t u = \partial_t f$. Therefore the induction hypothesis implies that

$$\|\partial_{y_j} u\|_{\mathcal{H}^s} + \|\partial_t u\|_{\mathcal{H}^{s-1}} \lesssim \|u(0)\|_{H^s} + \|\partial_t u(0)\|_{H^{s-2}} + \|f\|_{\mathcal{H}^{s-1}}$$

Thus, to control the $\mathcal{H}^{s+1}$ norm of u, only the L^2 norm of $\partial_x^{s+1} u$ is missing : all the norms of derivatives with at least one ∂_t are controlled by the $\mathcal{H}^{s-1}$ norm $\partial_t u$, and all the purely spatial derivatives with at least one ∂_{y_j} are controlled by the $\mathcal{H}^s$ norm $\partial_y u$. Using the equation, we write $-B_{d,d}\partial_x^{s+1}$

as the sum of $\partial_x^{s-2} fu$ plus $\partial_x^{s-2}\partial_t u$ and spatial derivatives with at least one tangential. Therefore, we obtain

$$\|u\|_{\mathcal{H}^{s+1}} \lesssim \|u(0)\|_{H^s} + \|\partial_t u(0)\|_{H^{s-2}} + \|f\|_{\mathcal{H}^{s-1}}$$

To finish the proof, we note that $\partial_t u = (G-\gamma)u + f \in \mathcal{H}^{s-1}$, and since $s-1 \geq 1$ we can take the trace at $t=0$ to obtain

$$\|\partial_t u(0)\|_{H^{s-2}} \lesssim \|u(0)\|_{H^s} + \|f(0)\|_{H^{s-2}} \lesssim \|u(0)\|_{H^s} + \|f\|_{\mathcal{H}^{s-1}}\,.$$

Adding up, this implies (3.2.14) at the order s. $\square$

3.2.3 Smooth solutions

We first discuss the compatibility conditions. With $s \geq 1$, let $u \in \mathcal{H}^{s+1}$ and $f = (\partial_t u - Gu) \in \mathcal{H}^{s-1}$. Then $\partial_t^j u \in C^0(H^{s-2j})$ for $j \leq s/2$ and $\partial_t^k f \in C^0(H^{s-2-2k})$ for $k \leq s/2-1$ when $s \geq 2$. Thus one can define the traces $u_j = \partial_t^j u(0)$ for $j \leq s/2$ and $f_j = \partial_t^j f(0)$ for $j \leq s/2-1$. Moreover, when $s \geq 2$, one has $u_j = Gu_{j-1} + f_{j-1}$ and thus,

$$u_j = G^j u_0 + \sum_{k=0}^{j-1} G^{j-1-k} f_k\,, \quad \text{when } \ 1 \leq j \leq s/2\,. \tag{3.2.17}$$

We use the following remark:

Lemma 3.2.10. *If $u \in \mathcal{H}^2 \cap \mathcal{H}_0^1$, which means that $u \in L^2([0,T]; H^2 \cap H_0^1(\mathbb{R}_+^d))$ and $\partial_t u \in L^2([0,T]\times\mathbb{R}_+^d)$, then $u \in C^0([0,T]; H_0^1(\mathbb{R}_+^d))$.*

When $u \in \mathcal{H}^{s+1} \cap \mathcal{H}_0^1$, then $\partial_t^j u \in \mathcal{H}^{s+1-2j} \cap \mathcal{H}_0^1$ when $j < s/2$. Thus $\partial_t^j u \in C^0([0,T]; H_0^1(\mathbb{R}_+^d))$. This leads to the following definition:

Definition 3.2.11. *Given $s \geq 1$, the data $u_0 \in H^s(\mathbb{R}_+^d)$ and $f \in \mathcal{H}^{s-1}$ satisfy the compatibility conditions up to order $\sigma \leq s-1$ if u_0 and the $u_j \in H^{s-2j}(\mathbb{R}_+^d)$ defined by* (3.2.17) *when $s \geq 2$, satisfy*

$$u_{j|x=0} = 0\,, \quad \text{for } j \leq \sigma/2\,. \tag{3.2.18}$$

When $s=0$, there are no compatibility conditions, as indicated by Theorem 3.2.5. When $s=1$, there is one compatibility condition, which reads

$$u_0 \in H_0^1(\mathbb{R}_+^d)\,. \tag{3.2.19}$$

When $s = 2$, $u_0 \in H^2$ and $u_1 \in L^2$ are defined, but there is still only one compatibility condition, (3.2.19). When $s = 3$, $u_0 \in H^3$ and $u_1 \in H^1$ and there are two compatibility conditions

$$u_0 \in H^3 \cap H^1_0(\mathbb{R}^d_+) \quad u_1 \in H^1_0(\mathbb{R}^d_+) \,. \tag{3.2.20}$$

The computations before Definition 3.2.11 show that the compatibility conditions are necessary:

Lemma 3.2.12. *Suppose that $u \in \mathcal{H}^{s+1} \cap \mathcal{H}^1_0$ with $s \geq 1$. Then $u_0 = u(0) \in H^s(\mathbb{R}^d_+)$ and $f = (\partial_t - G)u \in \mathcal{H}^{s-1}$ satisfy the compatibility conditions to order $s - 1$.*

Theorem 3.2.13. *Given $s \geq 1$, $u_0 \in H^s(\mathbb{R}^d_+)$ and $f \in \mathcal{H}^{s-1}$ satisfying the compatibility conditions up to order $s - 1$, the solution u of* (3.2.2) *given by Theorem* 3.2.5 *belongs to $\mathcal{H}^{s+1} \cap \mathcal{H}^1_0$.*

In particular, when $s = 1$, this implies :

Corollary 3.2.14. *Suppose that $f \in L^2([0,T] \times \mathbb{R}^d_+)$ and $u_0 \in H^1_0(\mathbb{R}^d_+)$. Then the solution $u \in \mathcal{H}^1_0$ of* (3.2.2) *belongs to $\mathcal{H}^2$.*

As in the previous section we write $u = e^{\gamma t}\tilde{u}$, $f = e^{\gamma t}\tilde{f}$ so that $\tilde{u}$ and $\tilde{f}$ satisfy

$$(\partial_t - G + \gamma)\tilde{u} = \tilde{f} \,, \quad \tilde{u}_{|t=0} = u_0 \,. \tag{3.2.21}$$

The traces $\tilde{u}_j$ and $\tilde{f}_k$ of $\tilde{u}$ and $\tilde{f}$ are related to those of u and f. The definition (3.2.17) is modified as follows

$$\tilde{u}_j = (G - \gamma)^j u_0 + \sum_{k=0}^{j-1} (G - \gamma)^{j-1-k} \tilde{f}_k \,, \tag{3.2.22}$$

and the compatibility conditions read

$$\tilde{u}_j \in H^{s-2j} \cap H^1_0 \,, \quad \text{for } j \leq (s-1)/2 \,. \tag{3.2.23}$$

We prove that for $\tilde{f} \in \mathcal{H}^{s-1}$ and $u_0 \in H^s$ satisfying the compatibility conditions to order $s - 1$, the solution $\tilde{u} \in \mathcal{H}^1_0$ of (3.2.21) belongs to $\mathcal{H}^{s+1}$. For simplicity of notations, below we drop the tildes. We first consider special data.

Proposition 3.2.15. *Suppose $s \geq 1$ and $f \in \mathcal{H}^{s-1}$. When $s \geq 2$, assume in addition that $\partial_t^k f(0) = 0$ for $k \leq (s-2)/2$. Then the solution $u \in \mathcal{H}^1_0$ of* (3.2.2) *with initial data $u_0 = 0$ belongs to $\mathcal{H}^{s+1}$.*

Proof. Denote by v^e the extension of v by 0 for $t < 0$. Since $u \in \mathcal{H}^1$ and $u(0) = 0$, $\partial_t u^e = (\partial_t u)^e$ and $u^e \in \mathcal{H}^1 \cap \mathcal{H}^1_0(]-\infty, T])$. Moreover, $(\partial_t - G + \gamma)u^e = f^e$. Since $f \in \mathcal{H}^{s-1}$ and $\partial_t^k f(0) = 0$ for $k \le (s-2)/2$, we have $\partial_t^k f^e = (\partial_t^k f)^e$ for $k \le (s-1)/2$ and $f^e \in \mathcal{H}^{s-1}(]-\infty, T])$.

Consider tangential mollifiers $\jmath_\varepsilon(t, y) = \varepsilon^{-d}\jmath(t/\varepsilon, y/\varepsilon)$ with $\jmath \in C_0^\infty(\mathbb{R}^d)$, $\jmath \ge 0$, $\int \jmath(t, y)dtdy = 1$ and $\jmath$ supported in $t > 0$. Consider then $u_\varepsilon = \jmath_\varepsilon * u^e$ and $f_\varepsilon = \jmath_\varepsilon * f^e$. The convolution is well defined thanks to the support condition on $\jmath$ and u_ε and f_ε vanish for $t < 0$. The convolution commutes with derivatives and $u_\varepsilon \to u$ in $\mathcal{H}^1$ and $f_\varepsilon \to f$ in $\mathcal{H}^{s-1}$. Moreover, $u_{\varepsilon|x=0} = \jmath_\varepsilon * (u^e_{|x=0}) = 0$. Thus

$$u_\varepsilon \in \mathcal{H}^1 \cap \mathcal{H}^1_0, \quad (\partial_t - G + \gamma)u_\varepsilon = f_\varepsilon, \quad u_\varepsilon(0) = 0.$$

In addition, u_ε and f_ε are infinitely smooth in (t, y). In particular, we know that $\partial_t^k \partial_y^\alpha u$, $\partial_t^k \partial_y^\alpha \partial_x u$, $\partial_t^k \partial_y^\alpha \partial_x^n f$ belong to $L^2([0, T] \times \mathbb{R}^d_+)$ for all k, α and $n \le s - 1$. Using the equation, we get that $\partial_t^k \partial_y^\alpha \partial_x^n u$ belongs to $L^2([0, T] \times \mathbb{R}^d_+)$ for all k, α and $n \le s + 1$ and therefore $u_\varepsilon \in \mathcal{H}^{s+1}$.

Thus we can apply Theorem 3.2.8 to u_ε and to differences $u_\varepsilon - u_{\varepsilon'}$. The estimate (3.2.14) and the convergence $f_\varepsilon \to f$ in $\mathcal{H}^{s-1}$ imply that u_ε is a Cauchy sequence and hence converges in $\mathcal{H}^{s+1}$. Since $u_\varepsilon \to u$ in $\mathcal{H}^1$, this implies that $u \in \mathcal{H}^{s+1}$. □

Next, we need an approximation lemma for compatible data.

Lemma 3.2.16. *Given $s \ge 1$, $u_0 \in H^s(\mathbb{R}^d_+)$ and $f \in \mathcal{H}^{s-1}$ satisfying the compatibility conditions up to order $s - 1$, there are sequences $u_0^n \in H^\infty$ and $f^n \in \mathcal{H}^\infty$* [1] *satisfying the compatibility conditions up to order s and such that $u_0^n \to u_0$ in $H^s(\mathbb{R}^d_+)$ and $f^n \to f$ in $\mathcal{H}^{s-1}$.*

Proof. Consider a sequence $f^n \in \mathcal{H}^\infty$ such that $f^n \to f$ in $\mathcal{H}^{s-1}$. The traces $f_k^n = \partial_t^k f^n(0)$ converge to $f_k = \partial_t^k f(0)$ in $H^{s-1-2k}(\mathbb{R}^d_+)$ for $k \le (s-1)/2$. Consider next a sequence $u_0^n \in H^\infty$ such that $u_0^n \to u_0$ in $H^s(\mathbb{R}^d_+)$. Thus, the u_j^n associated to u_0^n and f^n by (3.2.22) satisfy for $j \le s/2$:

$$u_j^n \in H^{s+2-2j}(\mathbb{R}^n_+), \quad u_j^n \to u_j \ \text{ in } H^{s-2j}$$

The compatibility conditions mean that $u_{j|x=0} = 0$ when $j < s/2$. The convergence above implies that

$$\text{(3.2.24)} \qquad \begin{cases} h_j^n := u^n_{j\,|x=0} \in H^\infty(\mathbb{R}^{d-1}) & \text{when } j \le s/2\,, \\ h_j^n \to 0 \ \text{ in } H^{s-2j-1/2}(\mathbb{R}^{d-1}) & \text{when } j < s/2\,. \end{cases}$$

[1] H^∞ or $\mathcal{H}^\infty$ denotes the intersection $\cap H^s$, $\cap \mathcal{H}^s$ of all spaces H^s or $\mathcal{H}^s$.

To prove the lemma, we look for modified initial data $u_0^n - v_0^n$. By (3.2.22), this modifies the traces u_j^n into $u_j^n - v_j^n$ with $v_j^n = (G-\gamma)^j v_0^n$. Therefore, it is sufficient to construct a sequence v_0^n such that

$$\left\{ \begin{array}{l} v_0^n \in H^\infty(\mathbb{R}^d_+)\,, \quad v_0^n \to 0 \ \text{ in } H^s\,, \\ \big\{(G-\gamma)^j v_0^n\big\}_{|x=0} = h_j^n \ \text{ for } j \le s/2\,. \end{array} \right. \tag{3.2.25}$$

We see that the equations on $\{x = 0\}$ determine by induction $\partial_x^{2j} v^n_{0\,|x=0}$ knowing $\partial_x^k v^n_{0\,|x=0}$ for $k < 2j$. We can choose arbitrarily the odd traces, for instance to be zero, and (3.2.25) is implied by

$$\left\{ \begin{array}{l} v_0^n \in H^\infty(\mathbb{R}^d_+)\,, \quad v_0^n \to 0 \ \text{ in } H^s\,, \\ \partial_x^{2j} v^n_{0\,|x=0} = \tilde h_j^n \quad \text{for } 2j \le s\,, \\ \partial_x^{2k+1} v^n_{0\,|x=0} = 0 \quad \text{for } 2k+1 < s\,, \end{array} \right. \tag{3.2.26}$$

where the $\tilde h_j^n$ are determined from the h_j^n and satisfy (3.2.24).

When s is odd, there are no $\tilde h_j$ with $2j = s$ and all the $\tilde h_j$ tend to zero in the appropriate space. In this case, the problem (3.2.26) is solved lifting the traces by standard operators.

When s is even, the term $\tilde h_j^n$ for $j = s/2$ is not controlled but we want to prove the convergence of v_0^n in H^s where the trace $v \mapsto \partial_x^s v_{|x=0}$ is not defined. Using classical Poisson operators to lift up the $2s-1$ first traces, one is reduced to solve (3.2.26) with $\tilde h_j = 0$ for $j < s/2$. To lift up the last trace, keeping the first $s-1$ equal to zero, we use a modified Poisson operator as in the proof of Proposition 2.4.9 of Chapter 2, see (2.4.11) $\square$

Proof of Theorem 3.2.13. Consider $u_0 \in H^s(\mathbb{R}^d_+)$ and $f \in \mathcal{H}^{s-1}$ satisfying the compatibility conditions up to order $s-1$, with $s \ge 1$. Introduce sequences $u_0^n \in H^{2s+1}$ and $f^n \in \mathcal{H}^{2s}$ as indicated in Lemma 3.2.16. We show that the equation

$$(\partial_t - G + \gamma) u^n = f^n\,, \quad u^n{}_{|t=0} = u_0^n \tag{3.2.27}$$

has a solution $u^n \in \mathcal{H}^{s+1} \cap \mathcal{H}^1_0$. By Theorem 3.2.8, applying (3.2.14) to $u^n - u^{n'}$, we see that u^n is a Cauchy sequence in $\mathcal{H}^{s+1} \cap \mathcal{H}^1_0$ and therefore converge to $u \in \mathcal{H}^{s+1} \cap \mathcal{H}^1_0$ which is solution of (3.2.21). By uniqueness in $\mathcal{H}^1_0$, this is the solution given by Theorem 3.2.5.

Thus it only remains to solve (3.2.27) in $\mathcal{H}^{s+1} \cap \mathcal{H}^1_0$. For $j \le s/2$, define the $u_j^n \in H^{2s+1-2j} \subset H^{s+1}$ by (3.2.22). The compatibility conditions to order s imply that $u_j^n \in H_0^1$ for $j \leq s/2$. Introduce

$$u^{n,a}(t,x) = \sum_{j \leq 2s} \frac{t^j}{j!} u_j^n(x) \in \mathcal{H}^{s+1} \cap \mathcal{H}_0^1 .$$

The traces of $f^{n,a} = (\partial_t - G + \gamma)u^a \in \mathcal{H}^{s-1}$ satisfy

$$\partial_t^k f^{n,a}{}_{|t=0} = u_{k+1}^{n,a} - (G-\gamma)u_k^n = f_k^n$$

when $k+1 \leq s/2$. Thus, by Proposition 3.2.15 the problem

$$(\partial_t - G + \gamma)v^n = f^n - f^{n,a}, \quad v^n{}_{|t=0} = 0 ,$$

has a solution $v^n \in \mathcal{H}^{s+1} \cap \mathcal{H}_0^1$. Thus $u^n = u^{n,a} + v^n \in \mathcal{H}^{s+1} \cap \mathcal{H}_0^1$ is a solution of (3.2.27), and the proof of the theorem is complete. □

3.3 Uniform estimates

The estimates given in the previous section depend strongly on the ellipticity constant c of (3.2.1), thus on the viscosity ε when one considers (3.1.2). In this section, we give the precise dependence of the constants with respect to ε. From now on, we suppose that Assumption 3.1.1 is satisfied.

3.3.1 Long time estimates

We start with giving estimates independent of time T for the solutions of (3.2.2) with $G = \sum A_j \partial_j - \sum B_{j,k} \partial_j \partial_k$, i.e. when $\varepsilon = 1$.

We denote by $\mathcal{H}^s([0,T])$ the spaces $\mathcal{H}^s$ of the previous section on $[0,T] \times \mathbb{R}_+^d$. For $s > 0$, we introduce the following notations for $u \in \mathcal{H}^{2s}([0,T])$,

$$\begin{aligned} \mathrm{n}_s(u;T) &= \sum_{2j+|\alpha| \leq 2s-1} \|\partial_t^j \partial_{y,x}^\alpha u(T)\|_{L^2(\mathbb{R}_+^d)} \\ \mathrm{N}'_s(u;T) &= \sum_{0<2j+|\alpha| \leq 2s} \|\partial_t^j \partial_{y,x}^\alpha u\|_{L^2([0,T]\times\mathbb{R}_+^d)} . \end{aligned} \tag{3.3.1}$$

For the source terms $f \in \mathcal{H}^{2s}([0,T])$, we use the norms

$$\begin{aligned} \mathrm{N}_s(f;T) &= \sum_{0 \leq 2j+|\alpha| \leq 2s} \|\partial_t^j \partial_{y,x}^\alpha f\|_{L^2([0,T]\times\mathbb{R}_+^d)} . \\ \mathrm{M}_s(f;T) &= \sum_{0 \leq 2j+|\alpha| \leq 2s} \|\partial_t^j \partial_{y,x}^\alpha f\|_{L^1([0,T];L^2(\mathbb{R}_+^d)} . \end{aligned} \tag{3.3.2}$$

The slight difference between N and N' is that the case $j = |\alpha| = 0$ is allowed in the former case.

Theorem 3.3.1. *For all $s \geq 0$, there is a constant C such that for all $T \geq 0$, all $f \in \mathcal{H}^{2s}([0,T])$ and $u_0 \in H^{2s+1}(\mathbb{R}^d_+)$ satisfying the compatibility conditions up to order $2s$, the solution $u \in \mathcal{H}^{2s+2}$ of (3.2.2) satisfies for all $t \in [0,T]$:*

$$\mathrm{n}_{s+1}(u;t) + \mathrm{N}'_{s+1}(u;t) \leq C\big(\mathrm{n}_{s+1}(u;0) + \mathrm{N}_s(f;t) + \mathrm{M}_s(f;t)\big) . \tag{3.3.3}$$

Remark 1. Integrating $\partial_t |\partial_{y,x} v|^2$, and using Plancherel's theorem, one obtains the estimate

$$\|\partial_{y,x} v(t)\|^2_{L^2} \leq \|\partial_{y,x} v(0)\|^2_{L^2} + 2\|\partial_t v\|_{L^2([0,t];L^2)} \|\partial^2_{y,x} v\|_{L^2[0,t];L^2)} , \tag{3.3.4}$$

for $v \in C_0^\infty([0,t] \times \mathbb{R}^d)$. Using an extension theorem, one proves that this estimate also holds on $[0,t] \times \mathbb{R}^d_+$ and for functions in $\mathcal{H}^2$. Easier is the estimate

$$\|v(t)\|^2_{L^2} \leq \|v(0)\|^2_{L^2} + 2\|\partial_t v\|_{L^2([0,t];L^2)} \|v\|_{L^2[0,t];L^2)} , \tag{3.3.5}$$

Suppose that $u \in \mathcal{H}^{2s+2}$ with $s \geq 0$. Consider j and α such that $2j + |\alpha| \leq 2s+1$. If $\alpha \neq 0$, we can write $\partial^\alpha = \partial^\beta \partial_k$ for some k. Consider $v = \partial_t^j \partial_{y,x}^\beta u$. The L^2 norms of $\partial_t v$ and $\partial^2_{y,x} v$ appear in $N'_{s+1}(u)$. Thus we can use (3.3.4) to estimate the L^2 norm of $D_k v(t) = \partial_t^j \partial_{y,x}^\alpha u(t)$. If $\alpha = 0$, then $j \leq s$. If $j > 0$, the L^2 norms of $v = \partial_t^j u$ and $\partial_t v$ appear in N'_{s+1}. In this case we apply (3.3.5). Adding up, we have the uniform estimate

$$\mathrm{n}'_{s+1}(u;t) \leq C\big(\mathrm{n}'_s(u;0) + \mathrm{N}'_{s+1}(u;t)\big) . \tag{3.3.6}$$

where n'_s is the sum of terms in n_s, except the first one $\|u(t)\|_{L^2}$. Therefore, it is sufficient to prove (3.3.3) with $\mathrm{n}_s(u;t)$ replaced by $\|u(t)\|_{L^2}$ in the left hand side.

However, as it is stated, the estimate (3.3.3) has the nice feature to give the same norm n_{s+1} at time t and at time zero.

One can also eliminate this last term, using an $L^1([0,t];L^2)$ norm of $\partial_t u$. What we have in $N'_{s+1}(u;t)$ is an $L^2(L^2)$ norm of $\partial_t u$. For a fixed T, the $L^1(L^2)$ norm is controlled by the $L^2(L^2)$ norm, but the control is not uniform in T.

Remark 2. The right hand side can be made explicit in terms of the data u_0 and f. When $s = 0$, this is immediate:

$$\mathrm{n}_1(u;0) = \|u_0\|_{H^1(\mathbb{R}^d_+)} . \tag{3.3.7}$$

When $s \geq 1$, using the equation, the traces $\partial_t^j u(0)$ are given by (3.2.17), and therefore

$$\mathrm{n}_{s+1}(u;0) \leq C\big(\|u_0\|_{H^{2s+1}(\mathbb{R}^d_+)} + \mathrm{n}_s(f;0)\big) . \tag{3.3.8}$$

Proof of Theorem 3.3.1. **a)** *The case* $s = 0$.

Since the matrices SA_j are symmetric, for $u \in \mathcal{H}^1_0$ there holds

$$\mathrm{Re}\,(SA_j \partial_j u, u)_{L^2} = 0\,.$$

By (3.2.6) and Proposition 3.2.4, we deduce that

$$\|u(t)\|^2_{L^2} + \|\partial_{y,x} u\|^2_{L^2([0,t]\times\mathbb{R}^d_+)} \lesssim \|u_0\|^2_{L^2} + \int_0^t \|f(t')\|_{L^2} \|u(t')\|_{L^2} dt'\,.$$

where $a \lesssim b$ means that there is a constant C, independent of T, such that $a \le Cb$. As in the proof of Proposition 2.4.1 this implies

$$\begin{aligned} &\|u(t)\|_{L^2(\mathbb{R}^d_+)} + \|\partial_{y,x} u\|_{L^2([0,t]\times\mathbb{R}^d_+)} \\ &\qquad\qquad \lesssim \|u_0\|_{L^2} + \int_0^t \|f(t')\|_{L^2(\mathbb{R}^d_+)} dt'\,. \end{aligned} \tag{3.3.9}$$

When $u_0 \in H^1_0$, that is when the first compatibility condition is satisfied, one can also estimate the second derivatives as in the proof of Lemma 3.2.9. Differentiating the equation in y, multiplying by $S\partial_y u$ and integrating over $[0,t] \times \mathbb{R}^d_+$ yields

$$\|\partial_y u(t)\|^2_{L^2} + \|\partial_{y,x}\partial_y u\|^2_{L^2([0,t]\times\mathbb{R}^d_+)} \lesssim \|\partial_y u_0\|^2_{L^2} + \int_0^t \|f(t')\|_{L^2} \|\partial^2_y u(t')\|_{L^2} dt'\,.$$

With Cauchy-Schwarz inequality, this implies

$$\|\partial_y u(t)\|^2_{L^2} + \|\partial_{y,x}\partial_y u\|^2_{L^2([0,t]\times\mathbb{R}^d_+)} \lesssim \|\partial_y u_0\|^2_{L^2} + \|f\|^2_{L^2([0,t]\times\mathbb{R}^d_+)}\,,$$

thus

$$\|\partial_y u(t)\|_{L^2} + \|\partial_{y,x}\partial_y u\|_{L^2([0,t]\times\mathbb{R}^d_+)} \lesssim \|\partial_y u_0\|_{L^2} + \|f\|_{L^2([0,t]\times\mathbb{R}^d_+)}\,. \tag{3.3.10}$$

Next, we multiply the equation by $S\partial^2_x u$ and integrate over $[0,t] \times \mathbb{R}^d_+$. We get

$$\|\partial_x u(t)\|^2_{L^2} + \|\partial^2_x u\|^2_{L^2([0,t]\times\mathbb{R}^d_+)} \lesssim \|\partial_x u_0\|^2_{L^2} + \int_0^t \|g\|_{L^2} \|\partial^2_x u(t')\|_{L^2} dt'\,.$$

thus

$$\|\partial_x u(t)\|_{L^2} + \|\partial^2_x u\|_{L^2([0,t]\times\mathbb{R}^d_+)} \lesssim \|\partial_x u_0\|_{L^2} + \|g\|_{L^2([0,t]\times\mathbb{R}^d_+)}\,, \tag{3.3.11}$$

where $g = f - \sum A_j \partial_j u + \varepsilon \sum B_{j,k} \partial^2_{j,k} u$ with $(j,k) \neq (d,d)$ in the last sum. The terms in $\partial_{y,x}\partial_y$ are controlled in L^2 by the previous estimate (3.3.10). The terms in $\partial_{y,x} u$ are controlled by (3.3.9). Hence $\|g\|_{L^2}$ is dominated by the sum of the right hand sides of (3.3.9) and (3.3.10). Adding up, we have proved that

$$\begin{aligned}\|u(t)\|_{L^2} + &\|\partial_{y,x} u(t)\|_{L^2} + \|\partial_{y,x} u\|_{L^2([0,t];L^2)} + \|\partial^2_{y,x} u\|_{L^2([0,t];L^2)} \\ &\lesssim \|u_0\|_{L^2} + \|\partial_{y,x} u_0\|_{L^2} + \|f\|_{L^1([0,t];L^2)} + \|f\|_{L^2([0,t];L^2)} \,.\end{aligned}$$

Since $\partial_t u = f + Gu$, we can add $\|\partial_t u\|_{L^2(L^2)}$ in the left hand side. This yields (3.3.3) for $s = 0$.

b) *The general case.*

We proceed by induction as in the proof of Theorem 3.2.8. Given $f \in \mathcal{H}^{2s}$ and $u_0 \in H^{2s+1}$, compatible to order $2s$, we know that there is a unique solution $u \in \mathcal{H}^{2s+2}$.

We consider $s \geq 1$ and assume that the estimate is proved at the order $s-1$. Thus we have L^2 bounds for $u^{(j,\alpha)} := \partial_t^j \partial^\alpha_{y,x} u$ for $2j + |\alpha| \leq 2s$. Thanks to (3.3.6), we only need bounds for the $u^{(j,\alpha)}$ when $2s+1 \leq 2j+|\alpha| \leq 2s+2$.

Differentiating the equations with respect to ∂_t, ∂_y and ∂_y^2, the induction hypothesis implies that the L^2 norm of the derivatives $\partial_t^j \partial_y^\beta \partial_x^k u$ is bounded by the right hand side of (3.3.3) for $2j + |\beta| + k \leq 2s + 2$ except for $j = 0$ and $k = 2s + 1$ or $k = 2s + 2$. Because $s \geq 1$, we can use the equation to express these derivatives as linear combinations of $\partial_x^{2s-1} f$, $\partial_y \partial_x^{2s-1} f$, $\partial_x^{2s} f$ and derivatives $\partial_t^j \partial_y^\beta \partial_x^k u$ with $j \in \{0,1\}$, $|\beta| \leq 3$, $k \leq 2s$ and $2j + |\beta| + k \leq 2s + 2$, which are already estimated. $\square$

3.3.2 Small viscosity estimates

We now consider the equations with small viscosity ε:

$$\tag{3.3.12} \begin{cases} Lu - \varepsilon P u = f & \text{on } [0,T] \times \mathbb{R}^d_+ \\ u_{|x=0} = 0 & \text{on } [0,T] \times \mathbb{R}^{d-1} \\ u_{|t=0} = u_0 & \text{on } \mathbb{R}^d_+ \end{cases}$$

For fixed ε, the existence and uniqueness of solutions follows from the analysis of section 2. In this section we discuss *uniform estimates* for the solutions.

Because of the boundary layers, the right hand side (and the solutions) are not uniformly smooth in the normal variable. For instance, we have seen in Chapter one, that solutions of the form $u_0(t,x) + e^{-ax/\varepsilon}\alpha(t)$ must be expected. For these solutions, the normal derivatives satisfy for $k > 0$:

$$\|\partial_x^k u\|_{L^2} \approx \varepsilon^{-k+1/2}\,. \tag{3.3.13}$$

The solutions of (3.3.12) are related to the solutions of (3.2.2): u is a solution of (3.3.12) on $[0,T]\times\mathbb{R}^d_+$, if and only if

$$\tilde u(\tilde t,\tilde y,\tilde x) = u(\varepsilon\tilde t,\varepsilon\tilde y,\varepsilon\tilde x)$$

is a solution of (3.2.2) on $[0,T/\varepsilon]\times\mathbb{R}^d_+$, with initial data $\tilde u_0$ and source term $\tilde f$ given by:

$$\tilde u_0(\tilde y,\tilde x) = u_0(\varepsilon\tilde t,\varepsilon\tilde y,\varepsilon\tilde x)\,,\quad \tilde f(\tilde y,\tilde x) = \varepsilon f(\varepsilon\tilde t,\varepsilon\tilde y,\varepsilon\tilde x)\,.$$

Thus the estimates for $\tilde u$ are immediately transposed for u. Up to a factor $\varepsilon^{-d/2}$, the norms (3.3.1) and (3.3.2) for $\tilde u$ and $\tilde f$ are equal to:

$$\begin{aligned}
\mathrm{n}_{s,\varepsilon}(u;T) &= \sum_{2j+|\alpha|\le 2s-1} \varepsilon^{j+|\alpha|}\|\partial_t^j\partial_{y,x}^\alpha u(T)\|_{L^2(\mathbb{R}^d_+)}\\
\mathrm{N}'_{s,\varepsilon}(u;T) &= \sum_{0<2j+|\alpha|\le 2s} \varepsilon^{j+|\alpha|-1/2}\|\partial_t^j\partial_{y,x}^\alpha u\|_{L^2([0,T]\times\mathbb{R}^d_+)}\,,\\
\mathrm{N}_{s,\varepsilon}(f;T) &= \sum_{0\le 2j+|\alpha|\le 2s} \varepsilon^{j+|\alpha|+1/2}\|\partial_t^j\partial_{y,x}^\alpha f\|_{L^2([0,T]\times\mathbb{R}^d_+)}\,,\\
\mathrm{M}_{s,\varepsilon}(f;T) &= \sum_{0\le 2j+|\alpha|\le 2s} \varepsilon^{j+|\alpha|}\|\partial_t^j\partial_{y,x}^\alpha f\|_{L^1([0,T];L^2(\mathbb{R}^d_+)}\,.
\end{aligned} \tag{3.3.14}$$

Theorem 3.3.1, immediately implies:

Theorem 3.3.2. *For all $s \ge 0$, there is a constant C such that for all $T \ge 0$, all $f \in \mathcal{H}^{2s}([0,T])$ and $u_0 \in H^{2s+1}(\mathbb{R}^d_+)$ satisfying the compatibility conditions up to order $2s$ for the problem* (3.3.12)*, the solution $u \in \mathcal{H}^{2s+2}$ of* (3.3.12) *satisfies for all $t\in[0,T]$:*

$$\mathrm{n}_{s+1,\varepsilon}(u;t)+\mathrm{N}'_{s+1,\varepsilon}(u;t) \le C\big(\mathrm{n}_{s+1,\varepsilon}(u;0)+\mathrm{N}_{s,\varepsilon}(f;t)+\mathrm{M}_{s,\varepsilon}(f;t)\big)\,. \tag{3.3.15}$$

For instance, for $s = 0$, $f \in L^2([0,T]\times\mathbb{R}^d_+)$ and $u_0 \in H^1_0(\mathbb{R}^d_+)$, the solution $u \in \mathcal{H}^2$ satisfies

$$\begin{aligned}
\|u(t)\|_{L^2} + \varepsilon\|\partial_{y,x}u(t)\|_{L^2} &+ \sqrt{\varepsilon}\|\partial_{t,y,x}u\|_{L^2([0,t],L^2)}\\
+\varepsilon^{3/2}\|\partial^2_{y,x}u\|_{L^2([0,t],L^2)} &\lesssim \|u_0\|_{L^2} + \varepsilon\|\partial_{y,x}u_0\|_{L^2}\\
&\quad+\|f\|_{L^1([0,t],L^2)} + \sqrt{\varepsilon}\|f\|_{L^2([0,t],L^2)}\,.
\end{aligned} \tag{3.3.16}$$

By (3.3.13), the factors $\varepsilon^{1/2}$ and $\varepsilon^{3/2}$ in front of the first and second normal derivatives are optimal.

3.4 Tangential and conormal estimates.

The estimates (3.3.15) are not suitable for nonlinear equations, since they do not provide uniform L^∞ estimates. Moreover, the boundary layer solutions are expected to be smooth in the tangential variables. This leads to introduce norms measuring the tangential smoothness.

3.4.1 Tangential regularity

For functions u on $\mathbb{R}^d_+$ define

$$\|u\|_{H^s_{tg}} = \sum_{|\alpha|\le s} \|\partial_y^\alpha u\|_{L^2(\mathbb{R}^d_+)} . \tag{3.4.1}$$

For functions on $[0,T]\times\mathbb{R}^d_+$, introduce

$$\|u\|_{H^s_{tg}([0,T])} = \sum_{j+|\alpha|\le s} \|\partial_t^j\partial_y^\alpha u\|_{L^2([0,T]\times\mathbb{R}^d_+)} = \sum_{j=0}^{s} \|\partial_t^j u\|_{H^{s-j}_{tg}} . \tag{3.4.2}$$

The importance of these spaces is apparent in the next lemma:

Lemma 3.4.1. *For $s>\frac{d}{2}$, there is C such that for all $u\in H^s_{tg}$ such that $\partial_x u\in H^{s-1}_{tg}$:*

$$\|u\|_{L^\infty(\mathbb{R}^d_+)} \le \|u\|^{1/2}_{H^s_{tg}} \|\partial_x u\|^{1/2}_{H^{s-1}_{tg}} . \tag{3.4.3}$$

Proof. Extend u for negative x as an even function. The extended function belongs to the H^s_{tg} space on $\mathbb{R}^d$ with derivative in H^{s-1}_{tg}.

Denoting by and $\hat u(\eta,\xi)$ the Fourier transform of $u(y,x)$, the L^∞ estimate on $\mathbb{R}\times\mathbb{R}^d$ follows from the Hölder's inequality:

$$\|\hat u\|^2_{L^1} \le \|\mu\|_{L^2} \|\lambda^s\hat u\|_{L^2} \|\lambda^{s-1}(1+|\xi|)\hat u\|_{L^2} .$$

with $\lambda=(1+|\eta|)$, $\mu=(1+|\eta|)^{-s}(1+|\xi|)^{-1/2}$.

□

There are easy a priori estimate in H^s_{tg} spaces. Introduce the notation

$$\mathrm{n}_{tg,s,\varepsilon}(u;t) = \sum_{|\alpha|\le s} \|\partial_{t,y}^\alpha u(t)\|_{L^2(\mathbb{R}^d_+)} + \varepsilon\|\partial_{t,y}^\alpha\partial_{y,x} u(t)\|_{L^2(\mathbb{R}^d_+)} . \tag{3.4.4}$$

For simplicity, we fix $T_0>0$ and restrict attention to $T\le T_0$.

Proposition 3.4.2. *Given s and $T_0 > 0$, there is C such that for all $T \in [0, T_0]$, all $\varepsilon \in]0, 1]$ and all $u \in C_0^\infty([0, T] \times \overline{\mathbb{R}}^d_+)$ with $u_{|x=0} = 0$, all $t \in [0, T]$*

$$
\begin{aligned}
\mathrm{n}_{tg,s,\varepsilon}(u, t) + \varepsilon^{1/2} \|\partial_{t,y,x} u\|_{H^s_{tg}([0,t])} + \varepsilon^{3/2} \|\partial^2_{y,x} u\|_{H^s_{tg}([0,t])} \\
\leq C \Big(\mathrm{n}_{tg,s,\varepsilon}(u; 0) + \|f\|_{H^s_{tg}([0,t])} \Big) .
\end{aligned} \tag{3.4.5}
$$

with $f = (L - \varepsilon P)u$.

Moreover, if $s \geq \frac{d}{2} + 1$, then

$$
\begin{aligned}
\|u\|_{L^\infty([0,T] \times \mathbb{R}^d_+)} \leq C \Big(\|u(0)\|_{L^\infty(\mathbb{R}^d_+)} \\
+ \mathrm{n}_{tg,s,\varepsilon}(u; 0) + \|f\|_{H^s_{tg}([0,t])} \Big) .
\end{aligned} \tag{3.4.6}
$$

Proof. The estimate for $s = 0$ follows immediately from (3.3.16), estimating the $L^1([0, t]; L^2)$ norm of f by its $L^2([0, t]; L^2)$ norm for $t \leq T \leq T_0$.

Differentiating the equation with respect to (t, y) immediately implies the estimate (3.4.5) for the tangential derivatives.

Consider $v = \varepsilon \partial_x u$. Then,

$$
\|v\|_{H^s_{tg}([0,T])} \leq \varepsilon^{1/2} R \quad \text{and} \quad \|\partial_x v\|_{H^s_{tg}([0,T])} \leq \varepsilon^{-1/2} R
$$

where R is the right hand side of (3.4.5). Thus $a_1(t) := \|v(t)\|_{H^s_{tg}(\mathbb{R}^d_+)}$ and $a_2(t) := \|\partial_x v\|_{H^s_{tg}(\mathbb{R}^d_+)}$ satisfy

$$
\varepsilon^{-1/2} \|a_1\|_{L^2([0,T])} + \varepsilon^{1/2} \|a_2\|_{L^2([0,T])} \leq R
$$

When $s > d/2$, we deduce from Lemma 3.4.1:

$$
a(t) := \|v(t)\|_{L^\infty(\mathbb{R}^d_+)} \leq \big(a_1(t) a_2(t) \big)^{1/2} .
$$

Hence, $\|a\|_{L^2([0,T])} \lesssim R$.

The equation for u reads:

$$
-\varepsilon \partial_x^2 u + A^\flat \partial_x u = g
$$

where $A^\flat = B^{-1}_{d,d} A_d$ and g is a linear combination of f, $\partial_{t,y} u$, $\varepsilon \partial_y^2 u$ and $\varepsilon \partial_x \partial_y u$. Thus,

$$
\|g\|_{H^{s-1}_{tg}([0,T])} \lesssim \|f\|_{H^{s-1}_{tg}([0,T])} + \|u\|_{H^s_{tg}([0,T])} + \varepsilon \|\partial_{y,x} u\|_{H^s_{tg}([0,T])} \lesssim R .
$$

Consider $w = A^\flat u - \varepsilon\partial_x u$. Then

$$\|w\|_{H^s_{tg}([0,T])} \leq R\,.$$

Since $\partial_x w = g$, Lemma 3.4.1 applied for fixed t as above, implies that $b(t) := \|w(t)\|_{L^\infty(\mathbb{R}^d_+)}$ satisfies $\|b\|_{L^2([0,T])} \lesssim R$. Since $u = (A^\flat)^{-1}(w+v)$, we have proved that for $s > d/2$, $c(t) := \|u(t)\|_{L^\infty(\mathbb{R}^d_+)}$ satisfies $\|c\|_{L^2([0,T])} \lesssim R$.

When $s > 1 + d/2$, we can apply the same analysis to $\partial_t u$ and conclude that $c_1(t) := \|\partial_t u(t)\|_{L^\infty(\mathbb{R}^d_+)}$ satisfies $\|c_1\|_{L^2([0,T]} \lesssim R$. Since

$$\|u\|_{L^\infty([0,T]\times\mathbb{R}^d_+)} \leq \|u(0)\|_{L^\infty(\mathbb{R}^d_+)} + \int_0^T \|\partial_t u(t)\|_{L^\infty(\mathbb{R}^d_+)} dt,$$

the estimate (3.4.6) follows. $\square$

There are difficulties to convert the estimates above into existence theorems in the corresponding spaces. This come from the fact that the initial value $\mathrm{n}_{tg,s}(u,0)$ cannot be expressed in terms of the data when they only belong to tangential spaces. Indeed, by (3.2.17) $\mathrm{n}_{tg,s}(u,0)$ involves high order ∂_x derivatives of u_0 and of the traces $\partial_t^j f(0)$. However, there is an easy case: when $u_0 = 0$ and the traces $\partial_t^j f(0)$ vanish.

Theorem 3.4.3. *Assume that $f \in H^s_{tg}([0,T])$ and*

$$\partial_t^j f_{|t=0} = 0\,, \qquad \text{for } j \leq s-1\,. \tag{3.4.7}$$

The solution $u \in \mathcal{H}^2$ of (3.3.12) *with initial data $u_0 = 0$ belongs to $H^s_{tg}([0,T])$ as well as $\partial_{t,y,x}u$ and $\partial^2_{y,x}u$. Moreover, u satisfies the estimates* (3.4.5) *and, when $s > 1 + d/2$,* (3.4.6).

Proof. Fix $\varepsilon > 0$. Thanks to (3.4.7), we can approximate f in $H^s_{tg}([0,T])$ by functions $f^n \in C^\infty_0(]0,T] \times \mathbb{R}^d_+)$. In this case, the f^n and $u_0 = 0$ are compatible at any order and the unique solution of (3.3.12) given by Theorem 3.2.5 is infinitely smooth. The estimates (3.4.5) show that the u^n form a Cauchy sequence in $H^s_{tg}([0,T])$ as well as $\partial_{t,y,x}u^n$ and $\partial^2_{y,x}u^n$, thus in $\mathcal{H}^2$. Therefore the limit u which is the unique solution of (3.3.12) in $\mathcal{H}^2$ has the tangential regularity as claimed. $\square$

3.4.2 Conormal regularity

The analysis above is simple but its application to variable coefficients or curved boundary leads to difficulties because it relies on good commutation properties of the fields $\partial_t, \partial_{y_j}$ with the equation. Note that this set of fields is *not* preserved by all the changes of variables which leave the boundary invariant. The boundary layer solutions we construct in Chapter four (see also Chapter one) are smooth functions of x for $x > 0$, with singularity only on $\{x = 0\}$. Typically, we expect solutions of the form

$$u^\varepsilon(t, y, x) = U(t, y, x, x/\varepsilon)$$

with $U(t, y, x, z)$ converging at an exponential rate when z tends to $+\infty$. So one can expect u^ε be tangentially smooth. But, the form above shows that one can also expect to apply vector fields $x\partial_x$ tu u^ε. In particular, away from the boundary, this allows to recover the usual isotropic Sobolev smoothness. To take care of this additional regularity, one introduces an extra vector field which is tangent to the boundary but independent of the $\partial_t, \partial_{y_j}$ when $x > 0$:

$$X = \frac{x}{1+x}\partial_x \, . \tag{3.4.8}$$

The choice of the function in front of ∂_x is far from unique: we ask it to be positive, to converge to a positive constant at infinity, and to be equivalent to x for small x. X behaves like ∂_x for $x \geq 1$ and like $x\partial_x$ for $x \leq 1$. We also introduce the notations

$$Z_0 = \partial_t \, , \quad Z_j = \partial_{y_j} \quad \text{for } j \in \{1, \ldots, d-1\} \, , \quad Z_d = X \, . \tag{3.4.9}$$

For a multi-index $\alpha = (\alpha_0, \ldots, \alpha_d) \in \mathbb{N}^{d+1}$, $Z^\alpha = Z_0^{\alpha_0} \cdots Z_d^{\alpha_d}$.

Definition 3.4.4. *For $s \in \mathbb{N}$, and $T \geq 0$, $H^s_{co}([0,T])$ denotes the space of functions $u \in L^2([0,T] \times \mathbb{R}^d_+)$ such that $Z^\alpha u \in L^2([0,T] \times \mathbb{R}^d_+)$ for all $\alpha \in \mathbb{N}^{d+1}$ with $|\alpha| \leq s$.*

As expected, for all $\delta > 0$, the function in H^s_{co} are in H^s on $\{x > \delta\}$. Moreover, this space is invariant by any change of variables which preserves the boundary.

The new difficulty is that X does not commute exactly to the equation. We use the following facts:

$$\partial_x X = X' \partial_x \, , \quad \text{with} \quad X' = X + \frac{1}{(1+x)^2} \, , \tag{3.4.10}$$

$$(X')^k = X^k + \sum_{l=0}^{k-1} c_{k,l} X^l \, , \tag{3.4.11}$$

where the $c_{k,l}(x)$ are bounded C^∞ functions with bounded derivatives. In commuting X with the operator $L - \varepsilon P$, the difficult term is $[X, \partial_x]$. This is why we compute $X'(L - \varepsilon P) - (L - \varepsilon P)X$. There holds

$$
(3.4.12) \qquad \begin{aligned} &(X')^k(L - \varepsilon P) - (L-\varepsilon P)X^k = \\ &\qquad \sum_{l=0}^{k-1} \Big(c\partial_{t,y} X^l + c\varepsilon\partial_y^2 X^l + \varepsilon\partial_x(cX^l\partial_x)\Big) \end{aligned}
$$

where c denotes various functions of x, bounded as well as all their derivatives.

Parallel to Proposition 3.4.2, there are a priori estimates in the conormal spaces. Introduce

$$
(3.4.13) \qquad \mathrm{n}_{co,s,\varepsilon}(u;t) = \sum_{|\alpha|\le s} \|Z^\alpha u(t)\|_{L^2(\mathbb{R}^d_+)} + \varepsilon\|\partial_{y,x} Z^\alpha u(t)\|_{L^2(\mathbb{R}^d_+)} \, ,
$$

Proposition 3.4.5. *Given s and $T_0 > 0$, there is C such that for all $T \in [0, T_0]$, all $\varepsilon \in]0, 1]$ and all $u \in C_0^\infty([0, T] \times \overline{\mathbb{R}}^d_+)$ with $u_{|x=0} = 0$, all $t \in [0, T]$*

$$
(3.4.14) \qquad \begin{aligned} &\mathrm{n}_{co,s,\varepsilon}(u, t) + \varepsilon^{1/2}\|\partial_{t,y,x} u\|_{H^s_{co}([0,t])} + \varepsilon^{3/2}\|\partial^2_{y,x} u\|_{H^s_{co}([0,t])} \\ &\qquad\qquad \le C\Big(\mathrm{n}_{co,s,\varepsilon}(u; 0) + \|f\|_{H^s_{co}(t)}\Big) \, . \end{aligned}
$$

with $f = (L - \varepsilon P)u$.

We need a modification of the basic L^2-estimate (3.3.16).

Proposition 3.4.6. *There is C such that for all $T \in [0, T_0]$, all $\varepsilon \in]0, 1]$ and all $u \in C_0^\infty([0, T] \times \overline{\mathbb{R}}^d_+)$ such that $u_{|x=0} = 0$ and*

$$
(3.4.15) \qquad (L - \varepsilon P)u = f + \sum_{j=0}^{d} \varepsilon\partial_j g_j \, ,
$$

the following energy estimate is satisfied

$$
(3.4.16) \qquad \begin{aligned} &\|u(t)\|_{L^2} + \varepsilon\|\partial_{y,x} u(t)\|_{L^2} + \sqrt{\varepsilon}\|\partial_{t,y,x} u\|_{L^2([0,t],L^2)} \\ &\quad + \varepsilon^{3/2}\|\partial^2_{y,x} u\|_{L^2([0,t],L^2)} \le C\Big(\|u_0\|_{L^2} + \varepsilon\|\partial_{y,x} u_0\|_{L^2} + \\ &\quad \|f\|_{L^2([0,t],L^2)} + \sum_j \sqrt{\varepsilon}\|g_j\|_{L^2([0,t],L^2)} + \varepsilon^{3/2}\|\partial_j g_j\|_{L^2([0,t],L^2)}\Big) \, . \end{aligned}
$$

Proof. First, we multiply the equation by Su. We estimate $(f, Su)_{L^2}$ as before by the L^2 norm of f and the $L^\infty(L^2)$ norm of u. Since u vanishes on the boundary,

$$\varepsilon(\partial_j g_j, Su)_{L^2} = -\varepsilon(g_j, \partial_j Su)_{L^2} = O\Big(\|\sqrt{\varepsilon} g_j\|_{L^2} \|\sqrt{\varepsilon}\partial_j u\|_{L^2}\Big).$$

This implies

$$\|u(t)\|_{L^2} + \sqrt{\varepsilon}\|\partial_{y,x} u\|_{L^2} \lesssim \|u_0\|_{L^2} + \|f\|_{L^2} + \sum_j \sqrt{\varepsilon}\|g_j\|_{L^2}.$$

Next, we differentiate the equation in ∂_{y_j} and multiply by $\varepsilon^2 S\partial_{y_j} u$. Since u vanishes on the boundary,

$$\big(\partial_{y_j}(f + \varepsilon\partial_k g_k), S\partial_{y_j} u\big)_{L^2} = -\big(f + \varepsilon\partial_k g_k, S\partial_{y_j}^2 u\big)_{L^2}$$

and thus

$$\varepsilon\|\partial_y u(t)\|_{L^2} + \varepsilon^{3/2}\|\partial_{y,x}\partial_y u\|_{L^2} \lesssim \varepsilon\|\partial_y u_0\|_{L^2} + \varepsilon^{1/2}\|f\|_{L^2} + \sum_j \varepsilon^{3/2}\|\partial_j g_j\|_{L^2}.$$

To get control of $\partial_x^2 u$, we multiply the equation by $\varepsilon^2 S\partial_x^2 u$, and obtain

$$\varepsilon\|\partial_x u(t)\|_{L^2} + \varepsilon^{3/2}\|\partial_x^2 u\|_{L^2} \lesssim \varepsilon\|\partial_x u_0\|_{L^2} + \|h\|_{L^2}$$

with

$$h = \sqrt{\varepsilon}\big(f - \partial_t u - \sum_{j\le d} A_j\partial_j u + \varepsilon \sum_{j+k<2d} B_{j,k}\partial_j\partial_k u\big).$$

The norm $\|h\|_{L^2}$ is estimated by the previous steps. At last, $\sqrt{\varepsilon}\partial_t u$ is estimated using the equation and the proposition follows. $\square$

Proof of Proposition 3.4.5. We have to bound the norms

$$\begin{array}{cc} \|u_{j,\beta,k}\|_{L^\infty(L^2)} & \varepsilon\|\partial_{y,x} u_{j,\beta,k}\|_{L^\infty(L^2)} \\ \sqrt{\varepsilon}\|\partial_{t,y,x} u_{j,\beta,k}\|_{L^2} & \varepsilon^{3/2}\|\partial_{y,x}^2 u_{j,\beta,k}\|_{L^2} \end{array} \tag{3.4.17}$$

of $u_{j,\beta,k} := \partial_t^j\partial_y^\beta X^k u$, for $j + |\beta| + k \le s$, by the right hand side of (3.4.14)

We proceed by induction on k. For $s = 0$, the estimate is true by (3.3.16) or by the proposition above. Differentiating in (t, y), we can bound the norms in (3.4.17) by the right hand side of (3.4.14) for $k = 0$ and $j + |\beta| \le s$.

Suppose that we have bounded the norms (3.4.17) up to the order $k-1$. We use the commutation relation (3.4.12) to write an equation for $u_{j,\beta,k}$:

$$
\begin{aligned}
(L-\varepsilon P)u_{j,\beta,k} =& \partial_t^j \partial_y^\beta (X')^k f \\
&+ \sum c u_{j',\beta',l} + \varepsilon \partial_{y,x}(c\partial_{y,x} u_{j',\beta',l}) + \varepsilon \partial_{y,x}(c u_{j',\beta',l})
\end{aligned}
$$

where the sum runs over indices (j', β', l) such that $j' + |\beta'| + l \le s$ and $l < k$. With Proposition 3.4.6, we can bound the norms in (3.4.17) at the order k by by the norms at order $l < k$ and the right hand side (3.4.14). $\square$

Parallel to Theorem 3.4.3, the energy estimates imply regularity of the solution when $u_0 = 0$ and the source term f vanishes at high order at $t = 0$.

Theorem 3.4.7. *Assume that $f \in H^s_{co}([0,T])$ and*

$$
\partial_t^j f_{|t=0} = 0\,, \qquad \text{for } j \le s-1\,. \tag{3.4.18}
$$

The solution $u \in \mathcal{H}^2$ of (3.3.12) *with initial data $u_0 = 0$ belongs to $H^s_{co}([0,T])$ as well as $\partial_{t,y,x} u$ and $\partial^2_{y,x} u$. Moreover, u satisfies the estimates* (3.4.14) *and, when $s > 1 + d/2$,* (3.4.6).

3.5 Nonlinear problems

In this section we consider the semilinear problem (3.1.2). We always suppose that Assumption 3.1.1 is satisfied and that F is a C^∞ function from $\mathbb{R}^N$ to $\mathbb{R}^N$ such that $F(0) = 0$.

3.5.1 Existence for fixed viscosity

We consider the semilinear equation

$$
(L-\varepsilon P)u = f + F(u)\,, \quad u_{|x=0} = 0\,, \quad u_{|t=0} = u_0\,. \tag{3.5.1}
$$

We also use the notation, $L = \partial_t + A$. *The compatibility conditions* must be modified as follows: first we define the functions $\mathcal{F}_j(u_0, \ldots, u_j)$ of the form

$$
\mathcal{F}_j(u_0, \ldots, u_j) = \sum_{k=1} \sum_{j^1+\ldots+j^k=j} c F^k(u_0)\big(u_{j^1}, \ldots, u_{j^k}\big)
$$

such that

$$
\partial_t^j F(u)_{|t=0} = \mathcal{F}_j(u_0, \ldots, u_j)\,, \quad u_j = \partial_t^j u_{|t=0}\,.
$$

Next, the definition (3.2.17) is modified as follows

$$u_j = (\varepsilon P - A)u_{j-1} + f_{j-1} + \mathcal{F}_{j-1}(u_0, \dots, u_{j-1})\,. \tag{3.5.2}$$

The multiplicative properties of Sobolev spaces in Proposition 2.5.2, imply the following result.

Lemma 3.5.1. *For $u_0 \in H^{2s+1}$ and $f \in \mathcal{H}^{2s}$ with $s > d/4$,* (3.5.2) *defines by induction functions $u_j \in H^{2s+1-2j}(\mathbb{R}^d_+)$ for $j \le s$.*

Definition 3.5.2. *The data $u_0 \in H^{2s+1}(\mathbb{R}^d_+)$ and $f \in \mathcal{H}^{2s}([0,T])$ satisfy the compatibility conditions to order $\sigma \le 2s$ if the u_j given by* (3.5.12) *satisfy*

$$u_{j\,|x=0} = 0\,, \quad j \in \{0, \dots, \sigma/2\}\,.$$

We first state a local in time existence theorem for a fixed ε.

Theorem 3.5.3. *Given $u_0 \in H^{2s+1}(\mathbb{R}^d_+)$ and $f \in \mathcal{H}^{2s}([0,T])$ compatible to order $2s$ with $s > d/4$, there is $T' \in]0,T]$ such that the equation* (3.5.1) *has a unique solution $u \in \mathcal{H}^{2s+2}([0,T'])$.*

Note that, in this theorem, T' may depend on ε.

Proof. We only sketch the argument since it is quite parallel to the proof of Theorem 2.5.1. We consider the iterative scheme

$$(L - \varepsilon P)u^n = f + F(u^{n-1})\,, \quad u^n{}_{|x=0} = 0\,, \quad u^n{}_{|t=0} = u_0\,. \tag{3.5.3}$$

a) Define the $u_j \in H^{2s+1-2j}(\mathbb{R}^d_+)$ by (3.5.2). Consider extensions in $H^{2s+1-2j}(\mathbb{R}^d)$ of the u_j. Then, denoting by $\hat{\ }$ the spatial Fourier transform define

$$\hat{u}^0(t, \eta, \xi) = \sum \frac{t^j}{j!} \chi(t\Lambda^2)\hat{u}_j(\eta, \xi)\,, \quad \Lambda = (1 + |\eta| + |\xi|)\,.$$

This function u^0 belongs to $\mathcal{H}^{2s+2}$ and $\partial_t^j u^0{}_{|t=0} = u_j$ for $j \le s$.

Starting with this u^0, we check by induction that for all $n \ge 0$, the data u_0 and source terms $f + F(u^{n-1})$ are compatible to order $2s$ and thus (3.5.3) has a solution $u^n \in \mathcal{H}^{2s+2}$ with satisfies $\partial_t^j u^n{}_{|t=0} = u_j$ for $j \le s$.

b) We use the norms n_s with $\varepsilon = 1$ defined in (3.3.14). Using the multiplicative properties of Sobolev spaces as in Proposition 2.5.2, we see that for $2s > d/2$

$$\mathrm{n}_s(F(u); t) \le C\big(\mathrm{n}_s(u; t)\big)\,. \tag{3.5.4}$$

Thus the energy estimate (3.3.3), one shows that there are $T' > 0$ and R such that for all n and $t \in [0, T']$, $\mathrm{n}_{s+1}(u^n; t) \le R$ and $\mathrm{N}_{s+1}(u^n; T')$ is bounded.

Next, one shows that the sequence u^n is a Cauchy sequence in $\mathcal{H}^{2s+2}([0, T'])$ and thus converge to a solution of (3.5.1).

The uniqueness follows form the L^2 energy estimate applied to a difference of solutions. $\square$

Next we state a continuation theorem, parallel to Proposition 2.5.10. The proof is based on Gagliardo-Nirenberg-Moser's inequalities. With Ω denoting $\mathbb{R}^d$, $\mathbb{R}^d_+$ or a strip $[0, T] \times \mathbb{R}^d_+$, with $T \ge T_1 > 0$, there holds (see the discussion after Proposition 2.5.4):

Proposition 3.5.4. *For all $s \in \mathbb{N}$, there is C such that for all $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ with $2j + |\alpha| \le 2s$ and all $u \in L^\infty(\Omega) \cap \mathcal{H}^s(\Omega)$, $\partial_t^j \partial_{y,x}^\alpha u \in L^p(\Omega)$ for $2 \le p \le 4s/(2j + |\alpha|)$ and*

$$\|\partial_t^j \partial_{y,x}^\alpha u\|_{L^p} \le C \|u\|_{L^\infty}^{1-2/p} \|u\|_{\mathcal{H}^{2s}}^{2/p}, \qquad \frac{2j + |\alpha|}{2s} \le \frac{2}{p} \le 1 \tag{3.5.5}$$

Corollary 3.5.5. *Given $T_1 > 0$ and $s \in \mathbb{N}$, there is a non decreasing function $C_F(\cdot)$ on $[0, \infty[$ such that for all $T \ge T_1$, for all $u \in L^\infty(\Omega) \cap \mathcal{H}^{2s}([0, T])$, one has $F(u) \in \mathcal{H}^{2s}(\Omega)$ and*

$$\|F(u)\|_{\mathcal{H}^{2s}([0,T])} \le C_F(\|u\|_{L^\infty}) \|u\|_{\mathcal{H}^{2s}([0,T])} . \tag{3.5.6}$$

Suppose that $f \in \mathcal{H}^{2s}([0, T_0] \times \mathbb{R}^d_+)$ and $u_0 \in H^{2s+1}(\mathbb{R}^d_+)$, with $s > d/4$, satisfy the compatibility conditions are satisfied at order $2s$. Let T^* denote the supremum of the set of $T \in]0; T_0]$ such that the problem (3.5.1) has a solution in $\mathcal{H}^{2s+2}([0, T])$. By uniqueness, there is a unique maximal solution u on $[0, T^*[$. Repeating the proof of Proposition 2.5.10, one obtains the following criterion for blow up:

Theorem 3.5.6. *If*

$$\limsup_{t \to T^*} \|u(t)\|_{L^\infty} < +\infty , \tag{3.5.7}$$

then $T^* = T_0$ and $u \in \mathcal{H}^{2s+2}([0, T_0])$.

3.5.2 Uniform existence theorem I

Our goal is now to prove the existence of solutions on interval of time independent of ε. Consider bounded families of initial data $\{u_0^\varepsilon\}_{\varepsilon \in]0,1]} \subset$

$H^{2s+1}(\mathbb{R}^d_+)$ and source terms $\{f^\varepsilon\}_{\varepsilon\in]0,1]}\subset\mathcal{H}^{2s}([0,T])$. By Theorems 3.5.3 and 3.5.6 we know that there are solutions $u^\varepsilon\in\mathcal{H}^{2s+2}([0,T(\varepsilon)[)$ where $T(\varepsilon)>0$ and $T(\varepsilon)=T$ or $u^\varepsilon\notin L^\infty([0,T(\varepsilon)]\times\mathbb{R}^d_+)$. The strategy is to use the L^∞ bounds of Proposition 3.4.2 to show that there is $T'>0$ such that for all $\varepsilon\in]0,1]$, $T(\varepsilon)\geq T'$.

Toward this end, we need uniform estimate on the norm $\mathrm{n}_{tg,s,\varepsilon}(u;0)$ of the traces $u_j=\partial_t^j u_{|t=0}$. We also need a good control of the source term

$$\|F(u)\|_{H^s_{tg}([0,t])}\lesssim C_F\big(\|u\|_{L^\infty}+\mathrm{n}_{tg,s,\varepsilon}(u;t)\big)\,. \tag{3.5.8}$$

with $C_F(\cdot)$ independent of time t. It turns out that theses estimates are delicate to prove: for instance application of Corollary 2.5.6 for fixed x, yields estimate with constants which blow up as $t\to 0$.

Consider initial data $u_0^\varepsilon\in H^{2s+1}(\mathbb{R}^d_+)$ and source terms $f^\varepsilon\in\mathcal{H}^{2s}([0,T])$. For $j\leq s$, define the $u_j^\varepsilon\in H^{2s+1-2j}$ by (3.5.2). We make the following assumption: there is C_1 such that for all $\varepsilon\in]0,1]$:

$$\sum_{j+|\alpha|\leq s}\Big(\|\partial_y^\alpha u_j^\varepsilon\|_{L^2(\mathbb{R}^d_+)}+\sqrt{\varepsilon}\|\partial_y^\alpha\partial_{y,x}u_j^\varepsilon\|_{L^2(\mathbb{R}^d_+)}\Big)\leq C_1\,, \tag{3.5.9}$$

$$\|f^\varepsilon\|_{H^s_{tg}([0,T])}\leq C_1\,. \tag{3.5.10}$$

We further assume that there are C_2 and a family $\tilde u^\varepsilon\in\mathcal{H}^{2s+2}([-1,0])$ such that

$$\begin{cases}\|\tilde u^\varepsilon\|_{H^s_{tg}([-1,T])}+\|\tilde u^\varepsilon\|_{L^\infty([-1,T]\times\mathbb{R}^d_+)}\leq C_2\,,\\ \partial_t^j\tilde u^\varepsilon{}_{|t=0}=u_j\,,\quad\text{for } j\leq s\,.\end{cases} \tag{3.5.11}$$

Example 3.5.7. The difficult part is to check the condition (3.5.9). Computing the u_j involves ∂_x derivatives, while the assumption on u_0 involves only ∂_y derivatives. However, there are several cases where the assumptions are easy to check. A first example, already occurs when $u_0^\varepsilon=0$ and f^ε satisfies

$$\partial_t^j f^\varepsilon_{|t=0}=0\,,\qquad\text{for } j\leq s-1\,. \tag{3.5.12}$$

In this case, all the u_j^ε vanish and the data are compatible to order 2s.

Example 3.5.8. Another interesting example occurs when f^ε is bounded in $H^{2s}([0,T]\times\mathbb{R}^d_+)$ and the u_0^ε are bounded in $H^{2s+1}(\mathbb{R}^d_+)$. In this case, the u_j^ε are bounded in $H^{2s+1-2j}(\mathbb{R}^d_+)$. Moreover, there are $\tilde u^\varepsilon$, bounded in $\mathcal{H}^{2s+2}(]-\infty,+\infty[)$, such that $\partial_t^j \tilde{u}^\varepsilon{}_{|t=0} = u_j^\varepsilon$. In particular, they are bounded in H^s_{tg}. If in addition one assumes that u_0^ε and f^ε vanish for $x < r$ and $x+t \le r$ respectively, then the u_j vanish for $x \le r$ and the compatibility conditions are satisfied.

Remark 3.5.9. The assumptions (3.5.9) (3.5.11) are well suited for continuation results : they are satisfied if solutions u^ε are given on $[-T_1, 0]$, such that they are bounded in L^∞ and the norms $\mathrm{n}_{tg,s\varepsilon}(u^\varepsilon, t))$ for $t \in [-T_1, 0]$ are bounded.

Note that one obtains equivalent conditions if one replaces in (3.5.11) the interval $[-1, T]$ by any interval $[-T_1, 0]$ or on $[0, T_1]$, for any $T_1 > 0$, since one can extend the $\tilde{u}^\varepsilon$ to $t \in \mathbb{R}$.

Theorem 3.5.10. *Assume that $s > 1 + \frac{d}{2}$ and consider families of initial data $u_0^\varepsilon \in H^{2s+1}(\mathbb{R}^d_+)$ and source terms $f^\varepsilon \in \mathcal{H}^{2s}([0,T])$ satisfying the compatibility conditions to order $2s$. Assume that* (3.5.9)(3.5.10)(3.5.11) *hold. There is $T' \in]0, T]$ such that for all $\varepsilon \in]0, 1]$ the problem* (3.5.1) *has a unique solution $u^\varepsilon \in \mathcal{H}^{2s+1}([0, T'])$. Moreover, the family u^ε is bounded in $H^s_{tg}([0,T]) \cap L^\infty([0,T] \times \mathbb{R}^d_+)$.*

Denote by $\mathcal{K}^\varepsilon([0,T'])$ the set of functions $u \in \mathcal{H}^{2s+2}([0,T'])$ such that $\partial_t^j u = u_j^\varepsilon$ for $j \le s$. We use the following estimate:

Lemma 3.5.11. *For $s > d/2$, there is constant C_3 and a function $C_F(\cdot)$ such that for all $t \in [0,T]$, all $\varepsilon \in]0,1]$ and all $u \in \mathcal{K}^\varepsilon([0,t])$, there holds*

$$\|F(u)\|_{H^s_{tg}([0,t])} \le C_3 + C_F\big(\|u\|_{L^\infty([0,t]\times\mathbb{R}^d_+)}\big)\big(1 + \|u\|_{H^s_{tg}([0,t])}\big)\,. \tag{3.5.13}$$

Proof. For $u \in \mathcal{K}^\varepsilon([0,T'])$ consider the function $\tilde{u}$ defined to be equal to $\tilde{u}^\varepsilon$ for $t \in [-1, 0]$ and equal to u for $t \in [0, T']$. Since $\tilde{u}^\varepsilon$ and u have the same traces at $t = 0$, the extended function belongs to $\mathcal{H}^{2s+2}([-1,T]) \subset H^s_{tg} \cap L^\infty([-1,T'] \times \mathbb{R}^d_+)$. Moreover,

$$\begin{aligned}
&\|\tilde{u}\|_{L^\infty([-1,T']\times\mathbb{R}^d)} \le C_2 + \|u\|_{L^\infty([0,T']\times\mathbb{R}^d_+)}\,,\\
&\|\tilde{u}\|_{H^s_{tg}([-1,T'])} \le C_2 + \|u\|_{H^s_{tg}([0,T'])}\,.
\end{aligned}$$

Applying Corollary 2.5.6 on $[-1, T'] \times \mathbb{R}^{d-1}$ with x as a parameter, and integrating in x, yields the estimate:

$$\|F(\tilde{u})\|_{H^s_{tg}([-1,T'])} \le C_F\big(\|\tilde{u}\|_{L^\infty([-1,T']\times\mathbb{R}^d_+)}\big)\|\tilde{u}\|_{H^s_{tg}([-1,T'])}\,. \tag{3.5.14}$$

The lemma follows. ∎

Proof of Theorem 3.5.10. By Theorems 3.5.3 and 3.5.6 we know that there are solutions $u^\varepsilon \in \mathcal{H}^{2s+2}([0,T(\varepsilon)[)$ where $T(\varepsilon) > 0$ and $T(\varepsilon) = T$ or $u^\varepsilon \notin L^\infty([0,T(\varepsilon)] \times \mathbb{R}^d_+)$.

Theorem 3.4.3 implies that for $t \in [0,T(\varepsilon)]$

$$\mathrm{n}_\varepsilon(t) = \mathrm{n}_{tg,s,\varepsilon}(u;t) \quad \text{and} \quad \mathrm{m}^n(t) = \|u^\varepsilon\|_{L^\infty([0,t]\times\mathbb{R}^d_+)}$$

satisfy

$$\mathrm{n}_\varepsilon(t) \le C_4 + C_F(\mathrm{m}_\varepsilon(t))\Big(\int_0^t \big(1+\mathrm{n}_\varepsilon(t')\big)^2 dt'\Big)^{1/2},$$
$$\mathrm{m}_\varepsilon(t) \le C_5 + C_F(\mathrm{m}_\varepsilon(t))\Big(\int_0^t \big(1+\mathrm{n}_\varepsilon(t')\big)^2 dt'\Big)^{1/2},$$

where C_4 and C_5 are independent of $t \in [0,T$ and $\varepsilon \in]0,1]$. There is $T' > 0$ such that

$$2C_4C_F(2C_5)\sqrt{T'} \le \min\{C_4, C_5\}\,.$$

The estimates above imply that

$$\forall t \le \min\{T(\varepsilon), T'\}: \quad \mathrm{n}_\varepsilon(t) \le 2C_4 \quad \text{and} \quad \mathrm{m}_\varepsilon(t) \le 2C_5\,.$$

By Theorem 3.5.6, this implies that $T(\varepsilon) \ge T'$. In addition, this shows that the u^ε are uniformly bounded in L^∞ and H^s_{tg} on $[0,T'] \times \mathbb{R}^d_+$. $\square$

3.5.3 Uniform existence theorem II

There are analogous results in spaces H^s_{co}. We make the following assumption: there is C_1 such that for all $\varepsilon \in]0,1]$:

$$\sum_{j+|\alpha|+k\le s}\Big(\|\partial_y^\alpha X^k u_j^\varepsilon\|_{L^2(\mathbb{R}^d_+)} + \sqrt{\varepsilon}\|\partial_y^\alpha\partial_{y,x}X^k u_j^\varepsilon\|_{L^2(\mathbb{R}^d_+)}\Big) \le C_1\,, \tag{3.5.15}$$

$$\|f^\varepsilon\|_{H^s_{co}([0,T])} \le C_1\,. \tag{3.5.16}$$

We further assume that there are C_2 and a family $\tilde u^\varepsilon \in \mathcal{H}^{2s+2}([-1,0])$ such that

$$\begin{cases} \|\tilde u^\varepsilon\|_{H^s_{co}([-1,T])} + \|\tilde u^\varepsilon\|_{L^\infty([-1,T]\times\mathbb{R}^d_+)} \le C_2\,,\\ \partial_t^j \tilde u^\varepsilon{}_{|t=0} = u_j\,, \quad \text{for } j \le s\,. \end{cases} \tag{3.5.17}$$

Examples 3.5.12. The assumptions (3.5.15) and (3.5.17) are satisfied when $u_0^\varepsilon = 0$ and f^ε satisfies (3.5.12). In this case, all the u_j^ε vanish and the data are compatible to order $2s$.

As in Example 3.5.8, the assumptions are satisfied when the f^ε are bounded in $H^{2s}([0,T] \times \mathbb{R}^d_+)$ and the u_0^ε are bounded in $H^{2s+1}(\mathbb{R}^d_+)$. In this case, the u_j^ε are bounded in $H^{2s+1-2j}(\mathbb{R}^d_+)$. If in addition u_0^ε and f^ε vanish for $x < r$ and $x + t \le r$ respectively, then the u_j vanish for $x \le r$ and the data are compatible to order $2s$.

Theorem 3.5.13. *Assume that $s > 1 + \frac{d}{2}$ and consider families of initial data $u_0^\varepsilon \in H^{2s+1}(\mathbb{R}^d_+)$ and source terms $f^\varepsilon \in \mathcal{H}^{2s}([0,T])$ satisfying the compatibility conditions to order $2s$. Assume that* (3.5.15)(3.5.16)(3.5.17) *hold. There is $T' \in]0,T]$ such that for all $\varepsilon \in]0,1]$ the problem* (3.5.1) *has a unique solution $u^\varepsilon \in \mathcal{H}^{2s+1}([0,T'])$. Moreover, the family u^ε is bounded in $H^s_{co}([0,T]) \cap L^\infty([0,T] \times \mathbb{R}^d_+)$.*

The proof is similar to the proof of Theorem 3.5.10. The analogue of (3.5.14) is

$$\|F(u)\|_{H^s_{co}([-1,T'])} \le C_F\big(\|u\|_{L^\infty([-1,T']\times\mathbb{R}^d_+)}\big)\|u\|_{H^s_{co}([-1,T'])}\big) . \tag{3.5.18}$$

It is a consequence of the following estimates on $[-1,T'] \times \mathbb{R}^d_+$:

$$\|Z^\alpha u\|_{L^p} \le C\|u\|_{L^\infty}^{1-2/p}\|u\|_{H^s_{co}}^{2/p} , \quad \frac{|\alpha|}{s} \le \frac{2}{p} \le 1 . \tag{3.5.19}$$

Chapter 4

Semilinear boundary layers

In this Chapter, we present the analysis of O.Guès ([Gu1]) in the case of constant coefficients systems, with noncharacteristic flat boundary. We first construct approximate solutions, using the existence theory of Chapter two. Next, we use the uniform estimates of Chapter three to solve the equation for the remainder.

4.1 Statement of the problem

In this Chapter we study the existence and the asymptotic behavior of solutions of the equations

$$\begin{cases} (L - \varepsilon P) u^\varepsilon = F(u) + f \,, \\ u^\varepsilon{}_{|x=0} = 0 \,, \\ u^\varepsilon{}_{|t=0} = h^\varepsilon \,, \end{cases} \tag{4.1.1}$$

with

$$Lu := \partial_t u + \sum_{j=1}^{d} A_j \partial_j u \,, \quad Pu := \sum_{j,k=1}^{d} B_{j,k} \partial_j \partial_k u \,.$$

The goal is twofold : first, prove the existence of the solutions u^ε on an interval of time $[0, T]$ independent of ε and second, give the asymptotic behavior of u^ε as ε tends to zero. In particular, it is expected that

$$u^\varepsilon - u = O(\varepsilon) \tag{4.1.2}$$

where u satisfies

$$
(4.1.3) \qquad \begin{cases} Lu = F(u) + f\,, \\ Mu_{|x=0} = 0\,, \\ u_{|t=0} = h\,, \end{cases}
$$

provided that $h^\varepsilon - h = O(\varepsilon)$. Part of the analysis is to determine the boundary conditions M. Throughout the chapter, we suppose that the following Assumption is satisfied:

Assumption 4.1.1. *i) There is a positive definite symmetric matrix $S = {}^tS \gg 0$ such that for all j the matrix SA_j is symmetric. Moreover, for all $\xi \neq 0$ the symmetric matrix $\sum \xi_j\xi_k \mathrm{Re}\, SB_{j,k}$ is definite positive.*

ii) The matrix A_d is invertible.

iii) F is a C^∞ mapping from $\mathbb{R}^N$ to $\mathbb{R}^N$, such that $F(0) = 0$.

To avoid the difficult questions of compatibility conditions, which are different for (4.1.1) and (4.1.3), we also assume that f vanishes on a neighborhood of the edge $\{t = x = 0\}$ and that the h^ε vanish on a fixed neighborhood of $\{x = 0\}$.

We first construct asymptotic and approximate solutions of (4.1.1) as power series of ε. This leads to solve ordinary differential equations for the inner layers and hyperbolic boundary values problems for the interior terms in $\{x > 0\}$. Next, we look for correctors to get exact solutions of the equations. There we use the analysis and estimates of Chapter three.

4.2 Asymptotic boundary layers

We look for solutions of (4.1.1) as formal series in powers of ε:

$$
(4.2.1) \qquad u^\varepsilon(t,y,x) \sim \sum_{n\geq 0} \varepsilon^n U_n\big(t,y,x,\frac{x}{\varepsilon}\big)\,.
$$

We look for the profiles U_n in the class $\mathcal{P}(T)$ of functions

$$
(4.2.2) \qquad U(t,y,x,z) = u(t,y,x) + U^*(t,y,x,z)
$$

with $u \in H^\infty([0,T]\times\mathbb{R}^d_+)$ and $U^* \in e^{-\delta z}H^\infty([0,T]\times\mathbb{R}^d_+\times\mathbb{R}_+)$ for some $\delta > 0$ (depending on U^*). Here, $H^\infty(\Omega) = \cap H^s(\Omega)$. In particular, functions in H^∞ are C^∞, bounded as well as all their derivatives.

For u^ε as in (4.2.1), the left hand side of (4.1.1) is (formally)

$$Lu^\varepsilon - \varepsilon Pu^\varepsilon \sim \sum_{n\geq -1} \varepsilon^n L_n(t,y,x,\frac{x}{\varepsilon}), \qquad L_n \in \mathcal{P} \tag{4.2.3}$$

with

$$L_{-1} = -B_{d,d}\partial_z^2 U_0 + A_d \partial_z U_0 \tag{4.2.4}$$

and, for $n > 0$:

$$\begin{aligned} L_{n-1} = -B_{d,d}\partial_z^2 U_n + A_d\partial_z U_n - P'(\partial_{y,x})\partial_z U_{n-1} \\ + L(\partial_{t,y,x})U_{n-1} - P(\partial_{y,x})U_{n-2} \end{aligned} \tag{4.2.5}$$

with $P'(\partial_{y,x}) = \sum (B_{d,j} + B_{j,d})\partial_j$ and $U_n = 0$ for $n < 0$. Formal series expansions yield

$$f + F\Big(\sum_{n\geq 0} \varepsilon^n U_n\Big) \sim \sum_{n\geq 0} \varepsilon^n F_n \qquad F_n \in \mathcal{P} \tag{4.2.6}$$

with $F_0 = F(U_0) + f$, $F_1 = F'(U_0)U_1$ etc. Thus for u^ε given by (4.2.1) the right hand side of (4.1.1) reads

$$F(u^\varepsilon) + f \sim \sum_{n\geq 0} \varepsilon^n F_n(t,y,x,\frac{x}{\varepsilon}). \tag{4.2.7}$$

The boundary condition is easily interpreted for formal series (4.2.1) since

$$u^\varepsilon{}_{|x=0} \sim \sum_{n\geq 0} \varepsilon^n U_n|_{x=0,z=0}. \tag{4.2.8}$$

Definition 4.2.1. *We say that $\sum \varepsilon^n U_n$ is a formal or a BKW solution of $(L - \varepsilon P)u^\varepsilon = f + F(u^\varepsilon)$ if and only if $L_{-1} = 0$ and $L_n = F_n$ for all $n \geq 0$. It is a formal solution of the boundary conditions $u^\varepsilon{}_{|x=0}$ if and only if for all $n \geq 0$:*

$$U_n|_{x=0,z=0} = 0. \tag{4.2.9}$$

Remark 4.2.2. The expansion (4.2.1) is not unique. For instance, one can perform a Taylor expansion with respect to the slow variable x in $U^*(t,y,x,z)$:

$$U^*(t,y,x,z) = U^*(t,y,0,z) + xV^*(t,y,x,z).$$

The function V^* is rapidly decaying in z, as well as $W^* = zV^*$. In this case

$$U^*(t,y,x,\frac{x}{\varepsilon}) = U^*(t,y,0,\frac{x}{\varepsilon}) + \varepsilon W^*(t,y,x,\frac{x}{\varepsilon})\,.$$

This means that, at first order, $U^*(t,y,x,z)$ and $U^*(t,y,0,z)$ define the same function. To fix the indeterminacy, one could restrict the class of profiles, as in Chapter one, to profiles of the form

$$U_n(t,y,x,z) = u(t,y,x) + U^*(t,y,z)\,. \tag{4.2.10}$$

The lack of uniqueness is not a difficulty. On the contrary it gives some flexibility. However, it implies that the expansions (4.2.3) (4.2.7) are not unique. Above we have made an explicit choice for the L_n and F_n given by (4.2.5) and (4.2.6). Note that the computation of profiles F_n of the form (4.2.10) for (4.2.7) is more complicated.

The aim of this section is to construct formal solutions of the mixed Cauchy problem (4.1.1) with initial condition

$$u^\varepsilon{}_{|t=0} = h^\varepsilon\,. \tag{4.2.11}$$

In this analysis, the initial data should also be given by formal series:

$$h^\varepsilon(y,x) \sim \sum_{n\geq 0} \varepsilon^n H_n(y,x,\frac{x}{\varepsilon})$$

However, to avoid the difficult question of compatibility conditions, we assume that the initial data has no rapid dependence on z:

$$h^\varepsilon(y,x) \sim \sum_{n\geq 0} \varepsilon^n h_n(y,x) \tag{4.2.12}$$

with $h_n \in H^\infty(\mathbb{R}^d_+)$. Moreover, we assume that that there is $r > 0$ such that

$$\begin{aligned} &\forall n \geq 0\,,: \quad h_n(y,x) = 0 \quad \text{for} \quad x \leq r\,,\\ &f(t,y,x) = 0 \quad \text{for} \quad x + t \leq r\,. \end{aligned} \tag{4.2.13}$$

This implies that the data are compatible at infinite order.

We say that $\sum \varepsilon^n U_n$ is a formal solution of the Cauchy condition (4.2.11) if

$$\forall n \geq 0\,,: \quad U_{n|t=0}(y,x,z) = h_n(y,x)\,, \tag{4.2.14}$$

that is

$$\forall n \geq 0\,,: \quad u_{n|t=0}(y,x) = h_n(y,x) \quad \text{and} \quad U^*_{n|t=0}(y,x,z) = 0\,.$$

Theorem 4.2.3. *For $f \in H^\infty([0,T]\times\mathbb{R}^d_+)$ and a formal initial data $\sum \varepsilon^n h_n$ satisfying* (4.2.13)*, there are $T' \in]0,T]$ and formal solutions $\sum \varepsilon^n U_n$ of* (4.1.1).

When $U(t,y,x,z)$ is a profile in $\mathcal{P}(T)$, we denote by

$$\underline{U}(t,y,x) = \lim_{z\to\infty} U(t,y,x,z)$$

its limit at $z = +\infty$ and by $U^* = U - \underline{U}$ the exponentially decaying part. In the decomposition (4.2.2), $\underline{U} = u$. We use this notation for the profiles the L_n and F_n and write:

$$L_n = \underline{L}_n + L_n^*\,, \quad F_n = \underline{F}_n + F_n^*\,.$$

By (4.2.4), L_{-1} is rapidly decaying and the limit $\underline{L}_{-1}$ vanishes. Therefore, it is equivalent to solve inductively for $n \ge 0$ the equations L_{n-1}^* and $\underline{L}_n$ together with the boundary condition (4.2.9) and the initial condition $U_n|t = 0 = h_n$. Thus, $\sum \varepsilon^n U_n$ is a formal solution if and only if

$$\text{(4.2.15)} \qquad \begin{cases} B_{d,d}\partial_z^2 U_0 + A_d \partial_z U_0 = 0\,, \\ L(\partial_{t,y,x})\underline{U}_0 = \underline{F}_0 = F(\underline{U}_0) + f\,, \\ U_{0|x=z=0} = 0\,, \\ U_{0|t=0} = h_0\,, \end{cases}$$

and for all $n > 0$

$$\text{(4.2.16)} \qquad \begin{cases} L_{n-1}^* = F_{n-1}^*\,, \\ \underline{L}_n = \underline{F}_n\,, \\ U_{n|x=z=0} = 0\,, \\ U_{n|t=0} = h_n\,. \end{cases}$$

4.3 The boundary layer ode and the hyperbolic boundary conditions

4.3.1 The inner layer o.d.e.

The first equation in (4.2.15) is a constant coefficient differential equation in z. More generally, consider the equation on $[0,\infty[$:

$$\text{(4.3.1)} \qquad -B_{d,d}\partial_z^2 U + A_d \partial_z U = F \in \mathcal{P}^*\,, \qquad U = \underline{U} + U^* \in \mathcal{P}\,.$$

Here, $\mathcal{P}^*$ denotes the space of exponentially decaying C^∞ functions on $\mathbb{R}_+$ and $\mathcal{P}$ the space generated by constants and $\mathcal{P}^*$. We add the boundary condition

$$\text{(4.3.2)} \qquad U(0) = 0\,.$$

Notation. For a $N\times N$ matrix G, $\mathbb{E}_+(G)$ [resp $\mathbb{E}_-(G)$] denotes the invariant subspace of $\mathbb{C}^N$ generated by the generalized eigenvectors associated to eigenvalues with positive [resp. negative] real part. Denote by $\Pi_\pm(G)$ the spectral projectors on $\mathbb{E}_\pm(G)$.

Note that Assumption 4.1.1 implies that $B_{d,d}$ and A_d are invertible. The next lemma is crucial in the analysis of (4.3.1).

Lemma 4.3.1. *The matrix $G_d = (B_{d,d})^{-1}A_d$ has no eigenvalues on the imaginary axis, thus $\mathbb{C}^N = \mathbb{E}_-(G_d)\oplus\mathbb{E}_+(G_d)$. Moreover, $\dim\mathbb{E}_-(G_d) = N_-$, the number of negative eigenvalues of A_d, and the matrix SA_d is negative definite on $\mathbb{E}_-(G_d)$.*

Proof. **a)** Suppose that λ is an eigenvalue of G_d and $h\neq 0$ satisfies $G_d h = \lambda h$. Thus,

$$\big(SB_{d,d}\lambda h, \lambda h\big)_{\mathbb{C}^N} = \overline{\lambda}\left(SA_d h, h\right)_{\mathbb{C}^N}$$

Assumption 4.1.1 implies that $\operatorname{Re} SB_{d,d}$ is definite positive and that SA_d is symmetric. Moreover, $\lambda\neq 0$ since G_d is invertible. Therefore

$$\operatorname{Re}\lambda\big(SA_d h, h\big)_{\mathbb{C}^N} > 0\,.$$

In particular, $\operatorname{Re}\lambda\neq 0$ and therefore $\mathbb{C}^N = \mathbb{E}_-(G_d)\oplus\mathbb{E}_+(G_d)$.
b) Suppose that $h\in\mathbb{E}_-(G_d)\setminus\{0\}$. Thus $u(z) = e^{zG_d}h$ is exponentially decaying at $+\infty$ and $B_{d,d}\partial_z u = A_d u$. Therefore

$$0 < 2\mathrm{Re}\int_0^\infty \big(SB_{d,d}\partial_z u(z), \partial_z u\big)_{\mathbb{C}^N} dz = -\big(SA_d h, h\big)_{\mathbb{C}^N}\,.$$

This shows that SA_d is definite negative on $\mathbb{E}_-(G_d)$. In particular, $\dim\mathbb{E}_-(G_d)\leq N_-$, where N_- is the number of negative eigenvalues of SA_d, which is the number of negative eigenvalues of A_d as seen in Chapter two.

Similarly, for $h\in\mathbb{E}_+(G_d)\setminus\{0\}$, there holds

$$0 < 2\mathrm{Re}\int_{-\infty}^0 \big(SB_{d,d}\partial_z u(z), \partial_z u\big)_{\mathbb{C}^N} dz = \big(SA_d h, h\big)_{\mathbb{C}^N}\,.$$

Thus SA_d is definite positive on $\mathbb{E}_+(G_d)$ and $\dim\mathbb{E}_+(G_d)\leq N_+ = N - N_-$. Since $\mathbb{C}^N = \mathbb{E}_-(G_d)\oplus\mathbb{E}_+(G_d)$, the two inequalities imply that $\dim\mathbb{E}_-(G_d) = N_-$. $\square$

4.3.2 Layers profiles

Lemma 4.3.2. *For $F \in \mathcal{P}^*$ the bounded solutions of equation* (4.3.1) *belong to $\mathcal{P}$. They are given by*

$$U(z) = \underline{U} + e^{zG_d} U^\flat + I(F)(z)\,, \quad \underline{U} \in \mathbb{R}^N\,, \quad U^\flat \in \mathbb{E}_-(G_d) \tag{4.3.3}$$

where $I(F)$ is an integral operator described below.

In this case, the boundary condition (4.3.2) *is equivalent to*

$$\begin{cases} \Pi_+(G_d)\underline{U} = -\Pi_+(G_d) I(F)(0)\,, \\ \qquad\quad U^\flat = -\Pi_-(G_d)\big(\underline{U} + I(F)(0)\big)\,. \end{cases} \tag{4.3.4}$$

Proof. The equation for $V = \partial_z U$ reads

$$\partial_z V = G_d V - B_{b,b}^{-1} F\,. \tag{4.3.5}$$

Dropping the indices d and setting $\Pi_\pm = \Pi_\pm(G_d)$, there is $\delta_0 > 0$ such that for $z \geq 0$:

$$|e^{zG}\Pi_-| \lesssim e^{-\delta_0 z}\,, \qquad |e^{-zG}\Pi_+| \lesssim e^{-\delta_0 z}\,. \tag{4.3.6}$$

Therefore, the solutions of (4.3.5) are

$$\begin{aligned} V(z) = e^{zG}V^\flat - &\int_0^z e^{(z-z')G}\Pi_- B^{-1} F(z')dz' \\ &+ \int_z^\infty e^{(z-z')G}\Pi_+ B^{-1} F(z')dz'\,. \end{aligned}$$

Thus

$$U(z) = \underline{U} + e^{zG_d} U^\flat + I(F)(z)\,. \tag{4.3.7}$$

with $U^\flat = G^{-1}V^\flat$ and $\underline{U}$ arbitrary and

$$\begin{aligned} I(F)(z) = -&\int_0^z e^{(z-z')G_d}\Pi_-(G_d) G_d^{-1} B_{d,d}^{-1} F(z')dz' \\ &+ \int_z^\infty e^{(z-z')G_d}\Pi_+(G_d) G_d^{-1} B_{d,d}^{-1} F(z')dz' \\ &- \int_z^\infty G_d^{-1} B_{d,d}^{-1} F(z')dz'\,. \end{aligned} \tag{4.3.8}$$

We note that the integrals are well defined thanks to (4.3.6) and that $I(F)$ is exponentially decaying if $F \in \mathcal{P}^*$. Thus the solution is bounded if and only if $U^\flat \in \mathbb{E}_-(G_d)$. In this case $U \in \mathcal{P}$.

By (4.3.7), $U(0) = \underline{U} + \Pi_-(G_d)U^\flat + I(F)(0)$ and (4.3.4) follows projecting by Π_+ and Π_-. $\square$

4.3.3 The hyperbolic boundary conditions.

From Lemma 4.3.2 the first equation in (4.2.15) is equivalent to

$$U_0^*(t,y,x,z) = e^{zG_d} U_0^\flat(t,y,x) \quad \text{with} \quad U_0^\flat(t,y,x) \in \mathbb{E}_-(G_d)\,. \tag{4.3.9}$$

and the boundary condition to

$$\Pi_+(G_d)\underline{U}_{0|x=0} = 0\,, \qquad U^\flat_{0|x=0} = -\Pi_-(G_d)\underline{U}_{0|x=0} \tag{4.3.10}$$

Thus, see that $\underline{U}_0$ must satisfy the boundary value problem

$$\begin{cases} L(\partial_{t,y,x})\underline{U}_0 = F(\underline{U}_0) + f\,, \\ \Pi_+(G_d)\underline{U}_{0|x=0} = 0 \end{cases} \tag{4.3.11}$$

By Lemma 4.3.1, the invariant space $\mathbb{E}_-(G_d) = \ker \Pi_+(G_d)$ is of dimension N_-. To match the notations of Chapter two, it is convenient to introduce a $N_- \times N$ matrix M such that

$$\ker M = \mathbb{E}_-(G_d)\,. \tag{4.3.12}$$

Thus, M is an isomorphism from $\mathbb{E}_+(G_d)$, the range of $\Pi_+(G_d)$, to $\mathbb{R}^{N_+}$. In particular, the first equation in (4.3.4) can be replaced by

$$M\underline{U} = -MI(F)(0)\,. \tag{4.3.13}$$

A key observation is that the limiting problem is well posed: Lemma 4.3.1 immediately implies

Proposition 4.3.3. *The boundary condition M is maximal dissipative for L.*

4.4 Solving the BKW equations

4.4.1 The leading term

We first solve (4.2.15). We have seen that $u_0 = \underline{U}_0$ must satisfy

$$\begin{cases} L(\partial_{t,y,x})u_0 = F(u_0) + f\,, \\ Mu_{0|x=0} = 0\,, \\ u_{0|t=0} = h_0 \end{cases} \tag{4.4.1}$$

Proposition 4.4.1. *There is $T' \in]0,T]$ and a unique solution of* (4.4.1), $u_0 \in H^\infty([0,T'] \times \mathbb{R}^d_+)$. *Moreover, there is $c > 0$ such that $u_0 = 0$ on $\Delta := [0,T] \times \mathbb{R}^d_+ \cap \{x + ct \le r\}$.*

Proof. By (4.2.13) the initial data are compatible to any order. Thus Theorem 2.5.1 implies that there is $T' > 0$ and for $s > d/2$ a unique solution $u \in W^s(T')$, which belongs to $W^\sigma(T')$ for all σ.

That u vanishes near the edge $x = t = 0$ is a consequence of the finite speed of propagation. For the sake of completeness we sketch a proof.

Fix $c \ge 1$ such that the symmetric matrix $\Sigma := S(c\mathrm{Id}+A_d)$ is nonnegative For C^1 functions, integration by parts over the domain $\Delta = [0,T] \times \mathbb{R}^d_+ \cap \{x + ct \le r\}$ yields

$$\gamma \| e^{-\gamma t} u \|^2_{L^2(\Delta)} + \int_{\partial\Delta} \big(e^{-2\gamma t} \Sigma u, u \big) \lesssim \| u(0) \|^2_{L^2(\{x \le r\})} + \frac{1}{\gamma} \| e^{-\gamma t} L u \|^2_{L^2(\Delta)}$$

where $\partial\Delta = [0,T] \times \mathbb{R}^d_+ \cap \{x + ct = r\}$. Since Σ is nonnegative the second term on the left hand side can be dropped.

Since u_0 is smooth, there is a constant C such that $|F(u_0)| \le C|u_0|$. In addition $f = 0$ on Δ and $u_0(0) = 0$ for $\{x \le r\}$. Thus there is C such that for all $\gamma > 0$:

$$\gamma \| e^{-\gamma t} u_0 \|^2_{L^2(\Delta)} \le \frac{C}{\gamma} \| e^{-\gamma t} u_0 \|^2_{L^2(\Delta)} .$$

Thus, for γ large enough, $\| e^{-\gamma t} u_0 \|^2_{L^2(\Delta)} = 0$, hence $u = 0$ on Δ. $\square$

Next, we choose $U_0^\flat \in H^\infty([0,T'] \times \mathbb{R}^d_+)$ such that (4.3.10) holds:

$$U^\flat_{0|x=0} = -\Pi_-(G_d) u_{0|x=0} , \quad U_0^\flat(t,y,x) = 0 \ \text{ for } ct \le r . \tag{4.4.2}$$

Note that the second condition can be fulfilled since $u_{0|x=0} = 0$ for $ct \le r$.

Then we define

$$\begin{aligned} U_0(t,y,x,z) &= u_0(t,y,x) + e^{zG_d} U_0^\flat(t,y,x) \\ &= u_0(t,y,x) + U_0^*(t,y,x,z) . \end{aligned} \tag{4.4.3}$$

Because $U_0^\flat$ vanishes for $t = 0$, there holds $U_0(0,y,x,z) = h_0(y,x)$. Adding up, we have proved:

Proposition 4.4.2. *There is $T' > 0$ such that the problem* (4.2.15) *has a solution $U_0 \in \mathcal{P}(T')$, given by* (4.4.3). *Moreover, $\underline{U}_0$ vanishes on Δ and U_0^* vanishes for $ct \le r$.*

Note that only the trace of $U^\flat_{0|x=0}$ is uniquely determined. The lifting of $U_0^\flat$ to $x > 0$ is arbitrary. This reflects the lack of uniqueness mentioned in Remark 4.2.2.

4.4.2 The full expansion

Next we solve (4.2.16) by induction on n. Suppose that $(U_0, \dots, U_{n-1})$ are known such that the equations are solved up to order $n-1$. Suppose in addition that the $\underline{U}_k$ vanish on Δ and the U_k^* vanishes for $ct \le r$. The n-th equation reads

$$
\tag{4.4.4}
\begin{cases}
-B_{d,d}\partial_z^2 U_n + A_d \partial_z U_n = \Phi_n^* \,, \\
L(\partial_{t,y,x})\underline{U}_n = \underline{\Phi}_n \,, \\
U_{n|x=z=0} = 0 \,, \\
U_{n|t=0} = h_n \,.
\end{cases}
$$

where Φ_n^* and $\underline{\Phi}_n$ are given by $(U_0, \dots, U_{n-1})$. They belong to $\mathcal{P}^*(T')$ and to $H^\infty([0, T'] \times \mathbb{R}^d +)$ respectively and vanish for $(t, x, y) \in \Delta$.

By Proposition 4.3.2 the first equation is equivalent to

$$
\tag{4.4.5}
U_n^*(t, y, x, z) = e^{zG_d} U_n^\flat(t, y, x) + I(\Phi_n^*) \,, \quad U_n^\flat(t, y, x) \in \mathbb{E}_-(G_d) \,,
$$

and the boundary condition to

$$
\tag{4.4.6}
\begin{aligned}
&M\underline{U}_{n|x=0} = g_n := -MI(\Phi_n^*)_{|x=z=0} \,, \\
&U^\flat_{0|x=0} = -\Pi_-(G_d)\underline{U}_{0|x=0} - \Pi_-(G_d)I(\Phi_n^*)_{|x=z=0}
\end{aligned}
$$

Thus, $u_n = \underline{U}_n$ must satisfy the boundary value problem

$$
\tag{4.4.7}
\begin{cases}
L(\partial_{t,y,x} x)u_n = \underline{\Phi}_n \,, \\
M\underline{u}_{n|x=0} = g_n \,, \\
u_{n|t=0} = h_n \,.
\end{cases}
$$

The computation of Φ_n^* shows that it involves the U_k and at least one U_k^*. Thus Φ_n^* and g_n vanish for $ct \le r$. Similarly, $\underline{\Phi}_n$ vanishes on Δ, and $h_n = 0$ for $x \le r$. Hence the data (Φ_n, g_n, h_n) satisfy the compatibility conditions to any order. Therefore, by Theorem 2.4.12 there is a unique solution $u_n \in H^\infty([0, T'] \times \mathbb{R}^d_+)$. Moreover, repeating the argument in the proof of Proposition 4.4.1, one shows that u_n vanishes on Δ.

Next, we choose $U_n^\flat \in H^\infty([0, T'] \times \mathbb{R}^d_+)$ such that

$$
\tag{4.4.8}
U^\flat_{n|x=0} = -\Pi_-(G_d)u_{n|x=0} \,, \quad U_n^\flat(t, y, x) = 0 \ \text{ for } ct \le r \,.
$$

Then we define

$$
\tag{4.4.9}
U_n(t, y, x, z) = u_n(t, y, x) + e^{zG_d} U_n^\flat(t, y, x) + I(\Phi_n^*) = u_n(t, y, x) + U_n^*(t, y, x, z) \in \mathcal{P}(T') \,.
$$

Since Φ_n^* vanishes for $ct \le r$, $I(\Phi_n^*)$ and hence U_n^* vanish on this set. In particular, $U_n(0,y,x,z) = h_n(y,x)$.

By construction, $U_n \in \mathcal{P}(T')$ is a solution of (4.4.4), $\underline{U}_n$ vanishes on Δ and U_n^* vanishes for $ct \le r$. This finishes the induction and the proof of Theorem 4.2.3 is complete.

4.5 Convergence and approximation theorems

Assume that $\sum \varepsilon^n U_n$ is a formal solution of the equations. The series is not likely to converge, and the question is to know what kind of information it gives on the existence and approximation of exact solutions of (4.1.1).

4.5.1 Approximate solutions

Given a formal solution $\sum \varepsilon^n U_n$ and a positive integer m, consider

$$u^\varepsilon_{app} = \sum_{n=0}^{m} \varepsilon^n U_n\big(t,y,x,\frac{x}{\varepsilon}\big)\,. \tag{4.5.1}$$

For all ε, u^ε_{app} is in $H^\infty([0,T]\times\mathbb{R}^d_+)$, but of course, the estimates are not uniform in ε. However, the u^ε_{app} satisfy for all $\alpha \in \mathbb{Z}^{d+1}$ and $k \in \mathbb{N}$:

$$\sup_{\varepsilon\in]0,1]} \|\varepsilon^k Z^\alpha \partial_x^k u^\varepsilon_{app}\|_{L^\infty\cap L^2([0,T]\times\mathbb{R}^d_+)} < +\infty\,. \tag{4.5.2}$$

When (4.5.1) is substituted in the equation, we see that

$$r^\varepsilon_{app} := (L-\varepsilon P)u^\varepsilon_{app} - f - F(u^\varepsilon_{app}) \tag{4.5.3}$$

is a function of the form $\varepsilon^{-1}R(\varepsilon,t,y,x,x/\varepsilon)$ where $R(\varepsilon,t,y,x,z)$ is a smooth function of its arguments. Moreover, in the Taylor expansion $R(\varepsilon,\cdot) \sim \varepsilon^n R_n(\cdot)$, the first $m+1$ terms $R_0,\dots,R_m$ vanish. Therefore,

$$r^\varepsilon_{app} = \varepsilon^m\phi^\varepsilon\,, \quad \phi^\varepsilon(t,y,x) = \Phi\big(\varepsilon,t,y,x,\frac{x}{\varepsilon}\big)$$

with $\{\Phi(\varepsilon,\cdot)\}$ bounded in $\mathcal{P}(T)$. Therefore, the ϕ^ε satisfy

$$\sup_{\varepsilon\in]0,1]} \|\varepsilon^k Z^\alpha \partial_x^k \phi^\varepsilon\|_{L^\infty\cap L^2([0,T]\times\mathbb{R}^d_+)} < +\infty\,. \tag{4.5.4}$$

The u^ε_{app} satisfy

$$u^\varepsilon_{app|x=0} = 0\,, \quad u^\varepsilon_{app|t=0} = h^\varepsilon_{app} := \sum_{n=0}^{m}\varepsilon^n h_n\,. \tag{4.5.5}$$

Moreover, if one assume that $U_n = \underline{U}_n + U_n^* \in \mathcal{P}(T)$, $\underline{U}_n = 0$ on a domain $\Delta = [0,T] \times \mathbb{R}^d_+ \cap \{x + ct \le r\}$ and $U_n^* = 0$ for $ct \le r$, for some $c > 0$ and $r > 0$, then u^ε_{app} and ϕ^ε also vanish on Δ.

Comment. u^ε_{app} is an approximate solution of the equation, in the sense that it satisfies the boundary condition and also satisfies the equation up to an error term which is $O(\varepsilon^m)$. The question is to know wether there exists an exact solution u^ε close to u^ε_{app}. Typically, one expects that $u^\varepsilon - u^\varepsilon_{app} = O(\varepsilon^m)$.

4.5.2 An equation for the remainder

We look for exact solutions of (4.1.1) with initial condition

$$u^\varepsilon{}_{|t=0} = h^\varepsilon = h^\varepsilon_{app} + \varepsilon^m \ell^\varepsilon \,. \tag{4.5.6}$$

We look for u^ε as a perturbation of u^ε_{app}:

$$u^\varepsilon = u^\varepsilon_{app} + \varepsilon^m v^\varepsilon \,. \tag{4.5.7}$$

Introduce the notation

$$F(u + v) = F(u) + F'(u)v + Q(u, v; v)$$

with $Q(u, v; w)$ quadratic in w. The equation for v^ε reads

$$\begin{cases} (L - \varepsilon P)v^\varepsilon = E^\varepsilon v^\varepsilon + \varepsilon^m G^\varepsilon(v^\varepsilon) + \phi^\varepsilon \,, \\ v^\varepsilon{}_{|x=0} = 0 \,, \\ v^\varepsilon{}_{|t=0} = \ell^\varepsilon \,, \end{cases} \tag{4.5.8}$$

with

$$E^\varepsilon = F'(u^\varepsilon_{app}) \,, \quad G^\varepsilon(v) = Q(u^\varepsilon_{app}, \varepsilon^m v; v) \,. \tag{4.5.9}$$

We use the notations $\mathcal{H}^s$, H^s_{co} of Chapter three and we fix $s > 1 + \frac{d}{2}$. By (4.5.4), the ϕ^ε belong to $\mathcal{H}^{2s}$ and are bounded in $H^s_{co}([0;T])$.

We consider initial data ℓ^ε such that

$$\sup_{\varepsilon \in]0,1]} \|\ell^\varepsilon\|_{H^{2s+1}(\mathbb{R}^d_+)} < +\infty \,, \quad \ell^\varepsilon{}_{|\{x \le r\}} = 0 \,. \tag{4.5.10}$$

Since $\phi^\varepsilon = 0$ for $x + ct \le r$, the data ϕ^ε and ℓ^ε are compatible to order $2s$.

Theorem 4.5.1. *If $m > 0$, there is $\varepsilon_0 > 0$, such that for all $\varepsilon \in]0, \varepsilon_0]$, the problem (4.5.8) has a solution $v^\varepsilon \in \mathcal{H}^{2s+2}([0,T])$ and the v^ε are uniformly bounded in $H^s_{co}([0,T] \cap L^\infty([0,T] \times \mathbb{R}^d_+)$.*

Consider the Taylor expansion at time $t = 0$ of the solutions. Using the notation $L = \partial_t + A$, the traces v^ε_j for $j \in \{1, \dots, s\}$ are given by induction by $v^\varepsilon_0 = \ell^\varepsilon$ and for $j \geq 1$:

$$(4.5.11) \quad v^\varepsilon_j = (\varepsilon P - A) v^\varepsilon_{j-1} + \phi_{j-1} + \sum_{k=0}^{j-1} E^\varepsilon_k v^\varepsilon_{j-k-1} + G_{j-1}(v^\varepsilon_0, \dots, v^\varepsilon_{j-1})$$

where G_j is a linear combination of terms of the form

$$\varepsilon^\kappa \tilde{Q}(a^\varepsilon_0, \varepsilon^m v^\varepsilon_0)\, a^\varepsilon_{k_1} \dots a^\varepsilon_{k_p}\, v^\varepsilon_{j_1} \dots v^\varepsilon_{j_q}$$

where $\tilde{Q}(a, v)$ is a multi-linear function depending smoothly on (a, v) and $\kappa \geq 0$, $p + q \leq j$, $k_1 + \dots + k_p + j_1 + \dots + j_q = j$ and

$$\phi^\varepsilon_j = \partial^j_t \phi^\varepsilon{}_{|t=0}\,, \quad E^\varepsilon_j = \partial^j_t E^\varepsilon{}_{|t=0}\,, \quad a^\varepsilon_j = \partial^j_t u^\varepsilon_{app|t=0}\,, \quad 0 \leq j \leq s-1\,.$$

Lemma 4.5.2. *i) For $j \leq s$, the v^ε_j are uniformly bounded in to $H^{2s+1-2j}$ and vanish for $x \leq r$.*

ii) There is a bounded family $\{v^{\varepsilon,0}\}_{\varepsilon \in]0,1]} \subset \mathcal{H}^{2s+2}$ such that for all $\varepsilon \in]0,1]$ and $j \in \{0, \dots, s\}$, $\partial^j_t v^{\varepsilon,0}{}_{|t=0} = v^\varepsilon_j$.

Proof. The first statement immediately follows from (4.5.2), (4.5.4) and (4.5.10).

Next, since the v^ε_j vanish for $x < r$, their extension by 0 for $x \leq 0$ belong and are bounded in $H^{2s+1-2j}(\mathbb{R}^d)$. The traces can be lifted up to $\mathcal{H}^{2s+2}$ as in the proof of Theorem 3.5.3: denoting by $\hat{\cdot}$ the spatial Fourier transform define

$$\hat{v}^{\varepsilon,0}(t, \eta, \xi) = \sum_{j=0}^{s} \frac{t^j}{j!} \chi(t\Lambda^2) \hat{v}^\varepsilon_j(\eta, \xi)\,, \quad \Lambda = (1 + |\eta| + |\xi|)\,.$$

$\square$

Starting from $v^{\varepsilon,0}$, we consider for $\nu \geq 1$ the iterative scheme

$$(4.5.12) \qquad \begin{cases} (L - \varepsilon P) v^{\varepsilon,\nu} = \phi^\varepsilon + E^\varepsilon v^{\varepsilon,\nu-1} + G^\varepsilon(v^{\varepsilon,\nu-1})\,, \\ v^{\varepsilon,\nu}{}_{|x=0} = 0\,, \qquad v^{\varepsilon,\nu}{}_{|t=0} = \ell^\varepsilon\,. \end{cases}$$

Theorem 4.5.1 follows from the next statement:

Proposition 4.5.3. *There is $\varepsilon_0 > 0$, such that for all $\varepsilon \in]0,1]$, the equations (4.5.12) define a Cauchy sequence $v^{\varepsilon,\nu}$ in $\mathcal{H}^{2s+2}([0,T])$ such that for all $j \in \{0,\dots,s\}$, $\partial_t^j v^{\varepsilon,\nu}{}_{|t=0} = v_j^\varepsilon$.*

Moreover, the family $\{v^{\varepsilon,\nu} : \varepsilon \in]0,\varepsilon_0], \nu \in \mathbb{N}\}$ is bounded in $L^\infty \cap H^s_{co}([0,T]\times\mathbb{R}^d_+)$.

Proof. **a)** We use the following notations, inspired with minor modifications from Chapter three:

$$
(4.5.13)\qquad
\begin{aligned}
\mathrm{n}_\varepsilon(u;t) &= \sum_{2j+|\alpha|\le 2s+1} \varepsilon^{j+|\alpha|} \|\partial_t^j \partial_{y,x}^\alpha u(T)\|_{L^2(\mathbb{R}^d_+)}\\
\mathrm{N}'_\varepsilon(u;t) &= \sum_{0<2j+|\alpha|\le 2s+2} \varepsilon^{j+|\alpha|-1/2} \|\partial_t^j \partial_{y,x}^\alpha u\|_{L^2([0,T]\times\mathbb{R}^d_+)}\,,\\
\mathrm{N}_\varepsilon(f;T) &= \sum_{0\le 2j+|\alpha|\le 2s} \varepsilon^{j+|\alpha|} \|\partial_t^j \partial_{y,x}^\alpha f\|_{L^2([0,T]\times\mathbb{R}^d_+)}\,,\\
\mathrm{n}_{co}(u;t) &= \sum_{|\alpha|\le s} \|Z^\alpha u(t)\|_{L^2(\mathbb{R}^d_+)}\,.
\end{aligned}
$$

By (4.5.2), E^ε satisfies for all k and α

$$
\sup_{\varepsilon\in]0,1]} \varepsilon^k \|Z^\alpha \partial_x^k E^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^d_+)} < +\infty\,.
$$

Therefore, there is a constant C such that for all $v \in \mathcal{H}^{2s}([0,T])$, $E^\varepsilon v \in \mathcal{H}^{2s}([0,T])$ and for all $t \le T$

$$
(4.5.14)\qquad
\begin{cases}
\mathrm{N}_\varepsilon(E^\varepsilon v;t) \le C\mathrm{N}_\varepsilon(v;t)\,,\\
\|E^\varepsilon v\|_{H^s_{co}([0,t])} \le C\|v\|_{H^s_{co}([0,t])} \le C\Big(\displaystyle\int_0^t \mathrm{n}_{co}(v;t')dt'\Big)^{1/2}.
\end{cases}
$$

Using the estimates (4.5.2) for u^ε_{app} and Corollary 3.5.5 on one hand and (3.5.18) on the other hand, one obtains that for $v \in \mathcal{H}^{2s}([0,T])$, $Q^\varepsilon(v) \in \mathcal{H}^{2s}([0,T])$ and there is a function $C_G(\cdot)$ such that

$$
(4.5.15)\qquad
\begin{cases}
\mathrm{N}_\varepsilon(Q^\varepsilon(v);T) \le C_G(R)\big(1+\mathrm{N}_\varepsilon(v;T)\big)\,,\\
\|Q^\varepsilon(v)\|_{H^s_{co}([0,T])} \le C_G(R)\big(1+\|v\|_{H^s_{co}([0,T])}\big)
\end{cases}
$$

with $R := \|v\|_{L^\infty([0,T]\times\mathbb{R}^d_+)}$.

b) Suppose that $v^{\varepsilon,\nu-1} \in \mathcal{H}^{2s+2}([0,T])$ is defined and satisfies the trace conditions $\partial_t^j v^{\varepsilon,\nu-1}{}_{|t=0} = v_j^\varepsilon$ for $j \le s$. By definition (4.5.11), we see that the traces associated to the equation (4.5.12) are $v_j^{\varepsilon,\nu} = v_j^\varepsilon$. Thus they vanish for $x < r$ and the compatibility conditions are satisfied. Moreover, the right hand side belongs to $\mathcal{H}^{2s}([0,T])$. Therefore, by Theorem 3.3.2 there is a unique solution $v^{\varepsilon,\nu} \in \mathcal{H}^{2s+2}([0,T])$ which satisfies the trace conditions.

Moreover, Proposition 3.4.5 and the estimate above imply that there are constants C_0 and C_1 and a function $C_G(\cdot)$, depending only on the data, such that for all $\varepsilon \in]0,1], \nu \geq 1$ and $t \in [0,T]$:

$$\begin{aligned} \mathrm{n}_{co}(v^{\varepsilon,\nu};t) \leq C_0 + & C_1 \|v^{\varepsilon,\nu-1}\|_{H^s_{co}([0,t])} \\ & + \varepsilon^m C_G(R^{\varepsilon,\nu-1})\big(1 + \|v^{\varepsilon,\nu-1}\|_{H^s_{co}([0,T])}\big)\,, \end{aligned} \tag{4.5.16}$$

$$\begin{aligned} R^{\varepsilon,\nu} \leq C_0 + & C_1 \|v^{\varepsilon,\nu-1}\|_{H^s_{co}([0,T])} \\ & + \varepsilon^m C_G(R^{\varepsilon,\nu-1})\big(1 + \|v^{\varepsilon,\nu-1}\|_{H^s_{co}([0,T])}\big)\,, \end{aligned} \tag{4.5.17}$$

with $R^{\varepsilon,\nu} := \|v^{\varepsilon,\nu}\|_{L^\infty([0,T]\times\mathbb{R}^d_+)}$. We show by induction that there are constants $\varepsilon_0 > 0$, C_2, R and K such that $\varepsilon \in]0,1], \nu \geq 1$ and $t \in [0,T]$:

$$\begin{cases} \mathrm{n}_{co}(v^{\varepsilon,\nu};t) \leq C_2 e^{Kt}\,, \\ R^{\varepsilon,\nu} \leq R\,. \end{cases} \tag{4.5.18}$$

The estimates are clearly satisfied for $\nu = 0$, provided that

$$C_2 \geq \sup_{t\in[0,T]} \mathrm{n}_{co}(v^{\varepsilon,0};t)\,, \quad R \geq \|v^{\varepsilon,0}\|_{L^\infty([0,T]\times\mathbb{R}^d_+)}\,. \tag{4.5.19}$$

These conditions only depend on the data. We choose successively C_2, K, R and ε_0 such that in addition to (4.5.19) there holds

$$\begin{aligned} & C_2 \geq 3C_0\,, \quad \sqrt{2K} \geq 3C_1\,, \quad R \geq 2C_0 + C_2 e^{KT}\,, \\ & \varepsilon_0^m C_G(R)\big(1 + C_2(2K)^{-1/2}e^{KT}\big) \leq C_0\,. \end{aligned} \tag{4.5.20}$$

Suppose that the estimate (4.5.18) is satisfied up to the order $\nu - 1$. Then

$$\|v^{\varepsilon,\nu-1}\|_{H^s_{co}([0,t])} = \Big(\int_0^t \big(\mathrm{n}_{co}(v^{\varepsilon,\nu-1};t')\big)^2 dt'\Big)^{1/2} \leq \frac{C_2}{\sqrt{2K}} e^{Kt}\,.$$

Therefore (4.5.16) implies:

$$\mathrm{n}_{co}(v^{\varepsilon,\nu};t) \leq C_0 + C_1 \frac{C_2}{\sqrt{2K}} e^{Kt} + \varepsilon^m C_G(R)\big(1 + C_2(2K)^{-1/2}e^{KT}\big)\,.$$

If $\varepsilon \le \varepsilon_0$, this implies that

$$\mathrm{n}_{co}(v^{\varepsilon,\nu};t) \le 2C_0 + \frac{C_2}{3}e^{Kt} \le C_2 e^{Kt}\,.$$

Similarly, (4.5.17) implies for $\varepsilon \le \varepsilon_0$:

$$R^{\varepsilon,\nu} \le C_0 + \frac{C_2}{3}e^{KT} + \varepsilon^m C_G(R)\big(1 + C_2(2K)^{-1/2}e^{KT}\big) \le R\,.$$

This shows that (4.5.18) holds at the order ν and the proof of Theorem 4.5.1 is complete. $\square$

4.5.3 Exact solutions and their asymptotic expansions

We can now gather the different results obtained above. Suppose that $f \in H^\infty([0,T_0] \times \mathbb{R}^d_+)$. Suppose that h^ε is a bounded family in $H^\infty(\mathbb{R}^d_+)$ with asymptotic expansion (4.2.12) in the sense that for all integer m

$$\varepsilon^{-m}\big(h^\varepsilon - \sum_{n=0}^{m} \varepsilon^n h_n\big)$$

is bounded in $H^\infty(\mathbb{R}^d_+)$. Assume in addition that the support conditions (4.2.13) are satisfied.

By Theorem 4.2.3, we construct a formal solution $\sum \varepsilon^m U_n$ on $[0,T]\times\mathbb{R}^d_+$, for some $T \in]0,T_0]$.

Theorem 4.5.4. *There is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0,\varepsilon_0]$, the problem* (4.1.1) *has a solution u^ε in $H^\infty([0,T] \times \mathbb{R}^d_+)$. Moreover, for all $m \ge 0$ and $s \ge 0$,*

$$r^{\varepsilon,m}(t,y,x) := u^\varepsilon(t,y,x) - \sum_{n=0}^{m} \varepsilon^n U_n(t,y,x,x/\varepsilon)$$

satisfies

$$\|v^{\varepsilon,m}\|_{L^\infty \cap H^s_{co}([0,T]\times\mathbb{R}^d_+)} = O(\varepsilon^{m+1})\,. \tag{4.5.21}$$

Proof. Applying Theorem 4.5.1 with $m = 1$ and s_0 large enough, gives a solution $u^\varepsilon \in \mathcal{H}^{2s_0+2}$ for $\varepsilon \le \varepsilon_0$. These solutions are bounded in $L^\infty([0,T] \times \mathbb{R}^d_+)$. Since the data are infinitely smooth and infinitely compatible, Theorem 3.5.6 implies that $u^\varepsilon \in \mathcal{H}^\infty = H^\infty$.

By uniqueness, Theorem 4.5.1 applied to any m and s large enough, implies that for ε small enough, $r^{\varepsilon,m} = u^\varepsilon - u^\varepsilon_{app}$ satisfies:

$$\|r^{\varepsilon,m}\|_{L^\infty\cap H^s_{co}} = O(\varepsilon^m)\,,$$

Since the last term in u^ε_{app} is $O(\varepsilon^m)$, this implies that $r^{\varepsilon,m-1} = O(\varepsilon^m)$ and the theorem follows. $\square$

Remark 4.5.5. One can consider finite expansions and in particular restrict attention to $u^\varepsilon(t, y, x) - U_0(t, y, x, x/\varepsilon)$. One could also consider profiles with finite smoothness, but the precise count of derivatives to get m terms in the expansion and to prove the approximation in H^s_{co} is delicate.

Chapter 5

Quasilinear boundary layers: the inner layer o.d.e.

In this Chapter, we start the analysis of quasilinear equations. We study the ordinary differential equation satisfied by stationary solutions which depend only on the normal variable. The admissible solutions w connect 0 at $z = 0$ to a bounded end state at $z = +\infty$. The set $\mathcal{C}$ of those reachable end states is given by a central manifold theorem. It determines the boundary conditions associated to the limiting hyperbolic system. The local structure of $\mathcal{C}$, depends on transversality conditions or equivalently on stability conditions of the o.d.e.

5.1 The equations

Consider a first order quasilinear system

$$L(u, \partial)u := \partial_t u + \sum_{j=1}^{d} A_j(u)\partial_j u = F(u) \tag{5.1.1}$$

The equation holds on $\mathbb{R} \times \mathbb{R}^d_+$. The unknown u is valued in $\mathbb{R}^N$. The space time variables are (t, y, x) as in Chapter 2.

Next, we consider a parabolic viscous perturbation of (5.1.1)

$$L(u, \partial)u - \varepsilon \sum_{1 \le j,k \le d} \partial_j \big(B_{j,k}(u)\partial_k u \big) = F(u)\,. \tag{5.1.2}$$

with Dirichlet boundary conditions:

$$u_{|x=0} = 0\,. \tag{5.1.3}$$

The parabolic solutions take values in $\mathcal{U}^* \subset \mathbb{R}^N$ while the hyperbolic solutions take values in $\mathcal{U} \subset \mathcal{U}^*$. It is assumed that $0 \in \mathcal{U}^*$.

Assumption 5.1.1.

(H0) *The A_j and $B_{j,k}$ are $N \times N$ real matrices, C^∞ for u in $\mathcal{U}^*$; F is a smooth function from $\mathcal{U}^*$ to $\mathbb{R}^N$.*

(H1) *There is $c > 0$ such that for all $u \in \mathcal{U}^*$ and all $\xi \in \mathbb{R}^d$ the eigenvalues of $\sum_{j,k=1}^d \xi_j \xi_k B_{j,k}(u)$ satisfy $\operatorname{Re}\mu \geq |\xi|^2$.*

(H2) *For all $u \in \mathcal{U}$, the eigenvalues of $\sum \xi_j A_j(u)$ are real and semi-simple. Moreover, the multiplicities are constant for $u \in \mathcal{U}$ and $\xi \in \mathbb{R}^d \backslash \{0\}$.*

(H3) *There is $c > 0$ such that for all $u \in \mathcal{U}$ and $\xi \in \mathbb{R}^d$ the eigenvalues of $\sum_{j=1}^d i\xi_j A_j(u) + \sum_{j,k=1}^d \xi_j \xi_k B_{j,k}(u)$ satisfy $\operatorname{Re}\mu \geq |\xi|^2$.*

(H4) *For all $u \in \mathcal{U}$, there holds $\det A_d(u) \neq 0$.*

Remarks 5.1.2. 1) The assumption (H1) means that for all $\varepsilon > 0$ the equation (5.1.2) is parabolic, for all state u in the large set $\mathcal{U}^*$. The first role of (H2) is to ensure that the first order equation (5.1.1) is hyperbolic for all state u in the domain $u \in \mathcal{U}$. It is important for applications to consider situations where the domain of hyperbolicity $\mathcal{U}$ is strictly smaller than the domain of definition $\mathcal{U}^*$ of the equation; for instance this occurs for Euler's equation with non monotone state laws.

The theory of hyperbolic boundary value problems is well developed in two cases: first in the case of dissipative boundary conditions for symmetric systems, as explained in Chapter two; second in the case of hyperbolic systems with constant multiplicity and boundary conditions satisfying a uniform Lopatinski condition (see [Kre], [Ch-Pi], [Maj], [Mé3]). In the second part of this book we have chosen to consider the second framework, first to give a new approach, and second to introduce to the sharp stability conditions which involve Lopatinski determinants. With (H4), we assume again that the boundary is noncharacteristic.

(H3) is a compatibility condition between the hyperbolic part and the parabolic singular perturbation. On one hand, letting ξ tend to zero, it implies that L is hyperbolic, but does not say anything about the multiplicities. On the other hand, letting ξ tend to infinity it implies that the condition in (H1) holds, but only for $u \in \mathcal{U}$.

2) For instance, if L satisfies (H2), then the assumptions (H1) (H2) are satisfied for the *artificial viscosity* perturbations:

$$L(u, \partial)u - \varepsilon \Delta u \,.$$

3) In the *symmetric case*, that is when there is a symmetric definite positive matrix $S(u)$ for $u \in \mathcal{U}^*$, such that SA_j and $SB_{j,k}$ are symmetric and $\sum \xi_j \xi_k SB_{j,k}$ is definite positive for $\xi \neq 0$ (see Assumption 3.1.1 of Chapter three), the assumptions (H1) and (H3) satisfied, as well as the first part of (H2). The full assumption (H2) requires that in addition the eigenvalues have constant multiplicity. On the other hand, in the analysis below, the assumptions on the boundary conditions are not restricted to the dissipative case. This is important because, in contrast with the linear or semilinear case, for large amplitude quasilinear layers, there is no analogue of Proposition 4.3.3, and for symmetric systems, the limiting inviscid boundary conditions are not necessarily maximal dissipative.

We denote by N_+ [resp. N_-] the number of positive [resp. negative] eigenvalues of $A_d(u)$ counted with their multiplicity. Then, by (H4), $N = N_+ + N_-$.

Lemma 5.1.3. *For $u \in \mathcal{U}$,*

i) The matrix $B_{d,d}(u)$ is invertible with eigenvalues in $\operatorname{Re}\mu \geq c$.

ii) The matrix $G_d(u) := B_{d,d}(u)^{-1}A_d(u)$ has no eigenvalue on the imaginary axis. N_+ of them, counted with their multiplicity, have positive real part and N_- have negative real part.

Proof. Taking $\xi = (0, \ldots, 1/\delta)$, (H3) implies that for $\delta \neq 0$, the spectrum of $B_{d,d}(u) + i\delta A_d(u)$ is contained in $\operatorname{Re}\mu \geq c$. Letting δ tend to zero implies *i)*.

If μ is an eigenvalue of $G_d(u)$, then $\mu \neq 0$ and 0 is an eigenvalue of $-\mu^{-1}A_d(u) + B_{d,d}(u)$. Thus, $\mu \notin i\mathbb{R}$.

Similarly, for $t \in [0,1]$, the spectrum of $tB_{d,d}(u) + (1-t)\mathrm{Id} + i\delta A_d(u)$ is contained in $\operatorname{Re}\mu > 0$ and $G(t) := \big(tB_{d,d}(u) + (1-t)\mathrm{Id}\big)^{-1}A_d(u)$ has no eigenvalue on the imaginary axis. Thus the number of eigenvalues of $G(t)$ in $\operatorname{Re}\mu > 0$ is constant for $t \in [0,1]$ and equal to N_+ when $t = 0$. $\square$

We use the following notation from Chapter four: given a matrix G, $\mathbb{E}_\pm(G)$ denotes the invariant space generated by the generalized eigenvectors of G associated to eigenvalues μ lying in $\{\pm\operatorname{Re}\mu > 0\}$. We denote by $\Pi_\pm(G)$ the corresponding spectral projectors. When G is real, then $\Pi_\pm$ are real so that the spaces $\mathbb{E}_\pm$ are real. In particular, for all $u \in \mathcal{U}$, we have

$$\mathbb{R}^N = \mathbb{E}_+(G_d(u)) \oplus \mathbb{E}_-(G_d(u))$$

and $\mathbb{E}_-(G_d) = \bigsqcup_{u \in \mathcal{U}} \mathbb{E}_-(G_d(u))$ form a fiber bundle over $\mathcal{U}$.

5.2 The inner-layer ode, and the hyperbolic boundary conditions

There are *exact* stationary solutions which depend only on the normal variable, scaled to describe the inner layer:

$$u^\varepsilon(t, y, x) = u(x/\varepsilon)\,. \tag{5.2.1}$$

In this case, the equation for u^ε is equivalent to the ordinary differential system:

$$A_d(u)\partial_z u - \partial_z\big(B_{d,d}(u)\partial_z u\big) = 0\,. \tag{5.2.2}$$

One can also consider *approximate* solutions, using BKW expansions

$$u^\varepsilon(t, y, x) = U_0(t, y, x, x/\varepsilon) + \varepsilon U_1(t, y, x, x/\varepsilon) + \varepsilon^2 \dots$$

with $U_n(t, y, x, z) = u_n(t, y, x) + U_n^*(t, y, z)$ and U_n^* rapidly decreasing when $z \to +\infty$. Plugging this expansion in the equation, the singular term in ε^{-1} yields the same equation (5.2.2) for $z \mapsto U_0(t, y, x, z)$. The boundary condition (5.1.3) reads

$$U_n(t, y, 0, 0) = 0$$

and the convergence at infinity implies that $w(z) = U_0(t, y, 0, z)$ must also satisfy the boundary conditions

$$w(0) = 0\,, \quad \lim_{z\to+\infty} w(z) = u_0(t, y, 0)\,.$$

In particular, we see that the supposed interior limit u_0 must satisfy the boundary condition

$$u_0(t, y, 0) \in \widetilde{\mathcal{C}} \tag{5.2.3}$$

where

Definition 5.2.1. *$\widetilde{\mathcal{C}}$ is the set of $p \in \mathcal{U}$ such that there exists a solution u on $[0, +\infty[$ of the profile equation*

$$A_d(u)\partial_z u - \partial_z\big(B_{d,d}(u)\partial_z u\big) = 0\,, \quad u(0) = 0\,, \quad \lim_{z\to+\infty} u(z) = p\,. \tag{5.2.4}$$

This section is devoted to the analysis of (5.2.4) and to the construction of *smooth* pieces $\mathcal{C}$ of $\widetilde{\mathcal{C}}$.

5.2.1 Example: Burgers equation

Consider in space dimension one, the Burgers-Hopf equation:

$$\partial_t u + u\partial_x u - \varepsilon \partial_x^2 u = 0\,. \tag{5.2.5}$$

In this case, the inner-layer o.d.e is

$$\partial_z^2 u = u\partial_z u\,, \quad u(0) = 0\,. \tag{5.2.6}$$

The equation can be integrated once yielding

$$\partial_z u = \frac{1}{2}u^2 + k\,, \quad u(0) = 0\,.$$

Depending on the sign of the constant k, there are two families of solutions

$$\begin{aligned} &1) \quad u(z) = -\lambda \tanh\big(\lambda z/2\big) \\ &2) \quad u(z) = \mu \tan\big(\mu z/2\big)\,. \end{aligned}$$

Changing λ into $-\lambda$ or μ into $-\mu$ does not change the solution, so we can assume that the parameters are nonnegative. The two families intersect only on the trivial solution $u = 0$.

Solutions of the second family, have a finite time of existence: they do not provide solutions of (5.2.5) on the half line.

Thus, we have to restrict attention to solutions of the first family, which are globally defined. In this case, we have

$$\lim_{z\to+\infty} u(z) = -\lambda \le 0\,.$$

The end state $-\lambda$ is non characteristic (i.e. satisfies (H4)) if $\lambda \neq 0$. Thus we have shown:

for the Burgers equation (5.2.5), *the set of end states* p *which satisfy (H4) and which can be connected to 0 by a solution of* (5.2.6) *is* $\widetilde{\mathcal{C}} =]-\infty, 0[$.

5.2.2 Example: the linear case.

Suppose that A_d and $B_{d,d}$ are constant (independent of u). This situation has been analyzed in Chapter 2. The o.d.e. reads

$$\partial_z^2 u - G_d \partial_z u\,. \tag{5.2.7}$$

The solutions of the o.d.e are

$$u(z) = p + e^{zG_d} a\,,$$

with $G_d = (B_{b,d})^{-1}A_d$ and arbitrary constants p and a. Because the eigenvalues of G_d are real and different from zero, the explicit formula implies the following results.

1. *The solution is bounded if and only if* $a \in \mathbb{E}_-(G_d)$.

2. *Bounded solutions of* (5.2.7) *converge at an exponential rate at infinity.*

3. *The bounded solutions of* (5.2.7) *form a manifold of dimension* $N + N_-$.

If we now add the initial condition $u(0) = 0$, we obtain

4. *The set of end states p which can be connected to* 0 *by a solution of* (5.2.4) *is equal to* $\mathbb{E}_-(G_d)$. *In particular, the set of bounded solutions of the boundary value problem form a manifold of dimension* N_-.

Remember that $p \in \mathbb{E}_-(G_d)$ appeared as the limiting boundary condition for the hyperbolic problem in Chapter four.

5.3 Solutions of the inner layer o.d.e.

We come back to the equation (5.2.2). Our goal is to extend the analysis performed in the linear example. First, we note that the constants are solutions of the o.d.e. We study solutions which converge at infinity.

Lemma 5.3.1. *Suppose that u satisfies the equation* (5.2.2) *on the interval* $[z_0, +\infty[$ *and* $u(z) \to p \in \mathcal{U}$ *as* $z \to +\infty$. *Then,* $\partial_z u(z)$ *tends to zero when z tends to infinity.*

Proof. With $v = B_{d,d}\partial_z u$, (5.2.2) implies that

(5.3.1)
$$\partial_z v = H(u)v\,.$$

with $H(u) := A_d(u)\big(B_{d,d}(u)\big)^{-1}$. Therefore,

$$|\partial_z v(z)| \le C|v(z)|\,, \quad C = \max_{[z_0,+\infty[} |H(u(z))|\,.$$

$$|\partial_z u(z)| \le C_1|v(z)|\,, \quad C_1 = \max_{[z_0,+\infty[} |B_{d,d}^{-1}(u(z))|\,.$$

For $z_1 \ge z_0$, consider the interval $I = [z_1, z_1+1]$ and $z_2 \in I$ such that

$$m := \max_I |v(z)| = |v(z_2)|\,.$$

There is a unit vector ℓ such that $\ell \cdot v(z_2) = m$. Since $|\partial_z v| \le Cm$ on I, there holds $\ell \cdot v(z) \ge m/2$ for $z \in I_1 := \{z \in I : |z - z_2| \le 1/(2C)\}$. Let $\delta = \min\{1, 1/(2C)\}$. Then I_1 contains an interval $[z_3, z_3+\delta]$ and

$$\ell \cdot B_{d,d}(u)\partial_z u \ge m/2 \quad \text{on } [z_3, z_3+\delta]\,.$$

Since u tends to p at infinity and $B_{d,d}$ is invertible, write

$$ {}^tB_{d,d}(u)\ell = \underline{\ell}' + \ell'(z) $$

with $\underline{\ell}' \neq 0$ and $\ell'(z) \to 0$ at infinity. Since ℓ is unitary, one has

$$ |\underline{\ell}'| \geq |{}^tB_{d,d}^{-1}(p)|^{-1}\,, \quad |\ell'(z)| \leq |{}^tB_{d,d}(p) - {}^tB_{d,d}(u(z))| $$

In particular, if z_1 is large enough, one has $|\ell'(z)| \leq 1/4C_1$ and therefore

$$ \underline{\ell}' \cdot \partial_z u \geq m/2 - |\ell'(z)||\partial_z u| \geq m/4 \quad \text{on } [z_3, z_3+\delta]\,. $$

Therefore, we have shown that there are $c > 0$ and $\delta > 0$ such that for all z_1 large enough, there is $\underline{\ell}'$ with $|\underline{\ell}'| \geq c$ and there is $z_3 \in I = [z_1, z_1+1]$ such that

$$ \underline{\ell}' \cdot (u(z_3+\delta) - u(z_3)) \geq m\delta/4\,. $$

Since u has a limit at infinity, the left hand side tends to zero as z_3 tends to infinity and therefore m must tend to zero as z_1 tends to infinity. $\square$

We continue the analysis of bounded solutions of (5.2.2). With $U = (u, v)$, $v = \partial_z u$, the equation is equivalent to the first order system

$$ \left\{ \begin{aligned} &\partial_z u = v\,, \\ &\partial_z v = G_d(u)v - B_{d,d}^{-1}(u)\big(v \cdot \nabla_u B_{d,d}(u)\big)v\,. \end{aligned} \right. \tag{5.3.2} $$

We look for solutions such that

$$ \lim_{z\to\infty} u(z) = p \in \mathcal{U}\,. \tag{5.3.3} $$

We consider (5.3.2) as a quadratic perturbation of

$$ \left\{ \begin{aligned} &\partial_z u = v\,, \\ &\partial_z v = G_d(p)v\,. \end{aligned} \right. $$

Set

$$ F(u,v) = \big(G_d(u) - G_d(p)\big)v - B_{d,d}^{-1}(u)\big(v \cdot \nabla_u B_{d,d}(u)\big)v $$

which is quadratic in $(u-p, v)$. More precisely, there holds

$$ F(u,v) = O\big(|u-p||v| + |v|^2\big)\,. \tag{5.3.4} $$

Suppose that v satisfies on $[z_0, +\infty[$

$$ \partial_z v = G_d(p)v + F\,, \tag{5.3.5} $$

then, denoting by $\Pi_\pm(p)$ the spectral projectors on $\mathbb{E}_\pm(G_d(p))$, one has for all z and z_1 in $[z_0, +\infty[$:

$$\begin{cases} \Pi_-(p)v(z) = e^{(z-z_0)G_d(p)}\Pi_-(p)v(z_0) + \displaystyle\int_{z_0}^{z} e^{(z-s)G_d(p)}\Pi_-(p)F(s)ds\,, \\ \Pi_+(p)v(z) = e^{(z-z_1)G_d(p)}\Pi_+(p)v(z_1) - \displaystyle\int_{z}^{z_1} e^{(z-s)G_d(p)}\Pi_+(p)F(s)ds\,. \end{cases}$$

Note that there is $\theta > 0$ such that

$$\begin{aligned} &\left|e^{(z-s)G}\Pi_-\right| \leq C_0 e^{-\theta(z-s)}\,, \qquad s \leq z\,, \\ &\left|e^{(z-s)G}\Pi_+\right| \leq C_0 e^{-\theta(s-z)}\,, \qquad s \geq z\,. \end{aligned} \tag{5.3.6}$$

Therefore, the operator

$$\begin{aligned} \mathcal{I}_{z_0}(F)(z) = &\int_{z_0}^{z} e^{(z-y)G_d(p)}\Pi_-(p)F(y)dy \\ &- \int_{z}^{\infty} e^{(z-y)G_d(p)}\Pi_+(p)F(y)dy\,. \end{aligned}$$

is bounded from $L^r([z_0, +\infty[)$ to itself for all $r \in [1, \infty]$. Thus, if $F \in L^r$ and v is a solution of (5.3.5) which tends to zero at infinity, letting z_1 tend to $+\infty$ in the representation of $\Pi_+(p)v(z)$ above, implies that v satisfies

$$v(z) = e^{(z-z_0)G_d(p)}\Pi_-(p)v(0) + \mathcal{I}_{z_0}(F)(z)\,.$$

In particular, together with Lemma 5.3.1, this implies the following result.

Lemma 5.3.2. *If u is a solution of* (5.2.2) *on* $[z_0, +\infty[$ *satisfying* (5.3.3), *then $v = \partial_z u$ tends to zero at infinity and satisfies the integral equation*

$$v(z) = e^{(z-z_0)G_d(p)}\Pi_-(p)v(0) + \mathcal{I}_{z_0}(F(u, v))(z)\,. \tag{5.3.7}$$

Next, we show that the solutions of (5.3.7) have exponential decay at infinity.

Lemma 5.3.3. *If u is a solution of* (5.2.2) *on* $[z_0, +\infty[$ *satisfying* (5.3.3), *then for all $\delta < \theta$, $e^{\delta z}\partial_z u$ is bounded.*

Proof. For $\delta \in [0, \theta[$, denote by $\|\cdot\|_\delta$ the norm

$$\|v\|_\delta = \sup_{z \geq 0}\left|e^{\delta z}v(z)\right| \geq \|v\|_0 = \|v\|_{L^\infty}\,.$$

Consider the mapping $w \mapsto f(w) = F(u,w)$. From (5.3.4), we deduce that for all $w \in e^{-\delta z}L^\infty$:

$$\|f(w)\|_\delta \le C(\|u-p\|_{L^\infty} + \|w\|_{L^\infty})\|w\|_\delta\,.$$

Similarly,

$$\|f(w_1) - f(w_2))\|_\delta \le C(\|u-p\|_{L^\infty} + \|w_1\|_{L^\infty} + \|w_2\|_{L^\infty})\|w_1 - w_2\|_\delta$$

Next, we note that for all $\delta \in [0,\theta[$, there is a constant C_δ such that for all $y \ge 0$ the integral operator $\mathcal{I}_y$ satisfies:

$$\|\mathcal{I}_y(f)\|_\delta \le C_\delta \|f\|_\delta\,.$$

Thus, the convergence (5.3.3) and the fixed point theorem imply that for all $\delta_1 < \theta$, there are R, $r > 0$ and z_1 such that for $\delta \in [0,\delta_1]$, all $y \ge z_1$ and all $|a| \le r$, the equation

$$w = e^{(z-y)G_d(p)}\Pi_-(p)a + \mathcal{I}_y(f(w)) \tag{5.3.8}$$

has a unique solution in $e^{-\delta z}L^\infty$ such that $\|w\|_\delta \le R$. The uniqueness in L^∞ implies that the solutions in L^∞ and $e^{-\delta z}L^\infty$ coincide. Therefore, the bounded solution belongs to $e^{-\delta_1 z}L^\infty$.

Since $\partial_z u \to 0$, for y large enough there holds $\|\partial_z u\|_{L^\infty([y,+\infty[)} \le R$ and $|\partial_z u(y)| \le r$. Since $\partial_z u$ is a solution of the integral equation (5.3.8) with $a = \partial_z u(y)$, we can apply the result above and conclude that $\partial_z u$ coincide with the exponentially decaying solution. $\square$

Corollary 5.3.4. *u is a solution of* (5.2.2) *on* $[z_0,+\infty[$ *satisfying* (5.3.3), *if and only if* $v = \partial_z u \in L^1([z_0,+\infty[)$ *and there is* $a \in \mathbb{E}_-(G_d(p))$ *such that*

$$\begin{cases} u(z) = p - \displaystyle\int_z^\infty v(y)dy := p - I(v)(z)\,, \\ v(z) = e^{(z-z_0)G_d(p)}a + \mathcal{I}_{z_0}\big(F(u,v)\big)(z)\,. \end{cases} \tag{5.3.9}$$

In addition, $a = \Pi_-(p)v(z_0) = \Pi_-(p)\partial_z u(z_0)$.

Proof. The direct part follows from the two lemmas above. Conversely, if (u,v) solves (5.3.9) and $v \in L^1$, then u satisfies (5.3.3), u and v are smooth, $v = \partial_z u$ and the definition of $\mathcal{I}_{z_0}$ shows that u is a solution of the o.d.e. (5.2.2). The definition of $\mathcal{I}_{z_0}$ also implies that $\Pi_-(p)v(z_0) = a$. $\square$

We now construct solutions of (5.2.2)(5.3.3) on $[0, \infty[$. According to Corollary 5.3.4, we add the boundary condition

$$\Pi_-(p)\partial_z u(0) = a\,, \tag{5.3.10}$$

and solve (5.3.9) with $z_0 = 0$.

Proposition 5.3.5. *Given ω a relatively compact open set in $\mathcal{U}$, there are $R > 0$ and $r > 0$ such that for all $p \in \omega$, all $a \in \mathbb{E}_-(G_d(p))$ with $|a| \le r$, the equation* (5.2.2) *has a unique solution $u = \Phi(\cdot, p, a)$ satisfying* (5.3.3) *and* (5.3.10) *and*

$$\|\partial_z u\|_{L^1} \le R \quad \text{and} \quad \|\partial_z u\|_{L^\infty} \le R\,. \tag{5.3.11}$$

The function Φ is a C^∞ function on $[0, +\infty[\times\Omega$, where Ω is the set of (p, a) with $p \in \omega$ and $a \in \mathbb{E}_-(G_d(p))$ with $|a| \le r$. It satisfies

$$\Phi(z, p, a) = p + e^{zG_d(p)} G_d^{-1}(p) a + O(|a|^2)\,.$$

Moreover, there are $\delta > 0$ and C such that for all z and $(p, a) \in \Omega$:

$$|\partial_z \Phi(z, p, a)| + |\Phi(z, p, a) - p| \le C e^{-\delta z}\,,$$

Proof. Choose $R_0 > 0$ such that for all $p \in \omega$, the closed ball of radius R_0 centered at p is contained in $\mathcal{U}^*$. For $R \le R_0$, denote by B_R the set of (u, v) such that $\|u - p\|_{L^\infty} \le R$, $\|v\|_{L^1} \le R$ and $\|v\|_{L^\infty} \le R$. Note that if $(u, v) \in B_R$, then u takes its values in $\mathcal{U}^*$. Then, for $(u, v) \in B_R$, one has

$$\|I(v)\|_{L^\infty} \le R\,, \quad \|F(u, v)\|_{L^1 \cap L^\infty} \le CR^2$$

With (5.3.6), this implies that

$$\|\mathcal{I}_0\big(F(u, v)\big)\|_{L^1 \cap L^\infty} \le C_1 \|F(u, v)\|_{L^1 \cap L^\infty} \le C_1 C R^2\,.$$

In addition, for $a \in \mathbb{E}_-(p)$, one has

$$\|e^{zG_d(p)} a\|_{L^1 \cap L^\infty} \le C_2 |a|\,.$$

Therefore, for R and r small enough and $|a| \le r$, the mapping

$$\mathcal{T} : (u, v) \mapsto \Big(p + I(v), e^{zG_d(p)} a + \mathcal{I}_0\big(F(u, v)\big)\Big)$$

maps B_R into itself. Similarly, for (u_1, v_1) and (u_2, v_2) in B_R, there holds

$$\begin{aligned}
&\|I(v_1 - v_2)\|_{L^\infty} \le \|I(v_1 - v_2)\|_{L^1}\,,\\
&\|F(u_1, v_1) - F(u_2, v_2)\|_{L^1 \cap L^\infty} \le C' R(\|u_1 - u_2\|_{L^\infty} + \|v_1 - v_2\|_{L^\infty})\,.
\end{aligned}$$

Thus, decreasing R if necessary, the mapping $\mathcal{T}$ is a contraction in B_R equipped with the distance $\|u_1 - u_2\|_{L^\infty} + 2\|v_1 - v_2\|_{L^1 \cap L^\infty}$. Therefore, the fixed point theorem implies that the equation (5.3.9) has a unique solution in B_R, and the first part of the proposition follows.

The fixed point theorem also implies that the solution depends smoothly on the parameters p and $a \in \mathbb{E}_-(p)$.

By construction, the solution satisfies $\|v\|_{L^1} = O(|a|)$, thus $\|u - p\|_{L^\infty} = O(|a|)$. Hence $\|v - e^{zG_d(p)}a\|_{L^1 \cap L^\infty} = O(|a|^2)$ and $\|u - p - e^{zG_d(p)}G_d^{-1}(p)a\|_{L^\infty} = O(|a|^2)$.

The exponential decay of $\partial_z u$ follows from Corollary 5.3.4 and Lemma 5.3.3, choosing a decay rate θ in (5.3.6) uniform in $p \in \omega$. One can also apply the fixed point theorem to construct solutions in spaces $e^{-\delta z}L^\infty$. $\square$

Next, we note that the proposition above allows to find all the solutions of (5.2.2) (5.3.3). Below, we assume that $\omega \subset \mathcal{U}$ is given and that the solutions $\Phi(z, p, a)$ are defined by Proposition 5.3.5 for $p \in \omega$, $a \in \mathbb{E}_-(p)$ with $|a| \leq r$ and $z \geq 0$. We still denote by $\Phi(z, p, a)$ their maximal extension for $z \leq 0$ as solutions of the o.d.e. (5.2.2).

Proposition 5.3.6. *u is a solution of (5.2.2), (5.3.3) with $p \in \omega$, if and only if there are $a \in \mathbb{E}_-(p)$ with norm $|a| \leq r$ and $z_0 \in \mathbb{R}$ such that $u(z) = \Phi(z - z_0, p, a)$.*

In this case, for all z_0 large enough, one has

$$u(z) = \Phi(z - z_0, p, a) \,, \quad \textit{with} \quad a = \Pi_-(p)\partial_z u(z_0) \,.$$

Proof. Since the equation is invariant by translation, for all $z_0 \in \mathbb{R}$, $u(z) = \Phi(z - z_0, p, a)$ is a solution which converges to p at infinity. It is defined on $]z_* + z_0, +\infty[$ where $]z_*, +\infty[$ is the maximal interval of existence of $\Phi(\cdot, p, a)$.

Conversely, by Lemma 5.3.3, if u is a solution of (5.2.2) (5.3.3) on $[z_1, +\infty[$, then for all $z_0 \geq z_1$ large enough, there holds

$$\|\partial_z u\|_{L^1([z_0, +\infty[)} \leq R \,, \quad |\Pi_-(p) D_z u(z_0)| \leq r \,,$$

where $R > 0$ and $r > 0$ are the constants determined at Proposition 5.3.5. The equation is invariant by translation, thus $u(z + z_0)$ is another solution with the same end point. By Corollary 5.3.4 and uniqueness in Proposition 5.3.5, one has

$$u(z + z_0) = \Phi(z, p, a) \quad \text{for } z \geq 0$$

with $a = \Pi_-(p)\partial_z u(z_0)$. By uniqueness of the Cauchy problem for (5.2.2), the identity extends to negative values of z, as long as $z + z_0$ remains in the domain of definition of u, and hence of $\Phi(\cdot, p, a)$. $\square$

5.4 Smooth hyperbolic boundary conditions

With now turn to the the boundary value problem (5.2.4) and construct smooth pieces of $\widetilde{\mathcal{C}}$. We first consider small values of p, assuming that 0 is an hyperbolic state.

5.4.1 Small amplitude layers

Proposition 5.4.1. *If $0 \in \mathcal{U}$, there are a neighborhood $\omega \subset \mathcal{U}$ of 0, a manifold $\mathcal{C} \subset \omega$ of dimension N_- and a smooth mapping W from $[0, +\infty[\times\mathcal{C}$ to $\mathcal{U}$ such that for all $p \in \mathcal{C}$, $W(\cdot, p)$ is a solution of (5.2.4) which converges at an exponential rate to p as z tends to infinity.*

Proof. We apply Proposition 5.3.5 to find Φ on a neighborhood of $p = 0, a = 0$. Next we use the implicit function theorem to solve the equation $\Phi(0, p, a) = 0$. The proposition implies

$$\Phi(0, p, a) = p + G_d^{-1}(p)a + O(|a|^2)\,.$$

Thus,

$$\nabla_{p,a}\Phi(0, 0, 0)(\dot{p}, \dot{a}) = \dot{p} + G_d^{-1}(0)\dot{a}$$

and $\nabla_{p,a}\Phi(0, 0, 0)$ is an isomorphism from the space of $(\dot{p}, \dot{a}) \in \mathbb{E}_+(0)\times\mathbb{E}_-(0)$ to $\mathbb{R}^N$. Therefore, the implicit function theorem implies that, locally near the origin, the equation $\Phi(0, p, a) = 0$ defines a manifold parametrized by $p_- = \Pi_-(0)p$, as $p = P(p_-)$, $a = A(p_-)$. We define locally the manifold $\mathcal{C}$ as $p = P(p_-)$, and for $p \in \mathcal{C}$, the profile $W(z, p)$ as $\Phi(z, p, A(p_-)$. □

5.4.2 Large amplitude layers

Below, we assume that $\omega \subset \mathcal{U}$ is given and that the solutions $\Phi(z, p, a)$ are defined by Proposition 5.3.5 for $p \in \omega$, $a \in \mathbb{E}_-(p)$ with $|a| \le r$ and $z \ge 0$. We still denote by $\Phi(z, p, a)$ their maximal extension for $z \le 0$ as solutions of the o.d.e. (5.2.2).

Remark 5.4.2. The boundary value problem (5.2.4) reduces to the equation

$$\Phi(z_0, p, a) = 0 \tag{5.4.1}$$

Indeed, if $\Phi(z, p, a)$ vanishes at z_0, then $z \mapsto \Phi(z+z_0, a, p)$ is a solution of the boundary value problem (5.2.4). Proposition 5.3.6 implies that conversely, if u is a solution of (5.2.4), then there is $a \in \mathbb{E}_-(p)$ with norm $|a| \le r$ and $z_0 \in \mathbb{R}$ such that $\Phi(\cdot, p, a)$ is defined for $z \ge z_0$, $\Phi(z_0, p, a) = 0$ and $u(z) = \Phi(z + z_0, p, a)$.

To solve (5.2.4) , that is (5.4.1), with large data, the analysis is much more delicate and depends on global properties of the differential system. There might be no solutions (see the example of Burgers equations with end states $p > 0$), there might be multiple solutions. What we study next is the structure of the set of solutions near a given solution $\underline{u}$.

Consider a particular solution $\underline{w}$ of (5.2.4) with end point $\underline{p}$. Then, by Proposition 5.3.6 there are $\underline{a}$ and $\underline{z}$ such that:

$$\underline{w}(z) = \Phi(z + \underline{z}, \underline{p}, \underline{a}) \,, \tag{5.4.2}$$

with $\Phi(\underline{z}, \underline{p}, \underline{a}) = 0$. To apply the implicit function theorem to the equation

$$\Phi(\underline{z}, p, a) = 0 \tag{5.4.3}$$

we differentiate Φ and introduce

$$\Phi'(z, \dot p, \dot a) = \dot p \nabla_p \Phi(z, \underline{p}, \underline{a}) + \dot a \nabla_a \Phi(z, \underline{p}, \underline{a}) \tag{5.4.4}$$

Proposition 5.4.3. *Suppose that*

$$\operatorname{rank} \nabla_a \Phi(\underline{z}, \underline{p}, \underline{a}) = N_- \,, \tag{5.4.5}$$

$$\operatorname{rank} \nabla_{a,p} \Phi(\underline{z}, \underline{p}, \underline{a}) = N \,. \tag{5.4.6}$$

Then, in a neighborhood of $\underline{p}$, there is a smooth manifold $\mathcal{C} \subset \mathcal{U}$ of dimension N_- and a smooth mapping W from $[0, +\infty[\times \mathcal{C}$ to $\mathcal{U}$ such that for all $p \in \mathcal{C}$, $W(\cdot, p)$ is a solution of (5.2.4) *which converges at an exponential rate to p as z tends to infinity.*

Proof. The assumptions implies that there are coordinates $p = (p_-, p_+) \in \mathbb{R}^{N_-} \times \mathbb{R}^{N_+}$ such that $\nabla_{a,p_+} \Phi(\underline{z}, \underline{p}, \underline{a})$ is an isomorphism. Therefore, the implicit function theorem implies that (5.4.3) defines near $(\underline{p}, \underline{a})$ a manifold of dimension N_- parametrized by $p_- = \Pi_-(0)p$, as $p = P(p_-)$, $a = A(p_-)$. We define locally the manifold $\mathcal{C}$ as $p = P(p_-)$, and for $p \in \mathcal{C}$, the profile $W(z, p)$ as $\Phi(z, p, A(p_-))$. $\square$

5.5 The linearized profile equation

In this subsection we study the linearized equations from (5.2.2) at $\underline{w}$. We assume that there are $\underline{p} \in \mathcal{U}$ and $\delta > 0$ such that

$$\underline{w}(z) - \underline{p} = O(e^{-\delta z}) \,, \quad \partial_z \underline{w}(z) = O(e^{-\delta z}) \,, \quad \partial_z^2 \underline{w}(z) = O(e^{-\delta z}) \,. \tag{5.5.1}$$

The linearized equation reads

$$\dot P \dot u := -\partial_z^2 \dot u + G^\sharp(z)\partial_z \dot u + E^\sharp(z)\dot u = \dot f\,. \tag{5.5.2}$$

where $G^\sharp(z) - G_d(\underline{w}(z))$ and $E^\sharp$ involve first and second order derivatives of $\underline{w}$. In particular:

$$G^\sharp(z) = G_d(\underline{p}) + O(e^{-\delta z})\,, \quad E^\sharp(z) = O(e^{-\delta z}) \tag{5.5.3}$$

As a first order system, (5.5.2) reads, with $\dot v = \partial_z \dot u$:

$$\partial_z U = \mathcal{G}^\sharp(z) U + F\,, \qquad \mathcal{G}^\sharp(z) = \begin{pmatrix} 0 & \mathrm{Id} \\ E^\sharp & G^\sharp \end{pmatrix} \tag{5.5.4}$$

$$U = \begin{pmatrix} \dot u \\ \dot v \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ \dot f \end{pmatrix}$$

We note that

$$\mathcal{G}^\sharp(z) = \mathcal{G}_d + O(e^{-\delta z})\,, \tag{5.5.5}$$

with

$$\mathcal{G}_d = \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & G_d(\underline{p}) \end{pmatrix}.$$

5.5.1 Conjugation to constant coefficients

An important idea is that variable coefficient systems like (5.5.4) with coefficients which converge at an exponential rate, are conjugated to constant systems.

Lemma 5.5.1. *There is a matrix $\mathcal{W}(z)$ such that*
i) $\mathcal{W}$ and $\mathcal{W}^{-1}$ are C^∞ and bounded with bounded derivatives,
ii) there are C and $\delta' > 0$ such that

$$|\mathcal{W}(z) - \mathrm{Id}| + |\partial_z \mathcal{W}(z)| \le C e^{-\delta' z}\,, \tag{5.5.6}$$

iii) $\mathcal{W}$ satisfies

$$\partial_z \mathcal{W} = \mathcal{G}^\sharp \mathcal{W} - \mathcal{W}\mathcal{G}_d\,. \tag{5.5.7}$$

Moreover, if $\mathcal{G}^\sharp$ depends smoothly on parameters, one can choose locally $\mathcal{W}$ depending smoothly on those parameters.

Proof. Consider (5.5.7) as an ordinary (linear) differential equation in the space of matrices. Because $\mathcal{G}$ converges exponentially to $\mathcal{G}_d$, it has the form

$$\partial_z \mathcal{W} = \mathcal{L}\mathcal{W} + \mathcal{G}'(z)\mathcal{W}\,,$$

where $\mathcal{L}$ is the constant coefficient operator $\mathrm{ad}\mathcal{G}_d(\mathcal{W}) := [\mathcal{G}^d,\ \mathcal{W}]$, and $\mathcal{G}'(z)$ is the left multiplication by $\mathcal{G}(z) - \mathcal{G}_d = O(e^{-\delta z})$. $\mathcal{W}$ is obtained as the solution of

$$\begin{aligned}\mathcal{W}(z) = \mathrm{Id} + &\int_{z_0}^{z} e^{(z-s)\mathcal{L}}\Pi_-(s)\mathcal{G}'(s)\mathcal{W}(s)ds\\ &- \int_{z}^{\infty} e^{(z-s)\mathcal{L}}\Pi_+\mathcal{G}'(s)\mathcal{W}(s)ds\end{aligned}$$

where Π_+ [resp Π_-] is the spectral projector on the sum of the generalized eigenspaces of $\mathcal{L}$ associated with eigenvalues in $\mathrm{Re}\,\mu > -\kappa$ [resp. $\mathrm{Re}\,\mu < -\kappa$] where κ is chosen in $]0, \delta[$ such that $\mathcal{L}$ has no eigenvalues on $\{\mathrm{Re}\,\mu = \kappa\}$. Arguing as in Proposition 5.3.5, using the fixed point theorem, we prove the existence of a solution on $[z_0, +\infty[$ such that $\mathcal{W} - \mathrm{Id} = O(e^{-\theta z})$ for some $\theta < \kappa$ and z_0 large enough. The solution of the linear equation (5.5.7) is next extended to $z \in [0, +\infty[$. This construction shows that one can choose $\mathcal{W}$ depending smoothly on parameters, as long as the eigenvalues of $\mathcal{L}$, which are differences of eigenvalues of $\mathcal{G}_d$, remain separated by a line $\mathrm{Re}\,\mu = \kappa$ for some $\kappa \in]0, \delta[$.

Consider $D(z) := \det \mathcal{W}(z)$. Then

$$\partial_z D(z) = \mathrm{tr}(\mathcal{G}(z) - \mathcal{G}_d)\, D(z)\,. \tag{5.5.8}$$

This clearly implies that $D(z)$ never vanishes on $[0, \infty[$. In addition, since $D(z) = 1 + O(e^{-\theta z})$, this also provides uniform bounds for $D(z)$ and $1/D(z)$. To prove (5.5.8), denote by $(W_1, \ldots, W_{2N})$ [resp denote by $(G_1, \ldots, G_{2N})$] the columns of $\mathcal{W}$ [resp. $\mathcal{G}$]. Then (5.5.7) implies that

$$\begin{aligned}\partial_z D = &\sum_j \det\big[W_1, \ldots, \mathcal{G}(z)W_j, \ldots W_{2N}\big]\\ &\qquad - \sum_j \det\big[W_1, \ldots, \mathcal{W}G_j(\infty), \ldots W_{2N}\big]\,.\end{aligned}$$

Next use the following algebraic identities for matrices $\mathcal{W}$ and $\mathcal{G}$ with columns $(W_1, \ldots, W_{2N})$ and $(G_1, \ldots, G_{2N})$:

$$\sum_j \det\big[W_1, \ldots, \mathcal{G}W_j, \ldots W_{2N}\big] = (\mathrm{tr}\,\mathcal{G}) \det \mathcal{W}\,,$$

$$\sum_j \det\big[W_1, \ldots, \mathcal{W}G_j, \ldots W_{2N}\big] = (\mathrm{tr}\,\mathcal{G}) \det \mathcal{W}\,,$$

which are quite clear when $(W_1, \dots, W_{2N})$ is a basis, and extend algebraically to general $\mathcal{W}$. $\square$

The Lemma 5.5.1 means that the variable coefficients equation (5.5.4) is conjugated to the constant coefficients equation

$$\partial_z U_1 = \mathcal{G}_d U_1 + F_1 \,. \tag{5.5.9}$$

Indeed, U satisfies (5.5.4) if and only if $U_1 = \mathcal{W}^{-1} U$ satisfies (5.5.9) with $F_1 = \mathcal{W}^{-1} F$. In particular, the solutions of the homogeneous equation $\partial_z U = \mathcal{G} U$ are

$$U(z) = \mathcal{W}(z) U_1(z) \,, \quad U_1(z) = e^{z\mathcal{G}_d} U_1(0) \,, \quad U_1(0) = \mathcal{W}^{-1}(0) U(0) \,.$$

and

$$U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} u_1(0) + \big(e^{zG_d(\underline{p})} - \mathrm{Id}\big) G_d^{-1}(\underline{p}) v_1(0) \\ e^{zG_d(\underline{p})} v_1(0) \end{pmatrix} .$$

The solution U, or equivalently U_1, is bounded if and only if $v_1(0) \in \mathbb{E}_-(G_d(\underline{p}))$. In this case, v_1 and v are exponentially decaying and

$$\lim_{z \to +\infty} u(z) = \lim_{z \to +\infty} u_1(z) = u_1(0) - G_d^{-1}(\underline{p}) v_1(0) \,.$$

This immediately implies the following result.

Lemma 5.5.2. *The space $\mathcal{S}$ of bounded solutions of the homogeneous equation $\dot{P}\dot{u} = 0$ has dimension equal to $N + N_-$. The subspace $\mathcal{S}_0$ of solutions which tend to zero at infinity has dimension equal to N_-.*

5.5.2 Transversality and the tangent space to $\mathcal{C}$

We suppose now that $\underline{w}$ is a particular solution of (5.2.4) with end point $\underline{p} \in \mathcal{U}$. Then, by Proposition 5.3.6 there are $\underline{a}$ and $\underline{z}$ such that: $\underline{w}(z) = \Phi(z + \underline{z}, \underline{p}, \underline{a})$. We now give equivalent formulations of conditions (5.4.5) (5.4.6).

Proposition 5.5.3. *If $\underline{w}$ is a particular solution of* (5.2.4) *with end point $\underline{p} \in \mathcal{U}$, the condition* (5.4.5) *is satisfied if and only if the problem*

$$\dot{P}\dot{u} = 0 \,, \quad u(0) = 0 \,, \qquad \lim_{z \to +\infty} u(z) = 0 \tag{5.5.10}$$

has no nontrivial solution.

The condition (5.4.6) *is satisfied if and only if for all* $u_0 \in \mathbb{R}^N$*, the problem*

$$\dot{P}\dot{u} = 0\,, \quad u(0) = u_0 \tag{5.5.11}$$

has a bounded solution.

Proof. For all (p, a) close to $(\underline{p}, \underline{a})$, $\Phi(\cdot + \underline{z}, p, a)$ is a solution of (5.2.2). Differentiating the equation in (p, a) implies that for all $(\dot{p}, \dot{a}) \in \mathbb{R}^N \times \mathbb{E}_-(\underline{p})$, $\Phi'(\cdot + \underline{z}, \dot{p}, \dot{a})$ is a solution of the homogeneous linearized equation. Next, Proposition 5.3.5 implies that

$$\lim_{z \to \infty} \Phi'(z, \dot{p}, \dot{a}) = \dot{p} \tag{5.5.12}$$

and the convergence holds at an exponential rate. Moreover,

$$\dot{a} \nabla_a \Phi(0, \underline{p}, \underline{a}) = \dot{a}\,.$$

Therefore, the mapping $(\dot{p}, \dot{a}) \mapsto \Phi'(\cdot + \underline{z}, \dot{p}, \dot{a})$ from $\mathbb{R}^N \times \mathbb{E}_-(\underline{p})$ into the space $\mathcal{S}$ of bounded solutions of the homogeneous equation, is injective. Since both space have dimension $N + N_-$, the mapping is an isomorphism and therefore

$$\mathcal{S} = \big\{ \Phi'(\cdot + \underline{z}, \dot{p}, \dot{a}) \,:\, (\dot{p}, \dot{a}) \in \mathbb{R}^N \times \mathbb{E}_-(\underline{p}) \big\}\,. \tag{5.5.13}$$

By (5.5.12), the function $u(z) = \Phi'(z + \underline{z}, \dot{p}, \dot{a})$ in $\mathcal{S}$ tends to zero at infinity if and only if $\dot{p} = 0$. Therefore, the dimension of the space of solutions of (5.5.10) is equal to the dimension of the space $\{\dot{a} \,:\, \dot{a} \nabla_a \Phi(\underline{z}, \underline{p}, \underline{a}) = 0\}$. It is equal to zero if and only if $\nabla_a \Phi(\underline{z}, \underline{p}, \underline{a})$ has maximal rank N_-.

Similarly, $\nabla_{p,a} \Phi(\underline{z}, \underline{p}, \underline{a})$ has maximal rank N, if and only if the mapping $u \mapsto u(0)$ from $\mathcal{S}$ to $\mathbb{R}^N$ is onto. $\square$

Definition 5.5.4. *We say that the the profile* $\underline{w}$ *solution of* (5.2.4) *is transversal if and only if the conditions* (5.4.5), (5.4.6)*, or their equivalent formulations given in Proposition* 5.5.3*, are satisfied.*

In analogy with the previous subsection, let $\dot{\mathcal{C}}$ denote the space of $\dot{p}$ such that there is a solution $\dot{u}$ of $\dot{P}\dot{u} = 0$ with $\dot{u}(z) \to \dot{p}$ as $z \to +\infty$.

Proposition 5.5.5. *Suppose that* $\underline{w}$ *is a transversal solution of* (5.2.4)*. Then* $\dim \dot{\mathcal{C}} = N_-$ *and* $\dot{\mathcal{C}}$ *is the tangent space to* $\mathcal{C}$ *at* $\underline{p}$*, where* $\mathcal{C}$ *denotes the local manifold constructed in Proposition* 5.4.3.

Proof. Recall from the proof of Proposition 5.4.3 that there are coordinates $(p_-, p_+) \in \mathbb{R}^{N_-} \times \mathbb{R}^{N_+}$ for p such that the manifold $\mathcal{C}$ is defined by $p = \pi(p_-)$. Moreover, the manifold of solutions (p, a) of $\Phi(\underline{z}, p, a) = 0$ is given by $p = \pi(p_-)$ and $a = \alpha(p_-)$. Therefore, the tangent space to $\mathcal{C}$ at $\underline{p}$ is determined by the differentiated equation $\dot{p}\nabla_p\Phi(\underline{z}, \underline{p}, \underline{a}) + \dot{a}\nabla_a\Phi(\underline{z}, \underline{p}, \underline{a}) = 0$, which by the transversality assumption determines $\dot{p} = \pi'(\underline{p\_})\dot{p}_-$ and $\dot{a} = \alpha'(\underline{p\_})\dot{p}_-$. By (5.5.13), this corresponds to end points $\dot{p}$ of solutions in $u \in \mathcal{S}$ such that $u(0) = 0$. $\square$

We now consider the inhomogeneous equation (5.5.2).

Proposition 5.5.6. *Suppose that $\underline{w}$ is a transversal solution of* (5.2.4) *and $f \in e^{-\delta z}L^\infty$, with $\delta > 0$ small enough. Then the equation* (5.5.2) *has solutions in $e^{-\delta z}W^{2,\infty}$.*

Furthermore, for all $u_0 \in \mathbb{R}^N$, the equation (5.5.2) *has bounded solutions u such that $u(0) = u_0$, u has a limit $\dot{p}$ at infinity, and $u - \dot{p} \in e^{-\delta z}W^{2,\infty}$ for some $\delta' > 0$.*

Proof. Consider $F = {}^t(0, f)$ and $F_1 = {}^t(f_1, g_1) = \mathcal{W}^{-1}F$. A solution of (5.5.9) is

$$v_1(z) = \int_0^z e^{(z-s)G_d(\underline{p})}\Pi_-(\underline{p})g_1(s)ds - \int_z^\infty e^{(z-s)G_d(\underline{p})}\Pi_+(\underline{p})g_1(s)ds\,,$$
$$u_1(z) = -\int_z^\infty \big(v_1(s) + f_1(s)\big)ds\,.$$

Thanks to the exponential estimates (5.3.6), for $\delta < \theta$, one has

$$\|e^{\delta z}U_1\|_{L^\infty} + \|e^{\delta z}\partial_z U_1\|_{L^\infty} \leq C\|e^{\delta z}F_1\|_{L^\infty}\,,$$

Thus the solution $U = \mathcal{W}U_1$ of (5.2.4) satisfies similar estimates. This provides us with an exponentially decaying solution u of (5.5.2).

To prove the second part of the proposition, it is sufficient to find a bounded solution $\dot{u}$ of $\dot{P}\dot{u} = 0$ such that $\dot{u}(0) = u_0 - u(0)$. By Proposition 5.5.3, the transversality conditions imply that this problem is solvable. The analysis before Lemma 5.5.2 shows that $\dot{u}$ has a limit $\dot{p}$ at infinity and that $\dot{u} - \dot{p}$ is exponentially decaying. $\square$

Chapter 6

Plane wave stability

In this chapter we analyze the plane wave stability of profiles $w(x)$. We start with general remarks about plane wave stability, deriving necessary conditions for energy estimates. These conditions are expressed in terms of a *Lopatinski determinant* in the constant coefficient case, and of an *Evans function* when the coefficients depend on the normal variable. We refer to the introduction for references concerning these notions. A key point in this Chapter is the theorem of F.Rousset ([Ro1]) asserting that the uniform Evans condition implies that the limiting hyperbolic boundary value problem satisfies the uniform Lopatinski condition (see also [ZS] for viscous shocks).

6.1 Statement of the problem

Consider a C^∞ profile w on $\mathbb{R}_+$ and an end state p such that for all $k \in \mathbb{N}$

$$|\partial_z^k (w(z) - p)| = O(e^{-\delta z}) \tag{6.1.1}$$

for some $\delta > 0$. It can be (and it will be in applications) a solution of (5.1.2), but we do not use this property here. In this Chapter, we always suppose that the equation (5.1.2) satisfies Assumption 5.1.1 of Chapter five and that $p \in \mathcal{U}$. The linearized equations of (5.1.2) (5.1.3) around $w(x/\varepsilon)$ read

$$\begin{cases} \partial_t u + \displaystyle\sum_{j=1}^{d} A_j^\sharp \partial_j u - \varepsilon \sum_{j,k=1}^{d} B_{j,k}^\sharp \partial_{j,k}^2 u + \frac{1}{\varepsilon} E^\sharp u = f \,, & x > 0 \,, \\ u_{|x=0} = 0 \,. \end{cases} \tag{6.1.2}$$

with

$$
(6.1.3)\qquad \begin{aligned}
A^\sharp_j v &= \widetilde{A}_j v - (v\cdot\nabla_u \widetilde{B}_{j,d})\partial_z w - (\partial_z w\cdot\nabla_u\widetilde{B}_{d,j})v\,,\\
B^\sharp_{j,k} &= \widetilde{B}_{j,k}\,,\\
E^\sharp v &= (v\cdot\nabla_u\widetilde{A}_d)\partial_z w - (v\cdot\nabla_u\widetilde{B}_{d,d}\cdot v)\partial_z^2 w - \nabla_u^2\widetilde{B}_{d,d}(v,\partial_z w)\partial_z w\,,
\end{aligned}
$$

where $\widetilde{A}$ denotes the function $A(u)$ evaluated at $u = w(x/\varepsilon))$. Note that all the coefficients $A^\sharp_j$, $B^\sharp_{j,k}$ and $E^\sharp$ are C^∞ functions of $z = x/\varepsilon$. Moreover, they converge with an exponential rate when z tends to $+\infty$. The limits are denoted by A^∞_j, $B^\infty_{j,k}$ and E^∞. They are given by

$$
(6.1.4)\qquad A^\infty_j = A_j(p)\,,\quad B^\infty_{j,k} = B_{j,k}(p)\,,\quad E^\infty = 0\,.
$$

There are C and $\delta > 0$, such that for all indices j and k:

$$
(6.1.5)\qquad |A^\sharp_j(z) - A^\infty| + |B^\sharp_{j,k}(z) - A^\infty| + |E^\sharp(z)| \le Ce^{-\delta z}\,.
$$

In this chapter we investigate the uniform (with respect to ε) well posedness of (6.1.2). Typically, we look for uniform *a priori* estimates for (6.1.2).

We also have to study the stability of the limiting hyperbolic problem. In this case, the unperturbed solution is the constant p, and the linearized operator reads

$$
(6.1.6)\qquad \begin{cases}
\partial_t u + \displaystyle\sum_{j=1}^d A_j(p)\partial_j u = f\,,\\
M u_{|x=0} = g\,,
\end{cases}
$$

where $M\dot p = 0$ is an equation of the tangent space $\dot C$ at p of the manifold $\mathcal{C}$, see Proposition 5.5.5. .

To avoid repetitions, we consider the general setting

$$
(6.1.7)\qquad \begin{cases}
\partial_t u + G^\varepsilon(x/\varepsilon,\partial_y,\partial_x)u = f\,,\quad \text{for } x\ge 0\,,\\
\Gamma u = g\,,\quad \text{for } x = 0\,,
\end{cases}
$$

where G^ε is a differential operator in (∂_y,∂_x) with coefficients depending only on $z = x/\varepsilon$ and the boundary operator Γ is constant.

We have two examples in mind: first, the hyperbolic case

$$
(6.1.8)\qquad G = \sum_{j=1}^d A_j(p)\partial_j\,,
$$

and second, the hyperbolic-parabolic case

$$G = \sum_{j=1}^{d} A_j^\sharp(z)\partial_j - \varepsilon \sum_{j,k=1}^{d} B_{j,k}^\sharp(z)\partial_j\partial_k\,. \tag{6.1.9}$$

Performing a Laplace-Fourier transform in (t, y), or applying the equations to the *plane waves*

$$u(t, y, x) = e^{(i\tau+\gamma)t+i\eta y}\hat{u}(x), \tag{6.1.10}$$

the equation (6.1.7) becomes

$$\begin{cases} (i\tau+\gamma)\hat{u} + G^\varepsilon(x/\varepsilon, i\eta, \partial_x)\hat{u} = \hat{f}\,, \quad \text{for } x \geq 0\,, \\ \Gamma\hat{u} = \hat{g}\,, \quad \text{for } x = 0\,, \end{cases} \tag{6.1.11}$$

This is a first order system of ordinary differential equations in x, depending on the parameters $\zeta = (\tau, \eta, \gamma)$. The main goal of the chapter is to link the well posedness of (6.1.7) to the well posedness of (6.1.11) and next to give "explicit" criteria for the later problem. In the constant coefficient case, i.e. when G is independent of z, the stability condition is naturally expressed in terms of a *Lopatinski determinant*. In the variable coefficient case, assuming that the coefficients converge at an exponential rate when $z \to +\infty$, the Lopatinski determinant is replaced by an *Evans function*.

6.2 Necessary conditions

6.2.1 General discussion

Definition 6.2.1. *We say that the equation* (6.1.7) *is* uniformly stable *if, for $T > 0$, there is a constant C such that for all $\varepsilon \in]0, 1]$ and all $u \in H^\infty(]-\infty, T] \times \mathbb{R}^d_+)$ vanishing for $t < 0$, there holds*

$$\begin{aligned} \|u\|_{L^2([0,T]\times\mathbb{R}^d_+)} + &\|u_{|x=0}\|_{L^2([0,T]\times\mathbb{R}^{d-1})} \\ &\leq C\Big(\|f\|_{L^2([0,T]\times\mathbb{R}^{1+d}_+)} + \|g\|_{L^2([0,T]\times\mathbb{R}^{d-1})}\Big)\,. \end{aligned} \tag{6.2.1}$$

with $f = \partial_t u + G^\varepsilon(x/\varepsilon, \partial_y, \partial_x)u$ and $g = \Gamma u_{|x=0}$.

We denote by ζ the frequency variables (τ, η, γ) and by $\mathcal{S}^{\zeta,\varepsilon}$ the set of solutions in H^∞ of the equation

$$(i\tau+\gamma)\phi + G^\varepsilon(x/\varepsilon, i\eta, \partial_x)\phi = 0\,. \tag{6.2.2}$$

Proposition 6.2.2. *If the equation* (6.1.7) *is uniformly stable, then there are constants $c \geq 0$ and C such that for all $\zeta \in \mathbb{R}^{d+1}$ with $\gamma > c$, and all $\varepsilon \in]0,1]$, the solutions ϕ of* (6.2.2) *satisfy*

$$|\phi(0)| \leq C|\Gamma\phi(0)|\,. \tag{6.2.3}$$

Proof. Introduce $\chi \in C^\infty(\mathbb{R})$ such that $\chi(t) = 0$ for $t \leq 0$ and $\chi(t) = 1$ for $t \geq T/2$. Suppose that $\phi \in \mathcal{S}^{\zeta,\varepsilon}$, and introduce

$$\widetilde{\phi}(t,y,x) = e^{(i\tau+\gamma)t+i\eta y}\phi(x)\,, \quad u(t,y,x) = \chi(t)e^{-\delta|y|^2}\widetilde{\phi}(t,y,x)\,.$$

Then $u \in H^\infty$ and vanishes for $t \leq 0$. There holds

$$(\partial_t + G^\varepsilon)u = \widetilde{\phi}\Big(\partial_t\chi e^{-\delta|y|^2} + \chi[G^\varepsilon, e^{-\delta|y|^2}]\Big).$$

Therefore

$$\begin{aligned} \|u\|_{L^2([0,T]\times\mathbb{R}^d_+)} &= c_0\delta^{-(d-1)/2}\|\phi\|_{L^2}\|e^{\gamma t}\chi\|_{L^2([0,T])}\,, \\ \|(\partial_t + G^\varepsilon)u\|_{L^2} &= c_0\delta^{-(d-1)/2}\|\phi\|_{L^2}\|e^{\gamma t}\partial_t\chi\|_{L^2([0,T])} \\ &\quad + \delta^{-(d-1)/2}O\big(\sqrt{\delta}\big)\|\phi\|_{H^2}\|e^{\gamma t}\chi\|_{L^2([0,T])}\,. \end{aligned}$$

In addition, $\|e^{\gamma t}\chi\|_{L^2([0,T])} \geq c_1 e^{\gamma 3T/4}$ and $\|e^{\gamma t}\partial_t\chi\|_{L^2([0,T])} \leq c_2 e^{\gamma T/2}$. Hence, there is γ_0 such that for $\gamma \geq \gamma_0$, one has

$$\|e^{\gamma t}\partial_t\chi\|_{L^2([0,T])} \leq \frac{1}{4C}\|e^{\gamma t}\chi\|_{L^2([0,T])}$$

where C is the constant in (6.2.1). Note that γ_0 depends only on the choice of χ and C. This implies that for δ small enough (depending on ϕ), there holds

$$\|(\partial_t + G^\varepsilon)u\|_{L^2} \leq \frac{1}{2C}\|u\|_{L^2([0,T])}\,.$$

Thus, (6.2.1) implies that for $\gamma \geq \gamma_0$ and δ small, there holds

$$\|u_{|x=0}\|_{L^2} \leq C\|\Gamma u_{|x=0}\|_{L^2}\,.$$

We have

$$\begin{aligned} \|u_{|x=0}\|_{L^2} &= c_0\delta^{-(d-1)/2}|\phi(0)|_{L^2}\|e^{\gamma t}\chi\|_{L^2([0,T])}\,, \\ \|\Gamma u_{|x=0}\|_{L^2} &= c_0\delta^{-(d-1)/2}|\Gamma\phi(0)|_{L^2}\|e^{\gamma t}\chi\|_{L^2([0,T])}\,. \end{aligned}$$

This implies (6.2.3). $\square$

6.2.2 The hyperbolic case: the Lopatinski Determinant

In the hyperbolic case (6.1.8), the equation and the space $\mathcal{S}^\zeta$ are independent of ε. The equation (6.2.2) reads

$$\partial_x \phi + A_d^{-1}(p)\Big((i\tau+\gamma)\mathrm{Id} + \sum_{j=1}^{d-1} i\eta_j A_j(p)\Big)\phi := \partial_x\phi - H(\zeta)\phi = 0\,. \tag{6.2.4}$$

We use here that the boundary is non characteristic for $p \in \mathcal{U}$, i.e. that $A_d(p)$ is invertible. We introduce the sign minus in front of H to be coherent with notations used in the sequel. Moreover, $\mu = i\xi$ is an eigenvalue of $H(\zeta)$ if and only if $\tau - i\gamma$ is an eigenvalue of $\sum \eta_j A_j + \xi A_d$. Therefore, the hyperbolicity assumption (H2) implies that for $\gamma \neq 0$, the matrix $H(\zeta)$ has no eigenvalue on the imaginary axis. Thus, for $\gamma > 0$, the space of L^2 or H^∞ solutions of (6.2.4) is the space of functions $e^{xH(\zeta)}\phi(0)$ with $\phi(0) \in \mathbb{E}_-\big(H(\zeta)\big)$, the invariant space generated by generalized eigenvectors associated to eigenvalues in $\{\mathrm{Re}\,\mu < 0\}$. By homogeneity, $\mathbb{E}_-(H(\lambda\zeta)) = \mathbb{E}_-(H(\zeta))$ for $\lambda > 0$, and the condition (6.2.3) is equivalent to

$$\forall \zeta \in S^d \text{ with } \gamma > 0\,, \quad \forall \phi \in \mathbb{E}_-(H(\zeta)) \ : \quad |\phi| \leq C|\Gamma\phi|\,. \tag{6.2.5}$$

where $S^d = \{\zeta \in \mathbb{R}^{d+1} : |\zeta| = 1\}$. In particular, for all $\gamma > 0$:

$$\ker\Gamma \cap \mathbb{E}_-(H(\zeta)) = \{0\}\,. \tag{6.2.6}$$

For a given ζ, the geometric property (6.2.6) implies the estimate in (6.2.5) with a constant C_ζ depending on ζ. The point in (6.2.5) is that one can choose a uniform constant C. By homogeneity, one can always restrict attention to ζ in the unit sphere.

Lemma 6.2.3. *For $\gamma > 0$,* $\dim \mathbb{E}_-(H(\zeta)) = N_+$.

Proof. The Assumption (H2) implies that for $\gamma \neq 0$, $H(\zeta)$ has no eigenvalues on the pure imaginary axis. Thus the dimension of $\mathbb{E}_-(H(\zeta))$ is constant for $\gamma > 0$. Taking $\tau = 0$ and $\eta = 0$, we see that this dimension is equal to N_+, the number of positive eigenvalues of $A_d(p)$. □

Suppose that we have

$$\dim\ker\Gamma = N_- = N - N_+\,. \tag{6.2.7}$$

Then, the condition (6.2.6) can be expressed using the determinant

$$D(\zeta) = \det(\mathbb{E}_-(H(\zeta)), \ker\Gamma) \tag{6.2.8}$$

where $\det(E, F)$ denotes the determinant formed by taking orthonormal bases in E end F, when E and F are subspaces of $\mathbb{C}^D$ with $\dim E + \dim F = D$. This determinant is independent of the choice of the bases. It vanishes if and only if $E \cap F \neq \{0\}$. It measures the angle between the two spaces:

Lemma 6.2.4. *Consider $E \subset \mathbb{C}^D$ with $\dim E = D_+$ and a $D_+ \times D$ matrix Γ such that $\dim E + \dim \ker \Gamma = D$. If*

$$|\det(E, \ker \Gamma)| \geq c > 0\,, \tag{6.2.9}$$

then

$$\forall e \in E \ : \quad |e| \leq C |\Gamma e| \tag{6.2.10}$$

with $C = c^{-1} |\Gamma^* (\Gamma \gamma^*)^{-1}|$.

Conversely, if (6.2.10) *holds, then* (6.2.9) *is satisfied with* $c = (C|\Gamma|)^{-D_+}$.

Proof. Consider the orthogonal projection π on $F := (\ker \Gamma)^\perp$. Diagonalizing the hermitian form $(\pi e, \pi e)$ on E, one obtains orthonormal bases $\{e_j\}$ and $\{f_j\}$ on E and F respectively such that $\pi e_j = \lambda_j f_j$ with $0 < \lambda_j \leq 1$. In this case,

$$\det(\ker \Gamma, E) = \det(F^\perp, E) = \det(e_j, f_k) = \prod \lambda_j\,.$$

If this determinant is larger than or equal to c, since $\lambda_k \leq 1$ for all k, then $\min \lambda_j \geq c$ and

$$c|e| \leq |\pi e| \leq |\Gamma^* (\Gamma \gamma^*)^{-1}| |\Gamma e|$$

since $\pi = \Gamma^* (\Gamma \gamma^*)^{-1} \Gamma$.

Conversely, if (6.2.10) is satisfied, then

$$|e| \leq C |\Gamma|\, |\pi e|$$

since $\Gamma e = \Gamma \pi e$. Therefore, $\lambda_j C |\Gamma| \geq 1$ for all j and the determinant is at least equal to $(C|\Gamma|)^{-D_+}$. $\square$

Definition 6.2.5. *The function D is called the Lopatinski determinant of the hyperbolic boundary value problem.*

The uniform Lopatinski condition holds if and only if

$$\forall \zeta \in S^d \text{ with } \gamma > 0 \ : \quad |D(\zeta)| \geq c\,. \tag{6.2.11}$$

Proposition 6.2.2 and Lemma 6.2.4 imply:

Proposition 6.2.6. *In the hyperbolic case and assuming* (6.2.7)*, if the problem* (6.1.7) *is uniformly stable then the uniform Loptinski condition is satisfied.*

The definition of the determinant depends on the choice of a scalar product on $\mathbb{C}^N$. The uniform Lopatinski condition does not depend on this choice. In particular, we note for future use the following result.

Lemma 6.2.7. *Suppose that W is an invertible $D \times D$ matrix. Then there is a constant C such that for all subspaces E and F of $\mathbb{C}^D$ such that $\dim E + \dim F = D$ there holds*

$$\frac{1}{C} \det(E, F) \le \det(WE, WF) \le C \det(E, F) .$$

Proof. Consider orthonormal bases $\{e_j\}$ and $\{f_k\}$ in E and F which is also orthogonal for the scalar product (Wh, Wh'). Then, $\tilde{e}_j = \alpha_j W e_j$ and $\tilde{f}_k = \beta_k W f_k$ are orthonormal bases of WE and WF respectively if $\alpha_j = |We_j|^{-1}$ and $\beta_k = |Wf_k|^{-1}$. Then

$$\begin{aligned} \det(WE, WF) &= \prod \alpha_j \prod \beta_k \det(We_j, Wf_k) \\ &= \prod \alpha_j \prod \beta_k \det W \det(E, F) . \end{aligned}$$

Since

$$|W|^{-1} \le \alpha_j, \beta_k \le |W^{-1}|$$

the lemma follows. $\square$

There are equivalent formulations of the uniform Lopatinski condition. For instance, one can show

Lemma 6.2.8. *Under Assumption (H2), for all $p \in \mathcal{U}$, the spaces $\mathbb{E}_-(H(\zeta))$ defined for $\gamma > 0$ have a continous extension $\widetilde{\mathbb{E}}_-(\zeta)$ to the set S^d_+ of $\zeta = (\tau, \eta, \gamma)$ in the unit sphere S^d with $\gamma \ge 0$.*

In particular, the Lopatinski determinant has a continuous extension $\widetilde{D}$ to S^d_+.

By homogeneity, continuity and compactness this implies:

Proposition 6.2.9. *The uniform Lopatinski condition condition* (6.2.5) *is equivalent to*

$$\forall \zeta \in S^d \text{ with } \gamma \ge 0 \ : \quad \ker \Gamma \cap \widetilde{\mathbb{E}}_-(\zeta) = \{0\} . \tag{6.2.12}$$

It holds if and only if

$$\forall \zeta \in S^d \text{ with } \gamma \ge 0 \ : \quad D(\zeta) \ne 0 . \tag{6.2.13}$$

6.3 Evans functions

In the hyperbolic-parabolic case, the analysis is similar but not identical.

6.3.1 Reduction to first order and rescaling

The equation (6.1.11) reads

$$(6.3.1) \qquad -\varepsilon\partial_x^2\hat{u} + \mathcal{A}(\frac{x}{\varepsilon}, \varepsilon\zeta)\partial_x\hat{u} + \frac{1}{\varepsilon}\mathcal{M}(\frac{x}{\varepsilon}, \varepsilon\zeta)\hat{u} = \hat{f}\,, \quad \hat{u}_{|x=0} = 0\,,$$

with

$$\begin{cases} \mathcal{A}(z,\zeta) = (B^\sharp_{d,d})^{-1}\Big(A^\sharp_1 - \sum_{j=1}^{d-1} i\eta_j(B^\sharp_{j,d} + B^\sharp_{d,j})\Big) \\ \mathcal{M}(z,\zeta) = (B^\sharp_{d,d})^{-1}\Big((i\tau+\gamma) + \sum_{j=1}^{d-1} i\eta_j A^\sharp_j + \sum_{j,k=1}^{d-1}\eta_j\eta_k B^\sharp_{j,k} + E^\sharp\Big)\,. \end{cases}$$

where the coefficients $A^\sharp_j$, $B^\sharp_{j,k}$ and $E^\sharp$ are functions of z defined at (6.1.3).

Write (6.3.1) as a first order system for $\hat{U} = \begin{pmatrix} \hat{u} \\ \varepsilon\partial_x\hat{u} \end{pmatrix}$

$$(6.3.2) \qquad \partial_x\hat{U} = \frac{1}{\varepsilon}\mathcal{G}(\frac{x}{\varepsilon}, \varepsilon\zeta)\hat{U} + F\,, \quad \Gamma\hat{U}_{|x=0} = \hat{u}_{|x=0} = 0\,.$$

where

$$\mathcal{G}(z,\zeta) = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathcal{M} & \mathcal{A} \end{pmatrix}\,.$$

It is convenient to eliminate the ε in (6.3.2) by setting

$$(6.3.3) \qquad \widetilde{U}(z) = \hat{U}(\varepsilon z)\,, \quad \widetilde{F}(z) = \varepsilon\hat{F}(\varepsilon z)\,, \quad \tilde{\zeta} = \varepsilon\zeta\,.$$

Then, (6.3.2) is transformed into

$$(6.3.4) \qquad \partial_z\widetilde{U} = \mathcal{G}(z,\tilde{\zeta})\widetilde{U} + \widetilde{F}\,, \quad \Gamma\widetilde{U}(0) = 0\,.$$

We recall from (6.1.5) that $\mathcal{G}(z,\zeta)$ converge at an exponential rate at infinity: with obvious notations, for ζ in any compact set, there holds

$$(6.3.5) \qquad |\mathcal{G}(z,\zeta) - \mathcal{G}^\infty(\zeta)| \le Ce^{-\delta z}$$

To take care of large frequencies, we have to take into account the *parabolic* homogeneity: We note that $\mathcal{A}$ is first order in η and $\mathcal{M}$ is first order in (τ,γ) and second order in η. This leads to introduce the weight:

$$(6.3.6) \qquad \langle\zeta\rangle = \Big(\tau^2 + \gamma^2 + |\eta|^4\Big)^{\frac{1}{4}}\,.$$

6.3.2 Spectral analysis of the symbol

Lemma 6.3.1. *i) There are $c > 0$ and $\rho_1 > 0$ such that for $|\zeta| \geq \rho_1$ with $\gamma \geq 0$, and $z \in [0, \infty[$, $\mathcal{G}(z, \zeta)$ has N eigenvalues, counted with their multiplicities, in $\operatorname{Re} \mu > 0$ and N eigenvalues in $\operatorname{Re} \mu < 0$. They satisfy $|\operatorname{Re} \mu| \geq c\langle\zeta\rangle$.*

ii) When $\zeta \neq 0$ and $\gamma \geq 0$, $\mathcal{G}^\infty(\zeta)$ has N eigenvalues, counted with their multiplicities, in $\operatorname{Re} \mu > 0$ and N eigenvalues in $\operatorname{Re} \mu < 0$.

iii) When $\zeta = 0$, $\mathcal{G}^\infty(0)$ has 0 as a semi-simple eigenvalue, of multiplicity N. The nonvanishing eigenvalues are those of $G_d(p) = (B_{d,d}(p))^{-1} A_d(p)$.

Proof. **a)** When ζ is large, we use the quasi-homogeneity to write

$$\mathcal{M} = \langle\zeta\rangle^2 \hat{M} + O(\langle\zeta\rangle)\,, \quad \mathcal{A} = \langle\zeta\rangle \hat{A} + O(1),$$

where

$$\begin{cases} \hat{M} = (B^\sharp_{d,d})^{-1}\Big((i\hat{\tau} + \hat{\gamma})\mathrm{Id} + \displaystyle\sum_{j,k\geq 1}^{d-1} \hat{\eta}_j \hat{\eta}_k B^\sharp_{j,k}\Big) \\ \hat{A} = -i \displaystyle\sum_{k=1}^{d-1} \hat{\eta}_k (B^\sharp_{d,d})^{-1} (B^\sharp_{k,d} + B^\sharp_{d,k}) \end{cases} \tag{6.3.7}$$

with

$$\hat{\tau} = \frac{\tau}{\langle\zeta\rangle^2}\,, \quad \hat{\gamma} = \frac{\gamma}{\langle\zeta\rangle^2}\,, \quad \hat{\eta} = \frac{\eta}{\langle\zeta\rangle}.$$

Thus

$$\begin{pmatrix} \langle\zeta\rangle \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \mathcal{G} \begin{pmatrix} \langle\zeta\rangle^{-1} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} = \langle\zeta\rangle \hat{\mathcal{G}} + O(1).$$

with

$$\hat{\mathcal{G}} := \begin{pmatrix} 0 & \mathrm{Id} \\ \hat{M} & \hat{A} \end{pmatrix}$$

Tracing back the definitions, $\hat{\mu}$ is an eigenvalue $\hat{\mathcal{G}}$ if and only if $-(i\hat{\tau} + \hat{\gamma})$ is an eigenvalue of

$$\sum_{j,k=1}^{d} \xi_j \xi_k B_{j,k}(w(z))$$

with $\xi_d = -i\hat{\mu}$ and $(\xi_1, \ldots, \xi_{d-1}) = \hat{\eta}$. If $\hat{\mu}$ belongs to imaginary axis, ξ_d is real and by (H1) one must have $\hat{\gamma} \leq -c|\xi|^2$. For $\hat{\gamma} \geq 0$, this implies that $\xi = 0$, and therefore that $\hat{\tau} - i\hat{\gamma} = 0$, which contradicts that $\langle\hat{\zeta}\rangle = 1$. Thus $\hat{\mathcal{G}}$ has no eigenvalues on the imaginary axis. Therefore, the number of eigenvalues in $\operatorname{Re}\mu > 0$ and in $\operatorname{Re}\mu < 0$ is independent of $\hat{\zeta}$ when $\hat{\gamma} \geq 0$. Moreover, when $\hat{\eta} = 0$, $\hat{\tau} = 0$ and $\hat{\gamma} = 1$, $\hat{\mathcal{G}}$ reduces to

$$\begin{pmatrix} 0 & \mathrm{Id} \\ \big(B_{d,d}(w(z))\big)^{-1} & 0 \end{pmatrix}.$$

In this case, the eigenvalues of $\hat{\mathcal{G}}$ are the square roots of the eigenvalues of $B_{d,d}^{-1}$, and therefore N of them are in $\operatorname{Re}\mu > 0$ and N in $\operatorname{Re}\mu < 0$.

By a standard perturbation argument, for $\langle\zeta\rangle$ large, the eigenvalues of $\mathcal{G}$ are $\mu = \langle\zeta\rangle\hat{\mu} + O(1)$, where $\hat{\mu}$ is an eigenvalue of $\hat{\mathcal{G}}$, and $i)$ of the lemma follows.

b) Similarly, tracing back the definitions and using (6.1.4), μ is an eigenvalue of $\mathcal{G}^\infty(\zeta)$ if and only if $-\tau + i\gamma$ is an eigenvalue of

$$\sum_{j=1}^{d} \eta_j A_j(p) - i \sum_{j,k=1}^{d} \xi_j \xi_k B_{j,k}(p)$$

with $\xi_d = -i\mu$ and $(\xi_1, \ldots, \xi_{d-1}) = \eta$. Since $p \in \mathcal{U}$, (H3) implies that if $\operatorname{Re}\mu = 0$ then ξ is real and $\gamma \leq -c(|\mu|^2 + |\eta|^2)$. For $\gamma \geq 0$, this implies that $\gamma = 0$, $\mu = 0$ and $\eta = 0$. Thus the matrix above vanishes, the eigenvalue $-\tau$ must be zero and therefore, $\zeta = 0$. This shows that $\mathcal{G}^\infty$ has no eigenvalues on the imaginary axis when $\zeta \neq 0$ and $\gamma \geq 0$.

The number of eigenvalues in $\operatorname{Re}\mu > 0$ and in $\operatorname{Re}\mu < 0$ is independent of (p, ζ) when $\zeta \neq 0$ and $\gamma \geq 0$. Letting z tend to ∞, $i)$ implies that for ζ large, N eigenvalues lie on each half plane.

c) When $\zeta = 0$, one has

$$\mathcal{G}^\infty(0) = \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & G_d(p) \end{pmatrix}.$$

By Lemma 5.1.3, G_d is invertible when $p \in \mathcal{U}$. Thus the eigenvalues of $\mathcal{G}^\infty$ are zero with multiplicity N, and the eigenvalues of $B_{d,d}^{-1} A_d$. $\square$

6.3.3 Conjugation to constant coefficients

Lemma 6.3.2. *For all $\underline{\zeta} \in \mathbb{R}^{d+1}$ with $\underline{\gamma} \geq 0$, there is a neighborhood ω of $\underline{\zeta}$ and there is a matrix $\mathcal{W}$ defined and C^∞ on $[0, \infty[\times \omega$ such that*

i) $\mathcal{W}^{-1}$ is uniformly bounded and there is $\theta > 0$ such that

$$|\mathcal{W}(z, \zeta) - \mathrm{Id}| \leq C e^{-\theta z} \tag{6.3.8}$$

ii) $\mathcal{W}$ *satisfies*

$$\partial_z \mathcal{W}(z,\zeta) = \mathcal{G}(z,\zeta)\mathcal{W}(z,\zeta) - \mathcal{W}(z,\zeta)\mathcal{G}^\infty(\zeta)\,. \tag{6.3.9}$$

Proof. This is a parameter dependent version of Lemma 5.5.1. □

The substitution $\widetilde{U} = \mathcal{W} U_1$ transforms the equation (6.3.4) into

$$\partial_z U_1 = \mathcal{G}^\infty(\zeta) U_1 + F_1\,, \quad \Gamma_1(\zeta) U_1(0) = 0 \tag{6.3.10}$$

with $F_1 = \mathcal{W}^{-1} F$ and

$$\Gamma_1(\zeta) = \Gamma \mathcal{W}(0,\zeta)\,. \tag{6.3.11}$$

We have won that (6.3.10) has constant coefficients, but the boundary condition now depends on the frequency ζ.

We introduce the following notations.

Definition 6.3.3. *We denote by* $\mathbb{E}_-(\zeta)$ *[resp.* $\mathbb{F}_-(\zeta)$*] the space of initial data* $U(0)$ *[resp.* $U_1(0)$*] such that the corresponding solution of* $\partial_z U = \mathcal{G}(z,\zeta)U$ *[resp.* $\partial_z U_1 = \mathcal{G}^\infty(\zeta)U_1$*] is bounded as* z *tends to infinity.*

Since the two equations are conjugated by $\mathcal{W}$, the two spaces are related and:

$$\mathbb{E}_-(\zeta) = \mathcal{W}(0,\zeta)\mathbb{F}_-(\zeta)\,. \tag{6.3.12}$$

Corollary 6.3.4. $\mathbb{E}_-(\zeta)$ *and* $\mathbb{F}_-(\zeta)$ *have dimension* N *and vary smoothly with* ζ *when* $\zeta \neq 0$ *and* $\gamma \geq 0$*.*

Proof. Since $\mathbb{F}_-$ is the spectral subspace for $\mathcal{G}^\infty$ associated to eigenvalues lying in $\operatorname{Re}\mu < 0$, it has dimension N by Lemma 6.3.1 and varies smoothly with ζ when $\zeta \neq 0$. □

6.3.4 Stability conditions

According to the general discussion of section 6.2, necessary stability conditions are that there hold estimates

$$\forall U \in \mathbb{E}_-(\zeta) : \quad |U| \leq C|\Gamma U|\,, \tag{6.3.13}$$

In order to measure the angle between the spaces $\mathbb{E}_-$ and $\ker\Gamma$, which are subspaces of dimension N in a space of dimension $2N$, we form the determinant

$$D(\zeta) = \det\big(\mathbb{E}_-(\zeta), \ker\Gamma\big) \tag{6.3.14}$$

obtained by taking orthonormal bases in each space. The result is independent of the choice of the bases. This is the *Evans' function* (see [Zum], [Ser]). D vanishes if and only if $\mathbb{E}_- \cap \ker \Gamma$ is not reduced to $\{0\}$.

To deal properly with the high frequencies, some appropriate scaling is required to recover the maximal parabolic estimates. With

$$\Lambda(\zeta) = \Big(1 + \tau^2 + \gamma^2 + |\eta|^4\Big)^{\frac{1}{4}} \tag{6.3.15}$$

introduce the space $\widetilde{\mathbb{E}}_-(\zeta) = J_\Lambda \mathbb{E}_-(\zeta)$ where J_Λ is the mapping $(u, v) \mapsto (u, \Lambda^{-1} v)$ in $\mathbb{C}^N \times \mathbb{C}^N$ and the "scaled" Evans' function

$$\widetilde{D}(\zeta) = \det\big(\widetilde{\mathbb{E}}_-(\zeta), \ker \Gamma\big)\,. \tag{6.3.16}$$

Note that $\ker \Gamma$ is invariant by J_Λ so that D vanishes if and only if $\widetilde{D}$ vanishes. Moreover, for bounded values of ζ, there is C such that $\frac{1}{C}|D| \le |\widetilde{D}| \le C|D|$, since, in the computation of the Evans' functions, the introduction of J_Λ only amounts to a change of scalar product in $\mathbb{C}^{2N}$.

The *weak* stability condition requires that $D \neq 0$ for $\zeta \neq 0$ with $\gamma \ge 0$. The *strong* or *uniform* reads

Definition 6.3.5 (Uniform Evans' condition). *We say that the linearized problem* (6.1.2) *satisfies the Uniform Evans condition, if there is a constant $c > 0$ such that for all for all $\zeta = (\tau, \gamma, \eta) \neq 0$ with $\gamma \ge 0$*

$$|\widetilde{D}(\zeta)| \ge c \tag{6.3.17}$$

For a fixed ζ, the condition $\widetilde{D}(\zeta) \neq 0$ is equivalent to the condition $D(\zeta) \neq 0$. It holds if and only if

$$\widetilde{\mathbb{E}}_-(\zeta) \cap \ker \Gamma = \{0\} \quad \text{or} \quad \mathbb{E}_-(\zeta) \cap \ker \Gamma = \{0\}\,. \tag{6.3.18}$$

Other equivalent conditions are: there are constants C_ζ or C'_ζ such that

$$\forall U \in \mathbb{E}_-(\zeta) : \quad |U| \le C_\zeta |\Gamma U| \quad \text{or} \quad \forall U \in \widetilde{\mathbb{E}}_-(\zeta) : \quad |U| \le C'_\zeta |\Gamma U|$$

The uniform condition (6.3.17) is equivalent to the fact that one can choose a uniform constant C'_ζ independent of ζ. Lemma 6.2.4 implies:

Lemma 6.3.6. *The uniform Evans' condition holds if and only if there is a constant $C > 0$ such that that for all for all $\zeta = (\tau, \gamma, \eta) \neq 0$ with $\gamma \ge 0$, there holds*

$$\forall U \in \widetilde{\mathbb{E}}_-(\zeta) : \quad |U| \le C |\Gamma U|\,. \tag{6.3.19}$$

Using the definition of $\widetilde{\mathbb{E}}_-$, (6.3.19) can be written

$$\forall (u, v) \in \mathbb{E}_-(\zeta) : \quad |v| \le C \Lambda(\zeta) |u|\,. \tag{6.3.20}$$

6.4 Low frequency analysis of the Evans condition

Consider a profile w solution of (5.2.2) such that

$$w(0) = 0 \quad \text{and} \quad \lim_{z \to +\infty} w(z) = p \in \mathcal{U}\,.$$

On one hand, we consider the linearized equation (6.1.2) around w. On the other hand, when the profile w is transversal in the sense of Definition 5.5.4, one can define a smooth manifold $\mathcal{C}$ near p and consider the linearized hyperbolic equation (6.1.6) at p where the boundary operator M is such that $Mu = 0$ is an equation of the tangent space $T_p\mathcal{C}$.

Theorem 6.4.1. *The uniform Evans condition* (6.3.17) *is satisfied on* $0 < |\zeta| \le \rho_0$ *for some* $\rho_0 > 0$, *if and only if the profile* w *is transversal and the hyperbolic boundary value problem* (6.1.6) *satisfies the uniform Lopatinski condition.*

6.4.1 Detailed spectral analysis of $\mathcal{G}^\infty$

By Lemma 6.3.2 there is $\mathcal{W}(z, \zeta)$ defined for ζ in a neighborhood ω_0 of 0 in $\mathbb{R}^{d+1}$ such that (6.3.8) and (6.3.9) are satisfied on ω_0.

Lemma 6.4.2. *There is a* C^∞ *invertible matrix* $\mathcal{V}(\zeta)$ *defined on a neighborhood* ω_0 *of* 0 *such that* $\mathcal{V}^{-1}\mathcal{G}^\infty\mathcal{V}$ *has the block diagonal form*

$$\mathcal{V}(\zeta)^{-1}\mathcal{G}^\infty(\zeta)\mathcal{V}(\zeta) = \begin{pmatrix} H(\zeta) & 0 \\ 0 & P(\zeta) \end{pmatrix} := \mathcal{G}_2(\zeta) \tag{6.4.1}$$

with $H(0) = 0$, $P(0) = G_d(p)(B_{d,d}(p))^{-1}A_d(p)$ *and*

$$\mathcal{V}(0) = \begin{pmatrix} \mathrm{Id} & (A_d(p))^{-1}B_{d,d}(p) \\ 0 & \mathrm{Id} \end{pmatrix} \tag{6.4.2}$$

The eigenvalues of P *satisfy* $|\mathrm{Re}\,\mu| \ge c$ *for some* $c > 0$ *and*

$$H = -(A_d(p))^{-1}\Big((i\tau + \gamma)\mathrm{Id} + \sum_{j=1}^{d-1} i\eta_j A_j(p)\Big) + O(|\zeta|^2). \tag{6.4.3}$$

A crucial remark is that the principal term in the right hand side of (6.4.3) is the symbol of the hyperbolic operator appearing in (6.2.4).

Proof. By Lemma 5.1.3, $G_d(p)$ is invertible. Lemma 6.3.1 implies that, on a small neighborhood ω_0 of the origin, there is a smooth family of matrices $\mathcal{V}$ such that (6.4.1) and (6.4.2) hold. Moreover, the eigenvalues of $P(0) = G_d(p)$ do not belong to the imaginary axis, and this remains true for $P(\zeta)$, ζ close to 0.

Next, note that

$$\mathcal{G}^\infty = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathcal{M}_1^\infty + O(|\eta|^2) & G_d + O(|\eta|) \end{pmatrix}$$

with

$$\mathcal{M}_1^\infty := (B_{d,d}(p))^{-1}\big((i\tau+\gamma)\mathrm{Id} + \sum_{j=1}^{d-1} i\eta_j A_j(p)\big)$$

Writing $\mathcal{V}(\zeta) = \mathcal{V}(0) + O(|\zeta|)$, one obtains that the left upper block of $\mathcal{V}^{-1}\mathcal{G}^\infty\mathcal{V}$ is $H = -G_d^{-1}\mathcal{M}_1^\infty + 0(|\zeta|^2)$, implying (6.4.3). $\square$

The lemma immediately implies that the negative space $\mathbb{F}_-(\zeta) = \mathbb{E}_-(\mathcal{G}^\infty(\zeta))$ is

$$\mathbb{F}_-(\zeta) = \mathcal{V}(\zeta)\big(\mathbb{E}_-(H(\zeta)) \times \mathbb{E}_-(P(\zeta))\big)\,. \tag{6.4.4}$$

In particular, for $\zeta \neq 0$ with $\gamma \geq 0$, the bounded solutions of the homogeneous equation $\partial_z U = \mathcal{G}U$ are

$$U(z) = \mathcal{W}(z,\zeta)\mathcal{V}(\zeta)\big(e^{zH(\zeta)}u_H, e^{zP(\zeta)}u_P\big) \tag{6.4.5}$$

with $u_H \in \mathbb{F}_-^H(\zeta) := \mathbb{E}_-(H(\zeta))$, $u_P \in \mathbb{F}_-^P(\zeta) := \mathbb{E}_-(P(\zeta))$.

Because $P(0)$ has no eigenvalues on the imaginary axis, the spaces $\mathbb{F}_\pm^P(\zeta)$ depend smoothly on $\zeta \in \omega_0$ and their value at $\zeta = 0$ are $\mathbb{F}_\pm^P(0) = \mathbb{E}_\pm(G_d(p))$. Thus $\dim \mathbb{F}_-^P(\zeta) = N_-$. For $\zeta > 0$ and $\gamma \geq 0$, the space $\mathbb{F}_-(\zeta)$ is well defined and $\dim \mathbb{F}_-(\zeta) = N$ by Lemma 6.3.1. Thus $\mathbb{F}_-^H(\zeta)$ depends smoothly on ζ for $\zeta > 0$ with $\gamma \geq 0$ and

$$\dim \mathbb{F}_-^H(\zeta) = N_+ \quad \text{for } \zeta > 0 \text{ with } \gamma \geq 0\,. \tag{6.4.6}$$

6.4.2 Proof of Theorem 6.4.1, necessary conditions

Introduce

$$\Gamma_2(\zeta) = \Gamma\mathcal{W}(0,\zeta)\mathcal{V}(\zeta)\,. \tag{6.4.7}$$

Since $\mathcal{W}$ and $\mathcal{V}$ are smooth, we deduce from Lemma 6.2.7 that there is a constant C such that for all $\zeta \in \omega_0$, with $\zeta \neq 0$ and $\gamma \geq 0$, there holds

$$\frac{1}{C} D(\zeta) \leq D_2(\zeta) \leq C D(\zeta) \tag{6.4.8}$$

with

$$D_2(\zeta) = \det \Big(\mathbb{F}_-^H(\zeta) \times \mathbb{F}_-^P(\zeta), \ker \Gamma_2(\zeta) \Big) . \tag{6.4.9}$$

Moreover, the uniform Evans condition (6.3.17) is satisfied on a neighborhood of the origin, if and only if there is a constant C such that for all ζ small with $\zeta \neq 0$ and $\gamma \geq 0$, there holds

$$\forall (u_H, u_P) \in \mathbb{F}_-^H(\zeta) \times \mathbb{F}_-^P(\zeta) : \quad |u_H| + |u_P| \leq C |\Gamma_2(\zeta)(u_h, u_P)| . \tag{6.4.10}$$

Proposition 6.4.3. *If the uniform Evans condition* (6.3.17) *is satisfied for ζ small, then the profile $w(z)$ is transversal.*

Proof. The linearized equation (6.3.4) at $\zeta = 0$ is exactly the linearized profile equation (5.5.4) considered in Chapter five. In particular, the bounded solutions of the homogeneous equation are

$$U(z) = \mathcal{W}(z, 0)\mathcal{V}(0)\big(q, e^{zP(0)} v\big) \tag{6.4.11}$$

with $v \in \mathbb{F}_-^P(0) = \mathbb{E}_-(G_d)$ and q arbitrary. The Dirichlet condition $\Gamma U(0) = 0$ corresponds to $\Gamma_2(0)(q, v) = 0$.

Applying (6.4.10) with $u_H = 0$, implies that for all $v \in \mathbb{F}_-^P(\zeta)$:

$$|v| \leq C |\Gamma_2(\zeta)(0, v)|.$$

Since $\mathbb{F}_-^P(\zeta)$ is smooth in ζ, this extends to $\zeta = 0$. Thus $v = 0$ if $\Gamma_2(0)(0, v) = 0$ and $v \in \mathbb{E}_-(G_d)$ proving that the equation (5.5.10) has no nontrivial solution.

If the uniform Evans condition holds, for $\zeta \neq 0$ small with $\gamma \geq 0$, the mapping $\Gamma_2(\zeta)$ from $\mathbb{F}_-^H(\zeta) \times \mathbb{F}_-^P(\zeta)$ to $\mathbb{C}^N$ is an isomorphism. Thus, for all $h \in \mathbb{C}^N$ there is a solution $U(\zeta) \in \mathbb{F}_-^H(\zeta) \times \mathbb{F}_-^P(\zeta)$ of $\Gamma_2(\zeta) U(\zeta) = h$. By (6.4.10), the $U(\zeta)$ are bounded, and extracting a subsequence, we can assume that $U(\zeta)$ converges to $U = (q, v)$ as $\zeta \to 0$ with $\gamma \geq 0$. By continuity of $\mathbb{F}_-^P(\zeta)$, the limit v belongs to $\mathbb{F}_-^P(0) = \mathbb{E}_-(G_d)$, and (q, v) solves $\Gamma_2(0)(q, v) = h$. This shows that the mapping $\Gamma_2(0)$ from $\mathbb{C}^N \times \mathbb{E}_-(D_d)$ to $\mathbb{C}^N$ is surjective, hence that the problem (5.5.11) has always a bounded solution. By Proposition 5.5.3, this implies that the profile w is transversal. $\square$

Suppose that the profile w is transversal. By Proposition 5.5.5 and the representation (6.4.11) of bounded solutions, the tangent space $\dot{\mathcal{C}}$ to $\mathcal{C}$ at p is

$$\dot{\mathcal{C}} := \pi\Big(\ker \Gamma_2(0) \cap \big(\mathbb{C}^N \times \mathbb{E}_-(G_d)\big)\Big)$$

where π is the mapping

$$\pi : (q, v) \mapsto q\,.$$

The transversality hypothesis implies that $\dim \mathcal{C} = N_-$ and there is a mapping K from $\dot{\mathcal{C}}$ to $\mathbb{E}_-(G_d)$ such that

$$\ker \Gamma_2(0) \cap \big(\mathbb{C}^N \times \mathbb{E}_-(G_d)\big) = \big\{(\varphi, K\varphi)\,;\, \varphi \in \dot{\mathcal{C}}\big\}\,.$$

By continuity, this extends to a neighborhood of the origin where the space $\pi(\mathbb{K}(\zeta))$ with

$$\mathbb{K}(\zeta) = \ker \Gamma_2(\zeta) \cap \big(\mathbb{C}^N \times \mathbb{F}^P_-(\zeta)\big)$$

has dimension N_- and there is a mapping $K(\zeta)$ from $\pi(\mathbb{K}(\zeta))$ to $\mathbb{F}^P_-(\zeta)$ such that

$$\mathbb{K}(\zeta) = \big\{(\varphi, K(\zeta)\varphi)\,;\, \varphi \in \pi(\mathbb{K}(\zeta))\big\}\,. \tag{6.4.12}$$

Consider next the space $\mathbb{E}_+(G_d)$. Then $\mathbb{C}^N = \mathbb{E}_-(G_d) \oplus \mathbb{E}_+(G_d)$. Consider the mapping

$$\varpi : (q, v_- + v_+) \mapsto v_+$$

from $\ker \Gamma_2(0) \subset \mathbb{C}^N \times \mathbb{C}^N$ to $\mathbb{E}_+(G_d)$. The kernel is $\ker \Gamma_2(0) \cap \big(\mathbb{C}^N \times \mathbb{E}_-(G_d)\big)$ and thus has dimension N_-. Thus the range has dimension $N - N_- = \dim \mathbb{E}_+(G_d)$, proving that ϖ is surjective. Therefore, there is map K' from $\mathbb{E}_-(G_d)$ to $\ker \Gamma_2(0)$ such that $\varpi K' = \mathrm{Id}$. The N_+ dimensional space $\mathbb{K}'(0) = K'\mathbb{E}_-(G_d)$ satisfies $\ker \Gamma_2(0) = \mathbb{K}(0) \oplus \mathbb{K}'(0)$. By continuity, this extends to a neighborhood of the origin where there is a mapping $K'(\zeta)$ from $\mathbb{E}_-(G_d)$ to $\ker \Gamma_2(\zeta)$ such that $\varpi K' = \mathrm{Id}$ and $\ker \Gamma_2(\zeta) = \mathbb{K}(\zeta) \oplus \mathbb{K}'(\zeta)$ where $\mathbb{K}'(\zeta) := K'(\zeta)\mathbb{E}_-(G_d)$.

Taking bases $\{e^H_j\}$, $\{e^P_k\}$, $\{\varphi_l, K\varphi_l)\}$ and $\{K'\psi_m, \psi_m)\}$ in $\mathbb{F}^H_-$, $\mathbb{F}^P_-$, $\mathbb{K}$ and $\mathbb{K}'$ respectively, we see that the determinant $D_2(\zeta)$ is, up to a permutation of columns, equal to

$$\begin{vmatrix} e^H_j & \varphi_l & 0 & K'_1\psi_m \\ 0 & K\varphi_l & e^P_k & K'_2\psi_m \end{vmatrix}$$

where we have written $K'\psi = (K'_1\psi, K'_2\psi) \in \mathbb{C}^N \times \mathbb{C}^N$. Since the $K\varphi_l$ belong to the space generated by the e^P_k, we can eliminate these terms in the determinant. Moreover, since $\varpi K' = \mathrm{Id}$ on $\mathbb{E}_+(G_d)$ and $\mathbb{C}^N = \mathbb{E}_-(G_d) \oplus \mathbb{E}_+(G_d)$, the determinant

$$\det\left(e_k^P, K_2'\psi_m\right)$$

does not vanish on a neighborhood of zero. Summing up, we have proved:

Proposition 6.4.4. *When the profile w is transversal, there is a neighborhood ω_0 of the origin in $\mathbb{R}^{d+1}$ such that the Evans determinant satisfies*

$$D(\zeta) = \beta(\zeta) \det\left(\mathbb{F}_-^H(\zeta), \pi\mathbb{K}(\zeta)\right). \tag{6.4.13}$$

where β is a non vanishing smooth function on a neighborhood of the origin.

Since $H(0) = 0$ the analysis of $\mathbb{F}_-^H(\zeta)$ is more delicate. We use *polar coordinates*:

$$\zeta = \rho\check{\zeta} = \rho(\check{\tau}, \check{\gamma}, \check{\eta}), \quad \text{with} \quad \rho = |\zeta|, \quad |\check{\zeta}| = 1. \tag{6.4.14}$$

Then, (6.4.3) implies that

$$H(\zeta) = H(\rho\check{\zeta}) = \rho\check{H}(\check{\zeta}, \rho) \tag{6.4.15}$$

where $\check{H}$ is a smooth function of $(\check{\zeta}, \rho) \in \mathbb{R}^{d+1} \times \mathbb{R}$ for $|\check{\zeta}| \leq 2$ and $|\rho| \leq \rho_0$ for some $\rho_0 > 0$. In addition:

$$\check{H}(\check{\zeta}, 0) = -(A_d(p))^{-1}\big((i\check{\tau} + \check{\gamma})\mathrm{Id} + \sum_{j=1}^{d-1} i\check{\eta}_j A_j(p)\big). \tag{6.4.16}$$

In particular, the negative space of $H(\zeta)$ is the negative space of $\check{H}(\check{\zeta}, \rho)$ which we denote by $\check{\mathbb{F}}_-(\check{\zeta}, \rho)$:

$$\mathbb{F}_-^H(\zeta) = \check{\mathbb{F}}(\check{\zeta}, \rho), \quad \text{when } \zeta = \rho\check{\zeta}. \tag{6.4.17}$$

By (6.4.6) this N_+ dimensional space is well defined for $\rho > 0$ small enough and $|\check{\zeta}| = 1$ with $\check{\gamma} \geq 0$.

Denote by S^d the unit sphere $\{\check{\zeta}; |\check{\zeta}| = 1\}$, by S_+^d the half sphere $\{\check{\gamma} > 0\}$ and by $\overline{S}_+^d$ the closed half sphere $\check{\gamma} \geq 0$.

Lemma 6.4.5. *The N_+ dimensional vector bundle $\check{\mathbb{F}}(\check{\zeta}, \rho)$ extends smoothly to $S_+^d \times \{0\}$.*

Proof. When $\rho = 0$, (6.4.16) and the hyperbolicity Assumption (H2) imply that $\check{H}(\check{\zeta}, 0)$ has no eigenvalues on the imaginary axis when $\check{\gamma} > 0$. Therefore, the negative space of $\check{H}(\check{\zeta}, 0)$ depends smoothly on $\check{\zeta}$ for $\check{\zeta} \in S^d_+$.

For a fixed $\underline{\check{\zeta}} \in S^d_+$, by continuity $\check{H}(\check{\zeta}, \rho)$ has no eigenvalues on the imaginary axis for $(\check{\zeta}, \rho)$ in a small neighborhood of $(\underline{\check{\zeta}}, 0)$ when $\check{\gamma} > 0$. Therefore the negative space $\check{\mathbb{F}}(\check{\zeta}, \rho)$ is smooth for $(\check{\zeta}, \rho)$ in a small neighborhood of $(\underline{\check{\zeta}}, 0)$. $\square$

Remark 6.4.6. Consider the hyperbolic boundary value problem (6.1.6). By definition $\ker M = \dot{\mathcal{C}} = \pi(\mathbb{K}(0))$ with the notations above. Therefore, the Lopatinski determinant of this problem is:

$$\check{D}(\check{\zeta}) = \det\left(\check{\mathbb{F}}(\check{\zeta}, 0), \pi(\mathbb{K}(0))\right). \tag{6.4.18}$$

Proposition 6.4.7. *If the system* (6.1.2) *satisfies the uniform Evans condition* (6.3.17) *for small* $\check{\zeta}$*, then the hyperbolic boundary value problem* (6.1.6) *satisfies the uniform Lopatinski condition.*

Proof. Using Propositions 6.4.3 and 6.4.4, the assumption implies that

$$\left|\det\left(\check{\mathbb{F}}(\check{\zeta}, \rho), \pi(\mathbb{K}(\rho\check{\zeta}))\right)\right| \geq c > 0 \tag{6.4.19}$$

for all $\check{\zeta} \in \overline{S}^d_+$ and $\rho > 0$ small enough.

The $\pi(\mathbb{K}(\zeta)$ form a smooth N_- dimensional bundle; by Lemma 6.4.5, the $\check{\mathbb{F}}(\check{\zeta}, \rho)$ are smooth up to $\rho = 0$ when $\check{\gamma} > 0$. Hence, the determinant above is smooth in $(\check{\zeta}, \rho)$, up to $\rho = 0$ when $\check{\gamma} > 0$. Therefore, the estimate implies that

$$\left|\det\left(\check{\mathbb{F}}(\check{\zeta}, 0), \pi(\mathbb{K}(0))\right)\right| \geq c > 0$$

for all $\check{\zeta} \in S^d_+$. $\square$

6.4.3 Proof of Theorem 6.4.1, sufficient conditions

To prove the converse of Proposition 6.4.7, the idea is to prove that the determinant in (6.4.19) extends continuously to $\rho = 0$ for $\check{\zeta}$ in the compact set $\overline{S}^d_+$. This follows from the next result which is much stronger than Lemma 6.4.5.

Theorem 6.4.8. *Under the Assumptions* 5.1.1*, the* N_+ *dimensional vector bundle* $\check{\mathbb{F}}(\check{\zeta}, \rho)$ *extends continuously to the* $\overline{S}^d_+ \times \{0\}$.

The proof of this theorem is postponed to the next Chapter, where we show that it is a consequence of the construction of symmetrizers, see [MZ2].

End of the proof of Theorem 6.4.1 If the profile is transversal, then the Evans function satisfies (6.4.13) for small values of $\rho = |\zeta|$. By Theorem 6.4.8, the determinant

$$\det\left(\check{\mathbb{F}}(\check{\zeta},\rho), \pi(\mathbb{K}(\rho\check{\zeta}))\right)$$

is continuous for $(\check{\zeta},\rho) \in \overline{S}^d_+ \times [0,\rho_0]$ for some $\rho_0 > 0$. The uniform Lopatinski condition states that this determinant is uniformly bounded from below for $\rho = 0$ and $\check{\zeta} \in S^d_+$. Thus the extension does not vanish for $\check{\zeta} \in \overline{S}^d_+$ and $\rho = 0$, and by compactness and continuity, it does not vanish for $(\check{\zeta},\rho) \in \overline{S}^d_+ \times [0,\rho_1]$ for some $\rho_1 > 0$.

Chapter 7

Stability estimates

In this chapter we prove that the uniform Evans condition of Definition 6.3.5 implies uniform estimates for the solutions of (6.1.2). Moreover, these estimates are optimal from the point of view of parabolic smoothness. The proof relies on the use of symmetrizers, which are constructed as Fourier multipliers. A corollary of the construction of symmetrizers, is the continuous extendability of the spectral spaces $\mathbb{E}_-$ stated in Lemma 6.2.8 and Theorem 6.4.8 (see [MZ2]). In this chapter, we always suppose that Assumption 5.1.1 are satisfied and consider the linearized equations (6.1.2) around a profile w which satisfies (6.1.1).

7.1 The estimates

They involve weighted norms. We consider the following weight functions : with $\zeta := (\tau, \gamma, \eta)$, let

$$\varphi = \begin{cases} \big(\gamma + \varepsilon|\zeta|^2\big)^{\frac{1}{2}} & \text{when} \quad |\varepsilon\zeta| \leq 1, \\ \approx \varepsilon^{-\frac{1}{2}} & \text{when} \quad 1 \leq |\varepsilon\zeta| \leq 2, \\ \frac{\Lambda(\varepsilon\zeta)}{\sqrt{\varepsilon}} \approx \big(\gamma + |\tau| + \varepsilon|\eta|^2\big)^{\frac{1}{2}} & \text{when} \quad |\varepsilon\zeta| \geq 2. \end{cases} \tag{7.1.1}$$

where Λ is defined at (6.3.15). Note that the three terms above have the same order when $|\varepsilon\zeta| \approx 1$.

Given a weight function $\psi(\tau, \eta)$, we introduce the norm

$$\big|u\big|_{(\psi)} = \Big(\int \psi(\tau,\eta)^2 \, |\hat{u}(\tau,\eta)|^2 \, d\tau d\eta\Big)^{\frac{1}{2}}, \tag{7.1.2}$$

where $\hat{u}$ is the Fourier transform of $u(t,y)$ defined on $\mathbb{R}^d$. When u also depends on the variable x, we denote by $\|u\|_{(\psi)}$ the norm

$$\|u\|_{(\psi)} = \Big(\int_0^\infty \big|u(x)\big|^2_{(\psi)} dx \Big)^{\frac{1}{2}} . \tag{7.1.3}$$

We use different weight functions, φ, φ^2, φ/Λ etc. In these case, the weights and the norms depend on the parameters ε and γ. For simplicity we do not reflect this dependence in the notation and write $\|\cdot\|_{(\varphi)}$ etc. When the weight is equal to 1, we obtain the usual L^2 norms on $\mathbb{R}^{1+d}_+$ and $\mathbb{R}^d$, denoted by $\|\cdot\|$ and $|\cdot|$ respectively.

Theorem 7.1.1. *Under the stability Assumption* 6.3.5*, there is a constant C such that for all u and f in $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ satisfying* (6.1.2)*, for all $\gamma > 0$ and all $\varepsilon \in]0,1]$, one has*

$$\|e^{-\gamma t}u\|_{(\varphi^2)} + \sqrt{\varepsilon}\|e^{-\gamma t}\partial_x u\|_{(\varphi)} + \varepsilon\big|e^{-\gamma t}\partial_x u_{|x=0}\big|_{(\varphi/\sqrt{\Lambda})} \le C\|e^{-\gamma t} f\| . \tag{7.1.4}$$

We first state a simplified version of the estimates:

Corollary 7.1.2. *Under the stability Assumption* 6.3.5*, there is C such that for all $\varepsilon \in]0,1]$, all $\gamma > 0$ and all test functions u, f satisfying* (6.1.2)*, one has*

$$\gamma\|e^{-\gamma t}u\| + \sqrt{\varepsilon\gamma}\|e^{-\gamma t}\nabla_{y,x}u\| + \varepsilon\|e^{-\gamma t}\nabla_y\nabla_{y,x}u\| \le C\|e^{-\gamma t}f\| \tag{7.1.5}$$

To simplify notations, we write below $a(\varepsilon,\zeta,u,f) \lesssim b(\varepsilon,\zeta,u,f)$, to mean that there is a constant C such that for all $\varepsilon \in]0,1]$, all ζ with $\gamma > 0$ and all u and f there holds $a(\varepsilon,\zeta,u,f) \le Cb(\varepsilon,\zeta,u,f)$.

Proof of Corollary 7.1.2. There holds

$$\sqrt{\gamma} + \sqrt{\varepsilon}|\eta| \lesssim \varphi$$

which by Plancherel's theorem implies that

$$\gamma\|u\| + \sqrt{\varepsilon\gamma}\|\partial_y u\| + \varepsilon\|\partial_y^2 u\| \lesssim \|u\|_{(\varphi^2)} , \quad \sqrt{\gamma}\|u\| + \sqrt{\varepsilon}\|\partial_y u\| \lesssim \|u\|_{(\varphi)} .$$

Thus (7.1.4) implies (7.1.5). $\square$

Introducing $u_* = e^{-\gamma t} u$, (6.1.2) is equivalent to

$$
(7.1.6) \qquad \begin{cases} (\partial_t + \gamma) u_* + \displaystyle\sum_{j=1}^{d} A_j^\sharp \partial_j u_* - \varepsilon \sum_{j,k=1}^{d} B_{j,k}^\sharp \partial_{j,k}^2 u_* + \frac{1}{\varepsilon} E^\sharp u_* = f_* \,, \\ u_{*|x=0} = 0 \,. \end{cases}
$$

with $f_* = e^{-\gamma t} f$. Thus (7.1.4) is equivalent to

$$
(7.1.7) \qquad \| u_* \|_{(\varphi^2)} + \sqrt{\varepsilon} \| \partial_x u_* \|_{(\varphi)} + \varepsilon \big| \partial_x u_{*|x=0} \big|_{(\varphi/\sqrt{\Lambda})} \lesssim \| f_* \| \,.
$$

for the solutions of (7.1.6).

Denote by $\hat{u}$ [resp. $\hat{f}$] the space-time tangential Fourier transform of u_* [resp. f_*], that is the partial Fourier transform with respect to the variables (t, y). The Fourier transform of (7.1.6) is pecisely the equation (6.3.1). Therefore, by Plancherel's theorem, the energy estimates (7.1.7) are implied by, and indeed equivalent to, the following estimates

$$
(7.1.8) \quad \varphi^2 \| \hat{u} \|_{L^2(\mathbb{R}_+)} + \sqrt{\varepsilon} \varphi \| \partial_x \hat{u} \|_{L^2(\mathbb{R}_+)} + \varepsilon (\varphi / \sqrt{\Lambda}) | \partial_x \hat{u}(0) | \lesssim \| \hat{f} \|_{L^2(\mathbb{R}_+)} \,.
$$

for the solutions of (6.3.1). We get rid of the ε 's using the rescaling (6.3.3)

$$
\tilde{u}(z) = \hat{u}(\varepsilon z) \,, \quad \tilde{f}(z) = \varepsilon \hat{f}(\varepsilon z) \,, \quad \tilde{\zeta} = \varepsilon \zeta \,.
$$

In this case, the equation (6.3.1) is transformed into

$$
(7.1.9) \qquad -\partial_z^2 \tilde{u} + \mathcal{A}(z, \tilde{\zeta}) \partial_z \tilde{u} + \mathcal{M}(z, \tilde{\zeta}) \tilde{u} = \tilde{f} \,, \quad \tilde{u}(0) = 0 \,.
$$

Introduce the weights
(7.1.10)

$$
h(\tilde{\zeta}) = \sqrt{\varepsilon} \varphi(\zeta) = \begin{cases} \big(\tilde{\gamma} + |\tilde{\zeta}|^2 \big)^{\frac{1}{2}} & \text{when } |\tilde{\zeta}| \le 1, \\ \approx 1 & \text{when } 1 \le |\tilde{\zeta}| \le 2, \\ \Lambda(\tilde{\zeta}) \approx \big(\tilde{\gamma} + |\tilde{\tau}| + |\tilde{\eta}|^2 \big)^{\frac{1}{2}} & \text{when } |\tilde{\zeta}| \ge 2, \end{cases}
$$

$$
\ell(\tilde{\zeta}) = h(\tilde{\zeta}) \Lambda(\tilde{\zeta})^{-1/2} \,.
$$

Therefore, the estimates (7.1.8) and Theorem 7.1.1 are consequences of the following estimates:

Theorem 7.1.3. *Suppose that the uniform Evans condition* 6.3.5 *is satisfied. Then, there is a constant C such that for all $\tilde{\zeta} \in \mathbb{R}^{1+d}$ with $\tilde{\gamma} > 0$ and for all $\tilde{u}$ and $\tilde{f}$ in $C_0^\infty(\overline{\mathbb{R}}_+)$ satisfying* (7.1.9)*, there holds*

$$
(7.1.11) \qquad h^2 \| \tilde{u} \|_{L^2(\mathbb{R}_+)} + h \| \partial_z \tilde{u} \|_{L^2(\mathbb{R}_+)} + \ell | \partial_z \tilde{u}(0) | \le C \| \tilde{f} \|_{L^2(\mathbb{R}_+)} \,.
$$

As in Chapter six we write the equation as a first order system: (6.3.4) for $\tilde{U} = {}^t(\tilde{u}, \tilde{v})$ with $\tilde{v} = \partial_z \tilde{u}$ and $\tilde{F} = {}^t(0, \tilde{f})$. The estimates to prove are

$$h^2 \|\tilde{u}\|_{L^2(\mathbb{R}_+)} + h\|\tilde{v}\|_{L^2(\mathbb{R}_+)} + \ell|\tilde{v}(0)| \;\le\; C\|\tilde{f}\|_{L^2(\mathbb{R}_+)} \,. \tag{7.1.12}$$

These estimates are proved using *symmetrizers*.

7.2 The method of symmetrizers

Recall now the essence of the "method of symmetrizers" as it applies to general boundary value problems

$$\partial_x u = G(x)u + f\,, \quad \Gamma u(0) = 0\,. \tag{7.2.1}$$

Here, u and f are functions on $[0, \infty[$ with values in some Hilbert space $\mathcal{H}$, and $G(x)$ is a C^1 family of (possibly unbounded) operators defined on $\mathcal{D}$, dense subspace of $\mathcal{H}$.

A symmetrizer is a family of C^1 functions $x \mapsto S(x)$ with values in the space of operators in $\mathcal{H}$ such that there are C_0, $\lambda > 0$, $\delta > 0$ and C_1 such that

$$\forall x\,, \quad S(x) = S(x)^* \quad \text{and} \quad |S(x)| \le C_0\,, \tag{7.2.2}$$

$$\forall x\,, \quad 2\mathrm{Re}\, S(x)G(x) + \partial_x S(x) \ge 2\lambda \mathrm{Id}\,, \tag{7.2.3}$$

$$S(0) \ge \delta Id - C_1 \Gamma^*\Gamma\,. \tag{7.2.4}$$

In (7.2.2), the norm of $S(x)$ is the norm in the space of bounded operators in $\mathcal{H}$. Similarly $S(x)^*$ is the adjoint operator of $S(x)$. The notation $\mathrm{Re}\, T = \frac{1}{2}(T + T^*)$ is used in (7.2.3) for the real part of an operator T. When T is unbounded, the meaning of $\mathrm{Re}\, T \ge \lambda$, is that all $u \in \mathcal{D}$ belong to the domain of T and satisfy

$$\mathrm{Re}\,\big(Tu, u\big) \ge \lambda |u|^2\,. \tag{7.2.5}$$

The property (7.2.3) has to be understood in this sense.

Lemma 7.2.1. *If there is a symmetrizer S, then for all $u \in C^1([0, \infty[; \mathcal{H}) \cap C^0([0, \infty; \mathcal{D})$ with compact support in time, there holds:*

$$\lambda \|u\|^2 + \delta |u(0)|^2 \le \frac{C_0^2}{\lambda} \|f\|^2 \; + C_1 |\Gamma u(0)|^2\,, \tag{7.2.6}$$

where $f := \partial_x u - Gu$.

Here, $|\cdot|$ is the norm in $\mathcal{H}$ and $\|\cdot\|$ the norm in $L^2([0,\infty[;\mathcal{H})$.

Proof. Taking the scalar product of Su with the equation (7.2.1) and integrating over $[0,\infty[$, (7.2.2) implies

$$
\begin{aligned}
-(S(0)u(0),u(0)) &= \int \partial_x(Su,u)dx \\
&= \int \big((2\mathrm{Re}\, SG + \partial_x S)u, u\big)dx + 2\mathrm{Re}\int \big(Sf,u\big)dx\,.
\end{aligned}
\tag{7.2.7}
$$

By (7.2.3), there holds:

$$
\int \big((2\mathrm{Re}\, SG + \partial_x S)u, u\big)dx \geq 2\lambda\|u\|^2\,.
$$

By (7.2.4),

$$
(S(0)u(0),u(0)) \geq \delta|u(0)|^2 - C_1|\Gamma u(0)|^2\,.
$$

By (7.2.2)

$$
2\Big|\int \big(Sf,u\big)dx\Big| \leq 2C_0\|f\|\,\|u\| \leq \frac{C_0^2}{\lambda}\|f\|^2 + \lambda\|u\|^2\,.
$$

Thus the identity (7.2.7) implies the energy estimate (7.2.6). □

To prove Theorem 7.1.3, it is sufficient to construct symmetrizers $\mathcal{S}(z,\zeta)$ for $\mathcal{G}(z,\zeta)$. Three different regimes appear in the construction: the high frequency regime, when $|\zeta|$ is large, the low frequency regime when $|\zeta|$ is small, and the intermediate regime when ζ is bounded and bounded away from zero. The three different constructions are developed in the next two sections.

Remark 7.2.2. In our application below, the operators are matrices and the Hilbert space $\mathcal{H}$ is finite dimensional. However, thinking of the matrices as Fourier multipliers, our computations apply in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ of functions of the variables (t,y). This is the correct approach to generalize them to variable coefficients operators, where the Fourier multipliers are replaced by pseudo-differential operators (see [MZ1]).

The construction of the symmetrizers has two parts. First, we construct families of symmetrizers $S_\kappa(z,\zeta)$ satisfying (7.2.2) and (7.2.3). This only uses the structural hyperbolicity-parabolicity Assumptions (5.1.1). Next we choose κ such that the third condition (7.2.4) holds. There we use the stability condition (6.3.5). We end this section with noticing a general recipe linking Evans-Lopatinski conditions to (7.2.4). Consider the following situation:

Suppose that we are given continuous vector bundles $\mathbb{E}_\pm(\zeta) \subset \mathbb{C}^D$ depending on parameters ζ in some set $\omega \subset \mathbb{R}^m$ and such that

$$\mathbb{C}^D = \mathbb{E}_-(\zeta) \oplus \mathbb{E}_+(\zeta)\,. \tag{7.2.8}$$

Denote by $\Pi_\pm(\zeta)$ the projectors associated to this decomposition. Consider a family of matrices $\Gamma(\zeta)$ depending continuously on $\zeta \in \omega$ and such that $\dim\ker\Gamma(\zeta) + \dim\mathbb{E}_-(\zeta) = D$.

Proposition 7.2.3. *Consider a family $S_\kappa(\zeta)$ of symmetric $D \times D$ matrices, with $\kappa \in \mathbb{R}_+$ and $\zeta \in \omega$. Suppose that there is a compact set $\underline{\omega} \subset \omega$ such that for all $\zeta \in \underline{\omega}$*

$$S_\kappa \geq \kappa \Pi_+^* \Pi_+ - \Pi_-^* \Pi_-\,. \tag{7.2.9}$$

and

$$\det\big(\mathbb{E}_-(\zeta), \ker\Gamma(\zeta)\big) \neq 0\,. \tag{7.2.10}$$

Then, for κ large enough, there are C_1, $\delta > 0$ and a neighborhood of $\underline{\omega}$ in ω such that for ζ in this neighborhood:

$$S_\kappa(\zeta) + C_1 \Gamma(\zeta)^* \Gamma(\zeta) \geq \delta \mathrm{Id}\,. \tag{7.2.11}$$

Proof. By continuity, there is $c > 0$ such that

$$\big|\det\big(\mathbb{E}_-(\zeta), \ker\Gamma(\zeta)\big)\big| \geq c$$

on a neighborhood of $\underline{\omega}$. By Lemma 6.2.4, this implies, that for ζ in a possibly smaller neighborhood:

$$|\Pi_- h|^2 \leq C_0 |\Gamma \Pi_- h|^2 \leq C_0'\big(|\Pi_+ h|^2 + |\Gamma h|^2\big)$$

with C_0 and C_0' independent of h and ζ. Thus, (7.2.9) implies that

$$(S_\kappa h, h) + C_1 |\Gamma h|^2 \geq (\kappa - C_1)|\Pi_+ h|^2 + (C_1/C_0' - 1)|\Pi_- h|^2\,.$$

If $C_1 > C_0'$ and $\kappa > C_1$, (7.2.11) holds for some $\delta > 0$. $\square$

In most of the applications, $\mathbb{E}_-$ and $\mathbb{E}_+$ will be the positive and negative space of $\mathcal{G}$, but near the so-called glancing modes, the spectral projectors on positive and negative spaces are not uniformly bounded. In these cases, $\mathbb{E}_+$ is chosen as a suitable supplementary space of the negative space $\mathbb{E}_-$.

7.3 High frequencies

We first consider the case where ζ is large. In this case the parabolic properties are dominant. As in the proof of Lemma 6.3.1, we introduce "parabolic polar coordinates at infinity

$$\begin{aligned} &\hat{\zeta} = (\hat{\tau}, \hat{\eta}, \hat{\gamma}) = (\lambda^2 \tau, \lambda\eta, \lambda^2\gamma) \quad \text{with} \\ &\lambda = \langle\zeta\rangle^{-1} = (\tau^2 + \gamma^2 + |\eta|^4)^{\frac{1}{4}}. \end{aligned} \tag{7.3.1}$$

and λ is small. Then

$$\mathcal{M}(z,\zeta) = \langle\zeta\rangle^2 \hat{\mathcal{M}}(z,\hat{\zeta},\lambda) \quad \mathcal{A}(z,\zeta) = \langle\zeta\rangle \hat{\mathcal{A}}(z,\hat{\zeta},\lambda)$$

with

$$\begin{aligned} &\hat{\mathcal{M}}(z,\hat{\zeta},\lambda) = \hat{M}(z,\hat{\zeta}) + \lambda \hat{M}^1(z,\hat{\zeta}) + \lambda^2 \hat{M}^2(z,\hat{\zeta}) \\ &\hat{\mathcal{A}}(z,\hat{\zeta},\lambda) = \hat{A}(z,\hat{\zeta}) + \lambda \hat{A}^1(z,\hat{\zeta}) \end{aligned} \tag{7.3.2}$$

where the $\hat{\mathcal{M}}^j$ and $\hat{\mathcal{A}}^j$ are smooth and bounded functions of z and $\hat{\zeta}$ in the "sphere" $\hat{S}^d := \{\langle\zeta\rangle = 1\}$. Moreover, the leading terms $\hat{M}$ and $\hat{A}$ are given by (6.3.7). $\lambda \in [-1,1]$. We denote by $\hat{S}^d_+$ the closed half sphere $\{\hat{\gamma} \geq 0\}$. It is convenient to reduce $\mathcal{G}$ to first order as in the proof of Lemma 6.3.1, introducing the change of unknowns

$$u_1 = \langle\zeta\rangle u, \quad v_1 = v. \tag{7.3.3}$$

Then, (6.3.4) is transformed into

$$\partial_z U_1 = \lambda^{-1} \hat{\mathcal{G}}_1(z,\hat{\zeta},\lambda) U_1 + F\,, \quad \Gamma U_1(0) = u_1(0) = 0\,, \tag{7.3.4}$$

$$\hat{\mathcal{G}}_1(z,\hat{\zeta},\lambda) := \begin{pmatrix} 0 & \mathrm{Id} \\ \hat{\mathcal{M}} & \hat{\mathcal{A}} \end{pmatrix}.$$

Proposition 7.3.1 (symmetrizers for high frequencies). *When the uniform Evans condition* (6.3.17) *holds, there are* $\lambda_0 > 0$, $c > 0$, $\delta > 0$, $C \geq 0$ *and a* C^∞ *self adjoint matrix* $\hat{\mathcal{S}}$ *on* $[0,+\infty[\times \hat{S}^d_+ \times [0,\lambda_0]$ *such that*

i) $\hat{S}$ *and its derivatives are uniformly bounded and converge with an exponential rate at* $z = +\infty$.

ii) $\lambda^{-1}\mathrm{Re}\,(\hat{\mathcal{S}}\hat{\mathcal{G}}_1) + \frac{1}{2}\partial_z \hat{\mathcal{S}} \geq c\lambda^{-1}\mathrm{Id}$.

iii) $\hat{\mathcal{S}}_{|z=0} + C\Gamma^*\Gamma \geq \delta\mathrm{Id}$.

Proof. With (7.3.2), one has $\hat{\mathcal{G}}_1(z,\hat\zeta,\lambda) = \hat{G}_1(z,\hat\zeta) + O(\lambda)$ with

$$\hat{G}_1 = \begin{pmatrix} 0 & \mathrm{Id} \\ \hat{M} & \hat{A} \end{pmatrix}$$

Thus the matrices $\mathcal{G}_1$ are uniformly bounded and by (6.3.5) they converge when z tends to infinity. Moreover, $\mathcal{G}_1(z,\hat\zeta,\lambda)$ has the same eigenvalues as $\lambda^{-1}\mathcal{G}(z,\zeta)$. Hence, by Lemma 6.3.1, for $\hat\zeta \in \hat{S}^d_+$, $\hat{\mathcal{G}}_1(z,\hat\zeta,\lambda)$ and $\hat{\mathcal{G}}_1^\infty(\hat\zeta,\lambda)$ have no purely imaginary eigenvalues. Thus, the spectrum of $\hat{\mathcal{G}}_1(z,\hat\zeta,\lambda)$ remains in a compact set which does not intersect the imaginary axis when $\hat\gamma \ge 0$. Therefore, the spectral projectors $\Pi_\pm(z,\hat\zeta,\lambda)$ on the N dimensional invariant spaces $\hat{\mathbb{F}}_\pm(z,\hat\zeta,\lambda)$ of $\hat{\mathcal{G}}_1(z,\hat\zeta,\lambda)$ associated to eigenvalues with positive/negative real part are well defined and smooth, bounded as well as their derivatives, for $z \ge 0$, $\hat\zeta \in \hat{S}^d_+$ and $\lambda \ge 0$.

Because the eigenvalues of $\hat{\mathcal{G}}_1\Pi_+$ have positive real part there are self-adjoint matrices $\hat{\mathcal{S}}_+$ such that

$$\operatorname{Re}(\hat{\mathcal{S}}_+\hat{\mathcal{G}}_1\Pi_+) \ge \Pi_+^*\Pi_+\,, \quad \Pi_+^*\Pi_+ \le \hat{\mathcal{S}}_+ \le C\Pi_+^*\Pi_+\,. \tag{7.3.5}$$

For instance we can choose

$$\hat{\mathcal{S}}_+ = 2\int_0^\infty \Pi_+^* e^{-s\hat{\mathcal{G}}_1^*} e^{-s\hat{\mathcal{G}}_1}\Pi_+\,ds\,. \tag{7.3.6}$$

Note that

$$e^{-s\hat{\mathcal{G}}_1}\Pi_+ = \frac{1}{2i\pi}\int_\alpha e^{-s\mu}(\mu - \hat{\mathcal{G}}_1)^{-1}d\mu$$

where α is a positively oriented circle in the right half plane $\{\operatorname{Re}\mu > 0\}$ surrounding the eigenvalues of $\hat{\mathcal{G}}_1$ in this half space. Thus, $e^{-s\hat{\mathcal{G}}_1}\Pi_+$ and its adjoint $\Pi_+^* e^{-s\hat{\mathcal{G}}_1^*}$ are exponentially decaying in s. With the choice (7.3.6), one has $\operatorname{Re}(\hat{\mathcal{S}}_+\hat{\mathcal{G}}_1\Pi_+) = \Pi_+^*\Pi_+$. Moreover, $\hat{\mathcal{S}}_+ \le C\Pi_+^*\Pi_+$ and multiplying $\hat{\mathcal{S}}_+$ by a positive constant we can achieve that $\hat{\mathcal{S}} \ge \Pi_+^*\Pi_+$. In addition, we note that the matrices $\hat{\mathcal{S}}_+(z,\hat\zeta,\lambda)$ are smooth functions of $z, \hat\zeta$ and λ, uniformly bounded as well as their derivatives. In particular, (7.3.5) holds with a constant C independent of $z \ge 0$, $\hat\zeta \in \hat{S}^d_+$ and $\lambda \in [0,1]$.

Similarly, there is $\hat{\mathcal{S}}_-(z,\hat\zeta,\lambda)$, such that

$$-\operatorname{Re}(\hat{\mathcal{S}}_-\hat{\mathcal{G}}_1\Pi_-) \ge \Pi_-^*\Pi_-\,, \quad \Pi_-^*\Pi_- \le \hat{\mathcal{S}}_- \le C\Pi_-^*\Pi_-\,. \tag{7.3.7}$$

One construct $\hat{\mathcal{S}}$ as

$$\hat{\mathcal{S}} = \kappa\hat{\mathcal{S}}_+ - \hat{\mathcal{S}}_-\,. \tag{7.3.8}$$

with $\kappa > 0$ to be chosen large enough. Property i) is clear. Moreover,

$$\hat{\mathcal{S}}\hat{\mathcal{G}}_1 = \hat{\mathcal{S}}\hat{\mathcal{G}}_1\Pi_+ + \hat{\mathcal{S}}\hat{\mathcal{G}}_1\Pi_- = \kappa\hat{\mathcal{S}}_+\hat{\mathcal{G}}_1\Pi_+ - \hat{\mathcal{S}}_-\hat{\mathcal{G}}_1\Pi_-$$

hence, there is $c > 0$ such that for all $z \geq 0$, all $\hat{\zeta} \in \hat{S}^d_+$, all $\lambda \in [0,1]$ and all $\kappa \geq 1$,

$$\operatorname{Re}\hat{\mathcal{S}}\hat{\mathcal{G}}_1 \geq \kappa\Pi_+^*\Pi_+ + \Pi_-^*\Pi_- \geq c\mathrm{Id}\,.$$

Since $\partial_z\mathcal{S}$ is bounded by $C(\kappa+1)$, this implies property ii) provided that λ is smaller than some $\lambda_0 > 0$, (possibly depending on κ).

To prove iii), we use the following result, concerning the spaces $\hat{\mathbb{F}}_-(0,\hat{\zeta},\lambda) = \ker\Pi_+(0,\hat{\zeta},\lambda)$.

Lemma 7.3.2. *Under Assumption* 6.3.5*, there holds*

$$\hat{\mathbb{F}}_-(0,\hat{\zeta},0) \cap \ker\Gamma = \{0\}\,. \tag{7.3.9}$$

Taking this lemma for granted, we finish the proof of Lemma 7.3.1. For $\hat{\zeta} \in \hat{S}^d_+$ there holds

$$\mathbb{C}^{2N} = \mathbb{F}_-(0,\hat{\zeta},0) \oplus \mathbb{F}_+(0,\hat{\zeta},0)$$

Moreover, (7.3.9) implies that the determinant $\det(\mathbb{F}_-(0,\hat{\zeta},0),\ker\Gamma)$ does not vanish. With (7.3.8), we are in position to apply Proposition 7.2.3 and there are κ and $\lambda_0 > 0$ such that the estimate iii) holds for $\hat{\zeta} \in \hat{S}^d_+$ and $\lambda \in [0,\lambda_0]$.

Hence, by continuity-compactness, it does not vanish for $\hat{\zeta} \in \hat{S}^d_+$ and λ small. This implies iii). It only remains to prove Lemma 7.3.2. $\square$

Proof of Lemma 7.3.2.

In order to use Assumption 6.3.5, we give a link between the spaces $\hat{\mathbb{F}}_-(0,\hat{\zeta},0)$ and the spaces $\widetilde{\mathbb{E}}_-(\zeta)$ introduced in Chapter 6, section 3. Recall that $\mathbb{E}_-(\zeta)$ is the set of initial data $U(0) = (u(0), v(0))$ for bounded solutions $U = (u,v)$ of the homogeneous equation (6.3.4) with $F = 0$. In addition to the change of unknows (7.3.3), rescale the variable $z = \lambda\hat{z}$ and introduce

$$u_2(\hat{z}) = u_1(\lambda\hat{z}) = \lambda^{-1}u(\lambda\hat{z})\,, \qquad v_2(\hat{z}) = v_1(\lambda\hat{z}) = v(\lambda\hat{z})\,.$$

Then the homogeneous equation (6.3.4) is transformed into

$$\partial_{\hat{z}}U_2 = \hat{\mathcal{G}}_2(\hat{z},\hat{\zeta},\lambda)U_2 \tag{7.3.10}$$

with

$$\hat{\mathcal{G}}_2(\hat{z}, \hat{\zeta}, \lambda) = \hat{\mathcal{G}}_1(\lambda\hat{z}, \hat{\zeta}, \lambda) \tag{7.3.11}$$

Denote by $\tilde{\mathbb{F}}_-(\hat{\zeta}, \lambda)$ the set of initial data for bounded solutions of (7.3.10). The computation above shows that $\tilde{\mathbb{F}}_-(\hat{\zeta}, \lambda) = \{(\lambda^{-1}u, v); (u, v) \in \mathbb{E}_-(\zeta)\}$. Therefore, by (6.3.20), Assumption 6.3.5 implies that there is C such that for all $\hat{\zeta} \in \hat{S}^d_+$ and all $\lambda \in [0, 1]$:

$$\forall U = (u, v) \in \tilde{\mathbb{F}}_-(\hat{\zeta}, \lambda) : \quad |v| \leq C|u| \, . \tag{7.3.12}$$

When λ tends to zero, (7.3.11) implies that

$$\hat{\mathcal{G}}_2(\hat{z}, \hat{\zeta}, \lambda) \to \hat{\mathcal{G}}_1(0, \hat{\zeta}, 0) = \begin{pmatrix} 0 & \mathrm{Id} \\ \hat{M}_0 & \hat{A}_0 \end{pmatrix}$$

where $\hat{M}_0$ and $\hat{A}_0$ are the evaluation at 0 of the functions $\hat{M}$ and $\hat{A}$ defined at (6.3.7). Since $\hat{\mathcal{G}}_1(0, \hat{\zeta}, 0)$ is constant, the space of initial data of bounded solutions of the equation

$$\partial_z U_2 = \hat{\mathcal{G}}_1(0, \hat{\zeta}, 0) \tag{7.3.13}$$

is the negative spectral space $\mathbb{F}^\infty_-(0, \hat{\zeta}, 0)$.

Using that $\hat{\mathcal{G}}_1$ has no eigenvalues on the imaginary axis, one shows that

$$\tilde{\mathbb{F}}_-(\hat{\zeta}, \lambda) \to \hat{\mathbb{F}}^\infty_-(0, \hat{\zeta}, 0)$$

as $\lambda \to 0$ The uniform estimate (7.3.12) implies that the estimate $|v| \leq C|u|$ extends to the limit space $\hat{\mathbb{F}}_-(0, \hat{\zeta}, 0)$, (7.3.9). $\square$

Remark 7.3.3. The transversality condition (7.3.9) is equivalent to the requirement that the problem

$$-\partial_z^2 u + \hat{A}_0 \partial_z u + \hat{M}_0 u = 0 \, , \qquad u(0) = 0 \tag{7.3.14}$$

has no nontrivial solution in $H^2([0, \infty[)$. Suppose that the parabolic problem is symmetric, i.e. that there is a smooth symmetric definite positive matrix $S(p)$ such that

$$\mathrm{Re} \sum \xi_j \xi_k S(p) B_{j,k}(p) \tag{7.3.15}$$

is definite positive. Recall the definition (6.3.7) of $\hat{A}$ and $\hat{M}$. If $u \in H^2$ satisfies (7.3.14), taking the real part of the scalar product in L^2 with $(S(0)B_{d,d}(0))u$ yields

$$\begin{aligned}\operatorname{Re}\big(SB_{d,d}\partial_z u, \partial_z u\big) + \sum_{j=1}^{d-1} \operatorname{Re} i\eta_j \big(S(B_{j,d} + B_{d,j})\partial_z u, u\big) \\ + \sum_{j,k=1}^{d-1} \operatorname{Re} \eta_j\eta_k \big(SB_{j,k}u, u\big) + \gamma\big(Su, u\big) \ = \ 0\,,\end{aligned}$$

where the matrices S and $B_{j,k}$ are evaluated at the state $p = 0$. The assumption on S implies that the sesquilinear form on the left hand side is coercive on the space $H^1_0([0, +\infty[)$, as easily seen by extending u by 0 for negative z and considering the Fourier transform in z. Thus (7.3.14) has no non trivial H^2 solution. Therefore, the symmetry of the Parabolic operator implies te transversality (7.3.9) and thus that the uniform stability condition is automatically satisfied for large ζ.

7.4 Medium frequencies

Consider $\underline{\zeta} = (\underline{\tau}, \underline{\eta}, \underline{\gamma}) \in \mathbb{R}^{d+1}$ with $\underline{\gamma} \geq 0$. We construct symmetrizers for ζ close to $\underline{\zeta}$. By Lemma 6.3.2 the system (6.3.4) is transformed into the constant coefficient system (6.3.10) through the change of unknows $\widetilde{U}(z) = \mathcal{W}(z, \zeta)U_1(z)$. Thus the main idea is to *construct symmetrizers for the constant coefficients system* (6.3.10). They will provide estimates for U_1, and thus for $\widetilde{U} = \mathcal{W}U_1$.

Proposition 7.4.1 (Symmetrizers for medium frequencies). *Suppose that the uniform stability condition* (6.3.17) *holds. Then, for all* $\underline{\zeta} \neq 0$ *with* $\underline{\gamma} \geq 0$ *there is a neighborhood* ω *of* $\underline{\zeta}$ *and there are constants* $c > 0$, $\delta > 0$, $C \geq 0$ *and a* C^∞ *matrix* $\mathcal{S}(\zeta)$ *on* ω *such that for all* $\zeta \in \omega$*:*

(7.4.1) $$\mathcal{S} = \mathcal{S}^*\,,$$

(7.4.2) $$\operatorname{Re}(\mathcal{S}\mathcal{G}^\infty) \geq c\mathrm{Id}\,,$$

(7.4.3) $$\mathcal{S} + C(\Gamma_1)^*\Gamma_1 \geq \delta\mathrm{Id}\,.$$

Proof. By Lemma 6.3.1, the eigenvalues of $\mathcal{G}^\infty(\underline{\zeta})$ are away from the imaginary axis. Hence, there is a smooth invertible matrix $\mathcal{V}$ on a neighborhood ω of $\underline{\zeta}$, such that

$$\mathcal{V}^{-1}\mathcal{G}^\infty\mathcal{V} = \begin{pmatrix} G_+ & 0 \\ 0 & G_- \end{pmatrix}$$

where $G_\pm$ have their spectrum in $\pm\mathrm{Re}\,\mu > 0$. Consider

$$\mathcal{S} = (\mathcal{V}^{-1})^* \begin{pmatrix} \kappa S_+ & 0 \\ 0 & -S_- \end{pmatrix} \mathcal{V}^{-1}$$

with $S_\pm$ symmetric, positive definite and such that

$$\mathrm{Re}\,(S_+G_+) \geq \mathrm{Id}\,, \quad -\mathrm{Re}\,(S_-G_-) \geq \mathrm{Id}\,.$$

For instance, as in (7.3.6), one can choose

$$S_+ = 2\int_0^\infty e^{-tG_+^*}e^{-tG_+}dt$$

and use a similar expression for S_-. For all $\kappa \geq 1$, (7.4.2) holds. Moreover, the form of $\mathcal{S}$ implies that there are constants c and C such that

$$(\mathcal{S}V, V) \geq c\kappa|\Pi_+V|^2 - C|\Pi_-V|^2\,. \tag{7.4.4}$$

where Π_+ [resp. Π_-] is the spectral projector of $\mathcal{G}^\infty$ on the space $\mathbb{F}_+$ [resp. $\mathbb{F}_-$] generated by generalized eigenvectors associated to eigenvalues in $\mathrm{Re}\,\mu > 0$ [resp. $\mathrm{Re}\,\mu < 0$]. These projectors are smooth functions of ζ in a neighborhood of $\underline{\zeta}$ since the two groups of eigenvalues remain separated, thus one has the smooth decomposition:

$$\mathbb{C}^{2N} = \mathbb{F}_-(\zeta) \oplus \mathbb{F}_+(\zeta)\,. \tag{7.4.5}$$

The space $\ker\Pi_+ = \mathbb{F}_-$ is the set of initial data such that the corresponding solution of $\partial_z U_1 = \mathcal{G}^\infty U_1$ is bounded. Thus, by (6.3.12), the Evans-Lopatinski stability condition (6.3.18) implies that

$$\mathbb{F}_-(\underline{\zeta}) \cap \ker\Gamma_1(\underline{\zeta}) = \{0\} \tag{7.4.6}$$

and this remains true on a neighborhood of $\underline{\zeta}$. Thus, with (7.4.4) and (7.4.5), we are in position to apply Proposition 7.2.3 and (7.4.3) follows for κ large enough. □

7.5 Low frequencies

We now turn to the most difficult case of low frequencies and work near $\underline{\zeta} = 0$. We use the change of unknows $U_2 = \mathcal{V}^{-1}U_1$ given by Lemma 6.4.2. It transforms the equation (6.3.10) into

$$\partial_z U_2 = \mathcal{G}_2(\zeta)U_2 + F_2\,, \quad \Gamma_2(\zeta)U_2(0) = 0\,, \tag{7.5.1}$$

with $F_2 = \mathcal{V}^{-1}F_1$ and $\Gamma_2(\zeta) = \Gamma_1(\zeta)\mathcal{V}(\zeta)$. The matrix $\mathcal{G}_2$ has the block diagonal form (6.4.1)

$$\mathcal{G}_2 = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix}.$$

In a neighborhood of $\underline{\zeta} = 0$, we construct a symmetrizer for the matrix $\mathcal{G}_2$:

$$\mathcal{S} = \begin{pmatrix} S_H & 0 \\ 0 & S_P \end{pmatrix} \tag{7.5.2}$$

where S_H and S_P are symmetrizers for H and P respectively. The construction of S_P is quite similar to the construction performed for medium frequencies, using that the spectrum of P is away from the imaginary axis. The construction of S_H is much more delicate. We use *polar coordinates*:

$$\zeta = \rho\check{\zeta} = \rho(\check{\tau}, \check{\gamma}, \check{\eta}) , \quad \text{with} \quad \rho = |\zeta| , \quad |\check{\zeta}| = 1 . \tag{7.5.3}$$

Recall that

$$H(\zeta) = \rho\check{H}(\check{\zeta}, \rho) \tag{7.5.4}$$

with $\check{H}(\check{\zeta}, 0)$ given by (6.4.16). As in Chapter 6, Section 4, S^d denotes the sphere $\{|\check{\zeta}| = 1\}$, S^d_+ the open half sphere where $\check{\gamma} > 0$ and $\overline{S}^d_+$ denotes the closed half sphere $\check{\gamma} \geq 0$.

Theorem 7.5.1 (Symmetrizers for low frequencies). *Suppose that the uniform stability condition* (6.3.17) *is satisfied. There are constants $c > 0$, $\delta > 0$ and C such that:*

i) there is a neighborhood ω of 0 *and there is a C^∞ $N \times N$ matrix S_P on ω such that for all $\zeta \in \omega$:*

$$S_P(\zeta) = (S_P(\zeta))^* , \quad \mathrm{Re}\,(S_P(\zeta)P(\zeta)) \geq c\mathrm{Id} ; \tag{7.5.5}$$

ii) there are $\rho_0 > 0$ and a C^∞ matrix $\check{S}_H$ on $\overline{S}^d_+ \times [0, \rho_0]$ such that for all $(\check{\zeta}, \rho) \in \overline{S}^d_+ \times [0, \rho_0]$ there holds:

$$\check{S}_H = (\check{S}_H)^* , \quad \mathrm{Re}\,(\check{S}_1\check{H}) \geq c(\check{\gamma} + \rho)\mathrm{Id} ; \tag{7.5.6}$$

iii) for all $\zeta = \rho\check{\zeta} \in \mathbb{R}^{1+d}$ with $\check{\zeta} \in \overline{S}^d_+$ and $\rho \in]0, \rho_0]$, the matrix

$$\mathcal{S}(\zeta) = \begin{bmatrix} \check{S}_H(\check{\zeta}, \rho) & 0 \\ 0 & S_P(\zeta) \end{bmatrix} \tag{7.5.7}$$

satisfies

$$\mathcal{S}(\zeta) + C\Gamma_2(\zeta)^*\Gamma_2(\zeta) \geq \delta\mathrm{Id} . \tag{7.5.8}$$

Since the eigenvalues of P are away from the imaginary axis, there are families of symmetrizers $S_P(\zeta)$ on a neighborhood of 0 in $\mathbb{R}^{d+1}$ such that

$$S_P = \kappa(\Pi_P^+)^* S_P^+ \Pi_P^+ - (\Pi_P^-)^* S_P^- \Pi_P^- , \quad \mathrm{Re}\,(S_P P) \geq c\mathrm{Id}\,, \tag{7.5.9}$$

where $\Pi_P^\pm$ are the spectral projectors associated to the spectral decomposition

$$\mathbb{C}^N = \mathbb{F}_-^P(\zeta) \oplus \mathbb{F}_+^P(\zeta) \tag{7.5.10}$$

associated to the splitting of eigenvalues of P in the half spaces $\{\pm\mathrm{Re}\,\mu > 0\}$ and $S_P^\pm$ are definite positive. Recall that $\dim \mathbb{F}_\pm^P = N_\pm$. For $\zeta \neq 0$ with $\gamma > 0$, Lemma 6.3.1 implies that $\mathcal{G}_2(\zeta)$ and hence $H(\zeta)$ and $\check H(\check\zeta, \rho)$ have no eigenvalues on the imaginary axis. Thus the positive and negative spaces $\check{\mathbb{F}}_\pm^H(\zeta)$ associated to $\check H$ satisfy

$$\dim \check{\mathbb{F}}_\pm^H(\check\zeta, \rho) = N_\mp\,. \tag{7.5.11}$$

To construct the $\check S_H$, one can argue locally near a given point $\underline{\check\zeta} \in \overline{S}_+^d$. The main argument is the following result which is proved in Chapter 8:

Theorem 7.5.2. *For all $\underline{\check\zeta} \in \overline{S}_+^d$, there are spaces $\underline{\mathbb{F}}_\pm$ satisfying*

$$\mathbb{C}^N = \underline{\mathbb{F}}_- \oplus \underline{\mathbb{F}}_+\,, \quad \dim \underline{\mathbb{F}}_\pm = N_\mp\,, \tag{7.5.12}$$

and such that for all $\kappa \geq 1$ there are a neighborhood $\check\omega$ of $(\underline{\check\zeta}, 0)$ in $\mathbb{R}^{d+1} \times \mathbb{R}$, a C^∞ mapping $\check S_H$ from $\check\omega$ to the space of $N \times N$ matrices and a constant $c > 0$ such that for all $(\check\zeta, \rho) \in \check\omega$,

$$\check S_H(\check\zeta, \rho) = \check S_H^*(\check\zeta, \rho)\,, \tag{7.5.13}$$

$$\big(\check S(\check\zeta, \rho)h, h\big) \geq \kappa|\underline{\Pi}_+ h|^2 - |\underline{\Pi}_- h|^2\,, \tag{7.5.14}$$

and for all $(\check\zeta, \rho) \in \check\omega$ with $\rho \geq 0$ and $\check\gamma \geq 0$:

$$\mathrm{Re}\,\big(\check S_H(\check\zeta, \rho)\check H(\check\zeta, \rho)h, h\big) \geq c(\check\gamma + \rho)|h|^2\,. \tag{7.5.15}$$

In (7.5.14), *$\underline{\Pi}_\pm$ denote the projectors on $\underline{\mathbb{F}}_\pm$ associated to the decomposition* (7.5.12).

Concerning the choice of spaces $\underline{\mathbb{F}}_\pm$, the first idea would be to take $\mathbb{F}_\pm = \check{\mathbb{F}}_\pm^H(\underline{\check\zeta}, 0)$. However we don't know yet that theses spaces are well defined when $\check\gamma = 0$, and the splitting (7.5.12) is not always true for all $\underline{\check\zeta}$. On the contrary, we can use this theorem to prove the continuous extendability of the $\check{\mathbb{F}}_-^H(\underline{\check\zeta}, \rho)$ to $\rho = 0$. We stress the fact the uniform stability condition is not required for this Theorem and for its following corollary:

Theorem 7.5.3. *The vector bundle $\check{\mathbb{F}}^H_-(\check\zeta,\rho)$ defined for $\check\zeta \in \overline{S}^d_+$ and $\rho \in]0,\rho_0]$ extends continuously to $\rho = 0$.*

Proof. **a)** Consider $\kappa > 2$ and $\check\omega$ given by Theorem (7.5.2). For $(\check\zeta,\rho) \in \check\omega$ with $\check\gamma \geq 0$ and $\rho > 0$, consider $h \in \check{\mathbb{F}}^H_-(\check\zeta,\rho)$ and

$$u(z) = e^{z\check H(\check\zeta,\rho)} h$$

This function is exponentially decaying at $+\infty$ and satisfies $\partial_z u = \check H(\check\zeta,\rho)u$. Therefore multiplying by $\check S(\check\zeta,\rho)$ and integrating by parts yield, thanks to (7.5.13):

$$\big(\check S h, h\big) + 2\mathrm{Re}\int_0^\infty \big(\check S\check H u(z), u(z)\big)dz = 0$$

By (7.5.15), the integral is nonnegative. Therefore $(\check S h, h)$ is nonpositive which by (7.5.14) implies that $\kappa|\underline{\Pi}_+ h|^2 \leq |\underline{\Pi}_- h|^2$. Thus

$$\forall h \in \check{\mathbb{F}}^H_-(\check\zeta,\rho) : \qquad |\underline{\Pi}_+ h| \leq \frac{1}{\sqrt{\kappa}-1}|h|\,. \tag{7.5.16}$$

This implies that the mapping $\underline{\Pi}_-$ from $\check{\mathbb{F}}^H_-(\check\zeta,\rho)$ into $\underline{\mathbb{F}}_-$ is one to one and since both spaces have dimension N, it is a bijection. Therefore, there is a mapping $\Phi(\check\zeta,\rho)$ from $\underline{\mathbb{F}}_-$ to $\underline{\mathbb{F}}_+$ such that

$$\check{\mathbb{F}}^H_-(\check\zeta,\rho) = \big\{u + \Phi(\check\zeta,\rho)u \,:\, u \in \underline{\mathbb{E}}_-\big\} \tag{7.5.17}$$

and

$$\forall u \in \underline{\mathbb{F}}_- : \qquad |\Phi(\check\zeta,\rho)u| \leq \frac{1}{\sqrt{\kappa}-2}|u|. \tag{7.5.18}$$

Since κ is arbitrarily large, this proves that

$$\underline{\mathbb{F}}_- = \widetilde{\lim}\,\check{\mathbb{F}}^H_-(\check\zeta,\rho), \tag{7.5.19}$$

where $\widetilde{\lim}$ means that $(\check\zeta,\rho)$ tends to $(\underline{\check\zeta},0)$ with $\rho > 0$, $\check\gamma \geq 0$.

b) The relation (7.5.19) implies that for all $\underline{\check\zeta} \in \overline{S}^d_+$ there is a unique space $\underline{\mathbb{F}}_-$ such that the properties listed in Theorem 7.5.2 are satisfied. We denote by $\widetilde{\mathbb{F}}^H_-(\underline{\check\zeta},\rho) = \underline{\mathbb{F}}_-$ the extension of $\check{\mathbb{F}}^H_-(\check\zeta,\rho)$ defined by (7.5.19) for $\rho = 0$. We prove that $\widetilde{\mathbb{F}}^H_-(\check\zeta,\rho)$ is continuous at $\rho = 0$.

Consider again a given point $\underline{\check\zeta} \in \overline{S}^d_+$. For $\kappa > 2$, let $\check\omega$ be given by Theorem 7.5.2. For all $(\check\zeta,\rho) \in \check\omega$, the estimate (7.5.16) holds. For all

$(\underset{\sim}{\check\zeta}', 0) \in \check\omega$, thanks to (7.5.18), we can let $(\check\zeta, \rho)$ tend $(\check\zeta', 0)$ in the sense of $\underset{\sim}{\lim}$. Therefore, passing to the limit in (7.5.16) implies that for all $(\check\zeta, \rho) \in \Omega$ with $\check\gamma \geq 0$ and $\rho \geq 0$:

$$\forall h \in \widetilde{\mathbb{F}}^H_-(\check\zeta, \rho) : \qquad |\underline{\Pi}_+ h| \leq \frac{1}{\kappa - 1}|h| .$$

Arguing as before, this implies that

$$\underline{\mathbb{F}}_- = \widetilde{\mathbb{F}}^H_-(\underline{\check\zeta}, 0) = \lim \widetilde{\mathbb{F}}^H_-(\check\zeta, \rho),$$

where the limit is taken for $(\check\zeta, \rho)$ tending to $(\underline{\check\zeta}, 0)$ in $\overline{S}^d_+ \times [0, \rho_0]$. This means that the bundle $\widetilde{\mathbb{F}}^H_-(\check\zeta, \rho)$ is continuous in $(\check\zeta, \rho) \in \overline{S}^d_+ \times [0, \rho_0]$ at $(\underline{\check\zeta}, 0)$. $\square$

Proof of Theorem 7.5.1 *assuming Theorem* 7.5.2. Consider $\underline{\check\zeta} \in \overline{S}^d_+$. We apply Proposition 7.2.3 to the symmetrizers

$$\check{\mathcal{S}}_\kappa(\check\zeta, \rho) = \begin{bmatrix} \check S_{H,\kappa}(\check\zeta, \rho) & 0 \\ 0 & S_{P,\kappa}(\rho\check\zeta) \end{bmatrix}$$

where $\check S_{H,\kappa}$ is given by Theorem 7.5.2 in a neighborhood of $(\underline{\check\zeta}, 0)$ which may depend on κ and $S_{P,\kappa}$ is given by (7.5.9). By (7.5.12), the spaces $\underline{\mathbb{F}}_\pm$ satisfy the property (7.2.8). By (7.5.9) and (7.5.14), the estimate (7.2.9) is satisfied at $(\underline{\check\zeta}, 0)$.

For $\rho > 0$, the negative space of $\mathcal{G}_2(\rho\check\zeta)$ is

$$\check{\mathbb{F}}_-(\check\zeta, \rho) := \check{\mathbb{F}}^H_-(\check\zeta, \rho) \times \mathbb{F}^P_-(\rho\check\zeta) .$$

Transporting the uniform stability condition by the change of unknows $\mathcal{V}(\zeta)$, we know that for all $\check\zeta \in \overline{S}^d_+$ and $\rho > 0$:

$$\left| \det \left(\check{\mathbb{F}}_-(\check\zeta, \rho), \ker \Gamma_2(\rho\check\zeta) \right) \right| \geq c > 0$$

By Theorem 7.5.3 this estimate extends to $\rho = 0$ and thus the determinant $\det \left(\widetilde{\mathbb{F}}_-(\check\zeta, \rho), \ker \Gamma(\rho\check\zeta) \right)$ does not vanish at $\zeta = \underline{\check\zeta}$ and $\rho = 0$. Thus Proposition 7.2.3 implies that there are κ, C and $\delta > 0$ and a neighborhood of ω of $(\underline{\check\zeta}, 0)$ in $S^d \times \mathbb{R}$ such that

$$\forall (\check\zeta, \rho) \in \omega : \quad \check{\mathcal{S}}_\kappa(\check\zeta, \rho) + C\Gamma_2(\zeta)^*\Gamma_2(\zeta) \geq \delta \mathrm{Id} .$$

By compactness, there are $\rho_0 > 0$ and a finite covering $\cup \omega_j$ of $\overline{S}^d_+ \times [0, \rho_0]$, parameters κ_j and constants C and $\delta > 0$ such that

$$\forall j\,,\ \forall (\check\zeta, \rho) \in \omega_j\ :\quad \check{\mathcal{S}}_{\kappa_j}(\check\zeta, \rho) + C\Gamma_2(\zeta)^*\Gamma_2(\zeta) \ge \delta \mathrm{Id}\,. \tag{7.5.20}$$

Consider a partition of unity $\sum \chi_j = 1$ on $\overline{S}^d_+ \times [0, \rho_0]$ with $\chi_j \in C_0^\infty(\omega_j)$. Let

$$\begin{aligned}
&\check S_H(\check\zeta, \rho) = \sum \chi_j \check S_{H,\kappa_j}\,,\\
&S_P(\zeta) = S_{P,\kappa^*}(\zeta)\,, \quad \kappa^* = \max \kappa_j\,,\\
&\check{\mathcal{S}}(\check\zeta, \rho) = \begin{bmatrix} \check S_H(\check\zeta, \rho) & 0 \\ 0 & S_P(\rho\check\zeta) \end{bmatrix}.
\end{aligned}$$

By (7.5.9), $S_{P,\kappa^*} \ge S_{P,\kappa_j}$ for all j. Hence $\mathcal{S} \ge \sum \chi_j \mathcal{S}_{\kappa_j}$ and therefore (7.5.20) implies that

$$\check{\mathcal{S}}(\check\zeta, \rho) + C\Gamma_2(\zeta)^*\Gamma_2(\zeta) \ge \delta \mathrm{Id}\,.$$

Thus property *iii)* in Theorem 7.5.1 is proved.

Properties *i)* and *ii)* directly follow from (7.5.9) and (7.5.15) respectively.

□

7.6 Proof of the L^2 estimates

We prove the estimate (7.1.12) in the three different regimes.

a) *Medium frequencies.* Lemmas 7.4.1 and 7.2.1 imply that for all $\underline\zeta \ne 0$ with $\underline\gamma \ge 0$, there is a neighborhood ω of $\underline\zeta$ such that for all $\zeta \in \omega$, the solutions of (6.3.10) satisfy

$$\|U_1\|^2_{L^2(\mathbb{R}_+)} + |U_1(0)|^2 \le \|F_1\|^2_{L^2(\mathbb{R}_+)}\,.$$

Shrinking ω if necessary, we can assume that the contjugation matrix $\mathcal{W}(z, \zeta)$ is defined for $\zeta \in \omega$. Therefore, the solutions $\widetilde U = \mathcal{W} U_1$ of (6.3.4) satisfy

$$\|\widetilde U\|^2_{L^2(\mathbb{R}_+)} + |\widetilde U(0)|^2 \le \|\widetilde F\|^2_{L^2(\mathbb{R}_+)}\,,$$

which implies (7.1.12) for $\zeta \in \omega$.

b) *Low frequencies.* We first consider the equation (7.5.1). We use the symmetrizers $S_H(\zeta) = \check S_H(\check\zeta, \rho)$ and $S_P(\zeta)$ given in polar coordinates $\zeta = \rho\check\zeta$ by Lemma 7.5.1. For $0 < \zeta \le \rho_0$ and $\gamma \ge 0$, the lemma implies that:

$$\mathrm{Re}\,(S_H H) \gtrsim \rho(\check\gamma + \rho) = c(\gamma + |\zeta|^2) \approx h^2\,, \quad \mathrm{Re}\,(S_P P) \gtrsim \mathrm{Id}\,.$$

Therefore, the components (u_2, v_2) and (f_2, g_2) of U_2 and F_2 respectively, satisfy

$$\begin{aligned} h^2\|u_2\|^2_{L^2(\mathbb{R}_+)} + \big(S_H u_2(0), u_2(0)\big) &\lesssim \frac{1}{h^2}\|f_2\|^2_{L^2(\mathbb{R}_+)}\,, \\ \|v_2\|^2_{L^2(\mathbb{R}_+)} + \big(S_P v_2(0), v_2(0)\big) &\lesssim C\|g_2\|^2_{L^2(\mathbb{R}_+)}\,. \end{aligned}$$

Adding up and using the third part of Lemma 7.5.1, we obtain that

$$h^2\|u_2\|^2_{L^2(\mathbb{R}_+)} + \|v_2\|^2_{L^2(\mathbb{R}_+)} + |U_2(0)|^2 \lesssim \frac{1}{h^2}\|f_2\|^2_{L^2(\mathbb{R}_+)} + \|g_2\|^2_{L^2(\mathbb{R}_+)}\,,$$

thus

$$\begin{aligned} h^2\|u_2\|_{L^2(\mathbb{R}_+)} + h\|v_2\|_{L^2(\mathbb{R}_+)} + h|U_2(0)| &\lesssim \|f_2\|_{L^2(\mathbb{R}_+)} + h\|g_2\| \\ &\lesssim \|F_2\|_{L^2(\mathbb{R}_+)}\,. \end{aligned}$$

Thanks to the special form (6.4.2) of $\mathcal{V}(0)$, $U_1 = \mathcal{V}U_2$ and $F_1 = \mathcal{V}F_2$ satisfy

$$u_1 = O(1)U_2\,, \quad v_1 = O(1)v_2 + O(\zeta)u_2\,, \quad F_2 = O(1)F_1\,.$$

On the neighborhood $|\zeta| \le \rho_0$ of the origin, h is bounded and $|\zeta| \lesssim h$. Hence $|\zeta|h \lesssim h^2$ and the solutions U_1 of (6.3.10) satisfy

$$h^2\|u_1\|_{L^2(\mathbb{R}_+)} + h\|v_1\|_{L^2(\mathbb{R}_+)} + h|U_1(0)| \le C\|F_1\|_{L^2(\mathbb{R}_+)}\,. \tag{7.6.1}$$

Decreasing ρ_0 if necessary, we can assume that the matrix $\mathcal{W}$ is defined for $|\zeta| \le \rho_0$. Since $\mathcal{W} = \mathrm{Id} + O(e^{-\theta z})$, $\widetilde{U} = \mathcal{W}U_1$ satisfies:

$$\tilde{u} = O(1)U_1\,, \quad \tilde{v} = O(1)v_1 + O(e^{-\theta z})u_1\,, \quad F_1 = O(1)F\,.$$

Therefore, the solutions $\widetilde{U}$ of (6.3.4) satisfy

$$\begin{aligned} h^2\|\tilde{u}\|_{L^2(\mathbb{R}_+)} + h\|\tilde{v}\|_{L^2(\mathbb{R}_+)} + h|\widetilde{U}(0)| &\lesssim h^2\|u_1\|_{L^2(\mathbb{R}_+)} \\ &\quad + h\|v_1\|_{L^2(\mathbb{R}_+)} + h\|e^{-\theta z}u_1\|_{L^2(\mathbb{R}_+)} + h|U_1(0)|\,. \end{aligned} \tag{7.6.2}$$

We use here the following inequality:

Lemma 7.6.1. *Given $\theta > 0$, there is a constant C such that all function $u \in H^1(\mathbb{R}_+)$ satisfies the inequality:*

$$\|e^{-\theta z}u\|_{L^2([0,\infty[)} \le C\Big(|u(0)| + \|\partial_z u\|_{L^2([0,\infty[)}\Big). \tag{7.6.3}$$

The proof is left as an exercise. We apply this estimate to u_1, noticing that the equation (6.3.10) implies that $\partial_z u_1 = v_1$. Thus

$$\|e^{-\theta z} u_1\|_{L^2(\mathbb{R}_+)} \leq C\big(|U_1(0)| + \|v_1\|_{L^2(\mathbb{R}_+)}\big) .$$

With (7.6.1) and (7.6.2), this implies that for $0 < |\zeta| \leq \rho_0$ with $\gamma \geq 0$, the solutions of (6.3.4) satisfy

$$h^2 \|\tilde{u}\|_{L^2(\mathbb{R}_+)} + h\|\tilde{v}\|_{L^2(\mathbb{R}_+)} + h|\widetilde{U}(0)| \lesssim \|F_1\|_{L^2(\mathbb{R}_+)} \lesssim \|\widetilde{F}\|_{L^2(\mathbb{R}_+)} .$$

This implies (7.1.12).

c) *High frequencies.* We use notations (7.3.3): $u_1 = \langle\zeta\rangle u$, $v_1 = v$. By Lemmas 7.3.1 and 7.2.1, for ζ large enough, the solution $U_1 = (u_1, v_1)$ of (7.3.4) satisfies

$$\langle\zeta\rangle \|U_1\|^2_{L^2(\mathbb{R}_+)} + |U_1(0)|^2 \lesssim \frac{1}{\langle\zeta\rangle} \|F\|^2_{L^2(\mathbb{R}_+)}$$

Thus, the solution $\widetilde{U}$ of (6.3.4) satisfies

$$\langle\zeta\rangle^2 \|\tilde{u}\|_{L^2(\mathbb{R}_+)} + \langle\zeta\rangle \|\tilde{v}\|_{L^2(\mathbb{R}_+)} + \langle\zeta\rangle |\tilde{v}(0)| \lesssim \|\widetilde{F}\|_{L^2(\mathbb{R}_+)} .$$

This implies (7.1.12).

d) *Endgame.* By steps b) and c), there are $\rho_0 > 0$ and $\lambda_0 > 0$ such that the estimate (7.1.12) is proved for $\zeta \neq 0$ with $\gamma \geq 0$ and either $|\zeta| \leq \rho_0$ or $|\zeta| \geq \lambda_0$. By step a), one can cover the compact set $\{\rho_0 \leq |\zeta| \leq \lambda_0 \ \gamma \geq 0\}$ by a finite number of open sets where the estimate holds, proving that (7.1.12) holds with a uniform constant C, independent of ζ. $\square$

Chapter 8

Kreiss Symmetrizers

This chapter is entirely devoted to the proof of Theorem 7.5.2. For strictly hyperbolic equations the construction of symmetrizers is due to O.Kreiss ([Kre] augmented with J.Ralston's note [Ral], see also [Ch-Pi]). It was then noticed by A.Majda and S.Osher ([Ma-Os], [Maj]) that the strict hyperbolicity can be somewhat relaxed and that the construction extends to systems satisfying *a block structure condition*. Finally, it is proved in [Mé3] that the block structure condition is satisfied for all hyperbolic systems with constant multiplicity. We discuss in this chapter the extension of Kreiss construction to hyperbolic-parabolic systems given in [MZ1].

8.1 Scheme of the construction

8.1.1 Notations

We denote here by

$$A(\eta, \xi) = \sum_{j<d} \eta_j A_j(p) + \xi A_d(p)\,, \quad (\eta, \xi) \in \mathbb{C}^{d-1} \times \mathbb{C}\,,$$

the symbol of the hyperbolic part of the equation evaluated at a given point $p \in \mathcal{U}$ and by

$$B(\eta, \xi) = \sum_{j,k<d} \eta_j \eta_k B_{j,k}(p) + \sum_{j<d} \xi \eta_j \big(B_{j,d}(p) + B_{d,j}(p)\big) + \xi^2 B_{d,d}(p)$$

the symbol of the parabolic part. In accordance with Assumption 5.1.1, we suppose in this chapter:

Assumption 8.1.1. *i) For all $(\eta, \xi) \in \mathbb{R}^d \backslash \{0\}$ the eigenvalues of $A(\eta, \xi)$ are real, semi-simple and have constant multiplicity. Moreover, A_d is non-singular.*

ii) There is $c > 0$ such that for all $(\eta, \xi) \in \mathbb{R}^d$, the eigenvalues of $iA(\eta, \xi) + B(\eta, \xi)$ satisfy $\operatorname{Re} \mu \geq c(|\eta|^2 + \xi^2)$.

Denoting by $\zeta = (\tau, \eta, \gamma) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$ the tangential Fourier-Laplace frequencies, the symbol called $\check{H}(\zeta, 0)$ in (6.4.16) reads

$$\mathcal{H}(\zeta, 0) = -A_d^{-1}\big((i\tau + \gamma)\mathrm{Id} + \sum_{j=1}^{d} i\eta_j A_j\big)\,. \tag{8.1.1}$$

With notations as in Chapter six, the block decomposition (6.4.1) shows that the perturbations $\check{H}(\check{\zeta}, \rho)$ occurring in (6.4.15) satisfy

$$\begin{aligned}\det\big(i\rho\xi\mathrm{Id} - \rho\check{H}(\check{\zeta}, \rho)\big)\,\det\big(i\rho\xi\mathrm{Id} - P(\rho\check{\zeta})\big) &= \det\big(i\rho\xi\mathrm{Id} - G^\infty(\rho\check{\zeta})\big)\\ &= \det(-B_{d,d}^{-1})\,\det\big((i\rho\check{\tau} + \rho\check{\gamma})\mathrm{Id} + i\rho A(\check{\eta}, \xi) + \rho^2 B(\check{\eta}, \xi)\big)\end{aligned}$$

Factoring out ρ^N implies that the perturbations $\check{H}(\check{\zeta}, \rho)$ satisfy the next hypothesis.

Assumption 8.1.2. *i) The $N \times N$ matrices $\mathcal{H}(\zeta, \rho)$ are smooth functions of ζ in the unit sphere S^d and $\rho \in [-\rho_0, \rho_0]$ such that $\mathcal{H}(\zeta, 0)$ is given by (8.1.1).*

ii) There holds

$$\begin{aligned}&\det\big((i\tau + \gamma)\mathrm{Id} + iA(\eta, \xi) + \rho B(\eta, \xi)\big)\\ &\qquad\qquad = e(\zeta, \xi, \rho)\,\det\big(i\xi\mathrm{Id} - \mathcal{H}(\zeta, \rho)\big)\end{aligned} \tag{8.1.2}$$

where e is a polynomial in ξ with smooth coefficients in (ζ, ρ).

Remark 8.1.3. When $\rho = 0$, (8.1.1) implies that

$$e(\zeta, \xi, 0) = \det A_d \neq 0\,. \tag{8.1.3}$$

Remark 8.1.4. The Assumptions 8.1.1 and 8.1.2 imply that for $\gamma \geq 0$, $\rho \geq 0$ and $\gamma + \rho > 0$, $\mathcal{H}(\zeta, \rho)$ has no eigenvalues in the imaginary axis $i\mathbb{R}$. Thus, the number of eigenvalues counted with their multiplicity in $\{\operatorname{Re} \mu > 0\}$ [resp. $\{\operatorname{Re} \mu < 0\}$] is therefore constant for $(\zeta, \rho) \in \overline{S}^d_+ \times [0, \rho_0]$ with $\gamma + \rho > 0$. It is equal to N_- [resp. N_+] the number of negative [resp. positive] eigenvalues of A_d.

Let us rephrase Theorem 7.5.2 in the new setting:

Theorem 8.1.5. *Suppose that Assumptions* 8.1.1 *and* 8.1.2 *are satisfied. For all* $\underline{\zeta} \in \overline{S}^d_+$, *there are spaces* $\underline{\mathbb{F}}_\pm$ *satisfying*

$$\mathbb{C}^N = \underline{\mathbb{F}}_- \oplus \underline{\mathbb{F}}_+ \,, \quad \dim \underline{\mathbb{F}}_\pm = N_\mp \,, \tag{8.1.4}$$

and such that for all $\kappa \geq 1$ *there are a neighborhood* ω *of* $(\underline{\zeta}, 0)$ *in* $S^d \times \mathbb{R}$, *a* C^∞ *mapping* $\mathcal{S}$ *from* ω *to the space of* $N \times N$ *matrices and a constant* $c > 0$ *such that for all* $(\zeta, \rho) \in \omega$,

$$\mathcal{S}(\zeta, \rho) = \mathcal{S}^*(\zeta, \rho) \,, \tag{8.1.5}$$

$$\big(\mathcal{S}(\zeta, \rho) h, h\big) \geq \kappa |\underline{\Pi}_+ h|^2 - |\underline{\Pi}_- h|^2 \,, \tag{8.1.6}$$

and for all $(\zeta, \rho) \in \omega$ *with* $\rho \geq 0$ *and* $\gamma \geq 0$*:*

$$\operatorname{Re} \big(\mathcal{S}(\zeta, \rho) \mathcal{H}(\zeta, \rho) h, h\big) \geq c(\gamma + \rho) |h|^2 \,. \tag{8.1.7}$$

In (8.1.6), $\underline{\Pi}_\pm$ *denote the projectors on* $\underline{\mathbb{F}}_\pm$ *associated to the decomposition* (8.1.4).

Note that, it is sufficient to check (8.1.6) at $(\underline{\zeta}, 0)$, since it will extend by continuity to a neighborhood, changing $\mathcal{S}$ to $\mathcal{S}/2$ and decreasing κ to $\kappa/4$.

8.1.2 Block reduction

Fix $\underline{\zeta} \in \overline{S}^d_+$. We split the eigenvalues of $\underline{\mathcal{H}} := \mathcal{H}(\underline{\zeta}, 0)$ into eigenvalues in $\{\pm \operatorname{Re} \mu > 0\}$ and in $\{\operatorname{Re} \mu = 0\}$. We denote by $\underline{\mu}_k = i \underline{\xi}_k$ the distinct eigenvalues located on the imaginary axis. By standard perturbation arguments, there are $\delta > 0$, a neighborhood ω_0 of $(\underline{\zeta}, 0)$ in $S^d \times \mathbb{R}$ and a smooth matrix $\mathcal{V}(\zeta, \rho)$ on ω such that

$$\mathcal{V}^{-1} \mathcal{H} \mathcal{V} = \begin{bmatrix} \mathcal{H}^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{H}^{\underline{k}} \end{bmatrix} \tag{8.1.8}$$

such that each block $\mathcal{H}_k(\zeta, \rho)$ has its spectrum either in $\{|\operatorname{Re} \mu| \geq 2\delta\}$ or in the ball of radius δ centered at $\underline{\mu}_k$. Moreover, we can assume that the balls of radius 2δ centered at the $\underline{\mu}_k$ do not intersect each other. This block decomposition corresponds to a smooth decomposition

$$\mathbb{C}^N = \mathbb{F}^1(\zeta, \rho) \oplus \cdots \oplus \mathbb{F}^{\underline{k}}(\zeta, \rho) \tag{8.1.9}$$

into invariant subspaces of $\mathcal{H}(\zeta,\rho)$. In particular, when k corresponds to a purely imaginary eigenvalue $\underline{\mu}_k$, $\mathbb{F}^k(\underline{\zeta},0)$ is the space spanned by the generalized eigenvectors of $\mathcal{H}(\underline{\zeta},0)$ associated to this eigenvalue. We denote by $N_k = \dim \mathbb{F}^k$ the dimension of the block $\mathcal{H}^k$. By Assumption 8.1.2, $\mathcal{H}$ and thus the $\mathcal{H}^k$ have no eigenvalues on the imaginary axis when $\rho \geq 0$, $\gamma \geq 0$ and $\rho + \gamma > 0$. Therefore, the number of eigenvalues of $\mathcal{H}^k$ in $\{\pm \mathrm{Re}\, \mu > 0\}$ is constant for $\rho \geq 0$, $\gamma \geq 0$ and $\rho + \gamma > 0$. We denote it by $N_{k,\pm}$. Because the total number of eigenvalues of $\mathcal{H}$ in $\{\pm \mathrm{Re}\, \mu > 0\}$ is $N_{\mp}$, we have

$$\sum N_{k,\pm} = N_{\mp} \,. \tag{8.1.10}$$

It is sufficient to construct symmetrizers $\mathcal{S}_k$ for each block $\mathcal{H}_k$ separately. With

$$\mathcal{S} = (\mathcal{V}^{-1})^* \begin{bmatrix} \mathcal{S}^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{S}^{\underline{k}} \end{bmatrix} \mathcal{V}^{-1}$$

Theorem 8.1.5 follows from the next result:

Proposition 8.1.6. *With notations as above, for all k, there are spaces $\underline{\mathbb{F}}^k_\pm$ satisfying*

$$\mathbb{C}^{N_k} = \underline{\mathbb{F}}^k_+ \oplus \underline{\mathbb{F}}^k_- \,, \quad \dim \underline{\mathbb{F}}^k_- = N_{k,-} \,, \tag{8.1.11}$$

and for all κ large enough, symmetrizers $\mathcal{S}^k$, C^∞ on a neighborhood of $(\underline{\zeta},0)$, such that

$$\mathcal{S}^k = (\mathcal{S}^k)^* \tag{8.1.12}$$

$$\big(\underline{\mathcal{S}}^k U, U\big) \geq \kappa |\underline{\Pi}^k_+ U|^2 - |\underline{\Pi}^k_- U|^2 \,, \tag{8.1.13}$$

$$\mathrm{Re}\, \mathcal{S}^k \mathcal{H}^k \geq c(\gamma+\rho)\mathrm{Id} \,, \quad \text{for } \rho \geq 0 \,, \ \gamma \geq 0 \,. \tag{8.1.14}$$

In (8.1.13)*, $\underline{\mathcal{S}}^k = \mathcal{S}^k(\check{\underline{\zeta}},0)$ and $\underline{\Pi}^k_\pm$ are the projectors associated to the decomposition* (8.1.11)*.*

The construction of the $\mathcal{S}^k$ depends on the nature of the spectrum of $\mathcal{H}^k$. The easy case is when the spectrum is away from the imaginary axis (elliptic modes). Next, we consider the case where the spectrum is purely imaginary when $\gamma = \rho = 0$ and semi-simple (hyperbolic modes). This case is studied in the next section. The most difficult case, considered in sections 3 and 4, occurs when Jordan blocks are present (glancing modes).

8.1.3 Elliptic modes

We now prove Proposition 8.1.6 when the spectrum of $\mathcal{H}^k$ lies in

$$\{\operatorname{Re}\mu > 2\delta\} \qquad [\text{resp. } \{\operatorname{Re}\mu < -2\delta\}\ ]\,. \tag{8.1.15}$$

In this case, there are self adjoint positive definite matrices $S^k(\check\zeta, \rho)$, defined and C^∞ on a neighborhood ω of $(\check{\underline\zeta}, 0)$ and such that

$$\operatorname{Re} S^k\mathcal{H}^k \geq c\mathrm{Id} \qquad [\text{resp. } \operatorname{Re} S^k\mathcal{H}^k \leq -c\mathrm{Id}\ ]\,.$$

with $c > 0$ independent of $(\zeta, \rho) \in \omega$. We set

$$\underline{\mathbb{F}}^k_- = \{0\} \qquad [\text{resp. } \ \underline{\mathbb{F}}^k_- = \mathbb{C}^{N_k}\ ]$$

and

$$\mathcal{S}^k = \kappa S^k \qquad [\text{resp. } \mathcal{S}^k = -S^k\ ]$$

and properties (8.1.5) to (8.1.7) are satisfied.

Remark 8.1.7. When $\underline\zeta$ belongs to the open half sphere S^d_+, that is when $\underline\gamma > 0$, $\mathcal{H}(\underline\zeta, 0)$ has no eigenvalues on the imaginary axis, and all the blocks $\mathcal{H}^k$ satisfy (8.1.15). Thus Proposition 8.1.6 is proved when $\underline\gamma > 0$.

8.2 Hyperbolic modes

8.2.1 Preliminaries

a) Introduce the characteristic polynomial

$$\Delta(\tau, \eta, \xi) := \det\big(\tau\mathrm{Id} + \sum_{j=1}^{d-1} \eta_j A_j + \xi A_d\big)\,. \tag{8.2.1}$$

The Assumption 8.1.1 implies that there are functions $\lambda_j(\eta, \xi)$, smooth and homogenenous of degree one in $\mathbb{R}^d \backslash \{0\}$, and fixed integers α_j such that

$$\lambda_1 < \lambda_2 < \ldots\,, \qquad \text{and} \quad \Delta(\tau, \eta, \xi) = \prod_{j=1}^{j} \big(\tau + \lambda_j(\eta, \xi)\big)^{\alpha_j}\,. \tag{8.2.2}$$

The roots λ_j are real analytic and therefore extend to the complex domain. In particular, there is $\delta > 0$ such that the λ_j are defined for complex ξ such that $|\mathrm{Im}\,\xi| \leq \delta(|\eta| + |\mathrm{Re}\,\xi|)$ and the factorization (8.2.2) extends to such ξ and $\tau \in \mathbb{C}$.

In addition, since the eigenvalues are semi-simple, the eigenprojectors $\Pi_j(\eta, \xi)$ are C^∞ functions, homogeneous of degree zero, of $(\eta, \xi) \in (\mathbb{R}^d \backslash \{0\})$. The dimension of the associated eigenspace is equal to the multiplicity α_j. By analytic continuation, the projectors Π_j extends analytically to ξ in the domain $|\mathrm{Im}\,\xi| \leq \delta(|\eta| + |\mathrm{Re}\,\xi|)$ and $A\Pi_j = \lambda_j \Pi_j$ on this domain.

b) For ρ small enough and (η, ξ) in a compact of $\mathbb{R}^d \backslash \{0\}$, the eigenvalues of $iA(\eta, \xi) + \rho B(\eta, \xi)$ remain close to the $i\lambda_j$ and the $\Pi_j(\eta, \xi)$ extend to smooth spectral projectors $\Pi_j(\eta, \xi, \rho)$ of $iA(\eta, \xi) + \rho B(\eta, \xi)$. Therefore, in a neighborhood of a given point $(\underline{\eta}, \underline{\xi}) \in \mathbb{R}^d \backslash \{0\}$ there is a smooth block decomposition

$$V^{-1}\big(A(\eta, \xi) + \rho B(\eta, \xi)\big)V = \begin{bmatrix} D_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{\underline{j}} \end{bmatrix}, \tag{8.2.3}$$

with D_j of dimension $\alpha_j \times \alpha_j$ of the form

$$D_j(\eta, \xi, \rho) = i\lambda_j(\eta, \xi) + \rho B'_j(\eta, \xi, \rho)\,. \tag{8.2.4}$$

By Assumption 8.1.1, the eigenvalues of D_j satisfy $\mathrm{Re}\,\mu \geq c\rho(|\eta|^2 + \xi^2)$ and thus the spectrum of B'_j is contained in $\{\mathrm{Re}\,\mu > c(|\eta|^2 + \xi^2)\}$.

Introduce the determinant

$$\widetilde{\Delta}(\tau, \eta, \xi, \rho) := \det\big(i\tau \mathrm{Id} + iA(\eta, \xi) + \rho B(\eta, \xi)\big)\,. \tag{8.2.5}$$

Near any point $(\underline{\eta}, \underline{\xi}) \in \mathbb{R}^d \backslash \{0\}$, the block decomposition (8.2.3) implies

$$\widetilde{\Delta}(\tau, \eta, \xi, \rho) = \prod \Delta_j(\tau, \eta, \xi, \rho) \tag{8.2.6}$$

with

$$\Delta_j(\tau, \eta, \xi, \rho) := \det\big(i(\tau + \lambda_j(\eta, \xi))\mathrm{Id} + \rho B'_j(\eta, \xi, \rho)\big). \tag{8.2.7}$$

8.2.2 Symmetrizers for hyperbolic modes

Let $\underline{\zeta} \in \overline{S}^d_+$. In the block decomposition (8.1.8), we consider now the case where the spectrum of $\underline{\mathcal{H}}^k$ is $\{i\xi_k\}$ and the spectrum of $\mathcal{H}^k(\zeta, \rho)$ is contained in the ball of radius δ centered at $\underline{\mu}_k = i\underline{\xi}_k$ when (ζ, ρ) remains in the neighborhood ω of $(\underline{\zeta}, 0)$. Note that necessarily $\underline{\gamma} = 0$.

Since $i\underline{\xi}_k$ is an eigenvalue of $\mathcal{H}(\underline{\zeta}, 0)$, $\underline{\tau}$ is an eigenvalue of $A(\underline{\eta}, \underline{\xi}_k)$. Note that $(\underline{\eta}, \underline{\xi}_k) \neq (0,0)$, since otherwise it would imply that $\underline{\tau} = 0$ which is impossible since $\underline{\zeta} \neq 0$. Therefore the factorization (8.2.2), implies that there is a unique eigenvalue λ_j such that

$$\underline{\tau} + \lambda_j(\underline{\eta}, \underline{\xi}_k) = 0\,. \tag{8.2.8}$$

We assume in this section that $\underline{\xi}_k$ is a simple root of this equation, that is

$$\ell := \partial_\xi \lambda_j(\underline{\eta}, \underline{\xi}_k) \neq 0\,. \tag{8.2.9}$$

Lemma 8.2.1. *Suppose that* (8.2.8) *and* (8.2.9) *hold. Then, the dimension* N_k *of the block* $\mathcal{H}^k$ *is equal to the multiplicity* α_k*. Moreover, there are a smooth scalar function* $q_k(\zeta)$ *and a smooth matrix* $\mathcal{R}^k(\zeta, \rho)$ *on a neighborhood* ω *of* $(\underline{\zeta}, 0)$*, such that*

$$\mathcal{H}^k(\zeta, \rho) = q^k(\zeta)\mathrm{Id} + \rho\mathcal{R}^k(\zeta, \rho)\,. \tag{8.2.10}$$

Furthermore, q^k *is purely imaginary when* $\gamma = 0$*,* $\dot{q}^k := \partial_\gamma \mathrm{Re}\, q^k(\underline{\zeta})$ *does not vanish and the spectrum of* $\dot{q}^k\mathcal{R}^k(\underline{\zeta}, 0)$ *is contained in the half space* $\{\mathrm{Re}\,\mu > 0\}$.

Proof. The eigenvalue λ_j is an analytic function of (η, ξ) and thus extends to complex values of ξ with small imaginary part. By (8.2.9), for $\zeta = (\tau, \eta, \gamma)$ close to $\underline{\zeta}$ the equation

$$\tau - i\gamma + \lambda_j(\eta, \xi) = 0 \tag{8.2.11}$$

has a unique solution $\xi_k(\zeta)$ close to $\underline{\xi}_k$. We define $q^k(\zeta) = i\xi_k(\zeta)$ so that $q^k(\underline{\zeta}) = \underline{\mu}_k = i\underline{\xi}_k$. The implicit function theorem also implies that $\xi_k(\zeta)$ is real when $\gamma = 0$. Moreover, there holds

$$\partial_\xi \lambda_j(\underline{\eta}, \underline{\xi}_k)\partial_\gamma \xi_k(\underline{\zeta}) = i\,.$$

Thus, using (8.2.9) yields

$$\dot{q}^k = \partial_\gamma q^k(\underline{\zeta}) = -1/\ell \neq 0\,. \tag{8.2.12}$$

Using Assumption 8.1.2 and Remark 8.1.4 on one hand and the factorization (8.2.6) on the other hand, we see that for (ζ, ξ, ρ) close to $(\underline{\zeta}, \underline{\xi}_k, 0)$, there holds

$$\begin{aligned} \widetilde{\Delta}(\tau - i\gamma, \eta, \xi, \rho) &= e(\zeta, \xi, \rho) \det\big(i\xi\mathrm{Id} - \mathcal{H}(\zeta, \rho)\big) \\ &= e_1(\zeta, \xi, \rho) \det\big(i\xi\mathrm{Id} - \mathcal{H}^k(\zeta, \rho)\big) \\ &= e_2(\zeta, \xi, \rho)\Delta_j(\tau - i\gamma, \eta, \xi, \rho), \end{aligned} \tag{8.2.13}$$

where e, e_1 and e_2 do not vanish in a neighborhood of $(\underline{\zeta}, \underline{\xi}_k, 0)$.

For $\rho = 0$, $\Delta_j(\tau, \eta, \xi, 0) = (\tau + \lambda_j(\eta, \xi))^{\alpha_j}$. Therefore, by (8.2.9), $\xi_k(\zeta)$ is a root of multiplicity α_j of $\Delta_j(\tau - i\gamma, \eta, \xi)$ and hence of $\det(i\xi \mathrm{Id} - \mathcal{H}(\zeta, 0))$. This shows that $q^k(\zeta)$ is an eigenvalue of algebraic mulitplicity α_j of $\mathcal{H}(\zeta, 0)$. Moreover, the kernel of $\xi_k(\zeta)\mathrm{Id} - \mathcal{H}(\underline{\zeta}, 0)$ is equal to the kernel of $(\tau - i\gamma)\mathrm{Id} + A(\eta, \xi_k(\zeta))$, which is of dimension α_j. This shows that the geometric multiplicity of the eigenvalue $q^k(\zeta)$ is equal to its algebraic multiplicity. Thus, this eigenvalue is semi-simple, showing that the block $\mathcal{H}^k$ has dimension $N_k = \alpha_j$ and that

$$\mathcal{H}^k(\eta, 0) = q^k(\eta)\mathrm{Id}\,. \tag{8.2.14}$$

This implies (8.2.10).

Moreover, by (8.2.9)

$$i(\underline{\tau} + \lambda_j(\underline{\eta}, \underline{\xi}_k + \xi'))\mathrm{Id} + \rho B_j'(\underline{\eta}, \underline{\xi}_k + \xi', \rho)\big) = i\ell\xi'\mathrm{Id} + \rho\underline{B}_j' + O\big(\xi'^2 + \rho^2\big)\,.$$

with $\underline{B}_j' = B_j'(\underline{\eta}, \underline{\xi}_k)$. Thus,

$$\Delta_j(\underline{\tau}, \underline{\eta}, \underline{\xi}_k + \xi', \rho) = \det\big(i\ell\xi'\mathrm{Id} + \rho\underline{B}_j'\big) + O\big((|\xi'| + |\rho|)^{\alpha_j + 1}\big)$$

Similarly, with (8.2.10)

$$i(\underline{\xi}_k + \xi')\mathrm{Id} - \mathcal{H}^k(\underline{\zeta}, \rho) = i\xi'\mathrm{Id} - \rho\underline{\mathcal{R}}^k + O(\rho^2)$$

with $\underline{\mathcal{R}}^k = \mathcal{R}^k(\underline{\zeta}, 0)$, hence

$$\det\big(i(\underline{\xi}_k + \xi')\mathrm{Id} - \mathcal{H}^k(\underline{\zeta}, \rho)\big) = \det\big(i\xi'\mathrm{Id} - \rho\underline{\mathcal{R}}^k\big) + O\big((|\xi'| + |\rho|)^{\alpha_j + 1}\big).$$

With (8.2.13), comparing the Taylor expansions implies

$$\det\big(i\ell\xi'\mathrm{Id} + \rho\underline{B}_j'\big) = \det\big(i\xi'\mathrm{Id} - \rho\underline{\mathcal{R}}^k\big). \tag{8.2.15}$$

Thus, using (8.2.12), the spectrum of $\dot{q}^k\underline{\mathcal{R}}^k = -\ell^{-1}\underline{\mathcal{R}}^k$ is equal to the spectrum of $\ell^{-2}\underline{B}_j'$ which is located in $\{\mathrm{Re}\,\mu > 0\}$. The proof of the lemma is now complete. □

Proof of Proposition 8.1.6 *for hyperbolic modes.*

When $\dot{q}_k > 0$ [resp. $\dot{q}_k < 0$], there is $c > 0$ such that the eigenvalue $q_k(\zeta)$ satisfies for ζ close to $\underline{\zeta}$ and $\gamma > 0$ small:

$$\mathrm{Re}\, q_k \geq c\gamma\,, \qquad [\text{resp. } \mathrm{Re}\, q_k \leq -c\gamma] \tag{8.2.16}$$

Thus, for $\gamma > 0$ small, (8.2.15) implies that the number of eigenvalues in $\{\operatorname{Re}\mu < 0\}$ of $\mathcal{H}^k(\zeta, 0)$ is 0 [resp. N_k]. The spectrum of $\mathcal{R}^k(\underline{\zeta}, 0) = \underline{\mathcal{R}}^k$ is contained in $\{\operatorname{Re}\mu > 0\}$ [resp. $\{\operatorname{Re}\mu < 0\}$] and this property extends to a neighborhood of $(\underline{\zeta}, 0)$. Therefore, there are self-adjoint positive definite matrices $S^k(\zeta, \rho)$ which depend smoothly on (ζ, ρ) in a neighborhood of $(\underline{\zeta}, 0)$, satisfying

$$\operatorname{Re} S^k \mathcal{R}^k \geq c\mathrm{Id}\,, \qquad [\text{resp. } -\operatorname{Re} S^k \mathcal{R}^k \geq c\mathrm{Id}]\,. \tag{8.2.17}$$

We set

$$\underline{\mathbb{F}}_-^k = \{0\}\,, \qquad [\text{resp. } \underline{\mathbb{F}}_-^k = \mathbb{C}^{N_k}]\,.$$

Next we choose

$$\mathcal{S}^k = \kappa S^k\,,\ \kappa \geq 1\,, \qquad [\text{resp. } \mathcal{S}^k = -S^k]$$

With this choice, properties (8.1.12) (8.1.13) are immediate. Moreover, (8.2.16) (8.2.17) imply that there is $c > 0$ such that for (ζ, ρ) in a neighborhood of $(\underline{\zeta}, 0)$ with $\gamma \geq 0$ and $\rho \geq 0$, there holds in both cases

$$\operatorname{Re} \mathcal{S}^k \mathcal{H}^k \geq c(\gamma + \rho)\mathrm{Id}\,.$$

This means that (8.1.14) is satisfied. □

8.3 The block structure property

Consider again $\underline{\zeta} \in \overline{S}^d_+$ with $\underline{\gamma} = 0$ and a purely imaginary eigenvalue $\underline{\mu}_k = i\underline{\xi}_k$ of $\mathcal{H}(\underline{\zeta}, 0)$. In the previous section we have shown that $(\underline{\eta}, \underline{\xi}_k) \neq 0$ and that there is a unique eigenvalue λ_j of $A(\eta, \xi)$ such that

$$\underline{\tau} + \lambda_j(\underline{\eta}, \underline{\xi}_k) = 0\,. \tag{8.3.1}$$

Since λ_j is real analytic in ξ, there is an integer $\nu \geq 1$ such that

$$\partial_\xi \lambda_j = \cdots = \partial_\xi^{\nu-1} \lambda_j = 0\,, \quad \partial_\xi^\nu \lambda_j = \nu! \ell \neq 0 \quad \text{at } (\underline{\eta}, \underline{\xi}_k). \tag{8.3.2}$$

In section 2, we investigated the case $\nu = 1$, see (8.2.9). From now, on we study the case $\nu > 1$, which corresponds to the so-called *glancing modes*. Note that ℓ is real. We denote by α_j the multiplicity of the eigenvalue λ_j such that (8.3.1) holds.

In this section, we prove that, locally, the block $\mathcal{H}^k(\zeta, \rho)$ can be put in a special form, first when $\rho = 0$ and next for $\rho \neq 0$. We use this special form in the next section to construct symmetrizers, finishing the proof of Proposition 8.1.6.

8.3.1 The hyperbolic case

The next result relies on the constant multiplicity hypothesis in Assumption 8.1.1.

Theorem 8.3.1. *With assumptions as above, there is a neighborhood ω of $\underline{\zeta}$ in $\mathbb{R}^{d+1}$ and there are matrices $\mathcal{T}(\zeta)$, C^∞ on ω such that*

$$\mathcal{Q}(\zeta) := \mathcal{T}(\zeta)^{-1}\mathcal{H}^k(\zeta,0)\mathcal{T}(\zeta) = \begin{bmatrix} Q & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & Q \end{bmatrix} \tag{8.3.3}$$

with α_j diagonal blocks all equal to the same matrix $Q(\zeta)$ of size $\nu \times \nu$. Moreover, at the base point

$$Q(\underline{\zeta}) = \underline{Q} := i(\underline{\xi}_k Id + J)\,, \quad J = \begin{bmatrix} 0 & 1 & 0 & \\ 0 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & \cdots & 0 \end{bmatrix} \tag{8.3.4}$$

where J is the Jordan's matrix of size ν.

In addition, Q has the form

$$Q(\zeta) = \underline{Q} + \begin{bmatrix} q_1(\zeta) & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ q_\nu(\zeta) & 0 & \cdots & 0 \end{bmatrix} \tag{8.3.5}$$

and the coefficients of Q are purely imaginary when $\gamma = 0$. Moreover,

$$\dot{q} := \partial_\gamma q_\nu(\underline{\zeta}) = \frac{-1}{\ell} \tag{8.3.6}$$

In the strictly hyperbolic case, $\alpha_j = 1$ and there is one block Q. In the general case, the main difficulty is to show that there exists a smooth block diagonal decomposition (8.3.3). Note that the theorem implies that the dimension N_k of the block $\mathcal{H}^k$ is necessarily

$$N_k = \nu\alpha_j\,. \tag{8.3.7}$$

First we study the structure of the characteristic polynomial of $\mathcal{H}^k(\zeta,0)$.

Lemma 8.3.2. *There is a neighborhood ω of $\underline{\zeta}$ and there is a monic polynomial in ξ, $D(z,\xi)$, of degree ν and with C^∞ coefficients in $\zeta \in \omega$ such that for all $\zeta \in \omega$,*

$$\mathcal{D}_k(\zeta,\xi) := \det\big(\xi Id + i\mathcal{H}^k(\zeta)\big) = \big(D(\zeta,\xi)\big)^{\alpha_j}, \tag{8.3.8}$$

The coefficients of D are real when $\gamma = 0$ and

$$\frac{\partial D}{\partial \gamma}(\underline{\zeta}, \underline{\xi}_k) = \frac{-i}{\ell}. \tag{8.3.9}$$

Furthermore, the set ω^* of points $\zeta \in \omega$ such that $D(\zeta,\cdot)$ has only simple roots is dense in ω.

Proof. The eigenvalue λ_j is real analytic in ξ and have an holomorphic extension to a complex neighborhood of $\underline{\xi}_k$ and the factorization (8.2.2) extends to the complex domain. In particular, there are neigborhood ω of $\underline{\zeta}$ and $O \subset \mathbb{C}$ of $\underline{\xi}_k$ such that for $\zeta \in \omega$ and $\xi \in O$, one has

$$\Delta(\tau - i\gamma, \eta, \xi) = \big(\tau - i\gamma + \lambda_j(\eta,\xi)\big)^{\alpha_j} E_1(\zeta,\xi),$$

where the function E_1 does not vanish on $\omega \times O$. By (8.1.1) (8.1.8), there holds

$$\Delta(\tau - i\gamma, \eta, \xi) = \det A_d \prod_m \det\big(\xi \mathrm{Id} + i\mathcal{H}^m(\zeta, 0)\big).$$

By definition of the $\mathcal{H}^m$, the roots of the $\mathcal{D}_m$, with $m \neq k$, are away from $\underline{\xi}_k$. Thus, shrinking the neighborhoods if necessary, one obtains that for $(\zeta,\xi) \in \omega \times O$

$$\mathcal{D}_k(\zeta,\xi) = \big(\tau - i\gamma + \lambda_j(\eta,\xi)\big)^{\alpha_j} E(\zeta,\xi), \tag{8.3.10}$$

where E is smooth in ζ, holomorphic in ξ and does not vanish on $\omega \times O$.

By (8.3.2) we are in position to apply the Weirstrass preparation theorem to the function $\tau + \lambda_j$: there is a monic polynomial of degree ν in ξ, $D(\zeta,\xi)$, with C^∞ coefficients in ζ, and there is a nonvanishing function $E_2(\zeta,\xi)$ defined for (ζ,ξ) in a neighborhood of $(\underline{\zeta}, \underline{\xi}_k)$, holomorphic in ξ, C^∞ in ζ and such that

$$\tau - i\gamma + \lambda_j(\eta,\xi) = E_2(\zeta,\xi)\, D(\zeta,\xi), \quad D(\underline{\zeta},\xi) = (\xi - \underline{\xi})^\nu. \tag{8.3.11}$$

For the convenience of the reader we include a short proof, which allows the introduction of parameters and non analytic regularity in ζ.

Suppose that $f(a,\xi)$ is holomorphic in ξ, C^∞ in the parameters a and satisfies the analogue of (8.3.2) at a given root $(\underline{a},\underline{\xi})$. Then, there is $r>0$ and there is a neighborhood ω of $\underline{a}$ such that for all $z\in\omega$, $f(a,\cdot)$ has ν roots in the disc $\{|\xi-\underline{\xi}|<r/2\}$. Thus, there is a unique decomposition

$$f(a,\xi)=p(a,\xi)\,e^{h(a,\xi)}$$

with p a polynomial of degree ν, with ν roots in the disc $\{|\xi-\underline{\xi}|<r/2\}$ and h holomorphic in $\{|\xi-\underline{\xi}|<2r\}$ such that $h(a,\underline{\xi})=0$. Thus necessarily, for $|\xi-\underline{\xi}|<r$, one has

$$\partial_\xi h(a,\xi)\ =\ \frac{1}{2i\pi}\int_{|w-\underline{\xi}|=r}\frac{\partial_\xi f(a,w)}{f(a,w)}\,\frac{dw}{w-\xi}\,.$$

This shows that $\partial_\xi h$ and therefore h and $p=fe^{-h}$ are C^∞ in a. Factoring out the coefficient of ξ^ν implies (8.3.11).

The Schwarz reflection principle implies that $\lambda_j(\eta,\overline{\xi})=\overline{\lambda_j(\eta,\xi)}$. Thus the explicit formula above shows that when $\gamma=0$, D has real coefficients.

Combining (8.3.10) and (8.3.11) implies that $\mathcal{D}_k=D^{\alpha_j}E_3$ on a neighborbood $\omega\times O$ of $(\underline{\zeta},\underline{\xi}_k)$ where E_3 does not vanish. Moreover, shrinking O if necessary, all the roots of the polynomials $\mathcal{D}_k(z,\cdot)$ and $D(z,\cdot)$ are in O. This implies that $E_3(z,\cdot)$ is a constant in ξ and, since both polynomials are monic (8.3.8) follows.

In (8.3.11), D and E_2 are real when $\gamma=0$. Moreover, together with (8.3.2), it implies that $E_2(\underline{\zeta},\underline{\xi}_k)=\ell$. Differentiating (8.3.11) in γ yields

$$-i=\ell\partial_\gamma D(\underline{\zeta},\underline{\xi}_k)$$

and the property (8.3.9) follows.

Shrinking the neighborhoods, one can assume that $\partial_\gamma D$ does not vanish on $\omega\times O$. Suppose that $\zeta_1\in\omega$ and $D(\zeta_1,\cdot)$ and has a multiple root at ξ_1 of multiplicity ν_1. Since $\partial_\gamma D$ does not vanish, for $\zeta=(\tau_1,\eta_1,\gamma_1+s)\in\omega$, one has $D=c_1(\xi-\xi_1)^{\nu_1}+O(\xi-\xi_1)^{\nu_1+1}+c_2s+O(s^2)$ with $c_2\neq 0$. Thus, for s small enough, the multiple root splits into simple roots. This proves that ω^* is dense in ω. $\square$

Next, we study the structure of $\mathcal{H}^k(\underline{\zeta},0)$. Let $\Pi(\eta,\xi)$ denote the eigenprojector associated to $\lambda_j(\eta,\xi)$. It has constant rank α_j and is a C^∞ function of (η,ξ) for $(\eta,\xi)\neq 0$. Denote by $\underline{\mathcal{H}}=\mathcal{H}(\underline{\zeta},0)$.

Lemma 8.3.3. *The operators* $\underline{P}_l = (\partial_\xi^l \Pi)(\underline{\eta}, \underline{\xi})$ *satisfy*

$$(\underline{\mathcal{H}} - \underline{\mu}_k)\underline{P}_0 = 0\,, \tag{8.3.12}$$

$$(\underline{\mathcal{H}} - \underline{\mu}_k)\underline{P}_l = il\underline{P}_{l-1} \quad for \quad l = 1, \dots \nu - 1\,. \tag{8.3.13}$$

Moreover, the generalized eigenspace of $\underline{\mathcal{H}}$ *associated to* $\underline{\mu}_k$ *is the direct sum*

$$\underline{\mathbb{K}} = \bigoplus_{l=0}^{\nu-1} \underline{P}_l \underline{\mathbb{K}}_0\,, \quad \underline{\mathbb{K}}_0 := \underline{P}_0 \mathbb{C}^N\,.$$

Proof. Freezing the coefficients at $\underline{\eta}$, (8.3.2) implies

$$\lambda(\underline{\eta}, \xi) + \underline{\tau} = O(\xi - \underline{\xi}_k)^\nu.$$

Moreover,

$$\begin{aligned}
0 &= \Big(\xi Id + \underline{A}_d^{-1}\big(\sum \underline{\eta}_j A_j - \lambda(\underline{\eta}, \xi) Id\big)\Big)\Pi(\underline{\eta}, \xi) \\
&= (i\underline{\mathcal{H}} + \xi Id)\Pi(\underline{\eta}, \xi) + O((\xi - \underline{\xi}_k)^\nu)\,.
\end{aligned}$$

Evaluating at $\xi = \underline{\xi}_k$ yields (8.3.12), since $\underline{\xi}_k = -i\underline{\mu}_k$. Taking the Taylor expansion at order $\nu - 1$ implies (8.3.13).

Introduce $\underline{\mathbb{K}}_0 := \underline{P}_0 \mathbb{C}^N$ and for $l = 1, \dots, \nu - 1$, $\underline{\mathbb{K}}_l := \underline{P}_l \underline{\mathbb{K}}_0$. Then (8.3.13) implies that

$$(\underline{\mathcal{H}} - \underline{\mu}_k)\underline{\mathbb{K}}_0 = 0\,, \quad (\underline{\mathcal{H}} - \underline{\mu}_k)\underline{\mathbb{K}}_l = \underline{\mathbb{K}}_{l-1}\,. \tag{8.3.14}$$

Note that $\mathbb{K}_0$ is the eigenspace of $A(\underline{\eta}, \underline{\xi}_k)$ associated to the eigenvalue $\lambda_j(\underline{\eta}, \underline{\xi}_k)$. Thus, $\dim \underline{\mathbb{K}}_0 = \operatorname{rank}\underline{\Pi}_0 = \alpha_j$ and $\dim \underline{\mathbb{K}}_l \le \operatorname{rank}\partial_\xi^l \Pi(\underline{\xi}) \le \alpha_j$. On the other hand, (8.3.14) implies that for $l \ge 1$, $\dim \underline{\mathbb{K}}_l \ge \dim \underline{\mathbb{K}}_{l-1}$. Therefore

$$\dim \underline{\mathbb{K}}_l = \alpha_j\,, \quad l \in \{0, \dots, \nu - 1\}\,. \tag{8.3.15}$$

Suppose that for $l \in \{0, \dots, \nu - 1\}$ there is $u_l \in \mathbb{K}_0$, that is such that $u_l = \underline{P}_0 u_l$ and assume that $\sum \underline{P}_l u_l = 0$. Applying $(\underline{\mathcal{H}} - \underline{\mu}_k)^{\nu-1}$ to this equation, implies that $0 = (\underline{\mathcal{H}} - \underline{\mu}_k)^{\nu-1}\underline{P}_{\nu-1}u_{\nu-1} = \underline{P}_0 u_{\nu-1} = u_{\nu-1}$. Inductively, one shows that all the u_l vanish. This proves that the sum $\underline{\mathbb{K}} := \underline{\mathbb{K}}_0 \oplus \dots \oplus \mathbb{K}_{\nu-1}$ is direct. In particular, $\dim \underline{\mathbb{K}} = \nu\alpha_j$.

By (8.3.15) $(\underline{\mathcal{H}} - \underline{\mu}_k)^\nu \underline{\mathbb{K}} = 0$, thus $\underline{\mathbb{K}}$ is contained in the generalized eigenspace $\mathbb{F}^k(\underline{\zeta}, 0)$ of $\underline{\mathcal{H}}$. By (8.3.8), the dimension N_k of this space, which is the degree of $\mathcal{D}_k$, is equal to $\nu\alpha_j$. Therefore $\underline{\mathbb{K}}$ is equal to $\mathbb{F}^k(\underline{\zeta}, 0)$ and the lemma is proved. $\square$

Let $(\underline{e}_{0,1}, \ldots, \underline{e}_{0,\alpha_j})$ be a basis of $\underline{\mathbb{K}}_0$. For $l = 1, \ldots, \nu - 1$, and $p = 1, \ldots, \alpha_j$, introduce

$$\underline{e}_{l,p} = \frac{1}{l!}\, \underline{P}_l \underline{e}_{0,p}\,. \tag{8.3.16}$$

Lemma 8.3.3 implies that

$$(\underline{\mathcal{H}} - \underline{\mu}_k) e_{0,p} = 0 \quad \text{and} \quad (\underline{\mathcal{H}} - \underline{\mu}_k) e_{l,p} = i e_{l-1,p} \quad \text{for } l \geq 1\,. \tag{8.3.17}$$

The $\{\underline{e}_{l,p}\}$ form a basis of $\underline{\mathbb{K}}$. We denote by $\underline{\mathbb{H}}_p$ the space generated by $(\underline{e}_{0,p}, \ldots, \underline{e}_{\nu-1,p})$. Thus

$$\underline{\mathbb{K}} = \underline{\mathbb{H}}_1 \oplus \ldots \oplus \underline{\mathbb{H}}_{\alpha_j}\,. \tag{8.3.18}$$

Taking this basis to write the conjugation (8.1.8), the relations (8.3.17) imply that the matrix of $\mathcal{H}^k(\underline{\zeta}, 0)$ has the following diagonal block structure

$$\mathcal{H}^k(\underline{\zeta}, 0) = \begin{bmatrix} \underline{Q} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \underline{Q} \end{bmatrix}$$

with $\underline{Q} = \mu_k Id + iJ = i(\underline{\xi}_k + J)$ where J is the $\nu \times \nu$ nilpotent matrix introduced in (8.3.4).

Our goal is to extend the splitting (8.3.14) to a neighborhood of $\underline{\zeta}$ with spaces $\mathbb{H}_p(\zeta)$ invariant by $\mathcal{H}(z)$ and smooth in ζ.

The eigenprojector $\Pi(\eta, \xi)$ extends analytically to a neighborhood $\omega_1 \times O$ of $(\underline{\eta}, \underline{\xi}_k)$; say that O contains the disc of radius $2r$ centered at $\underline{\xi}_k$. Since $D(\underline{\zeta}, \xi)$ vanishes only at $\xi = \underline{\xi}_k$, shrinking the neighborhood ω of $\underline{\zeta}$ if necessary, one can assume that for all $\zeta \in \omega$, the ν complex roots of $D(\zeta, \xi) = 0$ satisfy $|\xi_p - \underline{\xi}| \leq r/2$. Therefore, we can define for $\zeta \in \omega$ and $l \in \{0, \ldots, \nu - 1\}$,

$$P_l(\zeta) := \frac{l!(\nu - l - 1)!}{2i\pi\nu!} \int_{|\xi - \underline{\xi}| = r} \Pi(\eta, \xi)\, \frac{\partial_\xi^{l+1} D(\zeta, \xi)}{D(\zeta, \xi)}\, d\xi\,. \tag{8.3.19}$$

They are C^∞ functions on ω. Since $D(\underline{\zeta}, \xi) = (\xi - \underline{\xi}_k)^\nu$, Cauchy's formula implies that

$$P_l(\underline{\zeta}) = (\partial_\xi^l \Pi)(\underline{\eta}, \underline{\xi}) = \underline{P}_l\,. \tag{8.3.20}$$

Moreover, when $\zeta \in \omega^*$, the roots $(\xi_1, \dots, \xi_\nu)$ of $D(\zeta, \cdot)$ are simple and

$$P_l(\zeta) = \sum_{m=1}^{\nu} c_l(\zeta, \xi_m)\, \Pi(\eta, \xi_m) \tag{8.3.21}$$

where

$$c_l(\zeta, \xi_m) = \frac{l!(\nu - l - 1)!}{\nu!} \frac{\partial_\xi^{l+1} D(\zeta, \xi_m)}{\partial_\xi D(\zeta, \xi_m)} \in \mathbb{C}\,.$$

Recall that $\{\underline{e}_{0,p}\}_{1 \le p \le \alpha_j}$ is a basis of $\underline{\mathbb{K}}_0$. For $l \in \{0, \dots, \nu - 1\}$ and $p \in \{1, \dots, \alpha_j\}$, consider

$$e_{l,p}(\zeta) := \frac{1}{l!} P_l(\zeta) \underline{e}_{0,p} \tag{8.3.22}$$

and $\mathbb{H}_p(\zeta)$ the linear space spanned by $(e_{0,p}(\zeta), \dots, e_{\nu-1,p}(\zeta))$.

Lemma 8.3.4. *Shrinking the neighborhood ω if necessary, for $\zeta \in \omega$, the vectors $\{e_{l,p}(z)\}$ are linearly independent. Their span $\mathbb{K}(\zeta) = \mathbb{H}_1(z) \oplus \dots \oplus \mathbb{H}_{\alpha_j}(\zeta)$ is equal to the invariant space $\mathbb{F}^k(\zeta, 0)$ of $\mathcal{H}(\zeta, 0)$ associated to the eigenvalues close to $\underline{\mu}_k$.*

Moreover, for all p, $\mathbb{H}_p(\zeta)$ is invariant by $\mathcal{H}(\zeta, 0)$ and the matrix of $\mathcal{H}(\zeta, 0)_{|\mathbb{H}_p(\zeta)}$ in the basis $\{e_{l,p}(\zeta)\}_{0 \le l \le \nu - 1}$ is independent of p.

Proof. By (8.3.20), the definition (8.3.22) implies that $e_{l,p}(\underline{\zeta}) = \underline{e}_{l,p}$. Therefore, for ζ close to $\underline{\zeta}$, the vectors $e_{l,p}(\zeta)$ are linearly independent.

Suppose that $\zeta \in \omega^*$. In this case, $\mathcal{H}^k(\zeta, 0)$ has ν pairwise different eigenvalues in the disc $|\xi - \underline{\xi}_k| \le r/2$, $(i\xi_1, \dots, i\xi_\nu)$. They satisfy $D(\zeta, \xi_1) = \dots = D(\zeta, \xi_m) = 0$ and $\tau - i\gamma + \lambda_j(\eta, \xi_m) = 0$. Therefore the kernel $\mathbb{L}_m(\zeta)$ of $i\mathcal{H}(\zeta) + \xi_m \mathrm{Id}$ is the range of $\Pi(\eta, \xi_m)$ and

$$\mathcal{H}(\zeta, 0)\Pi(\eta, \xi_m) = i\xi_m \Pi(\eta, \xi_m)\,. \tag{8.3.23}$$

In particular, the dimension of $\mathbb{L}_m(\zeta)$ is α_j. Since the ξ_m are pairwise distinct the spaces $\mathbb{L}_m(\zeta)$ are in direct sum. Because, $N_k = \nu\alpha_j$, this implies that the eigenvalues $i\xi_m$ of $\mathcal{H}(\zeta, 0)$ are semi-simple and that

$$\mathbb{L}_1(\zeta) \oplus \dots \oplus \mathbb{L}_\nu(\zeta) = \mathbb{F}^k(\zeta, 0)\,.$$

For (ζ, ξ) close to $(\underline{\zeta}, \underline{\xi}_k)$, $\Pi(\eta, \xi)$ is close to $\Pi(\underline{\eta}, \underline{\xi}_k)$ and

$$\widetilde{e}_p(\eta, \xi) := \frac{1}{l!} \Pi(\eta, \xi) \underline{e}_{0,p}\,, \quad 1 \le p \le \alpha_j$$

form a basis of $\Pi(\eta,\xi)\mathbb{C}^N$. In particular, $\{\widetilde{e}_p(\eta,\xi_m\}_{1\le p\le\alpha_j}$ is a basis of $\mathbb{L}_m(\zeta)$. Since the $\mathbb{L}_m$ are in direct sum the $\{\widetilde{e}_p(\eta,\xi_m)\}_{p,m}$ are linearly independent and form a basis of $\mathbb{F}^k(\zeta)$. The identity (8.3.21) implies that for all l and p,

$$e_{l,p}(\zeta) \;=\; \sum_{m=1}^{\nu} c_l(\zeta,\xi_m)\,\widetilde{e}_p(h,\xi_m)\,. \tag{8.3.24}$$

Let $\widetilde{\mathbb{H}}_p(\zeta)$ denote the space spanned by $\{\widetilde{e}_p(\zeta,\xi_m)\}_{1\le m\le\nu}$. Then (8.3.24) implies that $\mathbb{H}_p(\zeta)\subset\widetilde{\mathbb{H}}_p(\zeta)$. Since they have the same dimension, they are equal and

$$\begin{aligned}\mathbb{H}_1(\zeta)\oplus\ldots\oplus\mathbb{H}_{\alpha_j}(\zeta) &= \widetilde{\mathbb{H}}_1(\zeta)\oplus\ldots\oplus\widetilde{\mathbb{H}}_{\alpha_j}(\zeta)\\ &= \mathbb{L}_1(\zeta)\oplus\ldots\oplus\mathbb{L}_\nu(\zeta) = \mathbb{F}^k(\zeta,0)\,.\end{aligned}$$

In addition, (8.3.23) implies that $\widetilde{\mathbb{H}}_p(\zeta)=\mathbb{H}_p(\zeta)$ is invariant by $\mathcal{H}(\zeta,0)$. The matrix of $\mathcal{H}(\zeta,0)_{|\mathbb{H}_p(\zeta)}$ in the basis $\{\widetilde{e}_p(\zeta,\xi_m)\}_{1\le m\le\nu}$ is diagonal with entries $\{i\xi_m\}$. It is independent of p. Since the coefficients $c_l(\zeta,\xi_m)$ in (8.3.24) are also independent of p, it follows that the matrix of $\mathcal{H}(\zeta,0)_{|\mathbb{H}_p(\zeta)}$ in the basis $\{e_{l,p}(\zeta)\}_{0\le l\le\nu-1}$ is independent of p.

The $e_{l,p}(\zeta)$ are smooth functions of ζ and ω^* is dense in ω. Therefore, it remains true for all $\zeta\in\omega$ that $\mathbb{H}_p(\zeta)$ is invariant by $\mathcal{H}(\zeta,0)$, that $\mathbb{H}_1(\zeta)\oplus\ldots\oplus\mathbb{H}_{\alpha_j}(\zeta)=\mathbb{F}^k(\zeta)$ and the matrix of $\mathcal{H}(\zeta,0)_{|\mathbb{H}_p(\zeta)}$ in the basis $\{e_{l,p}(\zeta)\}_{0\le l\le\nu-1}$ is independent of p. $\square$

Using the bases $\{e_{l,p}(\zeta)\}$, in the block decomposition

$$\mathbb{F}^k(\zeta)=\mathbb{H}_1(\zeta)\oplus\ldots\oplus\mathbb{H}_{\alpha_j}(\zeta),$$

the lemma implies that the matrix of $\mathcal{H}^k(\zeta,0)$ has the following diagonal block structure

$$\mathcal{H}^k(\zeta,0)=\begin{bmatrix} Q(\zeta) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & Q(\zeta)\end{bmatrix} \tag{8.3.25}$$

with $Q(\underline{\zeta})=\underline{Q}$ as in (8.3.4).

Lemma 8.3.5. *Shrinking the neighborhood ω if necessary, there are bases in the spaces $\mathbb{H}_p(\zeta)$ which are C^∞ in $\zeta\in\omega$ and such that*

i) (8.3.25) *holds and $Q(\underline{\zeta})=\underline{Q}$,*

ii) the matrix $Q(\zeta)$ has the special form (8.3.5) *and its coefficients are purely imaginary when $\gamma=0$,*

iii) the lower hand corner entry of $\partial_\gamma Q(\underline{\zeta})$ is equal to $-1/\ell$.

Proof. Consider the canonical basis $(e_1, \ldots, e_\nu)$ of $\mathbb{C}^\nu$. Using the notation $Q'(\zeta) = Q(\zeta) - \underline{Q}$, define the matrix T by $T(\zeta)e_\nu = e_\nu$ and inductively $T(\zeta)e_l = (J - i\underline{Q}')T(\zeta)e_{l+1}$ for $l < \nu$. Because $Q'(\underline{\zeta}) = 0$, there holds $T(\underline{\zeta})e_l = e_l$ for all l. Thus $T(\underline{\zeta}) = \mathrm{Id}$ and $T(\zeta)$ is invertible for ζ in a neighborhood of $\underline{\zeta}$. By contruction, $(J - iQ'(\zeta))Te_l = Te_{l-1} = TJe_l$ for $l \in \{2, \ldots, \nu\}$. Therefore, the matrix $\widetilde{Q}'(\zeta) = T^{-1}QT - \underline{Q}$ satisfies $\widetilde{Q}'(\zeta)e_l = 0$ for $l \geq 2$. This shows that

$$\widetilde{Q}(z) := T(z)^{-1}Q(z)T(z) \;=\; \underline{Q} \;+\; \begin{bmatrix} q_1(z) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_\nu(z) & 0 & \ldots & 0 \end{bmatrix}.$$

Lemma 8.3.2 and (8.3.25) imply that

$$\big(D(\zeta,\xi)\big)^{\alpha_j} = \det\big(i\mathcal{H}^k(\zeta) + \xi Id\big) = \Big(\det\big(iQ(\zeta) + \xi Id\big)\Big)^{\alpha_j}$$

and therefore the monic polynomials of degree ν $\det\big(iQ(\zeta) + \xi Id\big)$ and $D(\zeta,\xi)$ are equal. Thus,

$$\begin{aligned} D(\zeta,\xi) &= \det\big(iQ(\zeta) + \xi Id\big) = \det\big(i\widetilde{Q}(\zeta) + \xi Id\big) \\ &= \det\big((\xi - \underline{\xi}_k)Id - J + i\widetilde{Q}'(\zeta)\big) \\ &= (\xi - \underline{\xi}_k)^\nu + \sum_{l=1}^{\nu} iq_l(\zeta)\,(\xi - \underline{\xi})^{\nu-l}. \end{aligned} \tag{8.3.26}$$

Since D has real coefficients when $\gamma = 0$, this implies that the $iq_l(\zeta)$ are real and therefore that $\widetilde{Q}(\zeta)$ is purely imaginary when $\gamma = 0$.

In addition, (8.3.26) and (8.3.9) imply that

$$\frac{\partial D}{\partial \gamma}(\underline{\zeta}, \underline{\xi}_k) \;=\; i\frac{\partial q_\nu}{\partial \gamma}(\underline{\zeta}) = \frac{-i}{\ell}.$$

This implies (8.3.6) and the proof of Theorem 8.3.1 is complete. □

8.3.2 The generalized block structure condition

We now look for normal forms for $\mathcal{H}^k(\zeta,\rho)$ for $\rho \neq 0$.

Theorem 8.3.6. *There is a neighborhood ω of $(\underline{\zeta}, 0)$ in $\mathbb{R}^{d+1} \times \mathbb{R}$ and there are invertible matrices $\mathcal{V}(\zeta,\rho)$ C^∞ on ω such that*

$$\mathcal{V}^{-1}(\zeta,\rho)\mathcal{H}^k(\zeta,\rho)\mathcal{V}(\zeta,\rho) = \mathcal{Q}(\zeta) \;+\; \rho\mathcal{R}(\zeta,\rho). \tag{8.3.27}$$

where $\mathcal{Q}$ is given by Theorem 8.3.1 *and*

$$\mathcal{R} = \begin{bmatrix} R_{1,1} & \cdots & R_{1,\alpha_j} \\ \vdots & \ddots & \vdots \\ R_{\alpha_j,1} & \cdots & R_{\alpha_j,\alpha_j} \end{bmatrix}, \tag{8.3.28}$$

where the subblocks $R_{p,q}$ are $\nu\times\nu$ matrices. Moreover, at $\rho = 0$, the matrices $R_{p,q}$ have the special form

$$R_{p,q}(\underline{\zeta}, 0) = \begin{bmatrix} * & 0\ldots 0 \\ \vdots & 0\ldots 0 \\ r_{p,q} & 0\ldots 0 \end{bmatrix}. \tag{8.3.29}$$

In addition, denoting by $R^\flat$ the $\alpha_j \times \alpha_j$ matrix with entries $r_{p,q}$, the matrix $\dot{q}\mathrm{Re}\, R^\flat(\underline{\zeta}, 0)$ is definite positive, where $\dot{q} = \partial_\gamma \mathrm{Re}\, q_\nu(\underline{\zeta})$ as in Theorem 8.3.1.

Remarks 8.3.7. a) Part of this result was originally established in [Zum], under the additional assumption that $A(\eta, \xi)$ and $B(\eta, \xi)$ be simultaneously symmetrizable, under which the matrices $R^\flat$ may be chosen to be diagonal, see Observations 4.11–4.12 and equations (4.102)–(4.103) of that reference.

b) In some cases, it may happen that the eigenvalues of $iA(\eta, \xi) + \rho B(\eta, \xi)$ have constant multiplicity in (η, ξ) *and* ρ. For instance, this is the case of an artificial viscosity when $B = -\Delta_{y,x}\mathrm{Id}$, in which case the eigenvalues are $i\lambda_j(\eta, \xi) + \rho(|\eta|^2 + \xi^2)$. In this case, the analysis of the previous section can be extended and one can put $\mathcal{R}$ in a block diagonal form as well, that is $R_{p,q} = 0$ when $p \neq q$ and $R_{p,p} = R$. In, this case $R^\flat = r\mathrm{Id}$, where r is the lower left hand entry of R. However this extended constant multiplicity condition is not always satisfied.

Proof of Theorem 8.3.6*.*

a) With $\mathcal{T}$ given by Theorem 8.3.1, we have

$$\mathcal{T}^{-1}(\zeta)\mathcal{H}^k(\zeta, \rho)\mathcal{T}(\zeta) = \mathcal{Q}(\zeta) + \rho\mathcal{R}(\zeta, \rho)$$

for some matrix $\mathcal{R}(\zeta, \rho)$. Consider an additional change of basis is $Id + \rho\widetilde{\mathcal{T}}$. Then,

$$(Id + \rho\widetilde{\mathcal{T}})^{-1}(\mathcal{Q} + \rho\mathcal{R})(Id + \rho\widetilde{\mathcal{T}}) = \mathcal{Q} + \rho\widetilde{\mathcal{R}}, \quad \widetilde{\mathcal{R}} = \mathcal{R} + [\mathcal{Q}, \widetilde{\mathcal{T}}] + O(\rho).$$

Denoting by $T_{p,q}$ the blocks of $\widetilde{\mathcal{T}}(\underline{\zeta}, 0)$, the blocks $\widetilde{R}_{p,q}(\underline{\zeta}, 0)$ are $R_{p,q}(\underline{\zeta}, 0) + [iJ, T_{p,q}]$. Consider the canonical basis $(e_1, \ldots, e_\nu)$ of $\mathbb{C}^\nu$. Then $Je_1 = 0$

and $Je_l = e_{l-1}$ for $l \geq 2$. Define $T_{p,q}$ by $Te_\nu = 0$ and inductively $T_{p,q}e_l = JT_{p,q}e_{l+1} - iR_{p,q}e_{l+1}$ for $l < \nu$. Then $[T_{p,q}, J]e_l = -iR_{p,q}e_l$ for $l = 2, \dots \nu$. In this case, $\widetilde{R}_{p,q}e_l = 0$ for $l \geq 2$, showing that the blocks $\widetilde{R}_{p,q}$ have the form (8.3.29).

b) Thus, from now on we assume that (8.3.27) and (8.3.29) are satisfied. We compute the Taylor expansion at $(\underline{\zeta}, \underline{\xi}_k)$ of the characteristic polynomial of $\mathcal{H}^k$ using Assumption 8.1.2 and the factorization (8.2.6). With $\xi' = \xi - \underline{\xi}_k$ possibly complex, (8.3.2) implies

$$\big(i\underline{\tau} + \lambda_j(\underline{\eta}, \xi)\big)\mathrm{Id} + \rho B'_j = i\ell {\xi'}^\nu \mathrm{Id} + \rho \underline{B}'_j + O\big(|\xi'|^{\nu+1} + \rho|\xi'| + \rho^2\big)$$

with $\underline{B}'_j = B'_j(\underline{\eta}, \underline{\xi}_k, 0)$. Therefore, substituting in (8.2.7) yields

$$\begin{aligned} \Delta_j(\underline{\tau}, \underline{\eta}, \xi, \rho) = & \det\big(i\ell{\xi'}^\nu Id + \rho\underline{B}'_j\big) \\ & + O\big((|\rho| + |\xi'|)\,(|\xi'|^\nu + \rho)^{\alpha_j}\big)\,. \end{aligned} \tag{8.3.30}$$

Comparing (8.1.2) and (8.2.6) we see that

$$\begin{aligned} \Delta_j(\tau - i\gamma, \eta, \xi, \rho) = &\ E(\zeta, \xi)\ \det\big(i\xi Id - \mathcal{H}^k(\zeta)\big) \\ = &\ E(\zeta, \xi) \det\big(i\xi Id - \mathcal{Q}(\zeta) - \rho\mathcal{R}(\zeta, \rho)\big), \end{aligned} \tag{8.3.31}$$

with $E \neq 0$ near the $(\underline{\zeta}, \underline{\xi}_k)$. We now compare the Taylor expansion (8.3.30) of Δ_j to the Taylor expansion of the right hand side. There we use the following lemma, in which $\mathcal{J}$ is the block diagonal matrix

$$\mathcal{J} = \begin{bmatrix} J & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & J \end{bmatrix}.$$

Lemma 8.3.8. *Suppose that $\mathcal{M}(h)$ is a $\alpha_j\nu \times \alpha_j\nu$ matrix with blocks $M_{p,q}(h)$ depending smoothly on the parameter h, satisfying* (8.3.29) *and such that $\mathcal{M}(0) = 0$. Then there holds*

$$\begin{aligned} \det\big(\xi Id - \mathcal{J} + i\mathcal{M}(h)\big) = & \det(\xi^\nu Id + ih\partial_h M^\flat(0)) \\ & + O\big((|h| + |\xi|)\,(|\xi|^\nu + |h|)^{\alpha_j}\big), \end{aligned}$$

where $M^\flat$ is the $\alpha_j \times \alpha_j$ matrix with entries $m_{p,q}$ which are the lower left hand corner coefficient of $M_{p,q}$.

We apply this lemma with parameter $h = \rho$ and $\mathcal{M} = \rho\mathcal{R}(\underline{\zeta}, \rho)$. Recalling that $\mathcal{Q}(\underline{\zeta}) = i\underline{\xi}_k \mathrm{Id} + i\mathcal{J}$, it implies

$$\begin{aligned} \det\big(i\xi Id - \mathcal{Q}(\underline{\zeta}) - \rho\mathcal{R}(\underline{\zeta}, \rho)\big) & \\ &= i^{\nu\alpha_j} \det\big({\xi'}^{\nu} Id + i\rho\underline{R}^\flat\big) \; + h.o.t. \end{aligned} \tag{8.3.32}$$

where $R^\flat$ is the $\alpha_j \times \alpha_j$ matrix with entries $r_{p,q}$ and $\underline{R}^\flat$ its value at the base point. Thus, comparing the Taylor expansions (8.3.30) and (8.3.32), we find that

$$\det\big(\beta{\xi'}^{\nu}\mathrm{Id} - i\rho\underline{B}'_j\big) = E(\underline{\zeta}) i^{(\nu-1)\alpha_j} \det\big({\xi'}^{\nu}\mathrm{Id} + i\underline{R}^\flat\big) \, .$$

Therefore, the eigenvalues of $\underline{R}^\flat$ are the eigenvalues of $-\ell^{-1}\underline{B}'_j$. With (8.3.6), this implies

$$\mathrm{Spectrum}\big(\dot q \underline{R}^\flat\big) \subset \{\mathrm{Re}\, \mu > 0\} \, . \tag{8.3.33}$$

c) As already used, see for instance (7.3.6), (8.3.33) implies that there is a definite positive matrix Σ such that $\mathrm{Re}\, \dot q \Sigma R^\flat$ is definite positive. Therefore,

$$\mathrm{Re}\;\, T^{-1}\dot q \underline{R}^\flat T \quad \text{is definite positive}\, , \tag{8.3.34}$$

where $T = \Sigma^{-1/2}$. As $R^\flat$, T is a $\alpha_j \times \alpha_j$ matrix. Consider $\mathcal{T}$ the $\nu\alpha_j \times \nu\alpha_j$ matrix with blocks $T_{p,q} = t_{p,q}\mathrm{Id}$ of size $\nu \times \nu$, where the $t_{p,q}$ are the coefficients of T. Then $\mathcal{S} = \mathcal{T}^{-1}$ has blocks $S_{p,q} = s_{p,q}\mathrm{Id}$ where the $s_{p,q}$ are the entries of $S = T^{-1}$. Straightforward computations show that

$$\mathcal{T}^{-1}\mathcal{Q}\mathcal{T} = \mathcal{Q}\, ,$$

since the blocks of the first matrix in the left hand side are

$$\sum_n s_{p,n} Q t_{n,q} = Q \delta_{p,q} \, .$$

Next, the blocks of $\widetilde{\mathcal{R}} := \mathcal{T}^{-1}\mathcal{R}\mathcal{T}$ are

$$\widetilde{R}_{p,q} = \sum_{n,m} s_{p,n} t_{m,q} R_{n,m} \, . \tag{8.3.35}$$

At the base point $(\underline{\zeta}, 0)$ the columns 2 to ν of $\underline{R}_{n,m}$ vanish and the same property holds for $\underline{\widetilde{R}}_{p,q}$, showing that the form of the matrix $\widetilde{\mathcal{R}}$ at the base point is unchanged. Moreover, (8.3.35) implies that the matrix of the lower left hand corner elements in $\widetilde{\mathcal{R}}$ is $\widetilde{R}^\flat = T^{-1}R^\flat T$ and thus $\mathrm{Re}\, \cdot q \widetilde{R}^\flat$ is positive definite at the base point. $\square$

Proof of Lemma 8.3.8.

a) We start with a general remark. Consider a $N \times N$ matrix A with entries $a_{j,k}$ depending on variables x. Assume that

$$a_{j,k}(x) = \underline{a}_{j,k}(x) + h.o.t.$$

where $\underline{a}_{j,k}$ is homogeneous of degree $\mu_j - \nu_k$ and $h.o.t$ denotes terms of higher degree, here $O(|x|^{\mu_j - \nu_k + 1})$. Then

$$\det A(x) = \det \underline{A}(x) + h.o.t. \tag{8.3.36}$$

and $\det \underline{A}$ is homogeneous of degree $\mu := \sum \mu_j - \sum \nu_k$. Indeed,

$$\det A = \sum \epsilon(\sigma) a_{\sigma 1,1} \cdots a_{\sigma_N,N}$$

where the sum is extended over all the permutations σ of $\{1, \ldots, N\}$ and $\epsilon(\sigma)$ is the signature of σ. Each monomial is equal to the corresponding one with $\underline{a}$ in place of a plus higher order terms, and the term with the $\underline{a}$ is homogeneous of degree

$$\sum (\mu_{\sigma_k} - \nu_k) = \sum \mu_{\sigma_k} - \sum \nu_k = \sum \mu_j - \sum \nu_k = \mu.$$

b) In our case, we consider the matrix $\mathcal{A} = \xi Id - \mathcal{J} + i\rho\mathcal{M}$. Denote by $A_{p,q}$ the blocks in $\mathcal{A}$ and by $A_{p,a,p,b}$ the entries of $A_{p,q}$. Remember that $1 \leq p, q \leq \alpha$ and $1 \leq a, b \leq \nu$. We use a quasi-homogeneous version of (8.3.36) with weight 1 on the variable ξ and weight ν on the variable ρ. To be more specific, with ξ_0 and h_0 fixed, consider $\xi = t\xi_0$ and $h = t^\nu h_0$ with $t \in [0, 1]$. Introduce the weights

$$\mu_{p,a} = a + 1\,, \quad \nu_{q,b} = b\,.$$

The diagonal terms in $\mathcal{A}$ are equal to ξ, homogeneous of degree $1 = \mu_{p,a} - \nu_{p,a}$ in t. The entries $N_{p,a,q,b}$ of $\mathcal{N}$ are zero or equal to -1 when $p = q$ and $b = a+1$ which is homogeneous of degree $0 = \mu_{p,a} - \nu_{p,a+1}$. Introduce $\underline{\mathcal{M}} = \partial_h \mathcal{M}(0)$. Then the form (8.3.29) of the blocks $M_{p,q}$ of $\mathcal{M}$ implies that $M_{p,a,q,b}(th)$ vanishes when $b > 1$. When $b = 1$

$$M_{p,a,q,1}(th) = t^\nu h_0 \underline{M}_{p,a,q,b} + O(t^{2\nu})\,.$$

The leading term is homogeneous of degree ν which is strictly larger than $\mu_{p,a} - \nu_{q,1} = a$ if $a < \nu$, and exactly equal to $\mu_{p,a} - \nu_{q,1} = \nu$ if $a = \nu$. Thus, only the lower left hand corners of $M_{p,q}$ have a non vanishing principal part in the sense of a). Thus

$$\begin{aligned}\det\big(t\xi_0 Id - \mathcal{J} + i\mathcal{M}(t^\nu h_0)\big) = \\ \det\big(t\xi Id - \mathcal{J} + it^\nu h_0 \underline{\mathcal{M}}^\flat\big) + O(t^{\alpha\nu+1})\end{aligned} \tag{8.3.37}$$

where the leading term is homogeneous in t of degree $\alpha\nu$ and $\underline{\mathcal{M}}^\flat$ is the matrix with all entries equal to zero except $\underline{M}^\flat_{p,\nu,q,1} = \underline{m}_{p,q}$.

c) Grouping the indices the other way, i.e. considering the matrix $\mathcal{A}$ as a block matrix with blocks $\hat{A}_{a,b}$ with entries $\hat{A}_{p,a,q,b}$, we see that there is a permutation matrix P such that

$$P^{-1}\big(-\mathcal{J} + ih\underline{\mathcal{M}}^\flat\big)P = \begin{bmatrix} 0 & -Id & 0 & \cdots \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & -Id \\ ih\underline{M}^\flat & 0 & \cdots & 0 \end{bmatrix} := \widehat{\mathcal{M}}$$

where $\underline{M}^\flat$ is the matrix with entries $\underline{m}_{p,q}$. Thus $u \in \ker(\xi \mathrm{Id} - \mathcal{J} + ih\underline{\mathcal{M}}^\flat)$ if and only if $v = P^{-1}u \in \ker(\xi\mathrm{Id} + \widehat{\mathcal{M}})$, which means that the blocks components v_a of v satisfy $v_a = {\xi'}^{a-1}v_1$ and $v_1 \in \ker(h\underline{M}^\flat + {\xi'}^{\nu-1}Id)$. Therefore

$$\det\big(\xi Id - \mathcal{J} + ih\underline{\mathcal{M}}^\flat\big) = \det\big(\xi^\nu Id + ih\underline{M}^\flat\big)\,.$$

With (8.3.37) this implies that

$$\begin{aligned}\det\big(t\xi_0 Id - \mathcal{J} + i\mathcal{M}(t^\nu h_0)\big) = \\ \det\big((t\xi_0)^\nu Id + it^\nu h_0 \underline{M}^\flat\big) + O(t^{\alpha\nu+1})\end{aligned} \tag{8.3.38}$$

and the Lemma follows. $\square$

8.4 Construction of symmetrizers near glancing modes

8.4.1 Examples

We use examples to introduce the main three ingredients of the construction. Consider the $\nu \times \nu$ matrix

$$Q = iJ + \gamma K\,, \quad K = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ \sigma & 0 & \cdots & 0 \end{bmatrix}\,, \quad \sigma \in \{-1, +1\}, \tag{8.4.1}$$

where J is the Jordan matrix in (8.3.4). The characteristic polynomial is

$$\det(\mu Id - Q) = (\mu^\nu - i^{\nu-1}\gamma\sigma)\,. \tag{8.4.2}$$

When $\gamma > 0$, the eigenvalues are

$$\mu_l = i\gamma^{1/\nu} e^{i\theta_l}\,, \quad \theta_l = \frac{2l\pi}{\nu} - \frac{\sigma\pi}{2\nu}$$

There are no eigenvalues on the imaginary axis. The number of eigenvalues in $\{\mathrm{Re}\,\mu < 0\}$ is

$$\nu_- = \begin{cases} \nu/2 & \text{if } \nu \text{ is even,} \\ (\nu-1)/2 & \text{if } \nu \text{ is odd and } \sigma = +1 \\ (\nu+1)/2 & \text{if } \nu \text{ is odd and } \sigma = -1 \end{cases} \tag{8.4.3}$$

Eigenvectors associated to μ_l are $\phi_l := {}^t(1, -i\mu_l, \ldots, (-i\mu_l)^{\nu-1})$. They all converge to ${}^t(1, 0, \ldots, 0)$ as γ tends to zero.. Next, remark that $(\phi_1 - \phi_2)/(\gamma^{1/\nu})$ tends to ${}^t(0, e^{2i\pi/\nu}, 0, \ldots)$. Continuing the argument, one shows that $\mathbb{F}_-(\gamma)$, the space generated by eigenvectors associated to eigenvalues in $\{\mathrm{Re}\,\mu < 0\}$, has a limit $\underline{\mathbb{F}}_-$ as γ tends to zero and that

$$\underline{\mathbb{F}}_- = \mathbb{C}^{\nu_-} \times \{0\}^{\nu-\nu_-} \tag{8.4.4}$$

is the space generated by the first ν_- elements of the canonical basis in $\mathbb{C}^\nu$. This yields to *define* the space

$$\underline{\mathbb{F}}_+ = \{0\}^{\nu_-} \times \mathbb{C}^{\nu-\nu_-} \tag{8.4.5}$$

the space generated by the last $\nu - \nu_-$ elements of the canonical basis in $\mathbb{C}^\nu$. In particular,

$$\mathbb{C}^\nu = \underline{\mathbb{F}}_- \oplus \underline{\mathbb{F}}_+\,. \tag{8.4.6}$$

Remark 8.4.1. $\underline{\mathbb{F}}_+$ is *not* the limit of the space $\mathbb{F}_+(\gamma)$, generated by eigenvectors associated to eigenvalues in $\{\mathrm{Re}\,\mu > 0\}$. This limit is the space $\mathbb{C}^{\nu-\nu_-} \times \{0\}^\nu$ generated by the first $\nu - \nu_-$ vectors of the canonical basis.

We look for symmetrizers

$$S = E - i\gamma F \tag{8.4.7}$$

with

$$E = E^t \quad \text{real}\,, \tag{8.4.8}$$

$$F = -F^t \quad \text{real}\,, \tag{8.4.9}$$

This implies that $S = S^*$ is self adjoint. Moreover,

$$SQ = iEJ + \gamma(EK + FJ) - i\gamma^2 FK\,. \tag{8.4.10}$$

Our goal is to construct E and F such that

$$(Sw, w) \geq \kappa|\underline{\Pi}_+ w|^2 - |\underline{\Pi}_- w|^2\,, \tag{8.4.11}$$

$$\mathrm{Re}\,(iEJ) = 0\,, \tag{8.4.12}$$

$$\mathrm{Re}\,(EK + FJ) \geq c\mathrm{Id}\,, \quad \text{with } c > 0\,, \tag{8.4.13}$$

where $\underline{\Pi}_\pm$ are the projectors on $\underline{\mathbb{F}}_\pm$ in the decomposition (8.4.6). The last two properties imply that for γ small enough

$$\mathrm{Re}\,(SQ) \geq \frac{c}{2}\mathrm{Id}\,.$$

Conditions (8.4.8) (8.4.12) are satisfied when E has the form

$$E = \begin{bmatrix} 0 & \cdots & \cdots & 0 & e_1 \\ \vdots & & & \iddots & e_2 \\ \vdots & & \iddots & \iddots & \\ 0 & \iddots & \iddots & & \\ e_1 & e_2 & & & e_\nu \end{bmatrix}. \tag{8.4.14}$$

with coefficients $e_j \in \mathbb{R}$. Moreover,

$$EK + FJ = \begin{bmatrix} \sigma e_1 & F_{1,1} & \cdots & F_{1,\nu-1} \\ \vdots & \vdots & & \vdots \\ \sigma e_\nu & F_{\nu,1} & \cdots & F_{\nu,\nu-1} \end{bmatrix}. \tag{8.4.15}$$

In particular (8.4.13) requires that $\sigma e_1 > 0$. Conversely, there holds:

Lemma 8.4.2. *For all $\kappa \geq 1$, there is a matrix $E(\kappa)$ of the form* (8.4.14) *such that* (8.4.11) *holds and*

$$\sigma e_1 \geq \frac{1}{2}\,. \tag{8.4.16}$$

Next, the condition (8.4.16) implies that

$$\operatorname{Re}(EKw, w) \geq \frac{1}{4}|w_1|^2 - C|w'|^2 \tag{8.4.17}$$

where $(w_1, w_2, \ldots, w_\nu)$ are the components of $w \in \mathbb{C}^\nu$ and $w' = (w_2, \ldots, w_\nu)$. Thus, for (8.4.13) to hold, it is sufficient that

$$\operatorname{Re}(FJw, w) \geq (C+1)|w'|^2 - \frac{1}{8}|w_1|^2 . \tag{8.4.18}$$

The existence of such an F follows from the next lemma.

Lemma 8.4.3. *For all $C > 0$, there is a matrix F satisfying* (8.4.9) *such that*

$$\operatorname{Re}(FJw, w) \geq C|w'|^2 - |w_1|^2 . \tag{8.4.19}$$

Indeed, if F satisfies (8.4.19) with the constant $8(C+1)$, $F/8$ satisfies (8.4.18).

We now consider perturbations of (8.4.1):

$$Q = iJ + iQ'(\eta) + \gamma K \tag{8.4.20}$$

where Q' is a *real* matrix depending smoothly on the parameters η and such that $Q'(\underline{\eta}) = 0$. One look for symmetrizers which are perturbations of (8.4.7):

$$S = E + E'(\eta) - i\gamma F , \tag{8.4.21}$$

with $E'(\eta)$ real and symmetric, vanishing at $\underline{\eta}$. In this case

$$SQ = iEJ + iE'(J+Q') + iEQ' + \gamma((E+E')K + F(J+Q')) - i\gamma^2 FK . \tag{8.4.22}$$

The real part of the term in γ remains definite positive for η close to $\underline{\eta}$. The third ingredient in the construction of symmetrizers is the following result.

Lemma 8.4.4. *For all real matrices Q' and Q'' depending smoothly on the parameters η and such that $Q'(\underline{\eta}) = Q''(\underline{\eta}) = 0$, there exists on a neighborbood of $\underline{\eta}$, a real symmetric matrix $E'(\eta)$, C^∞ in η, vanishing at $\underline{\eta}$ and such that $E'(J + Q') + Q''$ is symmetric.*

Indeed, applying this lemma to $Q'' = EQ'$, provides E' such that

$$\operatorname{Re}\big(iE'(J+Q') + iEQ'\big) = 0 . \tag{8.4.23}$$

8.4.2 Proof of the main lemmas

Construction of E.

We start with examples and next consider the general case.

a) Consider the case $\nu = 2$. Then

$$E = \begin{bmatrix} 0 & e_1 \\ e_1 & e_2 \end{bmatrix}, \quad \underline{\mathbb{F}}_- = \mathbb{C}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{\mathbb{F}}_+ = \mathbb{C}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

One can choose $e_1 = \sigma$, and next $e_2 \geq \kappa + 1$ so that

$$(Ew, w) = e_2|w_2|^2 + 2e_1 \mathrm{Re}\, w_1\overline{w}_2 \geq \kappa|w_2|^2 - |w_1|^2.$$

b) Consider the case $\nu = 3$. We look for

$$E = \begin{bmatrix} 0 & 0 & e_1 \\ 0 & e_1 & 0 \\ e_1 & 0 & e_3 \end{bmatrix}.$$

The reader can check that one can introduced a non vanishing term e_2 in the matrix E, to the price of modifying the specific choice of the parameters below. There are two subcases:

b 1) $\sigma = +1$. Then

$$\underline{\mathbb{F}}_- = \mathbb{C}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{\mathbb{F}}_+ = \mathbb{C}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \oplus \mathbb{C}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then we choose $e_1 \geq \kappa$, so that $e_1\sigma \geq 1$, since $\sigma = 1$. Next we chose $e_3 \geq \kappa + e_1^2$ so that

$$\begin{aligned}(Ew, w) = e_3|w_3|^2 &+ e_1|w_2|^2 + 2e_1 \mathrm{Re}\, w_1\overline{w}_3 \\ &\geq \kappa|w_3|^2 + \kappa|w_2|^2 - |w_1|^2.\end{aligned}$$

b 2) $\sigma = -1$. Then

$$\underline{\mathbb{F}}_- = \mathbb{C}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \oplus \mathbb{C}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{\mathbb{F}}_+ = \mathbb{C}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then we choose $e_1 = -1$, so that $e_1\sigma = 1$. Next we choose $e_3 \geq \kappa + 1$ so that

$$\begin{aligned}(Ew, w) = e_3|w_3|^2 &- |w_2|^2 - 2\mathrm{Re}\, w_1\overline{w}_3 \\ &\geq \kappa|w_3|^2 - |w_2|^2 - |w_1|^2.\end{aligned}$$

c) We now come to the general case $\nu \geq 4$.
With E of the form (8.4.14), consider

$$\Phi_p(w) = \sum_{j,k \leq p} e_{j+k-\nu} w_j \overline{w}_k \,,$$

where it is agreed that $e_l = 0$ when $l \leq 0$. In particular, $\Phi_p(w) = 0$ if $2p \leq \nu$. Note that

$$(Ew, w) = \Phi_\nu(w)\,.$$

Then

$$\Phi_{p+1}(w) \geq \Phi_p(w) + e_{2p+2-\nu}|w_{p+1}|^2 - 2C_p \sum_{j=1}^{p} |w_j||w_{p+1}| \tag{8.4.24}$$

with

$$C_p = \max_{l \leq 2p+1-\nu} |e_l|\,.$$

Choose sequences α_p and β_p with $\alpha_{p+1} < \alpha_p$, $\beta_{p+1} > \beta_p$, $\alpha_{1+\nu_-} = 2$, $\beta_{1+\nu'} = 3/4$ and $\alpha_\nu = \beta_\nu = 1$. With ν_- given by (8.4.3) we show that one can choose the coefficients e_l so that (8.4.16) holds and for $p \geq \nu_- + 1$:

$$\Phi_p(w) \geq \alpha_p \kappa \sum_{j=\nu_-+1}^{p} |w_j|^2 - \beta_p \sum_{j=1}^{\nu_-} |w_j|^2\,. \tag{8.4.25}$$

We proceed by induction on p, getting (8.4.11) for $p = \nu$. Indeed, (8.4.24) implies that

$$\begin{aligned}
\Phi_{p+1}(w) \geq \alpha_p \kappa \sum_{j=\nu_-+1}^{p} |w_j|^2 + (e_{2p+2-\nu} - \frac{C_p^2}{\varepsilon})|w_{p+1}|^2 \\
- \beta_p \sum_{j=1}^{\nu_-} |w_j|^2 - \varepsilon \sum_{j=1}^{p} |w_j|^2 \\
\geq \alpha_{p+1}\kappa \sum_{j=\nu_-+1}^{p+1} |w_j|^2 - \beta_{p+1} \sum_{j=1}^{\nu_-} |w_j|^2
\end{aligned}$$

if ε is small enough and $e_{2p+2-\nu} - C_p^2/\varepsilon \geq \alpha_p \kappa$, which can be achieved since C_p depends only on the e_l with $l < 2p+2-\nu$. Thus it remains to prove (8.4.25) for $p = 1 + \nu_-$.

c1) If $\nu = 2\nu'$ is even, $\nu_- = \nu'$ and

$$\Phi_{\nu'+1}(w) = e_2|w_{\nu'+1}|^2 - 2e_1 \mathrm{Re}\, w_{\nu'}\overline{w}_{\nu'+1}\,.$$

We proceed as in step a): choose e_1 such that $e_1\sigma \geq 1/2$ and next choose e_2 such that

$$\Phi_{\nu'+1}(w) \geq 2\kappa|w_{\nu'+1}|^2 - \frac{1}{2}|w_{\nu'}|^2\,.$$

c2) If $\nu = 2\nu' + 1$ is odd and $\sigma = +1$, then $\nu_- = \nu'$ and

$$\Phi_{\nu'+1}(w) = e_1|w_{\nu'+1}|^2\,.$$

It is sufficient to choose $e_1 \geq 2\kappa$.

c3) If $\nu = 2\nu' + 1$ is odd and $\sigma = -1$, then $\nu_- = \nu' + 1$ and, if $e_2 = 0$,

$$\Phi_{\nu'+2}(w) = e_3|w_{\nu'+1}|^2 + e_1|w_{\nu'+1}|^2 + 2e_1 \mathrm{Re}\, w_{\nu'+1}\overline{w}_{\nu'+2}\,.$$

Choose $e_1 = -1/2$ and next e_3 large enough, so that

$$\Phi_{\nu'+2}(w) \geq 2\kappa|w_{\nu'+1}|^2 - \frac{3}{4}|w_{\nu'+1}|^2\,.$$

The proof of Lemma 8.4.2 is now complete. $\square$

Construction of F.

a) Start again with the case $\nu = 2$. In this case,

$$F = \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix}, \quad FJ = F\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}.$$

Thus,

$$\mathrm{Re}\,(FJw, w) = f|w_2|^2\,.$$

b) Consider the general case, $\nu \geq 3$. We look for F as a tridiagonal skew symmetric real matrix:

$$F = \begin{bmatrix} 0 & -f_1 & 0 & & & \\ f_1 & 0 & -f_2 & 0 & & \\ 0 & f_2 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & 0 & -f_{\nu-1} \\ & & & 0 & f_\nu & 0 \end{bmatrix}$$

with coefficients $f_j > 0$. Then

$$\mathrm{Re}\,(FJw, w) = \sum_{j=2}^{\nu} f_{j-1}|w_j|^2 - \sum_{j=1}^{\nu-2} \mathrm{Re}\, f_j w_{j+2}\overline{w}_j\,.$$

Thus, with $a_1 = f_1^2/4$ and next $a_j = f_j^2/2f_{j-1}$, there holds

$$\mathrm{Re}\,(FJw, w) \geq -|w_1|^2 + \sum_{j=2}^{\nu} \frac{1}{2} f_{j-1}|w_j|^2 - \sum_{j=3}^{\nu} a_{j-2}|w_j|^2\,.$$

Since a_{j-2} only depends of the f_l for $l < j-1$, one can choose the f_j inductively so that (8.4.19) is satisfied. $\square$

Proof of Lemma 8.4.4.

With $\Sigma = {}^t Q'' - Q''$ skew symmetric, vanishing at $\underline{\eta}$, we have to solve the equation

$$E'(\eta)(J + Q'(\eta)) - {}^t\big(J + Q'(\eta)\big)E'(\eta) = \Sigma(\eta)\,, \quad E'(\eta) = {}^t E'(\eta)\,. \tag{8.4.26}$$

The first equation is a linear system of $\nu(\nu-1)/2$ equation for $\nu(\nu+1)/2$ unknowns because of the symmetry of E'. The linear operator J : $E' \mapsto E'J - {}^t JE'$ maps the space of symmetric matrices to the space of skew symmetric matrices. Its kernel is the space of matrices of the form (8.4.14), thus its dimension is ν. Hence, J has rank equal to $\nu(\nu-1)/2$ and therefore it is onto.

More specifically, one can consider the space E$'$ of symmetric matrices of the form

$$E' = \begin{bmatrix} \check{E}' & 0 \\ 0 & 0 \end{bmatrix},$$

with $\check{E}'$ real, symmetric of dimension $(\nu-1)\times(\nu-1)$. This space of dimension $\nu(\nu-1)/2$ intersects the space of matrices (8.4.14) at $E' = 0$. Therefore, J is an isomorphism from E$'$ to the space of skew symmetric matrices. Hence, for η close to $\underline{\eta}$, the mapping $E' \mapsto E'(J + Q'(\eta)) - {}^t\big(J + Q'(\eta)\big)E'$ is still an isomorphism from E$'$ to the space of skew symmetric matrices and the lemma follows. $\square$

8.4.3 Proof of Proposition 8.1.6 near glancing modes

Consider $\underline{\zeta} \in S^d_+$ with $\underline{\gamma} = 0$ and a real root $\underline{\xi}_k$ of (8.3.1) satisfying (8.3.2) with $\nu \geq 2$. By Theorem 8.3.6, we know that there is a smooth matrix $\mathcal{V}(\zeta, \rho)$ on a neighborhood of $(\underline{\zeta}, 0)$ such that

$$\widetilde{\mathcal{H}}^k(\zeta,\rho) := \mathcal{V}^{-1}(\zeta,\rho)\mathcal{H}^k(\zeta,\rho)\mathcal{V}(\zeta,\rho) = \mathcal{Q}(\zeta) \, + \, \rho\mathcal{R}(\zeta,\rho)\,.$$

$\mathcal{Q}$ and $\mathcal{R}$ have the properties listed in Theorems 8.3.1 and 8.3.6 respectively.

It is sufficient to construct symmetrizers for $\widetilde{\mathcal{H}}^k$, since a symmetrizer $\mathcal{S}$ for $\widetilde{\mathcal{H}}^k$ provides a symmetrizer $(\mathcal{V}^{-1})^*\mathcal{S}\mathcal{V}^{-1}$ for $\mathcal{H}^k$.

We construct $\mathcal{S}$ in the block decomposition (8.3.3) (8.3.28) of $\mathcal{Q}$ and $\mathcal{R}$:

$$\mathcal{S} = \begin{bmatrix} S & 0 & \\ 0 & \ddots & 0 \\ & 0 & S \end{bmatrix}, \tag{8.4.27}$$

with $\nu \times \nu$ blocks S:

$$S(\zeta,\rho) = E + E'(\eta) - i\gamma F - i\rho F', \tag{8.4.28}$$

where E and E are real and symmetric matrices, and F and F' are real and skew symmetric. The idea is that $E + E' - i\gamma F$ is a good symmetrizer for Q, as suggested by the analysis above, and that $-i\rho F'$ is a perturbation analogous to $-i\gamma F$ which takes care of the perturbation $\rho\mathcal{R}$.

With obvious notations we denote by $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{F}$ and $\mathcal{F}'$ the block diagonal matrices

$$\mathcal{E} = \begin{bmatrix} E & 0 & \\ 0 & \ddots & 0 \\ & 0 & E \end{bmatrix}, \quad \ldots, \quad \mathcal{F}' = \begin{bmatrix} F' & 0 & \\ 0 & \ddots & 0 \\ & 0 & F' \end{bmatrix}. \tag{8.4.29}$$

By the analysis of the example (8.4.1), we have good candidates for the limit of the negative space $\mathbb{F}^k_-$. By (8.3.26), there holds

$$\det(\mu \mathrm{Id} - Q(\zeta)) = i^\nu \Big(\xi'^\nu + \sum_{l=1}^{\nu} i q_l(\zeta) \xi'^{\nu - l}\Big)$$

with $\mu = i(\xi' - \underline{\xi}_k)$. At $\eta = \underline{\eta}$, $q_\nu = \gamma\dot{q} + O(\gamma^2)$ and one checks that for $\gamma > 0$, the number ν_- of eigenvalues of Q in $\{\mathrm{Re}\,\mu < 0\}$ is given by (8.4.3), with $\sigma = \mathrm{sign}\,\dot{q}$. Consider

$$\underline{\mathrm{E}}_- = \mathbb{C}^{\nu_-} \times \{0\}^{\nu - \nu_-}\,, \quad \underline{\mathrm{E}}_+ = \{0\}^{\nu_-} \times \mathbb{C}^{\nu - \nu_-}\,. \tag{8.4.30}$$

Thus

$$\mathbb{C}^\nu = \underline{\mathrm{E}}_- \oplus \underline{\mathrm{E}}_+ .$$

In the block decomposition of $\mathbb{C}^{N_k}$ into α_j factors $\mathbb{C}^\nu$, let

$$\underline{\mathbb{F}}_\pm = \underline{\mathrm{E}}_\pm \oplus \cdots \oplus \underline{\mathrm{E}}_\pm . \tag{8.4.31}$$

Therefore, the splitting (8.1.11), $\mathbb{C}^{N_k} = \underline{\mathbb{F}}_- \oplus \underline{\mathbb{F}}_+$ holds and $\dim \underline{\mathbb{F}}_- = \alpha_j \nu_-$ is the number $N_{k,-}$ of eigenvalues of $\mathcal{Q}$ in $\{\operatorname{Re}\mu < 0\}$, when $\gamma > 0$, and thus also the eigenvalues of $\mathcal{H}^k$ in $\{\operatorname{Re}\mu < 0\}$ when $\gamma \geq 0$ and $\rho > 0$.

The order of the construction is as follows.

1. First one chooses E real and symmetric, using Lemma 8.4.2. The real coefficient e_1 satisfies (8.4.16) and adding up the estimates (8.4.11) in each block $\mathbb{C}^\nu$ yields:

$$(\mathcal{E}w, w) \geq \kappa |\underline{\Pi}_+ w|^2 - |\underline{\Pi}_- w|^2 ,$$

where $\underline{\Pi}_\pm$ are now the projectors on $\underline{\mathbb{F}}_\pm$. Thus (8.1.13) is satisfied.

2. Using (8.3.4), the Taylor expansion of Q at $\gamma = 0$ reads

$$Q(\zeta) = \underline{Q} + iQ'(\eta) + \gamma K(\zeta) . \tag{8.4.32}$$

By Theorem 8.3.1, $Q'(\eta)$ has real coefficients. Therefore, by Lemma 8.4.4, there are real symmetric matrices, $E'(\eta)$, depending smoothly on η and such that

$$E'(J + Q') + EQ' \quad \text{is symmetric.}$$

In the analysis of the example (8.4.1), we have seen that $\operatorname{Re}(iEJ) = 0$. Thus, since $\underline{Q} = i(\underline{\xi}_k \mathrm{Id} + J)$, we have

$$\operatorname{Re}\big((E + E')(\underline{Q} + iQ')\big) = 0 .$$

Therefore,

$$\operatorname{Re} \mathcal{S}\widetilde{\mathcal{H}}^k = \gamma \Phi(\zeta) + \rho \Phi'(\zeta, \rho) \tag{8.4.33}$$

with

$$\Phi(\underline{\zeta}) = \operatorname{Re}\big(\mathcal{E}\mathcal{K}(\underline{\zeta}) + \mathcal{F}\mathcal{J}\big) , \tag{8.4.34}$$

$$\Phi'(\underline{\zeta}, 0) = \operatorname{Re}\big(\mathcal{E}\mathcal{R}(\underline{\zeta}, 0) + \mathcal{F}'\mathcal{J}\big) . \tag{8.4.35}$$

Here, $\mathcal{K}$ [resp. $\mathcal{J}$] is the block diagonal matrix with diagonal entries equal to K [resp. J].

3. The special form (8.3.5) of Q and (8.4.16) imply that

$$\begin{aligned}\operatorname{Re}(EK(\underline{\zeta})w, w) &= e_1\dot{q}|w_1|^2 + \operatorname{Re}\sum_{j=2}^{\nu}\sum_{k=\nu-j+1}^{\nu} e_{j+k-\nu}\partial_\gamma q_k(\underline{\zeta})w_1\overline{w}_j \\ &\geq \frac{|\dot{q}|}{4}|w_1|^2 - C_0|w'|^2\,,\end{aligned}$$

with $w' = (w_2, \ldots, w_\nu)$. Here, C_0 is a constant which may depend on the coefficients e_l, thus on κ.

Applying Lemma 8.4.3 with $C = 8(C_0+1)/|\dot{q}|$, and multiplying by $|\dot{q}|/8$, one obtains a skew symmetric matrix F such that

$$\operatorname{Re}(FJw, w) \geq C_0|w'|^2 + \frac{|\dot{q}|}{8}(|w'|^2 - |w_1|^2)\,.$$

Therefore,

$$\operatorname{Re}\big(EK(\underline{\zeta}) + FJ\big) \geq \frac{|\dot{q}|}{8}\operatorname{Id}\,,$$

hence

$$\operatorname{Re}\big(\mathcal{E}\mathcal{K}(\underline{\zeta}) + \mathcal{F}\mathcal{J}\big) \geq \frac{|\dot{q}|}{8}\operatorname{Id}\,.$$

Thus, for ζ in a neighborhood of $\underline{\zeta}$, there holds:

$$\Phi(\zeta) \geq \frac{|\dot{q}|}{10}\operatorname{Id}\,. \tag{8.4.36}$$

4. Introduce a notation. In the decomposition $\mathbb{C}^{N_k} = \mathbb{C}^\nu \oplus \ldots \oplus \mathbb{C}^\nu$, a vector $w \in \mathbb{C}^{N_k}$ is broken into α_j blocks $w_p \in \mathbb{C}^{\nu_k}$, and the components of w_p are denoted by $w_{p,a}$. We denote by $R_{p,q}$ the $\nu \times \nu$ blocks of $\mathcal{R}$ and by $R_{p,a,q,b}$ their entries. The entries of E are denoted by $E_{a,b}$. By (8.3.29), $R_{p,a,q,b}(\underline{\zeta}, 0) = 0$ when $b > 1$. Then

$$\begin{aligned}\operatorname{Re}\big(\mathcal{E}\mathcal{R}(\underline{\zeta}, 0)w, w\big) &= \operatorname{Re}\sum E_{a,c}R_{p,a,q,1}(\underline{\zeta}, 0)w_{q,1}\overline{w}_{p,c} \\ &= \operatorname{Re}\sum e_1 r_{p,q}(\underline{\zeta}, 0)w_{q,1}\overline{w}_{p,1} + O(|w_{*,1}|\,|w'_*|),\end{aligned}$$

where $w_{*,1} \in \mathbb{C}^{\alpha_j}$ is the collection of the first components $w_{p,1}$ and w'_* denotes the other components. Moreover, $r_{p,q} = R_{p,\nu,q,1}$ the lower left hand corner entry of $R_{p,q}$. By Theorem 8.3.6, $\dot{q}R^\flat(\underline{\zeta}, 0)$ is definite positive. By (8.4.16), e_1 and $\dot{q}$ have the same sign. Therefore, the matrix $\operatorname{Re}(e_1 R^\flat(\underline{\zeta}, 0))$ is definite positive. Hence, there are $c > 0$ and C_1 such that

$$\operatorname{Re}\big(\mathcal{E}\mathcal{R}(\check{\underline{\zeta}}, 0)w, w\big) \geq c|w_{*,1}|^2 - C_1|w'_*|^2.$$

By Lemma 8.4.3, there is a skew symmetric matrix F' such that

$$\mathrm{Re}\,(F'Jw_p, w_p) \geq C_1 |w_p'|^2 + \frac{c}{2}(|w_p'|^2 - |w_{p,1}|^2)\,.$$

Hence, the block diagonal matrix $\mathcal{F}'$ satisfies

$$\mathrm{Re}\,(\mathcal{F}'\mathcal{J}w, w) \geq C_1 |w_*'|^2 + \frac{c}{2}(|w_*'|^2 - |w_{*,1}|^2)\,.$$

Therefore,

$$\mathrm{Re}\,\big(\mathcal{E}\mathcal{R}(\underline{\zeta}, 0) + \mathcal{F}'\mathcal{J}\big) \geq \frac{c}{2}\mathrm{Id}$$

and for (ζ, ρ) in a neighborhood of $(\underline{\zeta}, 0)$:

$$\Phi'(\zeta, \rho) \geq \frac{c}{4}\mathrm{Id}\,. \tag{8.4.37}$$

5. Summing up, we have constructed by definition (8.4.28) a self adjoint matrix $\mathcal{S}(\zeta, \rho)$, which satisfies (8.1.13) and, by (8.4.33), (8.4.36) (8.4.37), such that

$$\mathrm{Re}\,\mathcal{S}\widetilde{\mathcal{H}}^k \geq c'(\gamma + \rho)\mathrm{Id}\,, \quad c' > 0\,,$$

for $\gamma \geq 0$ and $\rho \geq 0$. This finishes the proof of Proposition 8.1.6, hence of Theorems 8.1.5 and 7.5.2.

Chapter 9

Linear and nonlinear stability of quasilinear boundary layers

In this chapter, we briefly describe the main results of [MZ1]. They extend to the multidimensional case the results obtained by E.Grenier and F.Rousset ([Gr-Ro]) in dimension one. They also extend the results of E.Grenier and O.Guès ([Gr-Gu]) and M.Gisclon and D.Serre ([Gi-Se]) which where obtained under a smallness assumption.

9.1 Assumptions

We consider on $\mathbb{R}^{1+d}_+$ the hyperbolic system (5.1.1)

$$L(u,\partial)u := \partial_t u + \sum_{j=1}^{d} A_j(u)\partial_j u = F(u) + f \tag{9.1.1}$$

and a parabolic viscous perturbation (5.1.2)

$$L(u,\partial)u - \varepsilon \sum_{1\leq j,k\leq d} \partial_j\big(B_{j,k}(u)\partial_k u\big) = F(u) + f\,. \tag{9.1.2}$$

with Dirichlet boundary conditions:

$$u_{|x=0} = 0\,. \tag{9.1.3}$$

We suppose that the hyperbolicity-parabolicity Assumption 5.1.1 is satisfied. Let us comment here the assumptions. The Assumption (H1) means that the perturbation

$$B(u, \partial) := \sum \partial_j \Big(B_{j,k}(u) \partial_k \cdot \Big)$$

is uniformly parabolic. (H2) means that L is hyperbolic, at least when the state u remains in the domain $\mathcal{U}$. The important Assumption (H4) means that the boundary $\{x = 0\}$ is noncharacteristic for L. The Assumption (H3) is a compatibility condition between L and B. For example, when $B = \Delta_x$ is the Laplacian, (H1) is trivial and (H3) follows immediately from (H2). When (9.1.1) is a system of conservation laws which admits a strictly convex entropy $\eta(u)$, the system is symmetric hyperbolic. If in addition, $\mathrm{Re}\,\big(\eta''(u) \sum \xi_j \xi_k B_{j,k}(u)\big)$ is definite positive for all $\xi \neq 0$, then the assumptions (H1) and (H3) are satisfied.

The solutions of (9.1.2) are expected to be of the form

$$u^\varepsilon(t, y, x) = U_0(t, y, x, x/\varepsilon) + \varepsilon U_1 \dots \tag{9.1.4}$$

where, for (t, y) in the boundary, $w(\cdot) = U_0(t, y, 0, \cdot)$ is a solution of the innerlayer ode (5.2.2), connecting 0 to

$$\lim_{z \to \infty} U_0(t, y, 0, z) = u_0(t, y, 0)$$

where $u_0(t, y, x)$ is the solution of the limiting hyperbolic boundary value problem. Following [Gr-Gu], if $u_0(t, y, 0)$ is small, there is a unique small profile w connecting 0 and u_0 if and only if $u_0(t, y, 0) \in \mathcal{C}$, where $\mathcal{C}$ is a smooth manifold of dimension N_-, see Proposition 5.4.1. Similarly, if $u_0(t, y, 0)$ remains close to $\underline{p}$ and $\underline{w}$ connects 0 to $\underline{p}$ and is transversal in the sense of Definition 5.5.4, there is a unique profile w connecting 0 and u_0, close to $\underline{w}$, for $u_0(t, y, 0) \in \mathcal{C}$, a smooth piece of manifold of dimension N_- near $\underline{p}$, see Proposition 5.5.5. In general, we have to assume that the connection w is given and this leads to the next assumption.

Assumption 9.1.1. *We are given a smooth manifold $\mathcal{C} \subset \mathcal{U}$ and a smooth function W from $\mathcal{C} \times [0, \infty[$ to $\mathcal{U}^*$, such that for all $p \in \mathcal{C}$, $w_p = W(p, \cdot)$ is a solution of* (5.2.2) *and $w_p(z)$ converges to p when z tends to $+\infty$, at an exponential rate, which can be chosen uniform on compact subsets of $\mathcal{C}$.*

Assumption 9.1.1 is the natural analog of assumption (H4), [Zum], made in the planar shock theory. For the limiting hyperbolic problem (9.1.1) one considers the boundary conditions:

$$u_{|x=0} \in \mathcal{C}\,. \tag{9.1.5}$$

For all $p \in \mathcal{C}$, we can form the linearized equation (6.1.2) around w_p, and the corresponding Evans function $D(p, \zeta)$ (6.3.14) or the scaled Evans function $\widetilde{D}(p, \zeta)$ (6.3.16). They are defined for $\zeta \neq 0$, $\zeta \in \overline{\mathbb{R}}^{d+1} := \{(\tau, \eta, \gamma) \in \mathbb{R}^{d+1}; \gamma \geq 0\}$. Similarly, we can form the linearized hyperbolic equation (6.1.6) around the constant solution p. Together with the linearized boundary conditions

$$\dot{u}_{|x=0} \in T_p\mathcal{C} \,,$$

we can define the Lopatinski determinant $\check{D}(p, \check{\zeta})$ (6.2.8), which is defined for $\check{\zeta} \in \overline{S}^d_+ := \{\check{\zeta} = (\check{\tau}, \check{\eta}, \check{\gamma}) \in \mathbb{R}^{d+1}; |\check{\zeta}| = 1 \,,\ \check{\gamma} \geq 0\}$.

According to Definition 6.3.5, the *strong or uniform* stability condition reads:

Assumption 9.1.2 (Uniform stability condition). *For all compact $\mathcal{K} \subset \mathcal{C}$, there is a constant $c > 0$ such that for all $p \in \mathcal{K}$ and $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \backslash \{0\}$, there holds:*

$$|\widetilde{D}(p, \zeta)| \geq c \tag{9.1.6}$$

Remarks 9.1.3. a) The stability conditions are conditions on the "frozen coefficient" planar boundary value problems associated with the inner layer solution. They are natural analogs of those defined in [Zum] for the planar shock case. In the one-dimensional boundary layer case, Assumption 9.1.2 reduces to the condition imposed by Grenier and Rousset [Gr-Ro]. For extensions to the multidimensional case, we refer to [GMWZ1], [GMWZ2].

b) The uniform stability conditions involves three regimes for ζ. For medium frequencies, it just means that $D(p, \zeta)$ does not vanish. For high frequencies, the analysis of section 7.3, shows that is is equivalent to the well posedness of the parabolic problem

$$\partial_t u - \sum_{j,k} \partial_j (B_{j,k}(0) \partial_k u) = f \,, \quad u_{|x=0} = 0 \,,$$

(see also [Zum], Lemma 4.28). In particular, by Remark 7.3.3 the condition (9.1.6) is satisfied for large ζ if the parabolic system is symmetric.

c) Theorem 6.4.1 gives equivalent conditions for the validity of the uniform condition (9.1.6) for low frequencies: it holds, if and only if the profile $w_p(\cdot)$ is transversal for $p \in \mathcal{C}$ and the limiting hyperbolic problem (9.1.1) (9.1.5) satisfies the uniform Lopatinski condition (see [Ro1] and also [ZS]).

d) When the system is symmetric and the parabolic term is the Laplace operator, it is proved in [Gr-Gu], that for then for small amplitude layers, i.e., for p in a suitably small neighborhood of 0, there is a unique manifold $\mathcal{C}$ and connection W having the properties above; moreover, the transversality condition is satisfied. They also prove that the boundary conditions (9.1.5) are maximally dissipative, when u is small. In the large, we substitute for maximal dissipativity the more general uniform Kreiss-Lopatinski-Evans condition.

9.2 Linear stability

By Theorem 6.4.1, under Assumptions 5.1.1, 9.1.1 and 9.1.2, the mixed problem (9.1.1) (9.1.5) satisfies the uniform Lopatinski conditions, and therefore the mixed initial boundary value problem can be solved for initial conditions which satisfy sufficiently many compatibility conditions (see [Maj], [Ra-Ma], [Mok], [Mé2]).

Consider a smooth enough function u_0 on $[-T,T]\times\mathbb{R}^d_+$, which is to be thought as a solution of the hyperbolic boundary value problem. For the moment, we only assume that u_0 satisfies the boundary condition (9.1.5). By definition of the boundary condition, there are profiles

$$w_0(t,y,z) = W(u_0(t,y,0),z) \tag{9.2.1}$$

connecting 0 to $u_0(t,y,0)$. To extend the definition to $x>0$, it is convenient to extend the definition of $W(p,z)$: consider a compact subset $\mathcal{K}\subset\mathcal{C}$ and assume that $u_0(t,y,0)\in\mathcal{K}$ for all $(t,y)\in[-T,T]\times\mathbb{R}^{d-1}$.

Lemma 9.2.1. *There is a neighborhood Ω of $\mathcal{K}$ in $\mathcal{U}$ and a C^∞ function $\widetilde{W}$ on $\Omega\times\mathbb{R}_+$ such that*

i) $\widetilde{W}(p,0)=0$,

ii) for all all multi-indices (α,k), there are $\delta>0$ and C such that

$$\forall p\in\Omega\,,\forall z\geq 0\ :\quad \left|\partial_p^\alpha\partial_z^k\big(\widetilde{W}(p,z)-p\big)\right|\leq Ce^{-\delta z}$$

iii) when $p\in\mathcal{C}$ then $\widetilde{W}(p,z)=W(p,z)$.

Proof. Near $\underline{p}\in\mathcal{C}$, one can use coordinates $p=(p',p'')$ such that $\mathcal{C}$ is given by the equations $p''=h(p')$. Then one can extend locally the function W as

$$\widetilde{W}(p,z) = W(p',z) + (p''-h(p'))\tanh z\,.$$

Gluing the pieces by a partition of unity yields the result. $\square$

9.2.1 L^2 stability

Assume that

$$u_0 \in W^{2,\infty}([-T,T]\times\mathbb{R}^d_+)\,, \quad u_{0|x=0} \in \mathcal{K}\,. \tag{9.2.2}$$

Introduce

$$u_0^\varepsilon(t,y,x) = \chi(x)\widetilde{W}(u_0(t,y,x),\varphi(x)/\varepsilon) + (1-\chi(x))u_0(t,y,x)\,, \tag{9.2.3}$$

where $\chi \in C^\infty(\mathbb{R})$ is equal to one on a small neighborhood of 0, so that $u_0(t,y,x) \in \Omega$ for x in the support of χ. By construction, it satisfies

$$\begin{cases} u^\varepsilon_{0|\partial\Omega} = 0\,. \\ u_0^\varepsilon - u_0 = O(e^{-\delta x/\varepsilon}). \end{cases} \tag{9.2.4}$$

Thus, u_0^ε is a perturbation of u_0 in the interior, that is for $x > 0$. The claim is that u_0^ε is close to a solution of (9.1.2) if u_0 is a solution of the hyperbolic problem. In this direction, the main step is to prove that the linearized equations from (9.1.1) around u_0^ε are stable. For applications, we need a little more. The BKW solutions will have the form $u^\varepsilon_{app} = u_0^\varepsilon + \varepsilon u_1^\varepsilon + \varepsilon^2 \ldots$. Since the term u_1^ε depends on the rapid variable x/ε, $\varepsilon u_1^\varepsilon$ is bounded in $W^{1,\infty}$ but is not a perturbation of u_0^ε is this space. To study the linear stability of u_0^ε and u^ε_{app}, we suppose that we are given a family of functions $v^\varepsilon \in W^{2,\infty}([-T,T]\times\mathbb{R}^d_+)$ such that

$$\sup_{\varepsilon\in]0,1]} \Big(\|v^\varepsilon\|_{L^\infty} + \|\varepsilon\nabla_{t,x}v^\varepsilon\|_{L^\infty} + \|\varepsilon^2\nabla^2_{t,x}v^\varepsilon\|_{L^\infty}\Big) \;<\; \infty\,. \tag{9.2.5}$$

Consider,

$$u_a^\varepsilon := u_0^\varepsilon + \varepsilon v^\varepsilon \tag{9.2.6}$$

and the linearized equation from (9.1.2) around u^ε_{app} reads

$$\mathcal{P}_{u_a^\varepsilon}(t,x,\partial_t,\partial_x)u \;=\; f\,, \quad u_{|x=0} = 0\,. \tag{9.2.7}$$

The differential operator $\mathcal{P}_{u_a^\varepsilon}$ is first order in t and second order in (y,x). it is given by (6.1.2) when $u_a^\varepsilon(t,y,x) = w(w/\varepsilon)$. In the general case, its coefficients depend on the fast variable x/ε as in (6.1.2), but in addition on the slow variables (t,y,x).

The first result of [MZ1] is that, under the Assumptions 5.1.1, 9.1.1 and 9.1.2, the equations (9.2.7) are well posed in L^2.

Theorem 9.2.2 (**L^2 stability).** *There are $C > 0$ and ε_0 such that for all $\varepsilon \in]0, \varepsilon_0]$ and $f \in L^2([-T,T] \times \Omega)$ vanishing for $t < 0$, the equation* (9.2.7) *has a unique solution which vanishes for $t < 0$. Moreover*

$$\|u\|_{L^2} + \sqrt{\varepsilon}\|\nabla_{y,x} u\|_{L^2} + \varepsilon^{3/2}\|\nabla^2_{y,x} u\|_{L^2} \leq C\|f\|_{L^2} \,. \tag{9.2.8}$$

The estimate (9.2.8) is the exact analogue of the basic uniform L^2 estimate of Theorem 3.3.2, see (3.3.16). However, the proof is much more delicate, even in the case of symmetric operators. Let us point out where the difficulty lies. The coefficients of $\mathcal{P}_{u_a^\varepsilon}$ depend on the rapid variable x/ε and thus are not (uniformly) Lipschitzean. Moreover, the zero-th order coefficient of u in $\mathcal{P}_{u_a^\varepsilon}$ has a factor $\frac{1}{\varepsilon}$ in front of it. Thus the usual energy method using integration by parts yields singular and unsigned terms, and therefore we get in the right hand side bad terms in

$$\frac{C}{\varepsilon} \int \alpha |u|^2 dtdydx \tag{9.2.9}$$

where the coefficient α depends on z-derivatives of the profile W. There is no way to absorb this term by a zero-th order term of the left hand side. However, on the left hand side there is a gain of

$$c\varepsilon \int |\partial_x u|^2 dtdydx \,. \tag{9.2.10}$$

Because α depends on rapid derivatives of W, it is exponentially decaying:

$$|\alpha(t, y, x)| \leq W^* e^{-\theta x/\varepsilon} \tag{9.2.11}$$

for some $\theta > 0$. The bound W^* is a measure of the strength of the boundary layer W. Because $u = 0$ when $x = 0$, there holds a Poincaré estimate

$$\frac{1}{\varepsilon} \int e^{-\theta x/\varepsilon} |u|^2 dtdydx \leq C_0 \varepsilon \int |\partial_x u|^2 dtdydx \,.$$

Indeed, this estimate is clear when $\varepsilon = 1$ and follows for $\varepsilon \in]0, 1]$ by scaling. Therefore, the bad term (9.2.9) is estimated by

$$CC_0 W^* \varepsilon \int |\partial_x u|^2 dtdydx \,.$$

It can be absorbed from the right to the left, that is controlled by the good term (9.2.10) if

$$W^* < \frac{c}{CC_0} \,.$$

This is exactly where the smallness assumption in [Gr-Gu] comes in. The main objective of Theorem 9.2.2 is to replace the smallness condition by the Assumption 9.1.2.

By localization in time, which can be obtained along the lines developed in Chapter two, the estimate (9.2.8) follows from a weighted estimates for solutions on $\mathbb{R}^{1+d}_+$. One can extend u_0 and the v^ε for $t \geq T$ and $t \leq T$, so that (9.2.2) and (9.2.5) hold on $\mathbb{R}^{1+d}_+$. We still denote by u_0^ε the extended function defined by (9.2.3) and by $u_a^\varepsilon = u_0^\varepsilon + \varepsilon v^\varepsilon$. Similarly, $\mathcal{P}_{u_a^\varepsilon}$ denotes the linearized operator around the extended u_a^ε. The theorem follows from the estimates:

$$\begin{aligned} \gamma \| e^{-\gamma t} \|_{L^2} + \sqrt{\varepsilon \gamma} \| e^{-\gamma t} \nabla_{y,x} u \|_{L^2} + \varepsilon \| e^{-\gamma t} \nabla_y \nabla_{y,x} u \|_{L^2} & \\ \leq C \, \| e^{-\gamma t} \mathcal{P}_{u_a^\varepsilon} u \|_{L^2} \, . & \end{aligned} \tag{9.2.12}$$

When u_a^ε is a profile $w(x/\varepsilon)$, the linearized equation is studied in Chapter seven and the estimate (9.2.12) is proved in Corollary 7.1.2. The proof is based on the use of symmetrizers, which are Fourier multipliers

$$S(x, D_t, D_y, \gamma) u = \mathcal{F}^{-1} \Big(S(x, \tau, \eta, \gamma) \mathcal{F} u(x, \tau, \eta) \Big)$$

where $\mathcal{F}$ denotes the tangential Fourier transform. In general, the coefficients of $\mathcal{P}_{u_a^\varepsilon}$ depend on the slow variables (t, y), so that the method of Fourier multipliers does not apply directly. However, there are known substitutes for it: this is the role of pseudodifferential calculus to extend the constant coefficients calculus of Fourier multipliers to the variable coefficients case. In short, after the analysis of Chapter seven provides us with symbols $S(t, y, x, \zeta)$ associated to planar layers obtained by freezing the slowly varying coefficients of $\mathcal{P}_{u_a^\varepsilon}$. By the pseudodifferential calulus, operators are associated to these symbols. More precisely, since the coefficients have a finite smoothness, we use the paradifferential calculus of J.M.Bony ([Bon])(see also [Mey], [Hör], [Tay] and [Mé1] [Mok] [Mé2] for the calculus with parameter γ). Indeed, because of the parabolic nature of the equation in the high frequency regime, we need extensions of the classical culculus. All the details are given in the Appendix B of [MZ1]. The main idea, is that the properties (7.2.2), (7.2.3) (7.2.4) of the symmetrizers as operators in L^2 follow from the similar properties of the symbols as matrices. We refer to [MZ1] for the details.

9.2.2 Conormal stability

The next step is to prove estimates for the derivatives of the solution u. As explained in Chapter three, one cannot expect uniform estimates in usual anisotropic Sobolev spaces. Instead, as in Chapter three, we prove estimates in the spaces with conormal regularity. Such spaces have already been widely used in the study of characteristic hyperbolic boundary value problems, see e.g. [Rau2], [Gu2]. Introduce again the vector fields $\{Z_k; k = 0, \dots, d\}$ as in (3.4.9) and the spaces $H^s_{co}([-T,T] \times \mathbb{R}^d_+)$ as in Definition 3.4.4:

$$
H^s_{co} := \big\{ u \in L^2 \ : Z_{k_1} \dots Z_{k_p} u \in L^2 \, , \quad \forall p \le m \, , \forall (k_1, \dots, k_p) \in \{0, \dots \underline{d}\}^p \, \big\} \tag{9.2.13}
$$

In order to solve nonlinear problems, we need work in Banach algebras which means here that we have to supplement the H^s_{co} estimates with L^∞ estimates. Introduce the following norms

$$
\|u\|_{W^\mu_{co}(U)} = \|u\|_{L^\infty} + \sum_{p=1}^{\mu} \sum_{0 \le k_1, \dots, k_p \le d} \|Z_{k_1} \dots Z_{k_p} u\|_{L^\infty} \, . \tag{9.2.14}
$$

Reinforcing (9.2.2) and (9.2.5), we now assume that m is a positive integer and that u_0 and v^ε satisfy on $[-T,T] \times \Omega$,

$$
\begin{cases} u_0 \in W^{m+2,\infty} \, , \qquad u_{0|x=0} \in \mathcal{K} \, , \\ \sup\limits_{\varepsilon \in]0,1]} \|v^\varepsilon\|_{W^m_{co}} + \varepsilon \|\nabla_{t,x} v^\varepsilon\|_{W^m_{co}} + \varepsilon^2 \|\nabla^2_x v^\varepsilon\|_{W^m_{co}} < \infty \, . \end{cases} \tag{9.2.15}
$$

Theorem 9.2.3. *There are $C > 0$ and ε_0 such that all $\varepsilon \in]0, \varepsilon_0]$ and all $f \in H^m_{co}([-T,T] \times \mathbb{R}^d_+)$ vanishing for $t < 0$, the solution of equation* (9.2.7) *which vanishes for $t < 0$, belongs to $H^m_{co}([-T,T] \times \mathbb{R}^d_+)$ and satisfies*

$$
\|u\|_{H^m_{co}} + \sqrt{\varepsilon} \|\partial_x u\|_{H^m_{co}} + \varepsilon^{3/2} \|\partial^2_x u\|_{H^m_{co}} \le C \|f\|_{H^m_{co}} \tag{9.2.16}
$$

If in addition $m \ge 2 + \frac{d+1}{2}$ and $f \in L^\infty([-T_0, T_0] \times \Omega)$, then the solution u also satisfies

$$
\|u\|_{W^2_{co}} + \varepsilon \|\partial_x u\|_{W^1_{co}} + \varepsilon^2 \|\partial^2_x u\|_{L^\infty} \le C \big(\|f\|_{\mathcal{H}^m} + \varepsilon \|f\|_{L^\infty} \big) \, . \tag{9.2.17}
$$

These estimates are parallel to the estimates (3.4.14) of Proposition 3.4.5. Usually, one derives the Sobolev estimates by tangential differentiation of the equations. This was used in Chapter three, as well as in Chapter two. Here, this procedure leads to difficulties: due to the fact that there is an ε^{-1} term in the equation, and the commutator of $[\mathcal{P}_{u^\varepsilon_a}, Z^\alpha]$ leads to extra singular terms which are not controlled. Instead, in [MZ1] we use again the symmetrizer technics, and prove directly the H^m_{co} estimates.

Knowing the H^m_{co} regularity, the L^∞ estimates follow as in Proposition 3.4.2.

9.3 Nonlinear stability

These results can be used to solve the nonlinear equations (9.1.2). In order to avoid technical discussions on compatibility conditions for the Cauchy data and the boundary conditions, we consider here the simple case where the Cauchy data for (9.1.1) and (9.1.2) are zero, but with a non trivial forcing term, see [Gr-Gu]. More precisely, we consider $F(u)$ such that $F(0) = 0$ and a source term f which vanishes in the past. Consider indices m and s_0 such that

$$m > \frac{d+1}{2}, \qquad s_0 > m + 3\frac{d+1}{2}. \tag{9.3.1}$$

Let

$$f \in H^{s_0}([-T_0, T_0] \times \mathbb{R}^d_+) \quad \text{with} \quad f_{|t<0} = 0. \tag{9.3.2}$$

Assume that the state $u = 0$ belongs the domain of hyperbolicity $\mathcal{U}$. Since the hyperbolic problem satisfies the uniform Lopatinski condition for states $u \in \mathcal{U}$, there is $T \in]0, T_0]$ such that (9.1.1) (9.1.5) has a solution u_0 in $H^{s_0}([-T_0, T] \times \mathbb{R}^d_+)$ which vanishes for $t < 0$. In this case, u_0^ε given by (9.2.3) vanishes for $t < 0$ and is an exact solution of (9.1.2) there. We show that this solution can be continued to $[0, T] \times \mathbb{R}^d_+$ and that u_0^ε is a good approximation.

Theorem 9.3.1. *There is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ the problem (9.1.2)(9.1.3) has a unique solution $u^\varepsilon \in H^m_{co} \cap L^\infty([-T, T] \times \mathbb{R}^d_+)$ which vanishes for $t < 0$. Moreover,*

$$\|u^\varepsilon - u_0^\varepsilon\|_{H^m_{co}} + \|u - u_0^\varepsilon\|_{L^\infty} = O(\varepsilon). \tag{9.3.3}$$

As in [Gr-Gu], one can construct BKW solutions, thanks to Theorem 6.4.1 which implies that both the hyperbolic equations and the inner layer o.d.e. are well posed. To prove the theorem, it is sufficient to construct a first corrector u_1^ε such that $u_1^\varepsilon = 0$ for $t < 0$, $u_1^\varepsilon = 0$ on $\{x = 0\}$ and $u_a^\varepsilon = u_0^\varepsilon + \varepsilon u_1^\varepsilon$ satisfies equation (9.1.2) up to an error $e = O(\varepsilon)$. Indeed, when one substitutes u_0^ε in (9.1.2), the $O(\varepsilon^{-1})$ term is killed by the choice (9.2.3) and because W satisfies (5.2.2) when the boundary condition is satisfied. The interior term vanishes since u_0 is a solution of the hyperbolic equation. However, it remains an $O(e^{-\delta x/\varepsilon})$ term (see the similar computations of Chapter four). The corrector u_1^ε, given by a formula analogous to (9.2.3), can be chosen to cancel this term (see the general discussion of BKW solutions in

[Gr-Gu] and Chapter four). Then the solution u^ε is constructed as $u_a^\varepsilon + \varepsilon v^\varepsilon$, where v^ε solves

$$
\mathcal{P}_{u_a^\varepsilon} v^\varepsilon + \varepsilon \mathcal{Q}(v^\varepsilon) = f := \varepsilon^{-1} e \tag{9.3.4}
$$

and $\mathcal{Q}$ is at least quadratic in v. Denoting by $\| \cdot \|_{\mathcal{X}_\varepsilon^m}$ [resp. $\| \cdot \|_{\mathcal{Y}_\varepsilon^m}$] the norm given by adding the left [resp right] hand sides of (9.2.16) and (9.2.17) one proves that

$$
\begin{aligned}
&\| \varepsilon \mathcal{Q}(v^\varepsilon) \|_{\mathcal{Y}_\varepsilon^m} \le \varepsilon^{1/4}\, C(M)\,, \\
&\| \varepsilon (\mathcal{Q}(v_1^\varepsilon) - \mathcal{Q}(v_2^\varepsilon)) \|_{\mathcal{Y}_\varepsilon^m} \le \varepsilon^{1/4} C(M)\, \| v_1 - v_2 \|_{\mathcal{X}_\varepsilon^m}\,,
\end{aligned} \tag{9.3.5}
$$

provided that

$$
\begin{aligned}
&\varepsilon \| v_1 \|_{L^\infty} \le 1\,, \qquad &&\varepsilon \| v_1 \|_{L^\infty} \le 1 \\
&\varepsilon \| v_1 \|_{\mathcal{X}_\varepsilon^m} \le M\,, \qquad &&\varepsilon \| v_1 \|_{\mathcal{X}_\varepsilon^m} \le M\,.
\end{aligned} \tag{9.3.6}
$$

Together with Theorem 9.2.3, this shows that the equation (9.3.4) can be solved in $\mathcal{X}^m$, provided that ε is small enough. We refer to [MZ1] for a detailed proof.

As a conclusion, we note that the results in Theorem 9.2.3 and 9.3.1 are not quite satisfactory. Because (9.1.2) is parabolic, one should expect the solutions to be smoother than the solutions of (9.1.1). Here we get a result going the wrong way. We start from a very smooth solution u_0 of (9.1.1) and we end up with less smooth solutions of (9.1.2). This is clearly related to the method of proof, and a direct proof of existence with uniform estimates for (9.1.2), without using the solution u_0 of (9.1.1) would be very interesting. In any case, the stability analysis in Theorem 9.2.2 is a key point

Bibliography

[Agm] S. Agmon, *Problèmes mixtes pour les équations hyperboliques d'ordre supérieur, Les Équations aux Dérivées Partielles.* Éditions du Centre National de la Recherche Scientifique, Paris (1962) 13–18.

[AGJ] J. Alexander-R. Gardner-C.K.R.T. Jones, *A topological invariant arising in the analysis of traveling waves.* J. Reine Angew. Math. 410 (1990) 167–212.

[AMPZ] A. Azevedo-D. Marchesin-B. Plohr-K. Zumbrun, *Nonuniqueness of solutions of Riemann problems.* Z. Angew. Math. Phys. 47 (1996), 977–998.

[BBB] C.Bardos-D.Brezis-H.Brezis, *Perturbations singulières et prolongement maximaux d'opérateurs positifs*, Arch.Rational Mech. Anal., 53 (1973), 69–100.

[Ba-Ra] C.Bardos-J.Rauch *Maximal positive boundary value problems as limits of singular perturbation problems*, Trans. Amer.Math.Soc., 270 (1982), 377–408.

[Bi-Br] S.Bianchini-A.Bressan, *Vanishing viscosity limit solutions to nonlinear hyperbolic systems*, preprint SISSA, 2001

[Bon] J.M.Bony, . *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann.Sc.E.N.S. Paris, 14 (1981) pp 209-246.

[Ch-Pi] J. Chazarain-A. Piriou, *Introduction to the theory of linear partial differential equations,* Translated from the French. Studies in Mathematics and its Applications, 14. North-Holland Publishing Co., Amsterdam-New York, 1982. xiv+559 pp. ISBN: 0-444-86452-0.

[Co-Me] R.Coifmann-Y.Meyer. *Au delà des opérateurs pseudo-différentiels*, Astérisque 57, (1978).

[DiP] R.DiPerna, *Convergence of approximate solutions to conservation laws,* Arch.Rat.Mech. Anal., 82 (1983), 27-70.

[Ev1] J.W. Evans, *Nerve axon equations: I. Linear approximations.* Ind. Univ. Math. J. 21 (1972) 877–885.

[Ev2] J.W. Evans, *Nerve axon equations: II. Stability at rest.* Ind. Univ. Math. J. 22 (1972) 75–90.

[Ev3] J.W. Evans, *Nerve axon equations: III. Stability of the nerve impulse.* Ind. Univ. Math. J. 22 (1972) 577–593.

[Ev4] J.W. Evans, *Nerve axon equations: IV. The stable and the unstable impulse.* Ind. Univ. Math. J. 24 (1975) 1169–1190.

[Fr1] K.0.Friedrichs, *Symmetric hyperbolic linear differential equations,* Comm. Pure Appl.Math., 7 (1954), pp 345-392.

[Fr2] K.0.Friedrichs, *Symmetric positive linear differential equations,* Comm. Pure Appl.Math., 11 (1958), pp 333-418.

[Fr-La] K.0.Friedrichs, P.Lax *Boundary value problems for first order operators,* Comm. Pure Appl.Math., 18 (1965), pp 355-388.

[Ga-Zu] R. Gardner-K. Zumbrun, *The Gap Lemma and geometric criteria for instability of viscous shock profiles.* Comm. Pure Appl. Math. 51 (1998), 797–855.

[Gi-Se] M. Gisclon-D. Serre, *Conditions aux limites pour un système strictement hyperbolique fournies par le schéma de Godunov.* RAIRO Modél. Math. Anal. Numér. 31 (1997), 359–380.

[Goo] J. Goodman, *Stability of viscous scalar shock fronts in several dimensions*, Trans. Amer. Math. Soc., 311. 1989, 683-695.

[Go-Xi] J.Goodman, and Z.Xin, *Viscous limits for piecewise smooth solutions to systems of conservation laws*, Arch. Rational Mech. Analysis 121. 1992, 235-265.

[Gr1] E.Grenier, *On the nonlinear instability of Euler and Prandtl equations,* Comm. Pure Appl.Math., 53 (2000), 1067-1091.

[Gr2] E.Grenier, *On the stability of boundary layers of incompressible Euler equations,* J.Diff.Equ., 164 (2000), 180-222.

[Gr-Gu] E. Grenier-O. Gues, *Boundary layers for viscous perturbations of noncharacteristic quasilinear hyperbolic problems*. J. Differential Equations 143 (1998), 110–146.

[Gr-Ro] E. Grenier-F. Rousset, *Stability of one-dimensional boundary layers by using Green's functions*. Comm. Pure Appl. Math. 54 (2001), 1343–1385.

[Gu1] O.Guès, *Perturbations visqueuses de problèmes mixtes hyperboliques et couches limites*, Ann.Inst.Fourier, 45 (1995),973–1006.

[Gu2] O.Guès, *Problème mixte hyperbolique quasilinéaire caractéristique*, Comm. in Part.Diff.Equ., 15 (1990), 595–645.

[GMWZ1] O.Gues, G.Metivier, M.Williams and K.Zumbrun, *Multidimensional viscous shocks I: degenerate symmetrizers and long time stability*, submitted, 2002, available at http://www.math.unc.edu/Faculty/williams/

[GMWZ2] O.Gues, G.Metivier, M.Williams and K.Zumbrun, *Multidimensional viscous shocks II: the small viscosity limit*, preprint 2002, available at http://www.math.unc.edu/Faculty/williams/

[Gu-Wi] O.Gues, M. Williams, *Curved shocks as viscous limits: a boundary problem approach*, Indiana Univ. Math.J., 51 (2002), 421-450.

[Ho-Zu] D. Hoff-K. Zumbrun, *Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow*. Indiana Univ. Math. J. 44 (1995), 603–676.

[Hör] L.Hörmander. *Lectures on Nonlinear Hyprbolic Differential Equations*, Mathématiques et Applications 26, Sringer Verlag, 1997.

[Jon] C.K.R.T. Jones, *Stability of the travelling wave solution of the FitzHugh–Nagumo system*. Trans. Amer. Math. Soc. 286 (1984), 431–469.

[Kap] T. Kapitula, *On the stability of travelling waves in weighted L^∞ spaces*. J. Diff. Eqs. 112 (1994), 179–215.

[Kre] H.O. Kreiss, *Initial boundary value problems for hyperbolic systems,* Comm. Pure Appl. Math. 23 (1970) 277-298.

[Kru] S.Kruzhkov, *First order quasilinear equations with several space variables,* Math.USSR Sbornik 10 (1970), 217-243.

[Lio] J.L.Lions, *Perturbations singulières dans les problèmes aux limites et en contrôle optimal*, Lectures Notes in Math., 323, Sringer Verlag, 1973.

[Maj] A. Majda, *The stability of multi-dimensional shock fronts – a new problem for linear hyperbolic equations.* Mem. Amer. Math. Soc. 275 (1983).

[Ma-Os] A. Majda, S.Osher, *Initial-Boundary Value Problem for Hyperbolic Equations with Uniformly Characteristic Boundary,* Comm. Pure Appl. Math., 23 (1975) 607-675.

[Mé1] G. Métivier, *Interaction de deux chocs pour un système de deux lois de conservation, en dimension deux d'espace.* Trans. Amer. Math. Soc. 296 (1986) 431–479.

[Mé2] G. Métivier, *Stability of multidimensional shocks.* Advances in the theory of shock waves, 25–103, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.

[Mé3] G.Métivier. *The Block Structure Condition for Symmetric Hyperbolic Problems*, Bull. London Math.Soc., 32 (2000), 689–702

[MZ1] Metivier, G. and Zumbrun, K., *Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems*, preprint, April 2002, available at http://www.math.u-bordeaux1.fr/ metivier/preprints.html

[MZ2] G.Métivier-K.Zumbrun, *Symmetrizers and Continuity of Stable Subspaces for Parabolic-Hyperbolic Boundary Value Problems*, preprint

[Mey] Y.Meyer,. *Remarques sur un théorème de J.M.Bony*, Supplemento al Rendiconti der Circolo Matematico di Palermo, Serie II, No1, 1981.

[Mok] A.Mokrane *Problèmes mixtes hyperboliques non linéaires*, Thesis, Université de Rennes 1, 1987.

[Ole] O.Oleinik *Discontinuous solutions of nonlinear differential equations,* Amer. Math.Soc Translations, 26 (1957), 95-172.

[Pe-We] R. L. Pego-M.I. Weinstein, *Eigenvalues, and instabilities of solitary waves.* Philos. Trans. Roy. Soc. London Ser. A 340 (1992), 47–94.

[Ral] J.V. Ralston, *Note on a paper of Kreiss,* Comm. Pure Appl. Math. 24 (1971) 759–762.

[Rau1] J.Rauch, *BV estimates fail for most quasilinear hyperbolic systems in dimensions greater than one*, Comm. Math. Phys., 106 (1986), 481–484.

[Rau2] J.Rauch, *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Amer.Math.Soc, 291 (1985), 167–185.

[Ra-Ma] J. Rauch-F. Massey, *Differentiability of solutions to hyperbolic intitial boundary value problems.* Trans. Amer. Math. Soc. 189 (1974) 303-318.

[Ro1] F. Rousset, *Inviscid boundary conditions and stability of viscous boundary layers.* Asymptot. Anal. 26 (2001), no. 3-4, 285–306.

[Ro2] F.Rousset, *Viscous limit for strong shocks of system of conservation laws,*

[Ser] D. Serre, *Sur la stabilité des couches limites de viscosité.* Ann. Inst. Fourier (Grenoble) 51 (2001), 109–130.

[St] E.Stein,. *Singular Integrals and Differentiability Propseties of Functions*, Princeton Univ. Press, Princeton, 1970.

[Tar] D.Tartakoff, *Regularity of solutions to boundary value problems for first order systems,* Indiana Univ.Math.J., 21 (1972), pp 1113-1129.

[Tay] M.Taylor. *Partial Differential Equations*III, Applied Mathematical Sciences 117, Springer, 1996.

[ZS] K. Zumbrun-D.Serre, *Viscous and inviscid stability of multidimensional planar shock fronts.* Indiana Univ. Math. J. 48 (1999), 937–992.

[ZH] K. Zumbrun-P. Howard, *Pointwise semigroup methods and stability of viscous shock waves.* Indiana Mathematics Journal V47 (1998), 741–871.

[Zum] K. Zumbrun, *Multidimensional stability of planar viscous shock waves*. Advances in the theory of shock waves, 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.