Chapter 9

Hyperbolic Mixed Problems

In this chapter, we discuss the classical theory of mixed Cauchy boundary value problem for symmetric hyperbolic systems see [Fr1], [Fr2], [**?**] and also [**?**], [**?**]. We follow closely the presentation in [**?**]. For simplicity, we consider here only constant coefficients equations, and flat boundaries, but all the technics can be adapted to variable coefficients and general smooth domains.

9.1 The equations

Consider a $N \times N$ system

$$Lu := \partial_t u + \sum_{j=1}^{d} A_j \partial_j u = F(u) + f \tag{9.1.1}$$

For simplicity, we assume that the coefficients A_j are constant. F is a C^∞ mapping from $\mathbb{R}^N$ to $\mathbb{R}^N$. The variables are $t \in \mathbb{R}$, $y = (y_1, \dots, y_{d-1}) \in \mathbb{R}^{d-1}$ and $x \in \mathbb{R}$. The derivations are $\partial_j = \partial_{y_j}$ for $j \in \{1, \dots, d-1\}$ and $\partial_d = \partial_x$.

For simplicity, we work in the class of symmetric hyperbolic operators:

Assumption 9.1.1.
(H1) *There is a positive definite symmetric matrix* $S = {}^tS \gg 0$ *such that for all* j, SA_j *is symmetric.*
(H2) $\det A_d \neq 0$

The matrix S is called a *symmetrizer* for L. The assumption (H2) means that the boundary is *not characteristic* for L. The eigenvalues of A_d are real and different from zero. We denote by N_+ [resp. N_-] the number of positive [resp. negative] eigenvalues of A_d. Then $N = N_+ + N_-$.

Lemma 9.1.2. *The matrix SA_d has only real eigenvalues. Counted with their multiplicities, N_+ are positive and N_- are negative.*

Proof. Dropping the subscript d, $SA = S^{1/2}\big(S^{1/2}AS^{1/2}\big)S^{-1/2}$ is conjugated to the symmetric matrix $A' := S^{1/2}AS^{1/2}$. Therefore the eigenvalues of SA are those of A', thus are real. In addition, A' has the same signature (N_+, N_-) as A. □

We consider the equations (9.1.1) on the half space $\{x \geq 0\}$ together with boundary conditions:

$$Mu_{|x=0} = Mg\,. \tag{9.1.2}$$

where M is a $N' \times N$ matrix.

In the theory of hyperbolic boundary problems, the simplest case occurs when the boundary conditions are *maximal dissipative*:

Definition 9.1.3. *The boundary condition* (9.1.2) *is maximal dissipative for L if and only if dim* $\ker M = N_-$ *and the symmetric matrix SA_d is definite negative on* $\ker M$*.*

In this Chapter we study the well-posedness of the hyperbolic boundary value problem (9.1.1) (9.1.2). We always assume that Assumption 9.1.1 holds and that the boundary condition is maximal dissipative. Restricting attention to the image of M, there is no loss of generality in assuming that $N' = N_+$, so that M is a $N_+ \times N$ matrix.

Remark 9.1.4. The number of boundary conditions is $N' = N_+$, and there is an easy way to see that this is the correct number of conditions. In space dimension one, consider a diagonal system $\partial_t + A\partial_x$ with $A = \mathrm{diag}(a_1, \ldots, a_N)$. The diagonal entries are real and do not vanish by Assumption (H2). By definition, among them $N+$ are positive and N_- are negative. We have seen in the first chapter, that a boundary condition is needed for $\partial_t + a_j\partial_x$ if and only a_j is positive. So, the total number of boundary conditions must be N_+.

Remark 9.1.5. The dissipativity condition is satisfied in many physical examples (wave equations with Dirichlet boundary conditions, Maxwell equations with usual boundary conditions, etc). However, it is far from being necessary (see the discussion in Chapter 6 for an approach to necessary conditions and elementary examples in [**?**]). In the analysis below, the dissipativity assumption appears as a trick to warranty the validity of good energy estimates: in applications these computations mean dissipation of a physical energy.

9.2 Hyperbolic boundary value problems

In this section we consider the problem

$$\text{(9.2.1)} \qquad \begin{cases} Lu = f & \textit{on } \mathbb{R}\times\mathbb{R}^d_+ \\ Mu_{|x=0} = g & \textit{on } \mathbb{R}\times\mathbb{R}^{d-1} \end{cases}$$

We use the notation $\mathbb{R}^d_+ = \{(y,x) \in \mathbb{R}^d : x > 0\}$. We assume that the Assumptions (H1) and (H2) are satisfied, that M is a $N_+ \times N$ matrix and that the boundary condition is maximal dissipative.

We first solve this equation in weighted spaces: we look for solutions $u = e^{\gamma t}\widetilde{u}$, assuming that $f = e^{\gamma t}\widetilde{f}$ and $g = e^{\gamma t}\widetilde{g}$, with $\widetilde{u}$, $\widetilde{f}$ and $\widetilde{g}$ at least in L^2. This yields the equations

$$\text{(9.2.2)} \qquad \begin{cases} (L+\gamma)\widetilde{u} = \widetilde{f} & \textit{on } \mathbb{R}\times\mathbb{R}^d_+ \\ M\widetilde{u}_{|x=0} = \widetilde{g} & \textit{on } \mathbb{R}\times\mathbb{R}^{d-1}\,. \end{cases}$$

The choice $\gamma > 0$ corresponds to the idea that the functions u, f and g vanish at $t = -\infty$ and thus to an orientation of time.

We first study (9.2.2), dropping the tildes. We denote by H^s the usual Sobolev spaces. We also use the notation $\mathbb{R}^{1+d}_+ = \mathbb{R}\times\mathbb{R}^d_+$.

9.2.1 The adjoint problem

The adjoint of L (in the sense of distributions) is $L^* := -\partial_t - \sum A_j^*\partial_j$. Thus $-L^*$ has the same form as L.

Lemma 9.2.1. *S^{-1} is a symmetrizer for $-L^*$.*

Proof. Since S is symmetric definite positive, S^{-1} is also definite positive. Moreover, $S^{-1}A_j^* = S^{-1}A_j^*SS^{-1} = S^{-1}SA_jS^{-1} = A_jS^{-1}$ is symmetric. □

For C^1 functions with compact support in $\overline{\mathbb{R}}^{1+d}_+$, one has

$$\text{(9.2.3)} \qquad \big(Lu, v\big)_{L^2} = \big(u, L^*v\big)_{L^2} - \big(A_d u_{|x=0}, v_{|x=0}\big)_{L^2}$$

where $(\cdot,\cdot)_{L^2}$ denotes the scalar product in L^2. Consider a space of dimension N_+ on which SA_d is definite positive. There is a $N_- \times N$ matrix M_1 such that this space is $\ker M_1$. Since M is maximal dissipative, SA_d is definite negative on $\ker M$ and therefore

$$\text{(9.2.4)} \qquad \mathbb{R}^N = \ker M \oplus \ker M_1$$

Lemma 9.2.2. *There are matrices R and R_1 of size $N_- \times N$ and $N_+ \times N$ respectively, such that for all vectors u and v in $\mathbb{R}^N$:*

$$\big(A_d u, v\big) = (Mu, R_1 v) + (M_1 u, Rv)\,. \tag{9.2.5}$$

Moreover, $\ker R = (A_d \ker M)^\perp$ *has dimension* N_+ *and* $S^{-1}A_d^*$ *is definite positive on* $\ker R$.

Proof. The identity (9.2.5) is equivalent to

$$\begin{aligned} (A_d u, v) &= (Mu, R_1 v), \quad \forall u \in \ker M_1 \\ (A_d u, v) &= (M_1 u, Rv), \quad \forall u \in \ker M\,. \end{aligned}$$

Since M is an isomorphism from $\ker M_1$ to $\mathbb{R}^{N_+}$, the first equation determines $R_1 v \in \mathbb{R}^{N_+}$. Similarly, the second equation determines $Rv \in \mathbb{R}^{N_-}$.

The identity (9.2.5) implies that $(A_d u, v) = 0$ when $u \in \ker M$ and $v \in \ker R$, thus $\ker R \subset (A_d \ker M)^\perp$. Because the two spaces have the same dimension, they are equal.

Suppose that $(S^{-1}A_d^* v, v) \le 0$ for some $v \in \ker R$. Then for all $u \in \ker M$, $(SA_d u, S^{-1}v) = 0$ by (9.2.5) and for all $\alpha \in \mathbb{R}$

$$\big(SA_d(u + \alpha S^{-1}v), u + \alpha S^{-1}v\big) = (SA_d u, u) + \alpha^2 (A_d S^{-1}v, v) \le 0\,.$$

Since $\ker M$ has maximal dimension among spaces on which SA_d is non positive, this implies that $S^{-1}v \in \ker M$. Because $\ker R$ and $A_d \ker M$ are orthogonal, one has $(A_d S^{-1}v, v) = (SA_d S^{-1}v, S^{-1}v) = 0$. Since SA_d is definite negative on $\ker M$, this shows that $S^{-1}v = 0$, hence $v = 0$. □

Definition 9.2.3. *The system L^* with boundary condition R is the adjoint problem of (L, M).*

Note that R is not unique, but the key object $\ker R = (A_d \ker M)^\perp$ is uniquely determined from L and M.

With (9.2.3), the lemma implies that for all u and v in $C_0^1(\overline{\mathbb{R}}_+^{1+d})$

$$\begin{aligned} \big((L+\gamma)u, v\big)_{L^2} = \big(u, (L^* + \gamma)v\big)_{L^2} &- \big(Mu_{|x=0}, R_1 v_{|x=0}\big)_{L^2} \\ &- \big(M_1 u_{|x=0}, Rv_{|x=0}\big)_{L^2}\,. \end{aligned}$$

In particular, if u is a solution of (9.2.2) and $Rv = 0$ on $\{x = 0\}$, one has

$$\big(f, v\big)_{L^2} = \big(u, (L^* + \gamma)v\big)_{L^2} - \big(g, R_1 v_{|x=0}\big)_{L^2}$$

This motivates the following definition of weak solutions.

Definition 9.2.4. *Given $f \in L^2(\mathbb{R}^{1+d}_+)$ and $g \in L^2(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^{1+d}_+)$ is a weak solution of* (9.2.2)*, if and only if for all $\Phi \in C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ such that $R\Phi_{|x=0} = 0$ one has*

$$\big(u, (L^* + \gamma)\Phi\big)_{L^2} = \big(f, \Phi\big)_{L^2} + \big(g, R_1\Phi_{|x=0}\big)_{L^2} . \tag{9.2.6}$$

We now discuss in which sense weak solutions are indeed solutions of (9.2.2). Introduce the spaces $H^{0,s}(\mathbb{R}^{1+d})$ of temperate distributions such that their Fourier transform satisfy

$$\int \big(1 + \tau^2 + |\eta|^2\big)^s |\hat{u}(\tau, \eta, \xi)|^2 d\tau d\eta d\xi < +\infty . \tag{9.2.7}$$

For $s \in \mathbb{N}$, this is the space of functions $u \in L^2$ such that their tangential derivatives $D^\alpha_{t,y}u$ of order $|\alpha| \le s$ belong to L^2. When s is a negative integer, this is the space of

$$u = \sum_{|\alpha| \le -s} \partial^\alpha_{t,y} u_\alpha , \qquad u_\alpha \in L^2 .$$

The space $H^{0,s}(\mathbb{R}^{1+d}_+)$ is the set of restrictions to $\{x > 0\}$ of functions in $H^{0,s}(\mathbb{R}^{1+d})$. When s is a positive or negative integer, there are equivalent definitions analogous to those given on the whole space.

Lemma 9.2.5. *For all $s \in \mathbb{R}$:*

i) the space $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ is dense in the space $H^{1,s}(\mathbb{R}^{1+d}_+)$ of functions $u \in H^{0,s+1}(\mathbb{R}^{1+d}_+)$ such that $D_x u \in H^{0,s}(\mathbb{R}^{1+d}_+)$;

ii) the mapping $u \mapsto u_{|x=0}$ extends continuously from $H^{1,s}(\mathbb{R}^{1+d}_+)$ to $H^{s+\frac{1}{2}}(\mathbb{R}^d)$.

Proof. The first part is proved by usual smoothing arguments. The details are left as an exercise.

Consider next $u \in C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ and denote by $\hat{u}(\tau, \eta, x)$ its partial Fourier transform with respect to the tangential variables (t, y). Integrating $\partial_x |\hat{u}|^2$ on $\mathbb{R}_+$, yields

$$|\hat{u}(\tau, \eta, 0)|^2 \le 2 \int_0^\infty |\partial_x \hat{u}(\tau, \eta, x)||\hat{u}(\tau, \eta, x|dx,$$

Thus, with $\Lambda = (1 + \tau^2 + |\eta|^2)^{1/2}$,

$$\Lambda^{2s+1} |\hat{u}(\cdot, 0)|^2 \le \Lambda^{2s} \int_0^\infty |\partial_x \hat{u}(\cdot, x)|^2 dx + \Lambda^{2s+2} \int_0^\infty |\hat{u}(\cdot, x)|^2 dx$$

Integrating in (τ, η) implies

$$\|u(\cdot,0)\|^2_{H^{s+1/2}(\mathbb{R}^d)} \leq \|\partial_x u\|^2_{H^{0,s}(\mathbb{R}^{1+d}_+)} + \|u\|^2_{H^{0,s+1}(\mathbb{R}^{1+d}_+)} = \|u\|^2_{H^{1,s}(\mathbb{R}^{1+d}_+)} \,.$$

Thus the mapping $u \mapsto u_{|x=0}$ extends by density and continuity to $H^{1,s}(\mathbb{R}^{1+d}_+)$ with values in $H^{s+1/2}(\mathbb{R}^d)$. $\square$

We apply this lemma to functions in the the space

$$\mathcal{D}(L) = \left\{ u \in L^2(\mathbb{R}^{1+d}_+) \,:\, Lu \in L^2(\mathbb{R}^{1+d}_+) \right\} \,. \tag{9.2.8}$$

Here Lu is computed in the sense of distributions on $\{x > 0\}$. This space is equipped with the norm $\|u\|_{L^2} + \|Lu\|_{L^2}$. Because A_d is invertible, for $u \in \mathcal{D}(L)$ one has

$$\partial_x u = A_d^{-1} Lu - A_d^{-1}\partial_t u - \sum_{j=1}^{d-1} A_d^{-1} A_j \partial_{y_j} u \tag{9.2.9}$$

and therefore $\mathcal{D}(L) \subset H^{1,-1}(\mathbb{R}^{1+d}_+)$. This shows that all $u \in \mathcal{D}(L)$ has a trace in $H^{-\frac{1}{2}}$.

Proposition 9.2.6. *i) $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ is dense in $\mathcal{D}(L)$*

ii) For all $u \in \mathcal{D}(L)$ and $v \in H^1(\mathbb{R}^{1+d}_+)$, there holds

$$\big(Lu, v\big)_{L^2} = \big(u, L^* v\big)_{L^2} - \big\langle A_d u_{|x=0}, v_{|x=0} \big\rangle_{H^{-1/2}\times H^{1/2}} \tag{9.2.10}$$

Proof. Consider a tangential mollifier $\jmath \in C_0^\infty(\mathbb{R} \times \mathbb{R}^{d-1})$, with $\jmath \geq 0$ and such that $\int \jmath(t,y)dtdy = 1$. For $\varepsilon > 0$, let

$$\jmath_\varepsilon(t,y) = \frac{1}{\varepsilon^d}\jmath\big(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\big) \,, \qquad t \in \mathbb{R} \,,\ y \in \mathbb{R}^{d-1} \,. \tag{9.2.11}$$

Denote by J_ε the convolution operator $\jmath_\varepsilon *$.

If $u \in \mathcal{D}(L)$ and $\Phi \in C_0^\infty(\mathbb{R}^{1+d})$ then $J_\varepsilon \Phi \in C_0^\infty(\mathbb{R}^{1+d})$ and in the sense of distributions

$$\big(u, L^* J_\varepsilon \Phi\big)_{L^2} = \big(Lu, J_\varepsilon \Phi\big)_{L^2} \,.$$

Note that we assume here that the support of Φ is contained in the open half space $\{x > 0\}$. Because J_ε commutes with differentiation and with multiplication by constants, $L^* J_\varepsilon \Phi = J_\varepsilon L^* \Phi$. Moreover, for all u and v in $L^2(\mathbb{R}^{1+d}_+)$, one has

$$\big(u, J_\varepsilon v\big)_{L^2} = \big(J_\varepsilon u, v\big)_{L^2}$$

Thus, there holds in the sense of distributions on $\{x > 0\}$:

$$LJ_\varepsilon u = J_\varepsilon Lu \; .$$

In particular $u_\varepsilon = J_\varepsilon u \in \mathcal{D}(L)$. Moreover, for all v in $L^2(\mathbb{R}^{1+d}_+)$, $J_\varepsilon v$ converges to v in L^2 when ε tends to zero. Thus, for $u \in \mathcal{D}(L)$, u_ε converges to u in $\mathcal{D}(L)$.

Next we note that for all v in L^2, $J_\varepsilon v \in H^{0,s}$ for all $s \in \mathbb{N}$, since for all $\alpha \in \mathbb{N}^d$, $\partial^\alpha_{t,y}(J_\varepsilon v) = (\partial^\alpha_{t,y} \jmath_\varepsilon) * v \in L^2$. Thus, $u_\varepsilon \in H^{0,s}$ for all s. Using (9.2.9) we see that $u_\varepsilon \in H^{1,s}$ for all s. In particular, $u_\varepsilon \in H^1(\mathbb{R}^{1+d}_+)$ and this shows that $H^1(\mathbb{R}^{1+d}_+)$ is dense in $\mathcal{D}(L)$. Since $C^\infty_0(\overline{\mathbb{R}}^{1+d}_+)$ is dense in $H^1(\mathbb{R}^{1+d}_+)$ this implies i).

By (9.2.9), we see that $\mathcal{D}(L) \subset H^{1,-1}$ and

$$\|u\|_{H^{1,-1}} \lesssim \|u\|_{L^2} + \|Lu\|_{L^2} \, .$$

Thus by the trace lemma, the trace $u_{|x=0}$ is well defined on $\mathcal{D}(L)$ and

$$\|u_{|x=0}\|_{H^{-1/2}} \lesssim \|u\|_{L^2} + \|Lu\|_{L^2} \, .$$

The identity (9.2.10) holds when u and v belong to $C^\infty_0(\overline{\mathbb{R}}^{1+d}_+)$. Both side are continuous for the norms of u in $\mathcal{D}(L)$ and v in H^1. Thus, the identity extends by density to $\mathcal{D}(L) \times H^1$. □

Corollary 9.2.7. *Given $f \in L^2(\mathbb{R}^{1+d}_+)$ and $g \in L^2(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^{1+d}_+)$ is a weak solution of* (9.2.2) *if and only if*

i) $u \in \mathcal{D}(L)$ and $Lu = f - \gamma u$ in the sense of distributions on $\{x > 0\}$,

ii) the trace $u_{|x=0}$ which is defined in $H^{-1/2}$ by i) satisfies $Mu_{|x=0} = g$.

Proof. If u is a weak solution, taking Φ with compact support in the open half space implies that $Lu + \gamma u = f$ in the sense of distributions. Thus $u \in \mathcal{D}(L)$.

Comparing (9.2.10) and (9.2.6) we see that for all $\Phi \in C^\infty_0(\overline{\mathbb{R}}^{1+d}_+)$ such that $R\Phi = 0$ on the boundary, there holds

$$\big(g, R_1\Phi_{|x=0}\big)_{L^2} = \big\langle A_d u_{|x=0}, \Phi_{|x=0}\big\rangle_{H^{-1/2}\times H^{1/2}}$$

Next we use Lemma 9.2.2, which means that $A_d = (R_1)^* M + R^* M_1$ to see that the right hand side is equal to

$$\big\langle M u_{|x=0}, R_1\Phi_{|x=0}\big\rangle_{H^{-1/2}\times H^{1/2}} \, .$$

For all $\phi \in C_0^\infty(\mathbb{R}^d)$ there is $\Phi \in C_0^\infty(\overline{\mathbb{R}}_+^{1+d})$ such that $\Phi_{|x=0} = \phi$. Thus, for all $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $R\phi = 0$,

$$\big(g, R_1\phi\big)_0 = \big\langle Mu_{|x=0}, R_1\phi\big\rangle_{H^{-1/2}\times H^{1/2}} .$$

Similar to (9.2.4), there is a splitting

$$\mathbb{R}^N = \ker R \oplus \ker R_1$$

Therefore, for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ with values in $\mathbb{R}^{N_+}$, there is $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $R\phi = 0$ and $R_1\phi = \varphi$. Thus for all $\varphi \in C_0^\infty(\mathbb{R}^d)$:

$$\big(g, \varphi\big)_0 = \big\langle Mu_{|x=0}, \varphi\big\rangle_{H^{-1/2}\times H^{1/2}} .$$

This means that $Mu_{|x=0} = g$.

Conversely, if $u \in \mathcal{D}(L)$ and $Lu + \gamma u = f$, for all test function Φ, one has

$$\begin{aligned}\big(u, (L^* + \gamma)\Phi\big)_0 - \big(f, \Phi\big)_0 =& \big\langle Mu_{|x=0}, R_1\phi\big\rangle_{H^{-1/2}\times H^{1/2}} \\ &+ \big\langle M_1 u_{|x=0}, R\phi\big\rangle_{H^{-1/2}\times H^{1/2}} .\end{aligned}$$

with $\phi = \Phi_{|x=0}$. Taking Φ such that $R\phi = 0$, we see that if $Mu_{|x=0} = g$ then u is a weak solution of (9.2.2). $\square$

9.2.2 Energy estimates. Existence of weak solutions

Lemma 9.2.8. *The symmetric matrix SA_d is definite negative on $\ker M$ if and only if there are constants $c > 0$ and C such that for all vector $h \in \mathbb{C}^N$:*

$$-(SA_d h, h) \geq c|h|^2 - C|Mh|^2 .$$

Proof. Since SA_d is definite negative on $\ker M$, there is $c > 0$ such that

$$\forall h \in \ker M \ : \quad -(SA_d h, h) \geq c|h|^2 .$$

Since SA_d is invertible, $\dim(SA_d \ker M) = \dim \ker M = N_-$, thus $K = (SA_d \ker M)^\perp$ has dimension $N - N_- = N_+$. In addition since SA_d is definite negative on $\ker M$, $K \cap \ker M = \{0\}$ and $\mathbb{R}^N = K \oplus \ker M$. In particular, there is C_0 such that for all $v \in K$, $|v| \leq C_0|Mv|$. By definition of K, if $h = v + w$ with $v \in K$ and $w \in \ker M$, there holds

$$\begin{aligned}-(SA_d h, h) = -(SA_d v, v) - (SA_d w, w) &\geq c|w|^2 - C|v|^2 \\ &\geq c(|w|^2 + |v|^2) - (C + c)C_0^2|Mv|^2 . \\ &\geq \frac{c}{2}|h|^2 - (C + c)C_0^2|Mh|^2 .\end{aligned}$$

The converse statement is clear. $\square$

Proposition 9.2.9 (Energy estimates)**.** *There is C such that for all $\gamma > 0$ and all test function $u \in H^1(\mathbb{R} \times \mathbb{R}^d_+)$, one has*

$$\gamma \|u\|^2_{L^2} + \|u_{|x=0}\|^2_{L^2} \leq C\big(\frac{1}{\gamma}\|(L+\gamma)u\|^2_{L^2} + \|Mu_{|x=0}\|^2_{L^2}\big) \tag{9.2.12}$$

$$\gamma \|v\|^2_{L^2} + \|v_{|x=0}\|^2_{L^2} \leq C\big(\frac{1}{\gamma}\|(L^*+\gamma)v\|^2_{L^2} + \|Ru_{|x=0}\|^2_{L^2}\big) \tag{9.2.13}$$

Proof. Both side of the estimates are continuous for the H^1 norm. Since $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ is dense in $H^1(\mathbb{R}^{1+d}_+)$ it is sufficient to make the proof when $u \in C_0^\infty$. Then, using that the SA_j are self adjoint and integrating by parts yields

$$2\mathrm{Re}\,\big(S(L+\gamma)u, u\big)_{L^2} = \gamma\big(Su, u\big)_{L^2} - \big(SA_d u_{|x=0}, u_{|x=0}\big)_{L^2}$$

By Lemma 9.2.8, there are $c > 0$ and $C \geq 0$ such that

$$-\big(SA_d u_{|x=0}, u_{|x=0}\big)_{L^2} \geq c\|u_{|x=0}\|^2_{L^2} - C\|Mu_{|x=0}\|^2_{L^2}\,.$$

Because S is definite positive, there is $c_1 > 0$ such that

$$\big(Su, u\big)_{L^2} \geq c_1\|u\|^2_{L^2}\,.$$

Therefore

$$c_1\gamma\|u\|^2_{L^2} + c\|u_{|x=0}\|^2_{L^2} \leq 2|S|\|(L+\gamma)u\|_{L^2}\|u\|_{L^2} + C\|Mu_{|x=0}\|^2_{L^2}\,.$$

This implies (9.2.12). The proof of (9.2.13) is similar. □

Proposition 9.2.10. *For all $\gamma > 0$, f and g in L^2, the problem* (9.2.2) *has a weak solution in L^2.*

Proof. Consider the space $\mathcal{H}$ of $\Phi \in H^1(\mathbb{R} \times \mathbb{R}^d_+)$ such that $R\Phi_{|x=0} = 0$. Let $\mathcal{H}_1 = (L^* + \gamma)\mathcal{H} \subset L^2$. By (9.2.13), the mapping $L^* + \gamma$ is one to one from $\mathcal{H}$ to $\mathcal{H}_1$ and the reciprocal mapping $\mathcal{F}$ satisfies

$$\gamma\|\mathcal{F}\varphi\|_{L^2} + \sqrt{\gamma}\|R_1\mathcal{F}\varphi_{|x=0}\|_{L^2} \leq C\|\varphi\|_{L^2}\,.$$

Thus the linear form

$$\varphi \quad\mapsto\quad \ell(\varphi) := \big(f, \mathcal{F}\varphi\big)_{L^2} + \big(g, R_1\mathcal{F}\varphi_{|x=0}\big)_{L^2}$$

is continuous on $\mathcal{H}_1$ equipped with the norm $\|\cdot\|_{L^2}$. Therefore it extends as a continuous linear form on L^2 and there is $u \in L^2$ such that $\ell(\varphi) = (u, \varphi)_0$. The definition of ℓ implies that u is a weak solution. □

9.2.3 Strong solutions

Definition 9.2.11. *Given f and g in L^2, $u \in L^2$ is a strong solution of* (9.2.2) *if there exist sequences (u_n, f_n) in $H^1(\mathbb{R}^{1+d}_+)$, and g_n in $H^1(\mathbb{R}^d)$ solutions of* (9.2.2) *and converging to (u, f) in $L^2(\mathbb{R}^{1+d}_+)$ and to g in $L^2(\mathbb{R}^d)$ respectively.*

By the density of $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ in $H^1(\mathbb{R}^{1+d}_+)$ and continuity from H^1 to L^2 of L and the traces, one obtains an equivalent definition if one requires that there is a sequence (u_n, f_n, g_n) in $C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$ solutions of (9.2.2) and converging to (u, f, g) in L^2.

Proposition 9.2.12 (Weak= strong)**.** *For all $\gamma > 0$, f and g in L^2, any weak solution of* (9.2.2) *in L^2 is a strong solution and*

$$\gamma \|u\|^2_{L^2} + \|u_{|x=0}\|^2_{L^2} \leq C\big(\frac{1}{\gamma}\|f\|^2_{L^2} + \|g\|^2_{L^2}\big) \tag{9.2.14}$$

In particular the weak=strong solution is unique.

Proof. Consider again the mollifiers $\jmath$ (9.2.11) and the convolution operator $J_\varepsilon u = \jmath_\varepsilon * u$.

Suppose that $u \in L^2$ is a weak solution of (9.2.2). For all test function Φ, $J_\varepsilon \Phi$ is also a test function and $R J_\varepsilon \Phi = 0$. Therefore,

$$\big(u, (L^* + \gamma) J_\varepsilon \Phi\big)_{L^2} = \big(f, J_\varepsilon \Phi\big)_{L^2} + \big(g, R_1 J_\varepsilon \Phi_{|x=0}\big)_{L^2}.$$

As in the proof of Proposition 9.2.6 this implies that

$$\big(J_\varepsilon u, (L^* + \gamma)\Phi\big)_{L^2} = \big(J_\varepsilon f, \Phi\big)_{L^2} + \big(J_\varepsilon g, R_1 \Phi_{|x=0}\big)_{L^2}.$$

This means that $u_\varepsilon = J_\varepsilon u$ is a weak solution of

$$\begin{cases} (L + \gamma) u_\varepsilon = f_\varepsilon\,, \\ \ M u_{\varepsilon|x=0} = g_\varepsilon\,. \end{cases} \tag{9.2.15}$$

with $f_\varepsilon = J_\varepsilon f$ and $g_\varepsilon = J_\varepsilon g$.

The proof of Proposition 9.2.6 shows that for all $\varepsilon > 0$, $u_\varepsilon \in H^1(\mathbb{R}^{1+d})$ and by Corollary 9.2.7 the equations (9.2.15) hold in L^2.

Since u_ε, f_ε and g_ε converge in L^2 to (u, f, g) respectively, this shows that u is a strong solution.

In addition, the energy estimates (9.2.12) hold for u_ε. Passing to the limit, we obtain that u satisfies (9.2.14). □

9.2.4 Regularity of solutions

We prove that if the data are regular, then the solution is regular. It is convenient to equip the spaces $H^s(\mathbb{R}^{1+d}_+)$ with a family of parameter dependent norms:

$$\|u\|_{s,\gamma} = \sum_{|\alpha|\leq s} \gamma^{s-|\alpha|}\|\partial^\alpha_{t,y,x}u\|_{L^2} \,. \tag{9.2.16}$$

We define similar norms on the spaces $H^s(\mathbb{R}^d)$, using only tangential derivatives $\partial^\alpha_{t,y}$.

Proposition 9.2.13. *Let s be a non negative integer. For $\gamma > 0$, $f \in H^s$ and $g \in H^s$ the solution of* (9.2.2) *belongs to H^s and*

$$\gamma\|u\|^2_{s,\gamma} + \|u_{|x=0}\|^2_{s,\gamma} \leq C\big(\frac{1}{\gamma}\|f\|^2_{s,\gamma} + \|g\|^2_{s,\gamma}\big) \tag{9.2.17}$$

Proof. First prove the tangential regularity. We use the mollified equation (9.2.15). Since $u_\varepsilon \in H^{1,s}$ for all s, we can differentiate this equation as many times as we want in (t, y) and $\partial^\alpha_{t,y}u_\varepsilon \in H^1(\mathbb{R}^{1+d}_+)$ satisfies

$$\begin{cases} (L+\gamma)\partial^\alpha_{t,y}u_\varepsilon = \partial^\alpha_{t,y}f_\varepsilon \,, \\ M\partial^\alpha_{t,y}u_{\varepsilon|x=0} = \partial^\alpha_{t,y}g_\varepsilon \,. \end{cases}$$

Proposition 9.2.12 implies that

$$\gamma\|u_\varepsilon\|^2_{H^{0,s}} + \|u_{\varepsilon|x=0}\|^2_{H^{0,s}} \leq C\big(\frac{1}{\gamma}\|f_\varepsilon\|^2_{H^{0,s}} + \|g_\varepsilon\|^2_{H^{0,s}}\big)$$

with C independent of ε.

Next we use the equation to recover the normal derivatives. We start from (9.2.9) which implies that

$$\|\partial_x u_\varepsilon\|_{H^{0,s-1}} \lesssim \|f_\varepsilon\|_{H^{0,s-1}} + \|u_\varepsilon\|_{H^{0,s}} \,.$$

In addition, since f_ε can be differentiated s times in x, we see by induction on $k \leq s$ that $\partial^k_x u_\varepsilon \in H^{0,s'}$ for all s' with

$$\partial^k_x u_\varepsilon = A_d^{-1}\partial^{k-1}_x f_\varepsilon - A_d^{-1}\partial^{k-1}_x\partial_t u_\varepsilon - \sum_{j=1}^{d-1} A_d^{-1}A_j\partial^{k-1}_x\partial_j u_\varepsilon \,.$$

Thus

$$\|\partial^k_x u_\varepsilon\|_{H^{0,s-k}} \lesssim \|\partial^{k-1}_x f_\varepsilon\|_{H^{0,s-k}} + \|\partial^{k-1}_x u_\varepsilon\|_{H^{0,s-k+1}}$$

Adding up, we see that $u_\varepsilon \in H^{s+1}$ and that there is C independent of ε and γ such that

$$\gamma \|u_\varepsilon\|^2_{s,\gamma} + \|u_{\varepsilon|x=0}\|^2_{s,\gamma} \le C\big(\frac{1}{\gamma}\|f_\varepsilon\|^2_{s,\gamma} + \|g_\varepsilon\|^2_{s,\gamma}\big)$$

This means that the u_ε satisfy (9.2.17). Similarly, the differences $u_\varepsilon - u_{\varepsilon'}$ satisfy (9.2.17). Hence the family u_ε is a Cauchy sequence in H^s, so that the limit u belongs to H^s and satisfy (9.2.17). □

9.2.5 Solutions of the boundary value problem (9.2.1)

We now turn to the original equation (9.2.1). Propositions 9.2.10, 9.2.12 and 9.2.13 imply the next result.

Theorem 9.2.14. *Suppose that $\gamma > 0$, $s \in \mathbb{N}$, $f \in e^{\gamma t} H^s$ and $g \in e^{\gamma t} H^s$. Then the problem* (9.2.1) *has a unique strong solution $u \in e^{\gamma t} H^s$ and*

$$\gamma \|e^{-\gamma t} u\|^2_{s,\gamma} + \|e^{-\gamma t} u_{|x=0}\|^2_{s,\gamma} \le C\big(\frac{1}{\gamma}\|e^{-\gamma t} f\|^2_{s,\gamma} + \|e^{-\gamma t} g\|^2_{s,\gamma}\big) \tag{9.2.18}$$

where C is independent of γ and u, f, g.

9.3 Solutions on $]-\infty, T]$ and the causality principle

In this section, we show that if the data of (9.2.1) vanish in the past, then the solution also does, and we solve the boundary value problem on $\{t \le T\}$.

First we note that we have a strong uniqueness result:

Lemma 9.3.1. *Assume that $f \in e^{\gamma_0 t} L^2 \cap e^{\gamma_1 t} L^2$ and $g \in e^{\gamma_0 t} L^2 \cap e^{\gamma_1 t} L^2$ with $0 < \gamma_0 < \gamma_1$. Then the solutions u_{γ_0} and u_{γ_1} given by Proposition* 9.2.13 *applied to $\gamma = \gamma_0$ and $\gamma = \gamma_1$ are equal.*

Proof. Note that $f \in e^{\gamma t} L^2$ and $g \in e^{\gamma t} L^2$ for all $\gamma \in [\gamma_0, \gamma_1]$. Therefore, for such γ (9.2.1) has a unique strong solution $u_\gamma \in e^{\gamma t} L^2$.

Introduce a function $\theta \in C^\infty(\mathbb{R})$ such that $\theta(t) = 1$ for $t \le 0$ and $\theta(t) = e^{-t}$ for $t \ge 1$. Thus $\partial_t \theta = h\theta$ with $h \in L^\infty$. With $\delta = \gamma - \gamma_0$, introduce

$$v = \theta(\delta t)\big(u_\gamma - u_{\gamma_0}\big)\,.$$

The properties of θ imply that $v \in e^{\gamma_0 t}L^2$ and

$$Lv = \delta\partial_t\theta(\delta t)\big(u_\gamma - u_{\gamma_0}\big) = \delta h(\delta t)v\,, \quad Mv_{|x=0} = 0\,.$$

Thus, by uniqueness in $e^{\gamma_0 t}L^2$, Theorem 9.2.14 applied to $\gamma = \gamma_0$, implies that there is a constant C, independent of the γ's, such that

$$\gamma_0\|e^{-\gamma_0 t}v\|_{L^2} \le C\delta\|e^{-\gamma_0 t}v\|_{L^2}$$

If $C\delta < \gamma_0$, this implies that $v = 0$. Summing up, we have proved that for $\gamma \le (1 + 1/2C)\gamma_0$ and $\gamma_0 \le \gamma \le \gamma_1$, one has $u_\gamma = u_{\gamma_0}$. By induction, this implies that for all integer $k \ge 1$, $u_\gamma = u_{\gamma_0}$ for $\gamma \in [\gamma_0, \gamma_1]$ with $\gamma \le (1 + 1/2C)^k\gamma_0$. Hence, $u_\gamma = u_{\gamma_0}$ for $\gamma \in [\gamma_0, \gamma_1]$. □

This implies *local* uniqueness:

Proposition 9.3.2. *If $f \in e^{\gamma t}L^2(\mathbb{R}^{1+d}_+)$ and $g \in e^{\gamma t}L^2(\mathbb{R}^d)$ with $\gamma > 0$ vanish for $t < T$, then the solution $u \in e^{\gamma t}L^2(\mathbb{R}^{1+d}_+)$ of* (9.2.1) *vanishes for $t < T$.*

Proof. Since f and g vanish for $t < T$, f and g belong to $e^{\gamma' t}L^2$ for all $\gamma' \ge \gamma$. Thus, by the lemma above, $u \in e^{\gamma' t}L^2$ for all γ' large and by Theorem 9.2.14 there is C such that for all $\gamma' \ge \gamma$:

$$\gamma'\|e^{-\gamma' t}u\|^2_{L^2} \le C\frac{1}{\gamma'}\|e^{-\gamma' t}f\|^2_{L^2} + C\|e^{-\gamma' t}g\|^2_{L^2}\,.$$

Thus

$$\begin{aligned}\gamma'\|u\|^2_{L^2(\{t\le T\})} \le \gamma'\|e^{\gamma'(T-t)}u\|^2_{L^2} &\le \frac{C}{\gamma'}\|e^{\gamma'(T-t)}f\|^2_{L^2} + C\|e^{\gamma'(T-t)}f\|^2_{L^2}\\ &\le \frac{C}{\gamma'}\|e^{\gamma(T-t)}f\|^2_{L^2} + C\|e^{\gamma(T-t)}f\|^2_{L^2}\,.\end{aligned}$$

The right hand side is bounded as γ' tends to infinity, thus $u_{|\{t\le T\}} = 0$. □

We now consider solutions of (9.2.1) on $]-\infty, T] \times \mathbb{R}^d_+$. First, we note that the traces are well defined.

Lemma 9.3.3. *Suppose that $u \in L^2(]T_1, T_2[\times\mathbb{R}^d_+)$ satisfies $Lu \in L^2(]T_1, T_2[\times\mathbb{R}^d_+)$. Then the trace $u_{|x=0}$ is well defined in $H^{-1/2}_{loc}(]T_1, T_2[\times\mathbb{R}^{d-1})$.*

Proof. Consider $\chi \in C^\infty_0(]T_1, T_2[)$. Then χu, extended by 0 belongs to $L^2(\mathbb{R}^{1+d}_+)$ and $L(\chi u)$, which is the extension by 0 of $\chi Lu + \partial_t\chi u$, also belongs to $L^2(\mathbb{R}^{1+d}_+)$. Thus, by Lemma 9.2.5, χu has a trace in $H^{-1/2}$ and the lemma follows. □

Therefore, for $u \in L^2(]T_1, T_2[\times \mathbb{R}^d_+)$ such that $Lu = f \in L^2(]T_1, T_2[\times \mathbb{R}^d_+)$ the equation $Mu_{|x=0} = g \in L^2$ makes sense.

Corollary 9.3.4. *Suppose that $\gamma > 0$ and $u \in e^{\gamma t} L^2(]-\infty, T] \times \mathbb{R}^d_+)$ satisfies*

$$\begin{cases} Lu = 0 & \text{on }]-\infty, T] \times \mathbb{R}_+) \\ Mu_{|x=0} = 0 & \text{on }]-\infty, T]\,. \end{cases}$$

Then $u = 0$.

Proof. For $\delta > 0$ choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = 1 \quad \text{for } t < T - \delta \qquad \text{and} \qquad \chi(t) = 0 \quad \text{for } t \geq T - \delta/2\,.$$

Extend $v = \chi(t)u$ by 0 for $t \geq T$. Then $v \in e^{\gamma t} L^2$, Mv vanishes on the boundary, and $f := Lv$ which is the extension of $(\partial_t \chi)u$ by 0 for $t \geq T$ vanishes for $t \leq T - \delta$ and belongs to $e^{\gamma t} L^2$. Thus, by Proposition 9.3.2 v and hence u vanish for $t \leq T - \delta$. Since δ is arbitrary, this implies that $u = 0$. □

Remark 9.3.5. If u and u_1 are two solutions in $e^{\gamma t} L^2$ of (9.2.1) on $]-\infty, T_1] \times \mathbb{R}^d_+$ associated to L^2 data (f, g) and (f_1, g_1) respectively, and if $f = f_1$ and $g = g_1$ for $t \leq T$, then $u = u_1$ for $t \leq T$. Thus the values of u for times $t \leq T$ only depend on the values of the data f and g for $t \leq T$. This means that the solutions constructed above satisfy the *causality principle*.

Theorem 9.3.6. *Suppose that $f \in e^{\gamma t} H^s(]-\infty, T] \times \mathbb{R}^d_+)$ and $g \in e^{\gamma t} H^s(]-\infty, T] \times \mathbb{R}^{d-1})$, for some $\gamma > 0$ and $s \in \mathbb{N}$. Then the problem* (9.2.1) *has a unique solution $u \in e^{\gamma t} H^s(]-\infty, T] \times \mathbb{R}^d_+)$.*

If f and g vanish for $t \leq T_1$, then the solution u also vanishes for $t \leq T_1$.

Moreover, estimates similar to (9.2.18) *are satisfied.*

Proof. Extend f and g for $t \geq T$ as $\widetilde{f} \in H^s(\mathbb{R} \times \mathbb{R}^d_+)$ and $\widetilde{g} \in H^s(\mathbb{R} \times \mathbb{R}^{d-1})$. We can choose the extension such that they vanish for $t \geq T + 1$. For instance, when $s = 0$, we can extend them by 0. Because $\widetilde{f} = f$ and $\widetilde{g} = g$ for $t \leq T$ and vanish for $t \geq T + 1$, $\widetilde{f}$ and $\widetilde{g}$ belong to $e^{\gamma t} H^s$. Therefore, by Theorem 9.2.14 the problem

$$L\widetilde{u} = f\,, \qquad M\widetilde{u}_{|x=0} = \widetilde{g} \tag{9.3.1}$$

has a unique solution $\widetilde{u} \in e^{\gamma t} H^s$. Its restriction to $\{t \leq T\}$ satisfies (9.2.1). This proves the existence part of the statement.

The uniqueness follows from Corollary 9.3.4. □

9.4 The mixed Cauchy problem

We now consider the mixed Cauchy-boundary value problem:

$$\left\{\begin{array}{rl} Lu = f & \textit{on } [0,T]\times\mathbb{R}^d_+ \\ Mu_{|x=0} = g & \textit{on } [0,T]\times\mathbb{R}^{d-1} \\ u_{|t=0} = u_0 & \textit{on } \mathbb{R}^d_+ \end{array}\right. \tag{9.4.1}$$

We first solve the problem in L^2 and next study the existence of smooth solutions.

When $u\in L^2([0,T]\times\mathbb{R}^d_+)$ and $Lu\in L^2([0,T]\times\mathbb{R}^d_+)$, the trace $u_{|x=0}$ is defined in $H^{-1/2}_{loc}(]0,T[\times\mathbb{R}^{d-1})$ thus the boundary condition makes sense. We will construct solution in the space $C^0([0,T];L^2(\mathbb{R}^d_+))$ identified with a subspace of $L^2([0,T]\times\mathbb{R}^d_+)$ and for such u the initial condition is meaningful.

9.4.1 L^2 solutions

The starting point is an energy estimate. Note that, by standard trace theorems (see also Lemma 9.2.5) all $u\in H^1([0,T]\times\mathbb{R}^d_+)$ belongs to $C^0([0,T];H^{1/2}(\mathbb{R}^d_+))\subset C^0([0,T];L^2(\mathbb{R}^d_+))$. In particular, for such u, the value of u at time $t\in[0,T]$, denoted by $u(t)$, is well defined in $L^2(\mathbb{R}^d_+)$.

Proposition 9.4.1. *There is a constant C such that for all $T>0$, all $u\in H^1([0,T]\times\mathbb{R}^d_+)$ and all $t\in[0,T]$, the following inequality holds:*

$$\begin{aligned}\|u(t)\|_{L^2(\mathbb{R}^d_+)} + \|u_{|x=0}\|_{L^2([0,t]\times\mathbb{R}^{d-1})} \le C\Big(&\|u_0\|_{L^2(\mathbb{R}^d_+)} \\ &+\int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)}ds + \|g\|_{L^2([0,t]\times\mathbb{R}^{d-1})}\Big).\end{aligned} \tag{9.4.2}$$

where $u_0=u(0)$, $f:=Lu$ and $g:=Mu_{|x=0}$.

Since $u\in H^1$, $f=Lu$ belongs to L^2, thus

$$\|f(t)\|_{L^2(\mathbb{R}^d_+)} = \Big(\int |f(t,y,x)|^2dydx\Big)^{1/2}$$

is well defined in $L^2([0,T])$, thus in $L^1([0,T])$.

Proof. By integration by parts, as in Proposition 9.2.9, there holds:

$$\begin{aligned}2\mathrm{Re}\,\big(Sf,u\big)_{L^2([0,t]\times\mathbb{R}^d_+)} = \big(Su(t),u(t)\big)_{L^2(\mathbb{R}^d_+)} &- \big(Su(0),u(0)\big)_{L^2(\mathbb{R}^d_+)} \\ &- \big(SA_du_{|x=0},u_{|x=0}\big)_{L^2([0,T]\times\mathbb{R}^{d-1})}.\end{aligned}$$

Since S is definite positive and using Lemma 9.2.8, this implies

$$\|u(t)\|^2_{L^2(\mathbb{R}^d_+)} + \|u_{|x=0}\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} \leq C\Big(\|u(0)\|^2_{L^2(\mathbb{R}^d_+)} + \|g\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} + \int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)}\|u(s)\|_{L^2(\mathbb{R}^d_+)}ds\Big).$$

Taking the supremum of these estimates for $t' \in [0,t]$, we can replace in the left hand side $\|u\|^2_{L^2(\mathbb{R}^d_+)}$ by $n^2(t)$ where $n(t) := \sup_{t'\in[0,t]} \|u(t')\|_{L^2(\mathbb{R}^d_+)}$. Moreover, the integral in the right hand side is smaller than

$$n(t)\int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)}ds \leq \varepsilon n^2(t) + \varepsilon^{-1}\Big(\int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)}ds\Big)^2.$$

Choosing ε small enough to absorb $C\varepsilon n^2$ from the right to the left, yields, with a new constant C:

$$n^2(t) + \|u_{|x=0}\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} \leq C\Big(\|u(0)\|^2_{L^2(\mathbb{R}^d_+)} + \|g\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} + \big(\int_0^t \|f(s)\|_{L^2(\mathbb{R}^d_+)}\big)^2 ds\Big)$$

and (9.4.2) follows. □

This estimate has consequences for *strong* solutions of (9.4.1).

Definition 9.4.2. *Given $f \in L^2([0,T]\times\mathbb{R}^d_+)$, $g \in L^2([0,T]\times\mathbb{R}^{d-1})$ and $u_0 \in L^2(\mathbb{R}^d_+)$, we say that $u \in L^2([0,T]\times\mathbb{R}^d_+)$ is a strong L^2-solution of (9.4.1) if there is a sequence $u^n \in H^1([0,T]\times\mathbb{R}^d_+)$ such that $u^n \to u$, $Lu^n \to f$, $M{u^n}_{|x=0} \to g$ and $u^n(0) \to u_0$ in L^2.*

Proposition 9.4.3. *If $u \in L^2([0,T]\times\mathbb{R}^d_+)$ is a strong L^2-solution of (9.4.1), then u satisfies the equations (9.4.1), $u \in C^0([0,T];L^2(\mathbb{R}^d_+))$, its trace $u_{|x=0}$ belongs to $L^2([0,T]\times\mathbb{R}^{d-1})$ and the energy inequalities (9.4.2) are satisfied.*

Proof. Suppose that u^n is a sequence in H^1 such that $u^n \to u$, $Lu^n \to f$, $M{u^n}_{|x=0} \to g$ and $u^n(0) \to u_0$ in L^2.

Applying the estimate (9.4.2) to differences $u^n - u^m$, we conclude that u^n is a Cauchy sequence in $C^0([0,T];L^2(\mathbb{R}^d_+))$ and that the traces ${u^n}_{|x=0}$ form a Cauchy sequence in $L^2([0,T]\times\mathbb{R}^{d-1})$. Hence u^n converges to a limit $v \in C^0([0,T];L^2(\mathbb{R}^d_+))$ and the traces ${u^n}_{|x=0}$ converge to a limit $h \in L^2([0,T]\times\mathbb{R}^{d-1})$. Since $u^n \to u$ in L^2, by uniqueness of the limit in the sense of distributions, $v = u$. Moreover, $Lu^n \to Lu$ in the sense of distributions, thus $Lu = f$. Using Lemmas 9.3.3 and 9.2.5, we get that the traces ${u^n}_{|x=0}$ converge to $u_{|x=0}$ in $H^{-1/2}_{loc}(]0,T[\times\mathbb{R}^{d-1})$, and since the traces converge to h in L^2, this implies that $u_{|x=0} = h \in L^2([0,T]\times\mathbb{R}^{d-1})$. In particular, $Mu_{|x=0} = \lim M{u^n}_{|x=0} = g$. Since $u^n \to u$ in $C^0([0,T];L^2)$, there holds $u^n(0) \to u(0)$ and thus $u(0) = u_0$ in L^2. This shows that u is a solution of (9.4.1) and that the trace on $\{x = 0\}$ is in L^2.

Knowing the convergences $u^n \to u$ in $C^0([0,T];L^2)$, $Lu^n \to f$, $u^n_{|x=0} \to u_{|x=0}$ in L^2, we can pass to the limit in the energy estimates for u^n, and so obtain that u satisfies (9.4.2). ☐

Remark 9.4.4. This statement applies to solutions of (9.2.1). Suppose that $f \in L^2(]-\infty,T]\times\mathbb{R}^d_+)$ and $g \in L^2(]-\infty,T]\times\mathbb{R}^{d-1})$ vanish for $t < 0$. The unique solution $u \in L^2(]-\infty,T]\times\mathbb{R}^d_+)$ of (9.2.1) which vanishes when $t < 0$, given by Theorem 9.3.6 is a strong solution by Proposition 9.2.12, or as seen by writing $f = \lim f^n$, $g = \lim g^n$ with $f^n \in H^1$, $g^n \in H^1$ vanishing when $t \leq 0$. Then, by Theorem 9.3.6, the solution u^n of (9.2.1) with data (f^n, g^n) belongs to H^1 and converge in L^2 to u. Since u^n vanishes for $t < 0$ and $u^n \in H^1$, the trace of u^n on $\{t = T_0\}$ vanishes, i.e. $u^n(T_0) = 0$ for all $T_0 \leq 0$. This shows that u, restricted to $\{t \geq T_0\}$ is a strong solution of (9.4.1) with vanishing initial data at time T_0. Thus, $u \in C^0(]-\infty,T];L^2(\mathbb{R}^d_+))$ and the estimates (9.4.2) hold.

We can now state the main theorem.

Theorem 9.4.5. *For all $u_0 \in L^2(\mathbb{R}^d_+)$, $f \in L^2([0,T]\times\mathbb{R}^d_+)$ and $g \in L^2([0,T]\times\mathbb{R}^{d-1})$, there is a unique solution $u \in C^0([0,T],L^2(\mathbb{R}^d_+))$ of* (9.4.1)*. It is a strong solution, its trace on $\{x = 0\}$ belongs to $L^2([0,T]\times\mathbb{R}^{d-1})$ and the energy estimate* (9.4.2) *is satisfied.*

Proof. **a)** *Existence.* Denote by $H^1_0(\mathbb{R}^d_+)$ the space of functions in $v \in H^1(\mathbb{R}^d_+)$ such that $v_{|x=0} = 0$. Since $H^1_0(\mathbb{R}^d_+)$ is dense in $L^2(\mathbb{R}^d_+)$, there is a sequence u^n_0 such that:

$$u^n_0 \in H^1_0(\mathbb{R}^d_+), \quad \|u^n_0 - u_0\|_{L^2} \to 0.$$

Considered as a function independent of t, u^n_0 belongs to $H^1([0,T]\times\mathbb{R}^d_+)$, its trace on $x = 0$ vanishes and $Lu^n_0 \in L^2([0,T]\times\mathbb{R}^{d+})$. By density of smooth functions with compact support in L^2, there is a function f^n such that

$$f^n \in H^1(]-\infty,T]\times\mathbb{R}^d_+), \quad f^n_{|t<0} = 0, \quad \|f^n - (f - Lu^n_0)\|_{L^2([0,T]\times\mathbb{R}^d_+)} \leq \frac{1}{n}.$$

Similarly, there is g^n such that

$$g^n \in H^1(]-\infty,T]\times\mathbb{R}^{d-1}), \quad g^n_{|t<0}=0, \quad \|g^n-g\|_{L^2([0,T]\times\mathbb{R}^{d-1})} \le \frac{1}{n}.$$

By Theorem 9.2.14, there is a unique function v^n, such that

$$v^n \in H^1(]-\infty,T]\times\mathbb{R}^d_+), \quad Lv^n = f^n, \quad v^n{}_{|t<0}=0, \quad Mv^n{}_{|x=0}=g^n.$$

In particular, since $v^n \in H^1$, $v^n \in C^0(]-\infty,T];L^2(\mathbb{R}^d_+))$, and since $v^n=0$ when $t<0$, this implies that $v^n(0)=0$.

Consider u^n the restriction on $[0,T]\times\mathbb{R}^d_+$ of $v^n+u^n_0$. It belongs to $H^1(]-\infty,T]\times\mathbb{R}^d_+)$, its trace on $\{x=0\}$ is equal to the trace of v^n, thus $Mu^n{}_{|x=0}=g^n\to g$ in L^2. Moreover, $u^n(0)=u^n_0\to u_0$ in L^2 and $Lu^n = f^n+Lu^n_0\to f$. Thus, applying the estimate (9.4.2) to differences u^n-u^m, we conclude that u^n is a Cauchy sequence in $C^0([0,T];L^2(\mathbb{R}^d_+))$. Thus u^n converges to a limit u in $C^0([0,T];L^2(\mathbb{R}^d_+))$, thus in $L^2([0,T]\times\mathbb{R}^d_+)$. The properties listed above show that u is a strong solution of (9.4.1), thus a solution which satisfies the estimates (9.4.2).

b) *Uniqueness.* Suppose that $u\in C^0([0,T];L^2(\mathbb{R}^d_+))$ satisfies $Lu=0$, $Mu_{|x=0}=0$ and $u(0)=0$. Consider a C^∞ non decreasing function $\chi(t)$ such that $\chi=0$ for $t<1$ and $\chi(t)=1$ for $t>2$. For $\delta>0$, let $\chi_\delta(t)=\chi(t/\delta)$. Consider u_δ the extension by 0 for $t\le 0$ of $\chi_\delta u$. Thus Lu_δ is the extension by 0 of $(\partial_t\chi_\delta)u$ and thus belongs to L^2. Moreover, the trace of u_δ is the extension of $\chi_\delta u_{|x=0}$. Thus $Mu_{\delta|x=0}=0$. Therefore, u_δ is a solution of (9.2.1) which vanishes in the past. By Remark 9.4.4, it is a strong solution and the energy estimates (9.4.2) are satisfied. Hence, for $t\ge 2\delta$

$$\|u(t)\|_{L^2} \le C\int_0^t (\partial_t\chi_\delta)(s)\|u(s)\|_{L^2}ds = C\int_1^2 (\partial_t\chi(s)\|u(\delta s)\|_{L^2}ds.$$

Since $u\in C^0([0,T];L^2)$ and $u(0)=0$, the right hand side converges to zero as δ tends to zero, implying that $u=0$. □

9.4.2 Compatibility conditions

In order to solve the mixed Cauchy problem in Sobolev spaces, compatibility conditions are needed. For instance, the initial and boundary conditions imply that necessarily

(9.4.3) $$Mu_{0|x=0} = g_{|t=0} = Mu_{|t=0,x=0},$$

provided that the traces are defined. Next, denote by A the operator

$$Au = \sum_{j=1}^{d} A_j \partial_j .$$

Thus, if $Lu = f$, $\partial_t u = f - Au$ and therefore

$$u_1 := \partial_t u_{|t=0} = -Au_0 + f_0$$

if $f_0 = f_{|t=0}$. Thus, provided that the traces are defined,

$$(9.4.4) \qquad Mu_{1|x=0} = M(f_0 - Au_0)_{|x=0} = g_1 := \partial_t g_{|t=0} = M\partial_t u_{|t=0,x=0} .$$

These conditions are *necessary* for the existence of a smooth solution. Continuing the Taylor expansions to higher order yields higher order condition as we now explain.

For u smooth enough denote by $u_j = \partial_t^j u_{|t=0}$ the traces at $t = 0$ of the derivatives of u. For instance, if $u \in H^s$, $s \geq 1$, they are defined for $j \leq s - 1$. Similarly, we note $f_j = \partial_t^j f_{|t=0}$ and $g_j = \partial_t^j g_{|t=0}$ when they are defined. If u is a solution of $Lu = f$, then for $j \geq 1$:

$$u_j = f_{j-1} - Au_{j-1}$$

By induction, this implies that

$$(9.4.5) \qquad u_j = (-A)^j u_0 + \sum_{l=0}^{j-1} (-A)^{j-l-1} f_l .$$

The boundary condition $Mu_{|x=0} = g$ implies that

$$Mu_{j\,|x=0} = g_j$$

Thus necessarily, for smooth enough functions, solutions of (9.4.1) must satisfy on the edge $\{t = 0, x = 0\}$:

$$(9.4.6) \qquad M\Big((-A)^j u_0 + \sum_{l=0}^{j-1} M(-A)^{j-l-1} f_l\Big)_{|x=0} = g_j .$$

Lemma 9.4.6. *For $s \geq 1$, $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$, the left and right hand sides of (9.4.6) are defined for $j \in \{0, \ldots, s-1\}$ and belong to $H^{s-j-1/2}(\mathbb{R}^{d-1})$.*

Proof. For $u_0 \in H^s$, $A^j u_0 \in H^{s-j}$ and the trace $(A^j u_0)_{|x=0}$ is defined for $j < s$ and belongs to $H^{s-j-1/2}(\mathbb{R}^{d-1})$. For $f \in H^s$, the traces f_l are defined for $l \le s-1$ and belong to $H^{s-l-1/2}$. Thus, $A^{j-l-1} f_l \in H^{s-j+1/2}$ and the traces $(A^{j-l-1} f_l)_{|x=0}$ are defined for $j < s$ and belong to $H^{s-j}(\mathbb{R}^{d-1})$. For $g \in H^s$, the traces g_j are defined for $j < s$ and belong to $H^{s-j-1/2}(\mathbb{R}^{d-1})$. □

The lemma shows that the following definition makes sense.

Definition 9.4.7. *The data $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ satisfy the compatibility conditions to order $\sigma \le s-1$ if the equations* (9.4.6) *hold for all $j \in \{0, \dots, \sigma\}$.*

For instance, the first two conditions, given by (9.4.3) and (9.4.4) are

$$Mu_{0|x=0} = g_{|t=0}\,, \tag{9.4.7}$$

$$(MAu_0)_{|x=0} = f_{0|x=0} - g_1\,. \tag{9.4.8}$$

When $s = 0$, there are no compatibility condition. When $s = 1$, there is only one, (9.4.7). When $s = 2$, there are two conditions, (9.4.7) and (9.4.8), etc.

Remark 9.4.8. Suppose that $f = 0$ and $g = 0$. In this case, the compatibility conditions read $M(A^j u_0)_{|x=0} = 0$. Considering the operator A with domain $D(A) = \{u \in L^2(\mathbb{R}^d_+); Au \in L^2(\mathbb{R}^d_+) \,\text{and}\, Mu_{|x=0} = 0\}$, the compatibility conditions of order s reads $u_0 \in D(A^s)$.

The next result is useful in the construction of smooth solutions.

Proposition 9.4.9. *Suppose that $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ are compatible to order $s-1$. Then there are sequences $u_0^n \in H^{s+1}(\mathbb{R}^d_+)$, $f^n \in H^{s+1}([0,T] \times \mathbb{R}^d_+)$ and $g^n \in H^{s+1}([0,T] \times \mathbb{R}^{d-1})$, compatible to order s, such that $u_0^n \to u_0$, $f^n \to f$ and $g^n \to g$ in H^s.*

Proof. **a)** Consider first the case $s = 0$. Then u_0, f and g are arbitrary data in L^2. One easily construct approximating sequences u_0^n, f^n, g^n arbitrarily smooth and compatible to any order, by approximating the data by C^∞ functions which vanish near $t = 0, x = 0$.

b) Suppose now that $s = 1$, u_0, f and g are data in H^1 which satisfy the first compatibility condition (9.4.7). Consider sequences u_0^n, f^n, g^n in H^2, which converge in H^1 to u_0, f and g respectively. By (9.4.7) and the continuity of the traces, $r_0^n := g^n_{|t=0} - Mu^n_{0|x=0}$ satisfies

$$r_0^n \in H^{3/2}(\mathbb{R}^{d-1})\,, \quad \|r_0^n\|_{H^{1/2}(\mathbb{R}^{d-1})} \to 0\,.$$

To construct H^2 data $(u_0^n + v^n, f^n, g^n)$ which are compatible to first order, it is sufficient to construct v^n such that:

$$v^n \in H^2(\mathbb{R}^d_+)\,, \quad \|v^n\|_{H^1} \to 0\,, \quad M v^n_{|x=0} = r^n_0\,, \quad M(Av^n)_{|x=0} = r^n_1\,,$$

with $r^n_1 = M(Au^n_0)_{|x=0} - f^n_{|x=t=0} - \partial_t g^n_{|t=0} \in H^{1/2}(\mathbb{R}^{d-1})$. Since M is onto, there is a $N \times N_+$ matrix, M', such that $MM' = \mathrm{Id}$. Thus is is sufficient to find v^n such that

$$\text{(9.4.9)} \qquad v^n \in H^2(\mathbb{R}^d_+)\,, \quad \|v^n\|_{H^1} \to 0\,, \quad v^n_{|x=0} = h^n_0\,, \quad (Av^n)_{|x=0} = h^n_1\,,$$

with $h^n_0 = M' r^n \in H^{3/2}$, $h^n_1 = M' r^n_1 \in H^{1/2}$. Moreover, $h^n_0 \to 0$ in $H^{1/2}$.

Note that (9.4.9) concerns only functions of $(y, x) \in \mathbb{R}^d$ and their traces on $\{x = 0\}$. We recall the classical construction of Poisson operators. Consider $\phi \in C_0^\infty(\mathbb{R})$, $\phi \geq 0$, such that $\phi(x) = 1$ for $|x| \leq 1$ Denoting here by $\hat{v}$ the Fourier transform with respect to y, consider the operator

$$K : h \mapsto Kh\,, \quad \widehat{Kh}(\eta, x) = \phi(x\langle\eta\rangle)\hat{h}(\eta)$$

with $\langle\eta\rangle = (1 + |\eta|^2)^{1/2}$. Then, K is bounded from $H^{1/2}(\mathbb{R}^{d-1})$ to $H^1(\mathbb{R}^d_+)$ and from $H^{3/2}(\mathbb{R}^{d-1})$ to $H^2(\mathbb{R}^d_+)$. Moreover, $(Kh)_{|x=0} = h$. Consider $v^n_0 = Kh^n$. Then, $v^n_0 \in H^2$, $v^n_0{}_{|x=0} = h^n_0$ and $v^n_0 \to 0$ in H^1. Therefore, to find a solution $v^n = v^n_0 + w^n$ of (9.4.9), it is sufficient to find w^n which satisfy the same properties with $h^n_0 = 0$ and h^n_1 replaced by $k^n_1 = h^n_1 - (Av^n_0)_{|x=0} \in H^{1/2}$. In addition, $A = A_d\partial_x + A'$ where $A' = \sum_{j<d} A_j\partial_j$. Thus, is is sufficient to find w^n such that

$$\text{(9.4.10)} \qquad w^n \in H^2(\mathbb{R}^d_+)\,, \quad \|w^n\|_{H^1} \to 0\,, \quad w^n_{|x=0} = 0\,, \quad \partial_x w^n{}_{|x=0} = k^n\,,$$

with $k^n = A_d^{-1} k^n_1 \in H^{1/2}$.

We use a Poisson operator P_n defined by

$$\widehat{P_n h}(\eta, x) = x\phi(\lambda_n x\langle\eta\rangle)\hat{h}(\eta)$$

where $\lambda_n \geq 1$ is to be chosen. We note that P_n maps $H^{1/2}(\mathbb{R}^{d-1})$ to $H^2(\mathbb{R}^d_+)$, that $(P_n h)_{|x=0} = 0$ and $(\partial_x P_n h)_{|x=0} = h$. Thus, $w^n = P_n k^n$ satisfies the first, third and fourth property in (9.4.10). It remains to show that one can choose the sequence λ_n such that $w^n \to 0$ in H^1.

Elementary computations using Plancherel's theorem, show that

$$\|P_n h\|^2_{H^1(\mathbb{R}^d_+)} \leq C \int \psi_n(\eta)|\hat{h}(\eta)|^2 d\eta$$

with C independent of n and h and

$$\psi_n(\eta) = \int_0^\infty \Big((x^2\langle\eta\rangle^2 + 1)|\phi(\lambda_n x\langle\eta\rangle)|^2 + \lambda_n^2 x^4 \langle\eta\rangle^2 |\phi'(\lambda_n x\langle\eta\rangle)|^2 \Big) dx\,.$$

For $\lambda_n \geq 1$, there holds

$$\psi_n(\eta) \leq \frac{C}{\lambda_n\langle\eta\rangle}$$

with C independent of n. Therefore

$$\|w^n\|_{H^1(\mathbb{R}^d_+)} \leq \frac{C}{\sqrt{\lambda_n}} \|k^n\|_{H^{-1/2}(\mathbb{R}^{d-1})}\,.$$

One can now choose λ_n such that the right hand side converges to zero, showing that w^n satisfies (9.4.10). This finishes the proof of the proposition when $s = 1$.

c) When $s \geq 2$, the proof is similar. One is reduced to find $v^n \in H^{s+1}(\mathbb{R}^d_+)$ such that $v^n \to 0$ in H^n and $(A^j v^n)_{|x=0} = h^n_j$ where the h^n_j are given in $H^{s-j+1/2}(\mathbb{R}^{d-1})$ for $j \leq s$ and converge to zero in $H^{s-j-1/2}(\mathbb{R}^{d-1})$ for $j \leq s-1$. We first lift up the $s-1$ first traces by a fixed Poisson operator, and reduce the problem to find $w^n \in H^{s+1}(\mathbb{R}^d_+)$ such that $w^n \to 0$ in H^n and $(\partial_x^j w^n)_{|x=0} = 0$ when $j \leq s-1$ and $(\partial_x^s w^n)_{|x=0} = k^n \in H^{1/2}(\mathbb{R}^{d-1})$. We lift up the traces using a Poisson operator

$$\widehat{P_n h}(\eta, x) = \frac{x^j}{j!}\phi(\lambda_n x\langle\eta\rangle)\hat{h}(\eta)\,, \tag{9.4.11}$$

and show that if the sequence λ_n is properly chosen $w^n = P_n k^n$ has the desired properties. The details are left as an exercise. □

9.4.3 Smooth solutions

Definition 9.4.10. *$W^s(T)$ denotes the space of $u \in C^0([0,T], H^s(\mathbb{R}^d_+))$ such that for all $j \leq s$, $\partial_t^j u \in C^0([0,T], H^{s-j}(\mathbb{R}^d_+))$.*

$W^s(T)$ is considered as a subspace of $H^s([0,T]\times\mathbb{R}^d_+)$ and $H^{s+1}([0,T]\times\mathbb{R}^d_+) \subset W^s(T)$. We also use the notation

$$|\!|\!| u(t) |\!|\!|_s = \sum_{j=0}^{s} \|\partial_t^j u(t)\|_{H^{s-j}(\mathbb{R}^d_+)}\,. \tag{9.4.12}$$

This function is bounded (and continuous) in time when $u \in W^s$ and in L^2 when $u \in H^s$.

We first state an a-priori estimate for smooth solutions.

Proposition 9.4.11. *There is a constant C such that for all $T > 0$, all $u \in H^{s+1}([0,T] \times \mathbb{R}^d_+)$ and all $t \in [0,T]$, the following inequality holds:*

$$\begin{aligned}
&|||u(t)|||_s + \|u_{|x=0}\|_{H^s([0,t]\times\mathbb{R}^{d-1})} \le C\Big(|||u(0)|||_s + \\
&\qquad + \int_0^t |||f(t')|||_s dt' + \|g\|_{H^s([0,t]\times\mathbb{R}^{d-1})}\Big) .
\end{aligned} \tag{9.4.13}$$

where $f := Lu$ and $g := Mu_{|x=0}$.

Proof. Consider the tangential derivatives $u_\alpha := \partial^\alpha_{t,y} u$ for $\alpha \in \mathbb{N}^d$, $|\alpha| \le s$. Since $u \in H^{s+1}$, they satisfy

$$Lu_\alpha = f_\alpha := \partial^\alpha_{t,y} f\,, \quad Mu_{\alpha|x=0} = g_\alpha := \partial^\alpha_{t,y} g\,.$$

Introduce the tangential norm

$$|||u(t)|||'_s := \sum_{|\alpha|\le s} \|\partial^\alpha_{t,y} u(t)\|_{L^2}\,.$$

The L^2 estimates (9.4.2) imply that

$$\begin{aligned}
|||u(t)|||'_s + \|u_{|x=0}\|_{H^s([0,t]\times\mathbb{R}^{d-1})} \le C\Big(&|||u(0)|||'_s \\
&+ \int_0^t |||f(t')|||'_s dt' + \|g\|_{H^s([0,t]\times\mathbb{R}^{d-1})}\Big)
\end{aligned}$$

which is dominated by the right hand side of (9.4.13). It remains to estimate the normal derivatives by tangential ones, using the equation (9.2.9). By induction, one proves that

$$|||u(t)|||_s \le C\big(|||u(t)|||'_s + |||f(t)|||_{s-1}\big)\,.$$

Since

$$|||f(t)|||_{s-1} \le |||f(0)|||_{s-1} + \int_0^t |||\partial_t f(t')|||_{s-1} dt'$$

and

$$|||f(0)|||_{s-1} \le |||u(0)|||_s\,, \quad |||\partial_t f(t')|||_{s-1} \le |||f(t')|||_s\,,$$

the estimate (9.4.13) follows. ∎

We can now prove the main theorem of this chapter.

Theorem 9.4.12. *For all* $u_0 \in H^s(\mathbb{R}^d_+)$*,* $f \in H^s([0,T] \times \mathbb{R}^d_+)$ *and* $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ *satisfying the compatibility conditions up to order* $s-1$*, there is a unique solution* $u \in W^s(T)$ *of* (9.4.1)*. Moreover, the trace of the solution* u *on* $\{x = 0\}$ *is in* $H^s([0,T] \times \mathbb{R}^{d-1})$ *and* u *satisfies the estimates* (9.4.13)*.*

Proof. When $s = 0$, this is Theorem 9.4.5. We suppose now that $s \geq 1$.
Step 1. Solve the equation with a loss of smoothness.

We prove that when u_0, f and g belong to H^{s+2} and satisfy the compatibility condition up to order s, there is a solution in $H^{s+1}([0,T] \times \mathbb{R}^{d+}) \subset W^{s+1}(T)$.

With $f_l = \partial_t^l f_{|t=0} \in H^{s+1-l}(\mathbb{R}^d_+)$, consider the functions $u_j \in H^{s+2-j}(\mathbb{R}^d_+)$ defined by (9.4.5) for $j \leq s+2$. Then, there is $u^a \in H^{s+2+1/2}(\mathbb{R} \times \mathbb{R}^d_+)$ such that

$$\partial_t^j u^a{}_{|t=0} = u_j \,, \qquad \text{for } j \leq s+2 \,. \tag{9.4.14}$$

We look for a solution as $u = u' + u^a$. The equation for u' reads

$$Lu' = f' := f - Lu^a \,, \quad Mu'_{|x=0} = g' := g - Mu^a{}_{|x=0} \,, \quad u'_{|t=0} = 0 \,.$$

We have $f' \in H^{s+2} - H^{s+3/2} \subset H^{s+1}$ and comparing (9.4.14) and (9.4.5) we see that

$$\partial_t^j f'_{|t=0} = 0 \qquad \text{for } j \leq s \,. \tag{9.4.15}$$

Moreover, $g' \in H^{s+2}$ and the compatibility conditions imply that

$$\partial_t^j g'_{|t=0} = 0 \qquad \text{for } j \leq s \,. \tag{9.4.16}$$

Denote by $\tilde{f}'$ and $\tilde{g}'$ the extensions of f' and g' by 0 for $t < 0$. Then, the trace conditions (9.4.15) and (9.4.16) imply that $\tilde{f}' \in H^{s+1}(]-\infty, T] \times \mathbb{R}^d_+)$ and $\tilde{g}' \in H^{s+1}(]-\infty, T] \times \mathbb{R}^{d-1})$. Thus, by Theorem 9.3.6, the boundary value problem

$$L\tilde{u}' = \tilde{f}' \,, \quad M\tilde{u}'_{|x=0} = \tilde{g}'$$

has a unique solution $\tilde{u}' \in H^{s+1}(]-\infty, T] \times \mathbb{R}^d_+)$ which vanishes when $t \leq 0$. Thus $\tilde{u}'(0) = 0$ and denoting by u' the restriction of $\tilde{u}'$ to $t \geq 0$, $u = u' + u^a$ is a solution of (9.4.1).

Step 2. H^s data.

Given $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ satisfying the compatibility conditions up to order $s-1$, by repeated applications of Proposition 9.4.9, there is a sequence $u_0^\nu \in H^{s+2}(\mathbb{R}^d_+)$, $f^\nu \in H^{s+2}([0,T] \times \mathbb{R}^d_+)$ and $g^\nu \in H^{s+2}([0,T] \times \mathbb{R}^{d-1})$ satisfying the compatibility conditions up to order $s+1$ and converging in H^s to u_0, f and g respectively.

We note that for solutions of (9.4.1),

$$\|u(0)\|_s = \sum_{j \le s} \|u_j\|_{H^{s-j}}$$

where the u_j are defined at (9.4.5). Thus $\|u^\nu(0) - u^\mu(0)\|_s$ tends to zero as μ and ν tend to infinity. Therefore, the energy estimates (9.4.13) imply that the sequence u^ν is a Cauchy sequence in $W^s(T)$ and therefore converges to $u \in W^s(T)$. Since $s \ge 1$, the limit u is clearly a solution of (9.4.1). The uniqueness follows from the L^2 uniqueness of Theorem 9.4.5. passing to the limit in the energy estimates for the u^ν implies that u also satisfies (9.4.13). $\square$

9.5 Nonlinear mixed problems

Consider the equation

$$\begin{cases} Lu = F(u) + f & \text{on } [0,T] \times \mathbb{R}^d_+ \\ Mu_{|x_d=0} = g & \text{on } [0,T] \times \mathbb{R}^{d-1} \\ u_{|t=0} = u_0 & \text{on } \mathbb{R}^d_+ \end{cases} \tag{9.5.1}$$

We assume that $F(0) = 0$, so that it makes sense to look for solutions vanishing at infinity and in Sobolev spaces H^s.

Theorem 9.5.1. *Let s be and integer $s > d/2$.*

i) Suppose that $f \in H^s([0,T_0] \times \mathbb{R}^d_+)$, $g \in H^s([0,T_0] \times \mathbb{R}^{d-1})$ and $u_0 \in H^s(\mathbb{R}^d_+)$. Suppose that the compatibility conditions of section 2.5.2 below are satisfied up to the order $s-1$. Then there is $T \in]0,T_0]$ such that the problem (9.5.1) has a unique solution $u \in W^s(T)$.

ii) If $\sigma > s$ and the data (f, g, u_0) belong to $H^\sigma([0,T] \times \mathbb{R}^d_+)$, $H^\sigma([0,T] \times \mathbb{R}^{d-1})$ and $H^\sigma(\mathbb{R}^d_+)$ respectively and satisfy the compatibility conditions to order $\sigma - 1$, then the solution u given by i) belongs to $W^\sigma(T)$.

9.5.1 Nonlinear estimates

Recall the following multiplicative properties of Sobolev spaces.

Proposition 9.5.2. *For non negative integers $s > d/2$ and j, k such that $j + k \leq s$ there is C such that for $u \in H^{s-j}(\mathbb{R}^d_+)$ and $v \in H^{s-k}(\mathbb{R}^d_+)$ the product $uv \in H^{s-j-k}(\mathbb{R}^d_+)$ and*

$$\|uv\|_{H^{s-j-k}} \leq C\|u\|_{H^{s-j}}\|v\|_{H^{s-k}} . \tag{9.5.2}$$

Corollary 9.5.3. *Let F be a C^∞ function such that $F(0) = 0$. For all $s > d/2$, there is a nondecreasing function $C(\cdot)$ on $[0, +\infty[$ such that for all $T > 0$ and $u \in W^s(T)$, $F(u) \in W^s(T)$ and for all $t \in [0, T]$:*

$$|\!|\!| u(t) |\!|\!|_s \leq R \quad \Rightarrow \quad |\!|\!| F(u)(t) |\!|\!|_s \leq C(R) . \tag{9.5.3}$$

Moreover, for all $u \in W^s(T)$ and $v \in W^s(T)$ with $|\!|\!| u(t) |\!|\!|_s \leq R$ and $|\!|\!| v(t) |\!|\!|_s \leq R$:

$$|\!|\!| \{F(u) - F(v)\}(t) |\!|\!|_s \leq C(R) |\!|\!| \{u - v\}(t) |\!|\!|_s . \tag{9.5.4}$$

Proof. Since $F(0) = 0$, there holds

$$\|F(u)(t)\|_{L^2} \leq \|\nabla_u F\|_{L^\infty(B_R)}\|u(t)\|_{L^2} , \quad \text{with } R = \|u(t)\|_{L^\infty} \leq |\!|\!| u(t) |\!|\!|_s ,$$

where B_R denotes the ball of radius R in the space of states u. The last inequality follows from Sobolev embedding $H^s(\mathbb{R}^d_+) \subset L^\infty(\mathbb{R}^d_+)$.

Next we estimate derivatives. For smooth functions u, there holds

$$\partial^\alpha F(u) = \sum_{k=1}^{|\alpha|} \sum_{\alpha^1+\ldots+\alpha^k=\alpha} c(k, \alpha^1, \ldots, \alpha^k) F^k(u)\big(\partial^{\alpha^1}u, \ldots, \partial^{\alpha^k}u\big) \tag{9.5.5}$$

where the $c(k, \alpha^1, \ldots, \alpha^k)$ are numerical coefficients. Since the derivative $\partial^{\alpha^j}u(t)$ belong to $H^{s-|\alpha^j|}(\mathbb{R}^d_+)$ and satisfy

$$\|\partial^{\alpha^j}u(t)\|_{H^{s-|\alpha^j|}} \leq |\!|\!| u(t) |\!|\!|_s ,$$

Proposition 9.5.2 implies that each term in the right hand side of (9.5.5) belongs to $C^0(L^2)$ with norm bounded by $C(|\!|\!| u(t) |\!|\!|)$ and the estimate (9.5.3) follows.

The estimate of differences is similar. □

Recall next the Gagliardo-Nirenberg-Moser's inequalities, which hold with Ω equal to an Euclidian space $\mathbb{R}^n$ or a half space of $\mathbb{R}^n$, or a quadrant :

Proposition 9.5.4. *For all $s \in \mathbb{N}$, there is C such that for all α of length $|\alpha| \leq s$, all $p \in [2, 2s/|\alpha|]$ and all $u \in L^\infty(\Omega) \cap H^s(\Omega)$, the derivative $\partial^\alpha u$ belongs to $L^p(\Omega)$, and*

$$\|\partial^\alpha u\|_{L^p} \leq C \|u\|_{L^\infty}^{1-2/p} \|u\|_{H^s}^{2/p} . \tag{9.5.6}$$

The condition on p reads $\frac{|\alpha|}{s} \leq \frac{2}{p} \leq 1$. Recall that the the proof when $\Omega = \mathbb{R}^n$ relies on the identity

$$0 = \int \partial_j (u |\partial_j u|^{p-2} \partial_j u) = \int |\partial_j u|^p + (p-1) \int u \partial_j^2 u |\partial_j u|^{p-2} .$$

With Hölder inequality, this implies that

$$\|\partial_j u\|_{L^p} \lesssim \|u\|_{L^{p'}}^{1/2} \|\partial_j^2 u\|_{L^{p''}}^{1/2} , \quad \frac{2}{p} = \frac{1}{p'} + \frac{1}{p''} .$$

The estimate (9.5.6) follows by induction on s. Note that the proof applies not only to the ∂_j but also to any vector field.

Using extension operators, the estimate holds on any smooth domain Ω, but the constant depends on the domain. For instance, if $\Omega = [0,T] \times \mathbb{R}^d_+$, the constant are unbounded as $T \to 0$. However, splitting $u = \chi(t) u + (1 - \chi(t)) u$ with $\chi \in C^\infty$, $\chi = 0$ for $t \geq 2T/3$ and $\chi = 1$ for $t \leq T/3$, reduces the problem to functions $\chi(t) u$ and $(1 - \chi(t)) u$ which can be extended in H^s by 0 for $t \geq T$ and $t \leq 0$ respectively, hence reducing the problem on quadrants $[0, +\infty[\times \mathbb{R}^d_+$ or $] - \infty, T] \times \mathbb{R}^d_+$. Therefore:

Lemma 9.5.5. *Given $T_0 > 0$, there is C such that for all $T \geq T_0$ the estimates* (9.5.6) *are satisfied on $\Omega = [0,T] \times \mathbb{R}^d_+$*

Corollary 9.5.6. *Let F be a C^∞ function such that $F(0) = 0$. For all $s \in \mathbb{N}$, and $T_0 > 0$, there is a non decreasing function $C_F(\cdot)$ on $[0, \infty[$ such that for all $T \geq T_0$, for all $u \in L^\infty(\Omega) \cap H^s(\Omega)$ where $\Omega = [0,T] \times \mathbb{R}^d_+$, one has $F(u) \in H^s(\Omega)$ and*

$$\|F(u)\|_{H^s} \leq C_F(\|u\|_{L^\infty}) \|u\|_{H^s} . \tag{9.5.7}$$

Proof. We estimate the L^2 norm as above :

$$\|F(u)\|_{L^2} \leq \|\nabla_u F\|_{L^\infty(B_R)} \|u\|_{L^2} , \quad \text{with } R = \|u\|_{L^\infty} ,$$

where B_R denotes the ball of radius R in the space of states u. Next we estimate derivatives using (9.5.5) which is valid at least for smooth u. Using the estimate (9.5.6) for $\partial^{\alpha^j} u$ with $2/p_j = |\alpha^j|/s$, we see that each term in the right hand side of (9.5.5) has an L^2 norm bounded by the right hand side of (9.5.7). The formula and the estimates extend to $u \in L^\infty \cap H^s$ by density (Exercise). $\square$

9.5.2 Compatibility conditions

For (9.5.1), the definition of traces u_j is modified as follows. First, with $u_j = \partial_t^j u_{|t=0}$, there holds

$$\partial_t^j F(u)_{|t=0} = \mathcal{F}_j(u_0, \dots, u_j) \tag{9.5.8}$$

with $\mathcal{F}_j$ of the form

$$\mathcal{F}_j(u_0, \dots, u_j) = \sum_{k=1}^{j} \sum_{j^1 + \dots + j^k = j} c(k, j^1, \dots, j^k) F^k(u_0)\big(u_{j^1}, \dots, u_{j^k}\big)$$

The definition (9.4.5) is modified as follows: by induction let

$$u_j = -A u_{j-1} + f_{j-1} + \mathcal{F}_{j-1}(u_0, \dots, u_{j-1})\,. \tag{9.5.9}$$

Then, for $u_0 \in H^s$ and $f \in H^s$ with $s > d/2$, using Proposition 9.5.2, we see that $u_j \in H^{s-j}(\mathbb{R}^d_+)$ for $j \le s$.

Definition 9.5.7. *The data $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T] \times \mathbb{R}^d_+)$ and $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ satisfy the compatibility conditions to order $\sigma \le s-1$ if the u_j given by* (9.5.9) *satisfy*

$$M u_{j|x=0} = \partial_t^j g_{|t=0}\,, \quad j \in \{0, \dots, \sigma\}\,.$$

9.5.3 Existence and uniqueness

We prove here the first part of Theorem 9.5.1. Below, it is always assumed that $s > d/2$, $f \in H^s([0,T_0] \times \mathbb{R}^d_+)$, $g \in H^s([0,T_0] \times \mathbb{R}^{d-1})$ and $u_0 \in H^s(\mathbb{R}^d_+)$.

Proposition 9.5.8. *Suppose that the compatibility conditions are satisfied up to the order $s-1$. Then there is $T \in]0, T_0]$ such that the problem* (9.5.1) *has a solution $u \in W^s(T)$.*

Proof. **a)** *The iterative scheme.*

Let $u_0 \in H^s(\mathbb{R}^d_+)$, $f \in H^s([0,T]\times\mathbb{R}^d_+)$ and $g \in H^s([0,T]\times\mathbb{R}^{d-1})$. Define the $u_j \in H^{s-j}(\mathbb{R}^d_+)$ by (9.5.9). Let $u^0 \in H^{s+1/2}(\mathbb{R}\times\mathbb{R}^d_+)$ such that

$$\partial_t^j u^0{}_{|t=0} = u_j\,, \quad 0 \le j \le s\,. \tag{9.5.10}$$

We can assume that u^0 vanishes for $|t| \ge 1$ and thus $u^0 \in W^s(T)$ for all T. There is C_0 depending only on the data such that

$$\sum_{j\le s} \|u_j\|_{H^{s-j}} \le C_0\,, \quad |\!|\!| u^0(t) |\!|\!|_s \le C_0\,.$$

For future use, we note that C_0 depends only on the data: there is a uniform constant C such that

$$C_0 \le C\|u_0\|_{H^s} + |\!|\!| f(0) |\!|\!|_{s-1}\,. \tag{9.5.11}$$

For $n \ge 1$, we solve by induction the linear mixed problems

$$Lu^n = f + F(u^{n-1})\,, \quad Mu^n{}_{|x=0} = g\,, \quad u^n{}_{|t=0} = u_0\,. \tag{9.5.12}$$

Suppose that u^{n-1} is constructed in $W^s(T_0)$ and satisfies

$$\partial_t^j u^{n-1}{}_{|t=0} = u_j\,, \quad j \le s\,. \tag{9.5.13}$$

This is true for $n = 1$. Then, by definition of the $\mathcal{F}_j$ and by (9.5.13), $\partial_t^j F(u^{n-1})_{|t=0} = \mathcal{F}_j(u_0,\ldots,u_j)$. Next, for the linear problem (9.5.12) we compute the u_j^n by (9.4.5). Comparing with the definition (9.5.9), we see that $u_j^n = u_j$. Thus, the compatibility conditions $Mu_{j|x=0} = g_j$ imply that the data $(f + F(u^{n-1}), g, u_0)$ are compatible for the linear problem. Therefore, Theorem 9.4.12 implies that (9.5.12) has a unique solution $u^n \in W^s(T_0)$ and that

$$\partial_t^j u^n{}_{|t=0} = u_j^n = u_j\,.$$

This shows that the construction can be carried on and thus defines a sequence $u^n \in W^s(T_0)$ satisfying (9.5.12)

b) *Uniform bounds*

We show that we can choose R and $T \in]0, T_0]$ such that for all n:

$$\forall t \in [0,T]\ :\quad |\!|\!| u^n(t) |\!|\!|_s \le R\,. \tag{9.5.14}$$

By (9.5.10), this estimate is satisfied for $n = 0$ if $R \ge C_0$.

Assume that (9.5.14) is satisfied at order $n-1$. Next, the energy estimate (9.4.13) and Corollary 9.5.3 imply that there is a constant C and a function $C_F(\cdot)$ such that for $t \le T$

$$\|\!|u^n(t)\|\!|_s \le C\big(\|\!|u^n(0)\|\!|_s + TC_F(R) + C_1\big)$$

with

$$C_1 = \|g\|_{H^s([0,T_0]\times\mathbb{R}^{d-1})} + \int_0^{T_0} \|\!|f(t')\|\!|_s dt' . \tag{9.5.15}$$

By (9.5.13) at order n and (9.5.10):

$$\|\!|u^n(0)\|\!| = \sum_{j\le s} \|u_j\|_{H^{s-j}} \le C_0 .$$

Thus, (9.5.14) holds provided that

$$R \ge C_0 , \quad R \ge C(C_0 + C_1 + 1) \quad \text{and} \quad TC(R) \le 1 . \tag{9.5.16}$$

This can be achieved, choosing R first and next T. For such a choice, by induction, (9.5.14) is satisfied for all n.

c) *Convergence*

Write the equation satisfied by $w^n = u^{n+1} - u^n$ for $n \ge 1$. By (9.5.13), there holds $\|\!|w^n(0)\|\!|_s = 0$. Knowing the uniform bounds (9.5.14), estimating the nonlinear terms by Corollary 9.5.3 and using the energy estimate (9.4.13) one obtains that for $n \ge 2$ and $t \le T$:

$$\|\!|w^n(t)\|\!|_s \le CC_F(R) \int_0^t \|\!|w^{n-1}(t')\|\!|_s dt'$$

Thus there is K such that for all $n \ge 1$ and $t \in [0, T]$:

$$\|\!|w^n(t)\|\!|_s \le K^{n+1} t^{n-1}/(n-1)! .$$

This implies that the sequence u^n converges in $W^s(T)$, thus in the uniform norm and the limit is clearly a solution of (9.5.1). □

Next we prove uniqueness.

Proposition 9.5.9. *If* $T \in]0, T_0]$ *and* u^1 *and* u^2 *are two solutions of* (9.5.1) *in* $W^s(T)$, *then* $u^1 = u^2$.

Proof. The traces at $\{t = 0\}$ necessarily satisfy

$$\partial_t^j u^1{}_{|t=0} = \partial_t^j u^2{}_{|t=0} = u_j \,.$$

Thus $w = u^2 - u^1$ satisfies $\|\!| w(0) |\!\|_s = 0$. Write the equation for w. Using bounds for the norms of u^1 and u^2 in W^s, the energy estimates and Corollary 9.5.3 to estimate the nonlinear terms, imply that there is C such that for all $t \in [0, T]$:

$$\|\!| w(t) |\!\|_s \leq C \int_0^t \|\!| w(t') |\!\|_s dt' \,.$$

Thus $w = 0$. $\square$

9.5.4 A criterion for blow-up

Suppose that $f \in H^s([0, T_0] \times \mathbb{R}^d_+)$, $g \in H^s([0, T_0] \times \mathbb{R}^{d-1})$ and $u_0 \in H^s(\mathbb{R}^d_+)$, with $s > d/2$. Suppose that the compatibility conditions are satisfied at order $s - 1$. We have proved that there is a local solution in $W^s(T)$. The question is how long can the solution be extended. Let T_* denote the supremum of the set of $T \in]0, T_0]$ such that the problem (9.5.1) has a solution in $W^s(T)$. By uniqueness, there is a unique maximal solution u on $[0, T_*[$. The proof of Proposition 9.5.8 above gives an estimate from below of T^*: since by (9.5.11), (9.5.15) and (9.5.16), there is a function $C(\cdot)$ such that the solution is $W^s(T)$ for

$$T = \min\{T_0, C(K)\} \tag{9.5.17}$$

with

$$K = \|u_0\|_{H^s} + \|\!| f(0) |\!\|_{s-1} + \|g\|_{H^s([0,T_0]\times\mathbb{R}^{d-1})} + \int_0^{T_0} \|\!| f(t') |\!\|_s dt' \,. \tag{9.5.18}$$

Proposition 9.5.10. *If $T^* < T_0$ or if $T^* = T_0$ but $u \notin W^s(T_0)$, then*

$$\limsup_{t \to T^*} \|u(t)\|_{L^\infty} = +\infty \,. \tag{9.5.19}$$

Proof. Suppose that (9.5.19) is not true. This means that $u \in L^\infty([0, T^*[\times \mathbb{R}^d_+)$. From Proposition 9.5.8 we know that $T^* > T_1$ for some T_1 depending only on the data. Thus, by Corollary 9.5.6 there is a constant C_1, depending only on the L^∞ norm of u such that for all $T \in [T_1, T^*[$:

$$\|F(u)\|_{H^s([0,T]\times\mathbb{R}^d_+)} \leq C_1 \|u\|_{H^s([0,T]\times\mathbb{R}^d_+)} \,.$$

The energy estimate (9.4.13) implies that

$$\|u(t)\|_s^2 \leq C_0 + C\Big(\int_0^t \|F(u)(t')\|_s dt'\Big)^2 ,$$

where C_0 only depends on the data and C depends only on the operator L. Thus, using Cauchy-Schwarz inequality, we get that there is C such that for all $t \in [T_1, T^*[$

$$\begin{aligned} \|u(t)\|_s^2 \leq C_0 + C\|F(u)\|^2_{H^s([0,t]\times\mathbb{R}^d_+)} &\leq C_0 + CC_1\|u\|^2_{H^s([0,t]\times\mathbb{R}^d_+)} \\ &\leq C_0 + CC_1 \int_0^t \|u(t')\|_s^2 dt' . \end{aligned}$$

This implies that there is a constant C_3, depending only on C_0, C, C_1 and the norm of u in $W^s(T_1)$, such that

$$\sup_{t<T^*} \|u(t)\|_s \leq C_3 . \tag{9.5.20}$$

Next we consider the Cauchy problem for (9.5.1) with initial data $u(T^* - \delta)$ at time $T^* - \delta$. Because $u \in W^s(T^* - \delta/2)$ is a solution, computing the traces from the equation we see that the compatibility conditions are satisfied up to order $s - 1$. Therefore, by Proposition 9.5.8 there is a solution $\tilde{u}$ in W^s on the interval $[T^* - \delta, T_2]$. By (9.5.17), we have an estimate from below for T_2:

$$T_2 = \min\{T_0, T^* - \delta + C(K)\}$$

with

$$K = \|u(T^* - \delta)\|_{H^s} + \|f(T^* - \delta)\|_{s-1} + \|g\|_{H^s([0,T_0]\times\mathbb{R}^{d-1})} + \int_0^{T_0} \|f(t')\|_s dt' .$$

Since f and g are given in H^s, the last three terms are bounded independently of $T^* - \delta$. By (9.5.20), the first term is bounded independently of T^* and δ. This shows that the increment $C(K)$ is bounded from below independently of T^* and δ.

If T^* were strictly smaller that T_0, we could choose $\delta = C(K)/2$ so that $T_2 > T^*$. By uniqueness, $\tilde{u}$ would be an extension of u, contradicting the definition of T^*. If $T_* = T_0$, choosing again $\delta = C(K)/2$, we see that $T_2 = T_0$ and thus $u \in W^(T_0)$. $\square$

9.5.5 Regularity of solutions

Suppose that $T > 0$ is given, $f \in H^s([0,T] \times \mathbb{R}^d_+)$, $g \in H^s([0,T] \times \mathbb{R}^{d-1})$ and $u_0 \in H^s(\mathbb{R}^d_+)$, with $s > d/2$. Suppose that the compatibility conditions are satisfied at order $s-1$ and $u \in W^s(T)$ is a solution of (9.5.1). The next result finishes the proof of Theorem 9.5.1.

Proposition 9.5.11. *Suppose that $\sigma > s$ and (f, g, u_0) belong to $H^\sigma([0,T] \times \mathbb{R}^d_+)$, $H^\sigma([0,T] \times \mathbb{R}^{d-1})$ and $H^\sigma(\mathbb{R}^d_+)$ respectively and satisfy the compatibility conditions to order $\sigma - 1$, then the solution u belongs to $W^\sigma(T)$.*

Proof. By Proposition 9.5.8 there is $T_1 \in]0, T]$ such that the problem has a solution $\tilde{u} \in W^\sigma(T_1)$. Denote by T^* the maximal time of existence of solutions in W^σ. By uniqueness in $W^s(T')$ for $T' < T^*$, $u = \tilde{u}$ for $t < T^*$. Since $u \in W^s(T)$ and $s > d/2$, $u \in L^\infty([0,T] \times \mathbb{R}^d_+)$ and thus $\tilde{u} \in L^\infty([0, T^*[\times \mathbb{R}^d_+)$. Therefore Proposition 9.5.10 implies that $T^* = T$ and $u = \tilde{u} \in W^\sigma(T)$. □

Bibliography

[Blo] N. Bloembergen, *Nonlinear Optics*, W.A. Benjamin Inc., New York, 1965

[Bon] J.M.Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann.Sc.E.N.S. Paris, 14 (1981) pp 209-246.

[Boy] R. Boyd, *Nonlinear Optics*, Academic Press, 1992

[Ch-Pi] J.Chazarain, A. Piriou, *Introduction à la théorie des équations aux dérivées partielles linéaires*, Bordas, Paris, 1981.

[Ch] H.Chihara, *Local existence for semilinear Schrödinger equations*, Math.Japonica 42 (1995) pp 35-52.

[CM] R.Coifman and Y.Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque 57 (1978).

[Co-Fr] R.Courant, K.O.Friedrichs, *Supersonic flow and shock waves,* Wiley-Interscience, New York, 1948.

[Co-Hi] R.Courant, D.Hilbert, *Methods of Mathematical Physics,* John Wiley & Sons, New York, 1962.

[Ehr] L.Ehrenpreis, *Solutions of some problems of division III*, Amer. J. Math, 78 (1956), pp 685–715.

[Fr1] K.0.Friedrichs, *Symmetric hyperbolic linear differential equations,* Comm. Pure Appl.Math., 7 (1954), pp 345-392.

[Fr2] K.0.Friedrichs, *Symmetric positive linear differential equations,* Comm. Pure Appl.Math., 11 (1958), pp 333-418.

[Fr3] K.0.Friedrichs *Nonlinear hyperbolic differential equations for functions of two independent variables,* Amer. J. Math., 70 (1948) pp 555-589.

[Går] L. Gårding, *Problèmes de Cauchy pour des systèmes quasi-linéaires d'ordre un, strictement hyperboliques,* dans *Les EDP,* Colloques Internationaux du CNRS, vol 117, (1963), pp 33-40.

[Hör] L.Hörmander, *Lectures on Nonlinear Hyprbolic Differential Equations*, Mathématiques et Applications 26, Sringer Verlag, 1997.

[La] P.Lax, *Hyperbolic systems of conservation laws*, Comm. Pure Appl. Math., 10 (1957), pp 537-566.

[Ma1] A.Majda, *The stability of multidimensional shock fronts*, Mem. Amer. Math. Soc., n^o 275, 1983.

[Ma2] A.Majda, *The existence of multidimensional shock fronts*, Mem. Amer. Math. Soc., n^o 281, 1983.

[Ma3] A.Majda, *Compressible fluid flows and systems of conservation laws,* Applied Math.Sc., 53, Springer Verlag, 1953.

[Mal] B.Malgrange, *Existence et approximation des solutions des équations aux dérives partielles et des équations de convolution,* Ann. Inst. Fourier 6 (1955-56) pp 271-355.

[Mé1] G. Métivier, *Stability of multidimensional shocks.* Advances in the theory of shock waves, 25–103, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.

[Mé2] G. Métivier, *Small viscosity and boundary layer methods. Theory, stability analysis, and applications.* Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston, MA, 2004.

[MZ] G. Métivier, K.Zumbrun, *Viscous Boundary Layers for Noncharacteristic Nonlinear Hyperbolic Problems*, Memoirs of the AMS, number 826 (2005).

[Mey] Y.Meyer, *Remarques sur un théorème de J.M.Bony*, Supplemento al Rendiconti der Circolo Matematico di Palermo, Serie II, No1, 1981.

[Miz] S. Mizohata, *On the Cauchy problem*, Notes and Reports in Mathematics in Science an Engineering, 3, Academic Press, 1985.

[NM] A. Newell, J. Moloney, *Nonlinear Optics* Addison-Wesley, Reading Mass., 1992.

[Ste] E.Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.

[Tay] M.Taylor. *Partial Differential Equations*III, Applied Mathematical Sciences 117, Springer, 1996.