# Chapter 5

# Smooth solutions of the nonlinear Cauchy problem

## 5.1 The results

We consider a first order $N \times N$ quasi-linear system

$$\begin{cases} \partial_t u + \sum_{j=1}^{d} A_j(t, x, u)\partial_j u = F(t, x, u), \\ u_{|t=0} = h. \end{cases} \tag{5.1.1}$$

We say that a function $a(t, x, u)$ belongs to $C_b^\infty[0, T] \times \mathbb{R}^d \times \mathbb{R}^N$ if it is infinitely differentiable on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^N$ and its derivatives at all order are bounded on the sets $[0, T] \times \mathbb{R}^d \times \{|u| \leq R\}$ for all $R$.

**Assumption 5.1.1.** *The matrices $A_j$ and the function $F$ belong to $C_b^\infty[0, T] \times \mathbb{R}^d \times \mathbb{R}^N$*

*Moreover, there is an invertible $N \times N$ matrix $S(t, x, u)$, such that $S$ and $S^{-1}$ belong to $C_b^\infty[0, T] \times \mathbb{R}^d \times \mathbb{R}^N$ and*

*i) $S(t, xu)$ is self adjoint, definite positive ;*

*ii) For all $(t, x, u)$ and all $j$, $S(t, x, u)A(t, x, \xi)$ is self-adjoint.*

We consider a Sobolev index $s > \frac{d}{2} + 1$ which is fixed throughout this section.

**Theorem 5.1.2.** *Suppose that $f = F(t, x, 0) \in C^0([0, T]; H^s(\mathbb{R}^d)$ and $h \in H^s(\mathbb{R}^d)$, there is $T' \in ]0, T]$ and a unique solution $u \in C^0([0, T']; H^s(\mathbb{R}^d))$ of the Cauchy problem* (5.1.1)*.*

An estimate from below of $T'$ is given in the proof of the theorem.

Uniqueness allows to define the maximal time of existence :

*$T^*$ is the supremum of $T' \in [0, T]$ such that the Cauchy problem has a solution $u \in C^0([0, T']; H^s(\mathbb{R}^d))$.*

The theorem implies that $T^* > 0$. By uniqueness, the solution $u$ is therefore defined for all $t < T^*$ and $u \in C^0([0, T^*[; H^s(\mathbb{R}^d))$.

**Theorem 5.1.3.** *Either $T^* = T$ or*

$$\limsup_{t \to T^*} \|u\|_{L^\infty([0,t] \times \mathbb{R}^d)} + \|\nabla_{t,x} u\|_{L^\infty([0,t] \times \mathbb{R}^d)} = +\infty. \tag{5.1.2}$$

In theses notes, the strategy is the following. We consider the regularized equations

$$\partial_t u + \sum J_\varepsilon A_j(t, x, u) \partial_{x_j} J_\varepsilon u = F(t, x, u), \quad u_{|t=0} = J_\varepsilon h \tag{5.1.3}$$

where $J_\varepsilon$ is a Friedrichs mollifier:

$$J_\varepsilon v = \jmath_\varepsilon \star v, \quad \jmath_\varepsilon(x) = \varepsilon^{-d} \jmath(x/\varepsilon) \tag{5.1.4}$$

where $\jmath \geq 0$ is smooth with compact support, of integral 1, and even so that the operator $J_\varepsilon$ is self adjoint in $L^2$:

$$\big(J_\varepsilon u, v\big)_{L^2(\mathbb{R}^d)} = \big(u, J_\varepsilon v\big)_{L^2(\mathbb{R}^d)}. \tag{5.1.5}$$

**Step 1.** Existence of solutions for the approximate equation. For all fixed $\varepsilon$, we consider (5.1.3) a nonlinear ode in $H^s$:

$$\partial_t u = \mathcal{F}_\varepsilon(u) \tag{5.1.6}$$

We will show that the smoothing properties of $J_\varepsilon$ imply

**Lemma 5.1.4.** *For all fixed $\varepsilon > 0$ the application $\mathcal{F}_\varepsilon$ is locally Lipschiztean from $H^s$ to $H^s$.*

Therefore, the Cauchy-Lipschitz theorem implies that for all $\varepsilon > 0$, there is $T_\varepsilon \in ]0, T]$ such that (5.1.3) has a unique solution $u^\varepsilon \in C^0([0, T_\varepsilon]; H^s(\mathbb{R}^d))$. One can introduce the maximal time of existence :

*$T^*_\varepsilon$ is the supremum of $T' \in [0, T]$ such that (5.1.3) has a solution $u \in C^0([0, T']; H^s(\mathbb{R}^d))$.*

Thus $T^*_\varepsilon > 0$ and by uniqueness, the solution $u - \varepsilon$ is therefore defined for all $t < T^*_\varepsilon$ and $u \in C^0([0, T^*_\varepsilon[; H^s(\mathbb{R}^d))$.

**Step 2.** Uniform estimates.

**Proposition 5.1.5.** *There are constants $C_0$, $\gamma$ and $T' \in ]0,T]$, such that for all $\varepsilon > 0$, and all $t \leq \min(T', T^*_\varepsilon)$*

$$\big\|u^\varepsilon(t)\big\|_{H^s(t)} \leq C_0 e^{\gamma t}\|h\|_{H^s} + C_0 \int_0^t e^{\gamma(t-t')}\|f(t')\|_{H^s} dt'. \tag{5.1.7}$$

**Step 3.** Passing to the limit $\varepsilon \to 0$. We will show that the estimates (5.1.7) imply that $T^*_\varepsilon > T'$ and that $u^\varepsilon$ converges to a solution $u$ of (5.1.1), which satisfies the estimates (5.1.7).

**Step 4.** The blow up theorem follows from the estimate (5.1.5) and the remark that $C_0$ and $\gamma$ only depend on the $W^{1,\infty}$ norm of $u$.

## 5.2 Nonlinear estimates

**Proposition 5.2.1.** *Sobolev embedding $H^s \subset L^\infty$ if $s > d/2$.*

**Theorem 5.2.2** (Gagliardo-Niremberg estimates)**.** *For $\frac{|\alpha|}{s} \leq \frac{2}{p} \leq 1$,*

$$\|\partial^\alpha u\|_{L^p} \leq C\|u\|_{L^\infty}^{1-2/p}\|u\|_{H^s}^{1/2p} \tag{5.2.1}$$

*Proof.* Use the identity

$$0 = \int \partial_x(u|\partial_x u|^{p-2}\partial_x u)dx = \int |\partial_x u|^p dx + (p-1)\int u|\partial_x u|^{p-2}\partial_x^2 u dx$$

Thus

$$\|\partial_x u\|_{L^p}^p \leq (p-1)\|u\|_{L^q}\||\partial_x u|^{p-2}\|_{L^s}\|\partial_x^2 u\|_{L^r}$$

if $1/q + 1/s + 1/r = 1$. We chose $s$ such that $s(p-2) = p$ so that

$$\||\partial_x u|^{p-2}\|_{L^s} = \||\partial_x u\|_{L^p}^{p/s} = \||\partial_x u\|_{L^p}^{p-2}$$

and after simplification,

$$\frac{1}{q} + \frac{1}{r} = \frac{2}{p} \quad \Rightarrow \quad \|\partial_x u\|_{L^p}^2 \leq (p-1)\|u\|_{L^q}\|\partial_x^2 u\|_{L^r}.$$

From this, we prove is by induction on $l$ that for

$$k \leq j \leq l, \qquad \frac{l}{p} = \frac{k}{r} + \frac{l-k}{r}$$

$$(5.2.2) \qquad \|\nabla^k u\|_{L^p} \le C\|u\|_{L^q}^{1-k/l}\|\nabla^l u\|_{L^r}^{k/l}$$

where

$$\|\nabla^j u\|_{L^p} = \sum_{|\alpha=j} \|\partial^\alpha u\|_{L^p}$$

First, we note that the estimate is true when $k = l$ and when $k = 0$.

Suppose that the estimate is proved up to $l$. We prove it at the order $l+1$. As already said it is true for $k = 0$ and $k = l+1$. We proceed by induction on $k$, for $1 \le k \le l$.

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**Corollary 5.2.3.**

$$(5.2.3) \qquad \|\partial^\alpha a\, \partial^\beta u\|_{L^2} \le C\big(\|a\|_{L^\infty}\|u\|_{H^s} + \|a\|_{H^s}\|u\|_{L^\infty}\big).$$

**Corollary 5.2.4.** *$H^s$ is an algebra if $s > d/2$.*

**Proposition 5.2.5.** *Let $F$ be a $C^\infty$ function such that $F(0) = 0$. For all $s$, there is a continuous function $\phi$ on $[0, +\infty[$ such that for all $u \in H^s \cap L^\infty$, $F(u) \in H^s$ and*

$$(5.2.4) \qquad \big\|F(u)\big\|_{H^s} \le \phi\big(\|u\|_{L^\infty}\big)\ \|u\|_{H^s}.$$

*Proof.* To estimate the $L^2$ norm of $F(u)$ we use the condition $F(0) = 0$ to write $F(u) = uG(u)$, so that

$$\|F(u)\|_{L^2} \le \phi\big(\|u\|_{L^\infty}\big)\ \|u\|_{L^2}.$$

with

$$\phi(r) = \sup_{|u|\le r} |G(u)|.$$

The derivative $\partial^\alpha F(u)$ is a linear combination of terms of the form

$$F^{(k)}(u)\partial^{\alpha_1}u \ldots \partial^{\alpha_k}u$$

with $\alpha_1 + \ldots + \alpha_k = \alpha$. We estimate the $L^\infty$ norm of $F^{(k)}(u)$ by a function of the $L^\infty$ norm of $u$ To estimate the $L^2$ norm of the product we note that, since $\sum |\alpha_j|/s \le 1$, we can choose exponents $p_j$ such that

$$\frac{|\alpha_j|}{s} \le \frac{2}{p_j}, \qquad \sum \frac{2}{p_j} = 1.$$

Then, by the Gagliardo Nirenberg estimates,

$$\|\partial^{\alpha_1}u \ldots \partial^{\alpha_k}u\|_{L^2} \le \prod \|\partial^{\alpha_j}u\|_{L^{p_j}} \le C\|u\|_{L^\infty}^{k-1}\|u\|_{H^s}$$

and the proposition follows. □

We end this section with a commutator estimate.

**Proposition 5.2.6.** *Suppose that $a \in C^\infty(\mathbb{R})$. For all $s$, there is a continuous function $C$ such that for all $u$ and $v$ in $W^{1,\infty} \cap H^s$, one for all $|\alpha| \leq s$: has*

$$\begin{aligned}\left\| \partial^\alpha (a(v)\partial_{x_j} u) - a(v)\partial^\alpha \partial_{x_j} u \right\|_{L^2} &\leq C(\|v\|_{L^\infty}) \\ \Big( (\|v\|_{L^\infty} + \|\nabla v\|_{L^\infty})\|u\|_{H^s} &+ (\|u\|_{L^\infty} + \|\nabla_x u\|_{L^\infty})\|v\|_{H^s} \Big).\end{aligned} \tag{5.2.5}$$

The proof is based on the following lemma

**Lemma 5.2.7.** *For $1 \leq |\alpha| \leq s-1$, $\rho \geq 0$ and $p \geq 2$ such that $\frac{|\alpha|-1}{s-1} \leq \frac{|\alpha|-\rho}{s} \leq \frac{2}{p} \leq \frac{2}{p} + \rho \leq 1$,*

$$\|\partial^\alpha u\|_{L^p} \leq C \|u\|_{L^\infty}^{1-2/p-\rho} \|\nabla u\|_{L^\infty}^{\rho} \|u\|_{H^s}^{1/2p}. \tag{5.2.6}$$

Note that the condition on the indices implies that

$$\rho \leq \frac{s-|\alpha|}{s-1} \leq 1. \tag{5.2.7}$$

*Proof.* The Gagliardo-Nirenberg estimates imply that

$$\begin{aligned}\|\partial^\alpha u\|_{L^q} &\leq C\|u\|_{L^\infty}^{1-2/q} \|u\|_{H^s}^{1/2q}, \quad \frac{2}{q} = \frac{|\alpha|}{s}, \\ \|\partial^\alpha u\|_{L^r} &\leq C\|\nabla u\|_{L^\infty}^{1-2/r} \|u\|_{H^s}^{1/2r}, \quad \frac{2}{r} = \frac{|\alpha|-1}{s-1}.\end{aligned}$$

By Hölder inequality, for non negative $\delta \in [0,1]$, one has

$$\|\partial^\alpha u\|_{L^p} \leq \|\partial^\alpha u\|_{L^q}^{1-\delta-\theta} \|\partial^\alpha u\|_{L^r}^{\delta}$$

with

$$\frac{1}{p} = \frac{1-\delta}{q} + \frac{\delta}{r} = \frac{|\alpha|}{2s} - \delta \frac{s-|\alpha|}{2s(s-1)},$$

implying (5.2.6) with

$$\rho = (1-2/r)\delta = \delta \frac{s-|\alpha|}{s-1}.$$

This proves the estimate when

$$\frac{|\alpha|-1}{s-1} \leq \frac{2}{p} = \frac{|\alpha|-\rho}{s} \leq \frac{|\alpha|}{s}, \tag{5.2.8}$$

since then $0 \leq \rho \leq (s-|\alpha|)/(s-1)$ and there is $\delta \in [0,1]$ such that $\rho = \delta(s-|\alpha|)/(s-1)$.

Moreover, the estimate follows immediately from the Gagliardo-Nirenberg estimates when

$$\frac{|\alpha|-1}{s-1} \leq \frac{2}{p}, \qquad \rho = 1 - \frac{2}{p} \tag{5.2.9}$$

$$\frac{|\alpha|}{s} \leq \frac{2}{p}, \qquad \rho = 0. \tag{5.2.10}$$

When $2/p \geq |\alpha|/s$, the estimate is proved for $\rho = 0$ and $\rho = 1-2/p$, and therefore holds for $\rho \in [0, 1-2/p]$. When $(|\alpha|-1)/(s-1) \leq 2/p \leq |\alpha|/s$, the estimate is proved for $\rho = 2s/p - |\alpha|$ and $\rho = 1-2/p$ and therefore holds for $\rho$ in the interval limited by these two values. □

*Proof of Proposition* 5.2.6. The term to estimate is a linear combination of terms of the form

$$a^{(k)}(v)\partial^{\beta_1} v \ldots \partial^{\beta_k} v \, \partial^{\beta_0} u$$

with $|\beta_1| + \ldots + |\beta_k| + |\beta_0| = |\alpha| + 1$ and all the $|\beta_j| \geq 1$.

The case $|\alpha| \leq s-1$ has already been treated in the proof of Proposition 5.2.5 and requires no $L^\infty$ estimates of the gradients.

Consider now the case where $\sum |\beta_j| = s+1$. We estimate the $L^\infty$ norm of $a^{(k)}(v)$ by a function of the $L^\infty$ norm of $v$ and it remains to estimate the $L^2$ norm of the product of the derivatives. Because the number of terms $k+1$ is at least 2, the sum $\sum (s-|\beta_j|)/(s-1)$ is larger than or equal to 1 and therefore, there are real numbers $\rho_j$ such that

$$0 \leq \rho_j \leq \frac{s-|\beta_j|}{s-1}, \qquad \sum \rho_j = 1.$$

Choosing $\frac{2}{p_j} = \frac{|\beta_j| - \rho_j}{s}$ we can use the estimates (5.2.6) and since $\sum \frac{2}{p_j} = 1$, we obtain

$$\begin{aligned} \left\| \partial^{\beta_1} v \ldots \partial^{\beta_k} v \, \partial^{\beta_0} u \right\|_{L^2} \leq C \|v\|_{L^\infty}^{k-2+\rho_0+2/p_0} \|\nabla v\|_{L^\infty}^{1-\rho_0} \|v\|_{H^s}^{1-2/p_0} \\ \|u\|_{L^\infty}^{1-\rho_0-2/p_0} \|\nabla u\|_{L^\infty}^{\rho_0} \|u\|_{H^s}^{2/p_0} \end{aligned}$$

and the proposition follows. □

## 5.3 The main estimate

In this section we assume that $u \in C^0([0,T'],H^s)$ is a solution of (5.1.3). Because $s > 1 + d/2$, the following quantities are finite:

$$R = \|u\|_{L^\infty([0,T']\times\mathbb{R}^d)}, \quad M = \|\nabla_x u\|_{L^\infty([0,T']\times\mathbb{R}^d)} \tag{5.3.1}$$

Below we denote by $C(\cdot)$ a continuous function on $\mathbb{R}_+$, which may vary from line to line.

**Lemma 5.3.1.** *There is a function $C_0(\cdot)$ such that*

$$\|\partial_t u\|_{L^\infty} \leq C_0(R)(1+M) \tag{5.3.2}$$

*Proof.* One has

$$\|F(u)\|_{L^\infty} \leq C(R)$$

and

$$\|J_\varepsilon(A_j J_\varepsilon \partial_j u)\|_{L^\infty} \leq C(R)\|\partial_j u\|_{L^\infty}$$

thus the estimate for $\partial_t u$ follows from the equation. $\square$

**Lemma 5.3.2.** *There are functions $C_0(\cdot)$ and $C_1(\cdot)$ such that for all $v \in C^0([0,T'];L^2)$ which satisfies*

$$g := \partial_t v + \sum J_\varepsilon(A_j(u)J_\varepsilon D_j v) \in L^2([0,T']\times\mathbb{R}^d), \tag{5.3.3}$$

*one has*

$$\|v(t)\|_{L^2} \leq C_0(R)e^{\gamma t}\|v(0)\|_{L^2} + \int_0^t C_0(R)e^{\gamma(t-t')}\|g(t')\|_{L^2}dt'. \tag{5.3.4}$$

*with $\gamma = C_1(R)M$.*

*Proof.* Multiply the equation by $S(u)$, so that

$$S(u)\partial_t v + \sum J_\varepsilon S(u)A_j(u)J_\varepsilon \partial_j v = S(u)g + \sum g_j := \tilde{g}.$$

where

$$j_j = [S(u),J_\varepsilon]A_j(u)J_\varepsilon\partial_j v.$$

Because $S(u)$ is Lipschtiz continuous, the commutator $[S(u),J_\varepsilon]$ is bounded from $L^2$ to $L^2$ with norm less than or equal to

$$\varepsilon\|\nabla S(u)\|_{L^\infty} \leq \varepsilon C(R)(1+M).$$

Thus

$$\|g_j(t)\|_{L^2} \le C(R)M\|\varepsilon J_\varepsilon \partial_j v(t)\|_{L^2}. \le C(R)M\|v(t)\|_{L^2}$$

Consider the energy

$$\mathcal{E}(t) = \big(Sv(t), v(t)\big)_{L^2}.$$

Using that the $J_\varepsilon$ are self adjoint and the matrices $SA_j$ are symmetric, we get that

$$\mathcal{E}(t) - \mathcal{E}(0) = 2\mathrm{Re}\,\big(\tilde{g}, v\big)_{L^2([0,t]\times\mathbb{R}^d)} + \big(Kv, v\big)_{L^2([0,t]\times\mathbb{R}^d)}$$

where

$$K = \partial_t S(u) + \sum J_\varepsilon \partial_j (SA_j) J_\varepsilon.$$

Using Lemma 5.3.1 we see that $\|K\|\|_{L^\infty} \le C(R)(1+M)$. Moreover, the positivity of $S$ and the bound of $S^{-1}$ imply that

$$|v|^2 \le C(R)(S(u)v, v) \le C^2(R)|v|^2$$

Hence we have

$$\|v(t)\|_{L^2} \le C(R)\|v(0)\|_{L^2}) + C(R)\int_0^t \|g(t')\|l\|v(t')\|dt' \\ + C(R)M\int_0^t \|v(t')\|^2 dt'.$$

We conclude by Gronwall's lemma. □

We now estimate the $H^s$ norm of $u$. Differentiate the equation (5.1.3) to find, for $|\alpha| \le s$:

(5.3.5) $$\partial_t \partial_x^\alpha u + \sum J_\varepsilon A_j J_\varepsilon \partial_x^\alpha u = \partial_x^\alpha F(u) + g_\alpha$$

where $g_\alpha$ is the commutator

$$g_\alpha = \sum_j J_\varepsilon [\partial_x^\alpha, A_j(u)] \partial_j J_\varepsilon u.$$

By Proposition 5.2.6

$$\|g_\alpha(t)\|_{L^2} \le C(R)M\|u(t)\|_{H^s}.$$

By Proposition 5.2.5 applied to $F(t,x,u) - F(t,x,0)$, we have

$$\|\partial_x^\alpha F(u(t))\|_{L^2} \le \|f(t)\|_{H^s} + C(R)\|u(t)\|_{H^s}$$

where $f(t,x) = F(t,x,0)$. Applying the lemma and summing in $\alpha$ we get that

$$\|u(t)\|_{H^s} \leq C_0 e^{\gamma t}\|u(0)\|_{H^s} + C_0(R)\int_0^t e^{\gamma(t-t')}\|f(t')\|_{H^s}dt' \\ + C_2(R)M\int_0^t e^{\gamma(t-t')}\|u(t')\|_{H^s}dt'.$$

Applying once more Gronwall's lemma, we have proved the following:

**Proposition 5.3.3.** *There are functions $C_1(\cdot)$ and $C_2(\cdot)$ such that if $u \in C^0([0,T'],H^s)$ is a solution of* (5.1.3)*, one has*

$$\|u(t)\|_{H^s} \leq C_1(R)e^{\gamma t}\|u(0)\|_{H^s} + \int_0^t C_1(R)e^{\gamma(t-t')}\|f(t')\|_{L^2}dt'. \tag{5.3.6}$$

*with $\gamma = C_2(R)(1+M)$, with $R$ and $M$ defined at* (5.3.1).

## 5.4 Solutions of the approximate equation

Let

$$\mathcal{F}_\varepsilon(t,u) = -\sum J_\varepsilon(A_j(t,x,u)\partial_j J_\varepsilon u) + F(t,x,u)$$

**Lemma 5.4.1.** *For all $\varepsilon > 0$, the mapping $u \mapsto \mathcal{F}_\varepsilon(u)$ is locally Lipschitzean from $H^s$ to $H^s$.*

*Proof.* Because $s > d/2$, $H^s$ is an algebra and is a consequence of the non-linear estimates that for all $R$, there is a constant $C$ such that $u$ and $v$ in $H^s(\mathbb{R}^d)$

$$\|u\|_{H^s} \leq R,\ \|v\|_{H^s} \leq R \quad \Rightarrow \quad \|\mathcal{F}_\varepsilon(t,u) - \mathcal{F}_\varepsilon(t,v)\|_{H^s} \leq C\varepsilon^{-1}\|u-v\|_{H^s}.$$

□

Thus by the Cauchy Lipschitz theorem, (5.1.3) has a solution $u \in C^0([0,T_\varepsilon],H^s)$ for some $T_\varepsilon > 0$ and the solution can be extended to a maximal interval $[0,T_\varepsilon^*[$ with either $T_\varepsilon = T$ or

$$\limsup_{t\to T_\varepsilon^*}\|u(t)\|_{H^s} = +\infty. \tag{5.4.1}$$

We now proceed to a choice of parameters. The functions $C_0$, $C_1$ and $C_2$ are those given at Lemma 5.3.1 and Proposition 5.3.3. We also introduce the Sobolev constant $C_S$ such that

$$\|u\|_{L^\infty} \leq C_S\|u\|_{H^{s-1}} \tag{5.4.2}$$

(recall that $s-1 > d/2$).

1. We fix $r >$ and set $\underline{R} = \|h\|_{L^\infty} + r$;

2. Let
$$\underline{C} = 1 + 2C_1(\underline{R})\|h\|_{H^s} + 2C_1(\underline{R}) \int_0^T \|f(t')\|_{H^s} dt';$$

3. Let $\underline{M} = C_S \underline{C}$;

4. We choose $T' \in ]0, T]$ such that
$$T' C_0(\underline{R})(1 + \underline{M}) \leq r; \qquad e^{T' C_2(\underline{R})(1+\underline{M})} \leq 2.$$

**Proposition 5.4.2.** *For all $\varepsilon > 0$, $T^*_\varepsilon > T'$ and for all $t \in [0, T']$*

$$\|u^\varepsilon(t)\|_{L^\infty} \leq \underline{R}, \quad \|\nabla_x u^\varepsilon(t)\|_{L^\infty} \leq \underline{M}, \quad \|u^\varepsilon(t)\|_{H^s} \leq \underline{C}. \tag{5.4.3}$$

*Proof.* At time $t = 0$, the estimates are satisfied with strict inequalities (remember that $C_1 \leq 1$). Thus, by continuity, they hold on a small interval $[0, T'_\varepsilon]$, $T'_\varepsilon > 0$.

Suppose that $T'' < \min(T', T^*_\varepsilon)$ is such that

$$\forall t \in [0, T''], \quad \|u^\varepsilon(t')\|_{H^s} \leq \underline{C}. \tag{5.4.4}$$

Then, by the Sobolev embedding, $M \leq \underline{M}$. With Lemma 5.3.1 we also have

$$R \leq \|h\|_{L^\infty} + T''\|\partial_T u\|_{L^\infty} \leq \underline{R}.$$

Therefore, Proposition 5.3.3 and the conditions on $T'$ imply that

$$\|u^\varepsilon(t')\|_{H^s} \leq 2C_1(\underline{R})\|h\|_{H^s} + 2C_1(\underline{R}) \int_0^T \|f(t')\|_{H^s} dt'$$

hence
$$\|u^\varepsilon(t')\|_{H^s} \leq \underline{C} - 1.$$

This implies that the blow up (5.4.1) cannot occur before $T'$. Hence $T^*_\varepsilon \geq T'$ and the bound (5.4.4) is valid on $[0, T']$. As shown, it implies the Lipschitz bound and the $L^\infty$ bound. $\square$

## 5.5 Proof of Theorem 5.1.2

We first prove the existence of solutions, passing to the limit in the equation.

**Proposition 5.5.1.** *There is a subsequence, still denoted by $u^\varepsilon$, which converges in $C^0([0,T'] \times \mathbb{R}^d])$ and the limit is a solution of* (5.1.1)*. Moreover, $u \in C^0([0,T']; H^s(\mathbb{R}^d)$, and $\partial_t u \in C^0([0,T]; H^{s-1}(\mathbb{R}^d)$.*

*Proof.* The $u^\varepsilon$ are bounded in $C^0([0,T]; H^s) \cap C^1([0,T']; H^{s-1})$. Thus there is a subsequence which converges in $C^0([0,T']; H^s_w)$, where $H^s_w$ is the space $H^s$ equipped with the weak topology. uniformly on compact subsets. Since $s > 1 + d/2$, this implies the convergence in $C^1$ on all compact subset of $[0,T'] \times \mathbb{R}^d$ and one can pass to the limit in the equation. Hence $u$ is a solution of (5.1.1)

To prove that $u$ belongs to $C^0([0,T']; H^s$ and not only to $C^0([0,T']; H^s_w)$ we differentiate the equation for $|\alpha| \le s$ and we get

$$\partial_t \partial_x^\alpha u + \sum A_j(u) \partial_{x_j} \partial_x^\alpha u = \partial_x^\alpha F(u) + g_\alpha \tag{5.5.1}$$

where $g_\alpha$ is the commutator

$$g_\alpha = \sum_j [\partial_x^\alpha, A_j(u)] \partial_j u.$$

Indeed, the identity

$$\partial_x^\alpha \big( A_j(u) \partial_{x_j} v \big) = A_j(u) \partial_{x_j} \partial_x^\alpha v + [\partial_x^\alpha, A_j(u)] \partial_{x_j} v$$

which is true for $v$ smooth, makes sense in $H^{-1}$ when $v \in H^s$. The estimate for the commutators can $g_\alpha$ can be repeated and the uniform bounds of $u(t)$ in $H^s$ imply that $g_\alpha \in L^2([0,T'] \times L^2(\mathbb{R}^d))$. Hence $\partial_x^\alpha u \in L^2([0,T'] \times \mathbb{R}^d)$ is a weak solution of (5.5.1). Thus by Friedrichs lemma, it is a strong solution on $[0,T'] \times \mathbb{R}^d$, and $\partial^\alpha u \in C^0([0,T']; L^2)$, proving that $u \in C^0([0,T']; H^s)$. □

To finish Theorem 5.1.2 it remains to prove uniqueness

**Proposition 5.5.2.** *The equation* (5.1.1) *has at mot one solution in* $C^0([0,T]; H^s(\mathbb{R}^d))$*.*

*Proof.* Suppose that $u$ and $v$ are two solutions. Then $w = u - v$ satisfies

$$\partial_t w + \sum_{j=1}^d A_j(u) \partial_j w = f, \qquad w_{|t=0} = 0 \tag{5.5.2}$$

with

$$(5.5.3) \qquad f = F(u) - F(v) + \sum_{j=1}^{d} (A_j(u) - A_j(v))\partial_j v.$$

Because $u$, $v$ and $\partial_j v$ are bounded, there is a constant $C$ such that $|f| \leq C(|u-v|)$ that is:

$$\forall (t,x), \quad |f(t,x)| \leq C|w(t,x)|.$$

The $L^2$ energy estimate can be applied to (5.5.2), and there are constants $C$ and $\gamma$ sucht that

$$\forall t, \quad \|w(t)\|_{L^2} \leq C \int_0^t e^{\gamma(t-t')} \|w(t')\|_{L^2} dt'.$$

By Gronwall's lemma, this implies that $\|w(t)\|_{L^2} = 0$, that is $w = 0$. □

## 5.6 Proof of Theorem 5.1.3

Repeating the proof of Proposition 5.3.3, one has :

**Proposition 5.6.1.** *There are functions $C_1(\cdot)$ and $C_2(\cdot)$ such that if $u \in C^0([0,T'], H^s)$ is a solution of* (5.1.1)*, one has for all $t \in [0,T']$:*

$$(5.6.1) \qquad \|u(t)\|_{H^s} \leq C_1(R) e^{\gamma t} \|u(0)\|_{H^s} + \int_0^t C_1(R) e^{\gamma(t-t')} \|f(t')\|_{L^2} dt'.$$

*with $\gamma = C_2(R)(1+M)$, with $R$ and $M$ defined at* (5.3.1)*.*

We can now proceed to the proof of Theorem 5.1.3. Suppose that the maximal time of existence $T^*$ is strictly smaller than $T$ and that

$$(5.6.2) \qquad R = \sup_{t<T^*} \|u(t)\|_{L^\infty} < +\infty, \quad M = \sup_{t<T^*} \|\nabla_x u(t)\|_{L^\infty} < +\infty.$$

Let

$$N = C_1(R) e^{\gamma T} \|h\|_{H^s} + C_1(R) \int_0^T e^{\gamma(t-t')} \|f(t')\|_{L^2} dt'.$$

One can apply Theorem 5.1.2 at any initial time $\tau \in [0,T]$, and by inspection of the proof one can see that the time of existence can be chosen independent of $\tau$, depending only on the size of the initial data in $H^s$:

**Lemma 5.6.2.** *There is $T' > 0$ such that for all initial time $\tau \in [0, T[$ and initial data $\tilde{h} \in H^s$ with $\|\tilde{h}\|_{H^s} \leq N$, the Cauchy problem* (6.1.1) *with initial data $\tilde{h}$ at time $\tau$ has a solution $u \in C^0([\tau, T'')]; H^s)$ with $T'' = \min(\tau + T', T)$.*

In particular, since $\|u(T^* - T'/2)\|_{H^s} \leq N$, the Cauchy problem with initial data $u(T^* - T'/2)$ at time $T^* - T'/2$ has a solution on $[T^* - T'/2, T''$ with $T'' = \min(T^* + T'/2, T) > T^*$. By uniqueness, this solution coincides with $u$ on $[T^* - T'/2, T^*[$, and thus extends $u$ to times larger than $T^*$, contradicting the definition of $T^*$.

## 5.7 An example of blow-up: the scalar case

Il is classical that the life span of smooth solutions of nonlinear equation is finite in general: consider for instance the ordinary differential equation

$$\partial_t u = u^2, \quad u_{|t=0} = h.$$

The solution is $h/(1 - th)$ and if $h > 0$, it blows up in finite time $T^* = 1/h$. This can be extended to semilinear equations, where the blow up occurs in the $L^\infty$ norm. We now illustrate, on a class of scalar equation, how the blow up can occur in the $L^\infty$ norm of the gradient of $u$.

Consider

$$\partial_t u + \sum_{j=1}^{d} a_j(u) \partial_{x_j} u, \qquad u(0, x) = h(x) \,. = 0\,, \tag{5.7.1}$$

with $a_j \in C^1(\mathbb{R}; \mathbb{R})$. We note $a = (a_1, \ldots, a_n) \in C^1(\mathbb{R}; \mathbb{R}^d)$.

**Proposition 5.7.1.** *$u \in C^1_b([0, T] \times \mathbb{R}^d)$ satisfies* (5.7.1) *if and only if $u$ satisfies the implicit equation*

$$F(t, x, u(t, x)) = 0\,, \tag{5.7.2}$$

*where $F(t, x, \lambda) = \lambda - h(x - ta(\lambda))$.*

*Proof.* Suppose that $u$ is $C^1$ and bounded on $[0, T] \times \mathbb{R}^d$ Consider the integral curves of

$$L = \partial_t + \sum_{j=1}^{n} a_j(u(t, x)) \partial_{x_j}$$

that is the solutions $X(s; t, x)$ of

$$\frac{dX}{ds} = a(u(s, X(s, t, x))), \qquad X(t, t, x) = x. \tag{5.7.3}$$

Because the $u \in C^1_b$, the flow $X$ is defined on $[0,T]\times[0,T]\times\mathbb{R}^d$. One has, for all $v \in C^1$,

$$\partial_s\big(v(s, X(s;t,x))\big) = (Lv)(s, X(s;t,x) \tag{5.7.4}$$

In particular, if $u$ is a solution of (5.7.1),

$$\partial_s\big(u(s, X(s;t,x))\big) = 0 \quad \Rightarrow \quad u(s, X(s;t,x)) = u(t,x).$$

Thus, $a(u(s, X(s,t,x))) = a(t,x)$, implying that the integral curves are lines

$$X(s;t,x) = x + (s-t)a(u(t,x)) \tag{5.7.5}$$

and that

$$u(s, x + (s-t)a(u(t,x))) = u(t,x). \tag{5.7.6}$$

At $s = 0$, this means

$$u(t,x) = h(x - ta(u(t,x)), \tag{5.7.7}$$

that is (5.7.2)

Conversely, suppose that $u \in C^1_b([0,T]\times\mathbb{R}^d)$ satisfies (5.7.2). For $t = 0$, this means that $u(0,x) = h(x)$. The derivatives of $F$ are :

$$\begin{aligned}
&\partial_t F(t,x,\lambda) = \sum_j a_j(\lambda)\partial_{x_j} h(x - ta(\lambda))\,,\\
&\partial_{x_j} F(t,x,\lambda) = -\partial_{x_j} h(x - ta(\lambda))\,,\\
&\partial_\lambda F(t,x,\lambda) = 1 + t\sum_j a'_j(\lambda)\partial_{x_j} h(x - ta(\lambda))\,.
\end{aligned}$$

Note that $\partial_\lambda F$ and $\nabla_x F \neq 0$ cannot vanish together. Differentiating (5.7.2), on has at $\lambda = u(t,x)$,

$$\begin{aligned}
\partial_t F(t,x,\lambda) + \partial_t u\,\partial_\lambda F(t,x,\lambda) &= 0\,,\\
\partial_{x_j} F(t,x,\lambda) + \partial_{x_j} u\,\partial_\lambda F(t,x,\lambda) &= 0\,.
\end{aligned} \tag{5.7.8}$$

In particular,

$$\partial_\lambda F(t,x,u(t,x)) \neq 0\,. \tag{5.7.9}$$

By (5.7.8),

$$\Big(\partial_t u + \sum_j a_j(u)\partial_{x_j} u\Big)\partial_\lambda F(t,x,u(t,x)) = 0\,.$$

With (5.7.9), this implies that $u$ satisfies the equation (5.7.1). $\square$

Note that $\partial_\lambda F \neq 0$ for small times. Therefore, the implicit function theorem can be applied to (5.7.2), yielding local solutions of (5.7.1). The next result gives a precise estimate of the life span of the solution, when the initial data $h \in C^1_b(\mathbb{R}^d)$. The form of $\partial_\lambda F$ leads to introduce the functions

$$g(x) = \sum_{j=1}^{n} a_j'(h(x))\partial_{x_j} h(x) \tag{5.7.10}$$

For $h \in C^1_b(\mathbb{R}^d)$, $g$ is bounded and one can introduce

$$\mu = \inf_{x\in\mathbb{R}^d} g(x) \in \mathbb{R}. \tag{5.7.11}$$

**Theorem 5.7.2.** *Soit $h \in C^1_b(\mathbb{R}^d)$. Let $T^* = +\infty$ if $\mu \geq 0$, and $T^* = -1/\mu$ si $\mu < 0$.*

*i) The Cauchy problem* (5.7.1) *has a unique solution $u \in C^1([0, T^*[\times\mathbb{R}^d)$; moreover,*

$$\forall (t, x) \in [0, T^*[\times\mathbb{R}^d , \quad |u(t, x)| \leq \|h\|_{L^\infty(\mathbb{R}^d)} . \tag{5.7.12}$$

*ii) For all $T < T^*$, $u \in C^1_b([0, T] \times \mathbb{R}^d)$ and*

$$\forall t < T^* , \quad \|\nabla_x u(t, .)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{1 + t\mu} \|\nabla_x h\|_{L^\infty(\mathbb{R}^d)} . \tag{5.7.13}$$

*iii) When $\mu < 0$, there is a constant $m > 0$ such that*

$$\forall t < T^* , \quad \|\nabla_x u(t, .)\|_{L^\infty(\mathbb{R}^d)} \geq \frac{m}{T^* - t} . \tag{5.7.14}$$

*Proof.* **a)** Let $(t, x) \in [0, T*[\times\mathbb{R}^d$. The function $\lambda \mapsto F(t, x, \lambda) = \lambda - h(x - ta(\lambda))$ is $C^1$; It is negative for $\lambda < -\|h\|_{L^\infty}$ and positive for $\lambda > \|h\|_{L^\infty}$. Therefore it vanishes. Moreover, when $F(t, x, \lambda) = 0$, on a

$$\partial_\lambda F(t, x, \lambda) = 1 + tg(x - ta(\lambda)) \geq 1 + t\mu > 0 .$$

Thus the root in $\lambda$ of $F(\lambda, t, x) = 0$ is unique. This determines uniquely $u(t, x)$ such that $F(t, x, u(t, x)) = 0$. Moreover, since $\partial_\lambda F(t, x, u(t, x) > 0$. the local implicit function theorem implies that $u$ is $C^1$ sur $[0, T^*[\times\mathbb{R}^d$. By Proposition 5.7.1 $u$ est solution de (5.7.1). Uniqueness also follows from Proposition 5.7.1 and the uniqueness of the solution of the implicit equation $F(t, x, \lambda) = 0$.

The $L^\infty$ bound (5.7.12) follows from the identity $u(t, x) = h(y)$ with $y = x - ta(u(t, x)$.

**b)** By (5.7.8),

$$\begin{cases} (1+tg(y))\,\partial_t u(t,x) = -a(h(y))\cdot\nabla_x h(y)\,, \\ (1+tg(y))\,\nabla_x u(t,x) = \nabla_x h(y)\,, \end{cases} \tag{5.7.15}$$

with $y = x - ta(u(t,x))$. Since $g \geq \mu$, the estimate of the derivatives follow. In particular, all the derivatives of $u$ are bounded on $[0,T]\times\mathbb{R}^d$, for all $T < T^*$.

**c)** Suppose that $\mu < 0$. Let $m = \sup|a'(h(y))| > 0$. For all $\mu' \in ]\mu, 0[$, there is $y \in \mathbb{R}^r$ tel que

$$0 < -\mu' \leq -g(y) = -a'(h(y))\cdot\nabla_x h(y) \leq m|\nabla_x h(y)|\,.$$

For $x = y + ta(h(y))$, one has $u(t,x) = h(y)$, and by (5.7.15)

$$|\nabla_x u(t,x)| \geq \frac{1}{1+t\mu'}|\nabla_x h(y)| \geq \frac{1}{1+t\mu'}\,\frac{|\mu'|}{m}\,.$$

Hence, for all $\mu' \in ]\mu, 0[$ and all $t \in [0, -1/\mu'[$:

$$\|\nabla_x u(t,.)\|_{L^\infty} \geq \frac{1}{1+t\mu'}\,\frac{|\mu'|}{m}$$

Hence, for all $t \in ]0, T^*[$, letting $\mu'$ tend to $\mu$, we see that

$$\|\nabla_x u(t,.)\|_{L^\infty} \geq \frac{1}{1+t\mu}\,\frac{|\mu|}{m} = \frac{1}{m(T^*-t)}\,.$$

The theorem is proved. □

**Corollary 5.7.3.** *Si* $\mu < 0$, (5.7.1) *has no solution in* $C^1_b([0,T[\times\mathbb{R}^d)$ *pour* $T > T*$.

**Remark 5.7.4.** When the infimum $\mu$ of $g$ est strictement is negative and reached at $y_0 \in \mathbb{R}^d$, one can choose this point in the proof above and for $t \in [0, T^*[$ and $x = y_0 + ta(h(y_0))$

$$|\nabla_x u(t,x)| = \frac{1}{1+t\mu}|\nabla_x h(y_0)|\,.$$

Because $\mu < 0$, $|\nabla_x h(y_0)| > 0$, and this formula shows that the gradient of $u$ blows up at the point $(T^*, y_0 + T^* a(h(y_0)))$. Therefore, the solution has no $C^1$ extension near this point.