Space Propagation of Instabilities in Zakharov Equations

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Abstract

In this paper we study an initial boundary value problem for Zakharov's equations, describing the space propagation of a laser beam entering in a plasma. We prove a strong instability result and prove that the mathematical problem is ill-posed in Sobolev spaces. We also show that it is well posed in spaces of analytic functions. Several consequences for the physical consistency of the model are discussed.

Contents

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1 Introduction

Zakharov's equations [12] model electronic plasma waves, describing the coupling between the slowly varying envelope of the electric field E and the low-frequency variation of the density of the ions n. A commonly used form of the equations reads

$$\text{(1.1)} \qquad \begin{cases} i(\partial_t + \dfrac{k_0 c^2}{\omega_0}\partial_z)A + \dfrac{c^2}{2\omega_0}\Delta_x A = \dfrac{\omega_{pe}^2}{2n_0\omega_0}\delta n\, A, \\ (\partial_t^2 - c_s^2\Delta_x)\delta n = \dfrac{\omega_{pe}^2}{4\pi m_i c^2}\Delta_x |A|^2, \end{cases}$$

where ω_0 is the frequency of the laser, k_0 its wave number and ω_{pe} the plasma electronic frequency; they are linked by the dispersion relation $\omega_0^2 = \omega_{pe}^2 + k_0^2 c^2$ where c is the speed of light; n_0 the mean density of the plasma, m_i is the mass of the ions and c_s the sound velocity in the plasma. The space variables are (z, x), $z \in \mathbb{R}$ and $x \in \mathbb{R}^2$; z is the direction of propagation of the laser beam and x are the directions transversal to the propagation. In this model, the transversal dispersion is neglected.

Reduced to a dimensionless form (see in Section 6), the equations become

$$\text{(1.2)} \qquad \begin{cases} i(\epsilon\partial_t + \partial_z)E + \Delta_x E = nE, \\ (\partial_t^2 - \Delta_x)n = \Delta_x |E|^2 \end{cases}$$

See [9] or [10] for the introduction of this kind of models for numerical simulation.

The local in time Cauchy problem for (1.2) is now well understood: in [6], it is proved that it is well posed, locally in time, for data in suitable Sobolev spaces. This extends previous results for the classical Zakharov system, where transversal dispersion is taken into account, that is when Δ_x is replaced by $\Delta_{(z,x)}$ (see [10, 5, 7] and references therein). However, the system (1.2) is quite different from the classical Zakharov system since the Cauchy problem for *periodic* data exhibits strong instabilities of Hadamard's type ([4]). Note that periodic data are considered in the quoted paper as a model for data which do not vanish at infinity. We refer to [4] for a discussion of the physical relevance of the different frameworks. In particular, we recall in Section 6, why the periodic context is well adpated to physical situations, where the envelop of the beam has rapid oscillations (speckles).

In this paper we consider a boundary value problem for (1.2) which models the the propagation of a laser beam entering the plasma at an interface $\{z = 0\}$. This approach is very common in physics where people is actually more interested in describing *the propagation in space* rather than in time, i.e. considering z as the propagation variable. Indeed, this is an underlying idea in many of the paraxial approximations, like the one which yields the Schrödinger equation in (1.2). This idea is also very common in numerical simulations. In this case, the system (1.2) is considered in the half space $\{z \geq 0\}$, for positive times $\{t \geq 0\}$ together with initial-boundary conditions

$$n_{|t=0} = 0, \quad \partial_t n_{|t=0} = 0, \quad E_{z=0} = E_0, \quad E_{t=0} = 0. \tag{1.3}$$

This approach is also very natural when the parameter ϵ is small. The solutions are expected to vanish for $t \leq z$, by finite speed of propagation. Changing t to $t - \epsilon z$, the system reads;

$$\begin{cases} i\partial_z E + \Delta_x E = nE, \\ (\partial_t^2 - \Delta_x)n = \Delta_x |E|^2, \end{cases} \tag{1.4}$$

$$\begin{cases} n_{|t=0} = 0, \quad \partial_t n_{|t=0} = 0, \\ E_{z=0} = E_0. \end{cases} \tag{1.5}$$

We look for solutions $U = (E, n)$ of (1.4) (1.5) which are periodic in x with period L. We denote by $\mathbb{T}$ the corresponding torus $(\mathbb{R}/2\pi L)^2$. As recalled in Section 6, periodicity is somewhat natural when the envelop of the laser beam has a transversal structure which involves length-scales that are large with respect to the length-scale of the laser but small compared to the width of the beam (see [4]). Periodicity is also natural for the use of spectral methods in numerical simulations.

The main result of this paper is a strong instability result for (1.4), in the spirit of [4]. It says that *the boundary value problem* (1.4) (1.5) *is ill-posed in Sobolev spaces*. Prior to this result, we make a detailed analysis of *amplification properties* of the linearized equations : at a space frequency of length k, the amplification is exponential in $e^{c\sqrt{k}}$. This implies a Hadamard's type instability and the ill-posedness in Sobolev spaces, as well a the local well-posedness in spaces of real analytic functions. For physical interpretations the frequencies must be considered as bounded while for numerical applications they are filtered by one way or another. Thus is is important to give a numerical value to the amplification rate and to be evaluate the amplification effect on definite values of the fields. This is done in Section 6.

Let us describe now the main contents of the paper. We first consider the linearized equations. Consider here a constant solution

$$\underline{U} = (\underline{E}, 0), \quad \underline{E} \neq 0, \tag{1.6}$$

which satisfies (1.4) (1.5) with boundary data $E_0 = \underline{E}$. The homogeneous linearized equations near $\underline{U}$, at the frequency $\xi \in \mathbb{R}^2$ read:

$$\begin{cases} i\partial_z u - k^2 u - \underline{E} n = 0, \\ i\partial_z v + k^2 v + \overline{\underline{E}} n = 0, \\ (\partial_t^2 + k^2) n + k^2 \underline{E}(u + v) = 0, \end{cases} \tag{1.7}$$

where $k := |\xi|$ and u [resp. v] [resp n] denote the the Fourier coefficient at the frequency ξ of the variation of the field E [resp. $\overline{E}$] [resp of the density]. They are supplemented with initial boundary boundary conditions:

$$n_{|t=0} = 0, \quad \partial_t n_{|t=0} = 0, \quad u_{z=0} = u_0, \quad v_{z=0} = v_0. \tag{1.8}$$

The equations (1.7) (1.8) form a well posed hyperbolic Goursat problem (see [1]). Our concern is to understand its behavior for large k. The fundamental solution is studied in details in Section 2, where we also construct blowing up solutions:

Theorem 1.1. *There are initial data $|u_0| \leq 1$ and $v_0 = 0$, such that for k large, $t \geq \frac{1}{2}$ and $\rho := \sqrt{2k|\underline{E}|^2 zt}$ large, the solution of the homogeneous equation* (1.7) *satisfies*

$$|n(t,z)| \gtrsim k\rho^{-\frac{5}{2}} e^{\rho}. \tag{1.9}$$

The exponential amplification in $e^{c\sqrt{k}}$ is the signal of a strong instability. We construct solutions on domains

$$\Omega = \{(t, z, x) \in [0, T] \times [0, Z] \times \mathbb{T}^2 \ ; \ zt \le \delta\} \tag{1.10}$$

and prove that they blow up on the part of the boundary

$$\Gamma = \{(t, z, x) \in \Omega \ ; \ zt = \delta\} \tag{1.11}$$

which is not empty if $TZ > \delta$.

Theorem 1.2. *For all s, $T > 0$ $L > 0$ and $\underline{E} \neq 0$, there are sequences δ_k and Z_k and families of solutions $U_k = \underline{U} + (e_k, n_k)$ of (1.4), in $C^0(\Omega_k; H^s(\mathbb{T}))$ such that*

$$\|e_{k|z=0}\|_{H^s([0,T]\times\mathbb{T}^2)} \ \to \ 0, \tag{1.12}$$

$$Z_k \to 0, \quad \delta_k \to 0, \quad TZ_k > \delta_k, \tag{1.13}$$

$$\sup_{(t,z)\in\Gamma_k} \|n_k(t, z)\|_{L^2(\mathbb{T})} \ \to \ \infty. \tag{1.14}$$

This theorem is proved in Section 3, with technical details postponed to Section 4. This nonlinear instability result is pretty strong: not only the amplification $\|e\|_{L^2}/\|e_{|z=0}\|_{H^s}$ is arbitrarily large, in arbitrarily small distance Z, with arbitrary loss of derivatives s, but there is an effective *blow up* of the norm of n_k.

This analysis reveals the importance of the amplification factor, $\rho := \sqrt{2k|\underline{E}|^2 zt}$, and indeed there is a good uniform stability for a filtered system at frequencies $|\xi| \le k$, on the domain $\{(t, z, x) \in [0, T] \times [0, Z] \times \mathbb{T}^2\}$ provided that

$$kZT|\underline{E}|^2 \ll 1. \tag{1.15}$$

Instead of filtering the frequencies, another mathematical approach is to counterbalance the amplification by an exponential decay of the Fourier coefficients. This means that one works in spaces of real analytic functions. In this framework, we prove in Section 5 a local existence theorem by an easy adaptation of proof of the existence of analytic solutions to the hyperbolic Goursat problem (see [11]). The interesting point is that the length Z of propagation satisfies an estimate which is very similar to (1.15):

$$RZT\|\underline{E}_0\|^2_{\mathbb{E}_R} \ll 1. \tag{1.16}$$

where $\mathbb{E}_R$ is a space of analytic functions for the boundary data on $[0, T] \times \mathbb{T}^2$ and R^{-1} measures the width of the complex domain where E_0 can be extended. Indeed, for $E_0(t,x) = e_0(t)e^{i\xi\cdot x}$ with $|\xi| = k$, and $R \approx k$, there holds $\|E_0\|_{\mathbb{E}_R} \lesssim \underline{E} = \|e\|_{L^\infty}$ and (1.16) is equivalent to (1.16).

This shows that (1.15) *or* (1.16) *can be seen as stability criteria for the Goursat problem for Zakharov equations* (1.2).

Section 6 is devoted to a qualitative discussion of the results. We discuss several points.

- Taking $\underline{E}$ to be a constant is not physically realistic : it would mean that the envelop of the mean electric field has a jump. The case where $\underline{E}$ is a smooth function of time will be briefly discussed. It yields additional technical difficulties but does not change qualitatively the results.

- The boundary data in Theorem 1.1 are very particular. Thus, it is important to understand better how general solutions behave. Actually, the rate of amplification also depends on the time frequencies. The underlying phenomenon is a *resonance* between plasma waves governed by the Schrödinger equation and acoustic waves for n. In other words, *only the acoustic oscillations* $e^{-i\omega t + i\xi x}$ with $\omega^2 = |\xi|^2$ which present in the boundary data E_0 are effectively amplified at the given exponential rate. Mathematically, in general, these acoustic frequencies have a nonvanishing amplitude because the signals exactly vanish in the past and thus their Laplace-Fourier transform has no lacuna. Physically, the acoustic frequencies, even when absent from the main scene, can be present in a background noise. By a standard plane wave analysis, we will also give an amplification rate for oscillations which are not exactly acoustic, but this is not a correct approach for the Gourset problem.

- The blow up in Theorem 1.2 is totally unphysical, since n is a variation of density and thus must remain bounded. However, for physical interpretations, one must keep in mind that that (1.1) is only a *model* which has a limited range of validity. In particular, it is tacitly assumed that the variations of the ion density are not too large and that the paraxial approximation for the envelop is valid. Moreover, the physical frequencies k are *bounded.* One crucial question is to know *wether the factor* $\rho = \sqrt{2k|\underline{E}|^2 zt}$ *is small or large.* In Section 6, we will give standard physical data for laser-plasma propagation showing that this *factor can be large* (10 to 10^2) for not very intense fields E of order 10^9 or 10^{10} Wm^{-1}. In this case, that the amplification e^ρ ranges from 10^4 to 10^8. Thus, the acoustic boundary oscillations can be ignored only if their relative value is $\ll e^{-\rho}$, which is much beyond the usual admissible errors. This seems to indicate that the model, as is it, is not well adapted to the propagation of intense laser beams.

- For numerical simulations, the analysis shows that increasing the number of Fourier modes, which is natural to improve the accuracy of computations, may introduce strong instabilities. Of course, one can eliminate most of them by filtering out the bad acoustic oscillations, but then the question is the relevance of the computations with respect to the model.

2 The linear instability

Consider the linearized equation from (1.4) around $(\underline{E}, 0)$:

$$\begin{cases} i\partial_z e + \Delta_x e - \underline{E} n = f, \\ (\partial_t^2 - \Delta_x) n - \Delta_x(\overline{\underline{E}} e + \underline{E}\overline{e}) = \Delta_x h \end{cases} \tag{2.1}$$

$$n_{|t=0} = 0, \quad \partial_t n_{|t=0} = 0, \quad e_{z=0} = e_0. \tag{2.2}$$

Multiplying e by a constant phase factor $e^{i\theta}$, there is no restriction in assuming that $\underline{E}$ is real. Taking $u = e$ and $v = \overline{e}$ as (independent) unknowns, the system reads

$$\begin{cases} i\partial_z u + \Delta_x u - \underline{E} n = f, \\ i\partial_z v - \Delta_x v + \underline{E} n = g, \\ (\partial_t^2 - \Delta_x) n - \underline{E} \Delta_x (u + v) = \Delta_x h, \end{cases} \tag{2.3}$$

with $g = -\overline{f}$.

Performing a Fourier series expansion in x (or a Fourier transform), and still denoting by u, v and n the Fourier coefficients, the equations at the frequency $\xi \in \mathbb{R}^2$ read with $k := |\xi|$:

$$\begin{cases} i\partial_z u - k^2 u - \underline{E} n = f, \\ i\partial_z v + k^2 v + \overline{\underline{E}} n = g, \\ (\partial_t^2 + k^2) n + k^2 \underline{E}(u + v) = -k^2 h, \end{cases} \tag{2.4}$$

together with initial boundary boundary conditions:

$$n_{|t=0} = 0, \quad \partial_t n_{|t=0} = 0, \quad u_{z=0} = u_0, \quad v_{z=0} = v_0. \tag{2.5}$$

For L-periodic functions, the frequencies $\xi \in \frac{2\pi}{L}\mathbb{Z}^2$.

The Goursat problem (2.4) (2.5) is well posed ([1]). The main purpose of this section is to prove estimates for the fundamental solutions (Propositions 2.5 and 2.6 below), and give an example of a solution of the homogeneous equation which is amplified at indicated rate indicated in Theorem 1.1. (see also Theorem 2.8 below for a more precise statement).

2.1 The fundamental solution

Extend the functions by 0 for $t < 0$ and perform a Fourier-Laplace transform in time; this amounts to replace ∂_t by $i\zeta$ with ζ lying in the lower half plane $\{\operatorname{Im}\zeta < 0\}$: the third equation in (2.4) and the homogeneous initial conditions for n imply that

$$\hat{n} = \frac{k^2}{\zeta^2 - k^2}\big(\underline{E}(\hat{u} + \hat{v}) + \hat{h}\big). \tag{2.6}$$

We denote here by $\hat{\phi}(z, \zeta)$ the Fourier-Laplace transform of $\phi(z, t)$. We end up with the system

$$\partial_z \hat{U} = iA\hat{U} + \hat{F}, \qquad \hat{U}_{|z=0} = \hat{U}_0 \tag{2.7}$$

for

$$\hat{U} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}, \qquad \hat{F} = \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix} + i\frac{k^2 \underline{E}\hat{h}}{\zeta^2 - k^2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{2.8}$$

with

$$A(\zeta) = \begin{pmatrix} -k^2 - a & -a \\ a & a + k^2 \end{pmatrix}, \qquad a = \frac{k^2 |\underline{E}|^2}{\zeta^2 - k^2}. \tag{2.9}$$

Note that $A(\zeta)$ is bounded and holomorphic for $\{\operatorname{Im}\zeta \le -\gamma\}$ for all $\gamma > 0$ and therefore, by inverse Laplace transform:

Lemma 2.1. *The solution of the homogeneous system* (2.4) *with initial-boundary values* (2.5) *where* $U_0 = {}^t(u_0, v_0) \in C_0^\infty(\overline{\mathbb{R}}_+)$ *on* $\{z = 0\}$, *is*

$$U(t, z) = \frac{1}{2\pi}\int_{\mathbb{R} - i\gamma} e^{i\left(t\zeta + zA(\zeta)\right)}\hat{U}_0(\zeta)\, d\zeta,$$

$$n(t, z) = \frac{1}{2\pi}\int_{\mathbb{R} - i\gamma} \ell \cdot e^{i\left(t\zeta + zA(\zeta)\right)}\hat{U}_0(\zeta)\frac{k^2 E}{\zeta^2 - k^2} d\zeta,$$

where $\ell := (1, 1)$ *and* γ *is any positive real number.*

Note that the integrals above are convergent, as a consequence of the estimates given below.

The fundamental solution is therefore linked to the distribution

$$\mathcal{E}(t, z) = \frac{1}{2\pi}\int_{\mathbb{R} - i\gamma} e^{i\left(t\zeta + zA(\zeta)\right)}\, d\zeta, \tag{2.10}$$

where the integral is taken in the sense of an inverse Laplace transform. More generally, we are led to consider integrals of the form

$$\mathcal{I}(t,z) = \frac{1}{2\pi}\int_{\mathbb{R}-i\gamma} e^{i\left(t\zeta+zA(\zeta)\right)} p(\zeta)\; d\zeta, \tag{2.11}$$

for rational functions p.

Note that a and A and thus $e^{i\left(t\zeta+zA(\zeta)\right)}$ are holomorphic in $\mathbb{C}\backslash\{-k,+k\}$. In addition, $a(\zeta) = O(|\zeta|^{-2})$ at infinity, and thus

$$A(\zeta) - A(\infty) = O(|\zeta|^{-2}), \qquad e^{izA(\zeta)} - e^{izA(\infty)} = O(|\zeta|^{-2}) \tag{2.12}$$

where

$$A(\infty) = \begin{pmatrix} -k^2 & 0 \\ 0 & k^2 \end{pmatrix}. \tag{2.13}$$

This implies the following

Lemma 2.2. *If p is a rational function with poles in $\{\operatorname{Im}\zeta \geq 0\}$, and $p(\zeta) = O(|\zeta|^{-2})$ at infinity, then the integral in* (2.8) *is absolutely convergent for all $t \in \mathbb{R}$. It vanishes for $t < 0$ and for $t \geq 0$,*

$$\mathcal{I}(t,z) = \frac{1}{2\pi}\int_{\Gamma} e^{it\zeta+izA(\zeta)} p(\zeta)\; d\zeta \tag{2.14}$$

where Γ is any simple contour oriented positively winding around the poles of p and a.

Proof. The integral is clearly convergent and letting γ tend to $+\infty$ implies that it vanishes when $t < 0$. When $t \geq 0$, the integral over a large half circle in the upper half space tends to 0 as the radius tends to infinity, allowing to close the integration path. $\square$

Similarly, we can split $\mathcal{E}$ into two parts:

$$\mathcal{E}(t,z) = \frac{1}{2\pi}\int_{\mathbb{R}-i\gamma} e^{i\left(t\zeta+zA(\infty)\right)}\; d\zeta + \frac{1}{2\pi}\int_{\mathbb{R}-i\gamma} e^{it\zeta}\left(e^{izA(\zeta)} - e^{izA(\infty)}\right)\; d\zeta.$$

Thanks to (2.12), the second integral can be deformed to an integral on a closed contour Γ, on which the integral of the entire function $e^{it\zeta}e^{izA(\infty)}$ vanishes. Therefore,

Lemma 2.3. *The distribution $\mathcal{E}$ defined in* (2.10) *is equal to*

$$\mathcal{E}(t,z) = e^{izA(\infty)}\ \delta_{t=0} + \mathcal{E}_0(t,z) \tag{2.15}$$

where $\mathcal{E}_0$ vanishes for $t<0$ and is equal to

$$\mathcal{E}_0(t,z) = \frac{1}{2\pi}\int_\Gamma e^{it\zeta + izA_k(\zeta)}\ d\zeta \tag{2.16}$$

when $t \geq 0$, where Γ is any simple contour oriented positively surrounding the poles of a.

2.2 Estimates

We give sharp upper bounds for the contour integrals in (2.14) and (2.16). The matrix A is traceless, therefore $A^2 = -(\det A)\mathrm{Id} = k^2(k^2+2a)\mathrm{Id}$ and

$$e^{izA} = \cos(kz\lambda)\mathrm{Id}\ +\ \frac{\sin(kz\lambda)}{k\lambda}A$$

where

$$\lambda = \sqrt{k^2+2a}. \tag{2.17}$$

The choice of the square root is irrelevant in the expression above of e^{izA}. Therefore,

$$\begin{aligned} e^{it\zeta+izA} =& \frac{1}{4}e^{it\zeta-ik\lambda z}\begin{pmatrix} 2+\frac{\lambda}{k}+\frac{k}{\lambda} & \frac{\lambda}{k}-\frac{k}{\lambda} \\ \frac{k}{\lambda}-\frac{\lambda}{k} & 2-\frac{\lambda}{k}-\frac{k}{\lambda}\end{pmatrix} \\ &+\frac{1}{4}e^{it\zeta+ik\lambda z}\begin{pmatrix} 2-\frac{\lambda}{k}-\frac{k}{\lambda} & \frac{k}{\lambda}-\frac{\lambda}{k} \\ \frac{\lambda}{k}-\frac{k}{\lambda} & 2+\frac{\lambda}{k}+\frac{k}{\lambda}\end{pmatrix}. \end{aligned} \tag{2.18}$$

This reveals the phases

$$\varphi = t\zeta - zk\lambda, \qquad \psi = t\zeta + zk\lambda \tag{2.19}$$

Lemma 2.4. *For $k \geq \max\{1, 4|\underline{E}|^2\}$, choosing the principal determination of the square root in* (2.17) *defines $\lambda(k+w)$ as an holomorphic function of w for $1 \leq |w| \leq \frac{1}{2}k$. Moreover, for all $t \geq 0$, $z \geq 0$ and w in this annulus, there holds*

$$\begin{aligned} \Big|\varphi(k+w) - \big(kt - k^2z + tw - \frac{kz|\underline{E}|^2}{2w}\big)\Big| &\leq z\big(\tfrac{1}{3}|\underline{E}|^2 + \tfrac{1}{5}|\underline{E}|^4\big), \\ \Big|\psi(k+w) - \big(kt + k^2z + tw + \frac{kz|\underline{E}|^2}{2w}\big)\Big| &\leq z\big(\tfrac{1}{3}|\underline{E}|^2 + \tfrac{1}{5}|\underline{E}|^4\big). \end{aligned}$$

Proof. For $|w| \le \frac{1}{2}k$,

$$a(k+w) = \frac{k^2|\underline{E}|^2}{w(w+2k)} = \frac{k|\underline{E}|^2}{2w} - \frac{|\underline{E}|^2}{4(1+\frac{w}{2k})}$$

Thus

$$\Big|a - \frac{k|\underline{E}|^2}{2w}\Big| \le \frac{1}{3}\ |\underline{E}|^2$$

If in addition $|w| \ge 1$ and $k \ge \max\{1, 4|\underline{E}|^2\}$, we have

$$\Big|\frac{a}{k}\Big| \le |\underline{E}|^2, \qquad \Big|\frac{2a}{k^2}\Big| \le \frac{2|\underline{E}|^2}{k} \le \frac{1}{2}.$$

Thus we can choose λ to be the principal determination of the square root of $k^2 + 2a$ and

$$k\lambda = k^2 + a + \frac{a^2}{k^2}G\big(\frac{2a}{k^2}\big) \tag{2.20}$$

where G is holomorphic on the unit disc. Substituting, we find that

$$\big|k\lambda - k^2 - \frac{k|\underline{E}|^2}{2w}\big| \le \frac{1}{3}|\underline{E}|^2 + |\underline{E}|^4 \sup_{|\zeta|\le\frac{1}{2}} |G(\zeta)|. \tag{2.21}$$

The supremum is $|G(-\frac{1}{2})| \le \frac{1}{5}$ and the lemma follows. ☐

Proposition 2.5. *Then there is a constant* C*, such that for all* $\underline{E}$*,* k*,* t *and* z *satisfying*

$$|\underline{E}| \le 1, \quad k \ge \max\{1, 4|\underline{E}|^2\}, \quad t \in [0,1], \quad z \in [0, \frac{1}{2}], \tag{2.22}$$

the contour integral in (2.16) *satisfies :*

$$\big|\mathcal{E}_0(t,z)\big| \le C\frac{r\ e^{\rho}}{\sqrt{1+\rho}}$$

with $\rho = \sqrt{2k|E|^2zt}$ *and* $r = 1 + \frac{k|\underline{E}|^2z}{1+\rho}$.

Proof. **a)** We choose Γ to be the union of the circles

$$\Gamma_1 = \{|\zeta - k| = r\}, \qquad \Gamma_2 = \{|\zeta + k| = r\},$$

with $r \in [1, \frac{1}{2}k]$ to be chosen depending on t and z.

By (2.21) and (2.22), we have the following bounds:

$$\Big|\frac{\lambda}{k} - 1\Big| \le \frac{|\underline{E}|^2}{2kr} + \frac{|\underline{E}|^2 z}{k^2} \le \frac{|\underline{E}|^2}{k} \le \frac{1}{4}.$$

Furthermore

$$\Big|\frac{k}{\lambda} - 1\Big| \le \frac{|\underline{E}|^2}{2|\lambda| r} + \frac{|\underline{E}|^2 z}{k|\lambda|} \le \frac{2|\underline{E}|^2}{k} \le \frac{1}{2}.$$

Thus

$$\Big|\frac{\lambda}{k} + \frac{k}{\lambda}\Big| \le 3, \quad \Big|\frac{\lambda}{k} - \frac{k}{\lambda}\Big| \le \frac{3|\underline{E}|^2}{k}, \quad \Big|2 - \frac{\lambda}{k} - \frac{k}{\lambda}\Big| \le \frac{3|\underline{E}|^2}{k}.$$

Therefore, all the entries of the two matrices present in (2.18) are bounded by 5.

b) If

$$k|E|^2 zt \ge 2, \tag{2.23}$$

then we choose

$$r = \sqrt{\frac{kz|E|^2}{2t}}. \tag{2.24}$$

By (2.23) and (2.22), $r \in [1, \frac{1}{2}k]$. For $\zeta = k + re^{i\theta} \in \Gamma_1$, Lemma 2.4 implies that the imaginary parts of the phases satisfy

$$\big|\operatorname{Im}\varphi(k + re^{i\theta}) - \rho\sin\theta\big| \le z|\underline{E}|^2, \tag{2.25}$$

$$\big|\operatorname{Im}\psi(k + re^{i\theta}\big| \le z|\underline{E}|^2. \tag{2.26}$$

Therefore, the integral over Γ_1 contributes to $\mathcal{E}_0$ to a matrix whose entries are bounded by

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{5}{4} e^{z|\underline{E}|^2}\big(1 + e^{-\rho\sin\theta}\big)\ r\ d\theta.$$

By symmetry, the integral over Γ_2 is estimated similarly, and therefore, for $\rho \ge 2$, the entries of $\mathcal{E}_0$ are bounded by

$$\frac{5r}{4\pi} e^{z|\underline{E}|^2} \int_0^{2\pi} \big(1 + e^{-\rho\sin\theta}\big)\ d\theta.$$

The integral is bounded by

$$\begin{aligned} 3\pi + \int_{-\pi/2}^{\pi/2} e^{\rho\cos\theta}\ d\theta &= 3\pi + 2e^{\rho}\int_{-\pi/4}^{\pi/4} e^{-2\rho\sin^2 y}\ dy \\ &\le 3\pi + 2e^{\rho}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} e^{-2\rho t^2}\ \frac{dt}{\sqrt{1-t^2}} \le 3\pi + \frac{2\sqrt{\pi}e^{\rho}}{\sqrt{\rho}}. \end{aligned}$$

Therefore, for $\rho \geq 2$, the entries of $\mathcal{E}_0$ are bounded by

$$(2.27)\quad \left|\mathcal{E}_0(t,z)\right| \leq 5\ e^{z|\underline{E}|^2}\sqrt{\frac{|\underline{E}|^2 kz}{t}}\ \Big(\frac{3}{4}+\frac{e^\rho}{2\sqrt{\pi\rho}}\Big) \leq 5\ \sqrt{e}\sqrt{\frac{|\underline{E}|^2 kz}{t}}\ \frac{e^\rho}{\sqrt{\pi\rho}}$$

c) If

$$(2.28)\qquad \frac{1}{2}\rho^2 = k|E|^2 zt \leq 2,$$

we choose

$$(2.29)\qquad r = \max\{1, k|E|^2 z\} \in [1, \frac{1}{2}k].$$

Consider first the integral on Γ_1. The entries of the corresponding matrix The imaginary part of the phases are bounded by

$$\left|\operatorname{Im}\varphi\right| \leq tr + zk\frac{|\underline{E}|^2}{r} \leq \max\{t, tk|E|^2 z\} + 1 \leq 2+\rho.$$

Therefore, the integral are bounded by

$$\frac{5r}{4}e^{2+\rho} \leq 10\ e^\rho.$$

The analysis of the integral over Γ_2 is similar, and combining the estimates above, one obtains (2.19) □

Proposition 2.6. *Denote by $\mathcal{E}_1$ [resp. $\mathcal{E}_2$] the contour integral in* (2.11) *with $p = \frac{k^2}{\zeta^2-k^2}$ [resp. $p = \frac{k^4}{(\zeta^2-k^2)^2}$]. There is a constant C, such that for $|\underline{E}| \leq 1$, $k \geq \max\{1, 4|\underline{E}|^2\}$, $t \in [0,1]$ and $2|\underline{E}|^2 z \leq 1$ there holds*

$$(2.30)\qquad \left|\mathcal{E}_1(t,z)\right| \leq C\frac{k\ e^\rho}{\sqrt{1+\rho}}$$

$$(2.31)\qquad \left|\mathcal{E}_2(t,z)\right| \leq C\frac{k^2\ e^\rho}{\tilde{r}\sqrt{1+\rho}}.$$

with $\frac{1}{\tilde{r}} = \frac{1}{k} + \frac{t}{1+\rho}$.

Proof. For $\mathcal{E}_1$ we proceed as for $\mathcal{E}_0$, noticing that $|\frac{k^2}{\zeta^2-k^2}| \approx \frac{k}{r}$ on the integration path.

For (2.31), we make the same choice of radius r when $k|E|^2zt \geq 2$. Noticing that $|\frac{k^4}{(\zeta^2-k^2)^2}| \approx \frac{k^2}{r^2}$ on the integration path yields the estimate

$$\left|\mathcal{E}_2(t,z)\right| \leq C\sqrt{\frac{t}{|\underline{E}|^2kz}}\,\frac{k^2e^\rho}{\sqrt{1+\rho}}.$$

When $k|E|^2zt \leq 2$, the length Γ is $O(r)$, we maximize r in the range $[1, \frac{1}{2}k]$ instead of minimizing it as we did in (2.29), so that the imaginary parts of the phases remain bounded. Namely, we choose

$$\tilde{r} = \min\{\frac{1}{t}, \frac{1}{2}k\}$$

and get

$$\left|\mathcal{E}_2(t,z)\right| \leq C\frac{k^2}{\tilde{r}}.$$

in this zone. Combining the estimates yields (2.31) $\square$

2.3 Integral formulas for the solutions of (2.4) (2.5)

At the given frequency k, the system (2.4) (2.5) is a classical hyperbolic Goursat problem in dimension 2, and therefore has a unique smooth solution for smooth data (see [1]), which respects the causality principle in time. We will use the following description:

Proposition 2.7. *For $F = {}^t(f,g)$ and h in $C^\infty([0,T]\times[0,Z])$ and the problem (2.4) (2.5) with $U_0 = {}^t(u_0,v_0) = 0$, has a unique solution $U = {}^t(u,v)$, n in $C^\infty([0,T]\times[0,Z])$ and*

$$\begin{aligned}U(t,z) =& i\int_0^z e^{i(z-z')A(\infty)}F(t,z')dz' \\ &+\int_0^z\!\!\int_0^t \mathcal{E}_0(t',z')F(t-t',z-z')dz'dt' \\ &+\int_0^z\!\!\int_0^t \mathcal{E}_1(t',z')\begin{pmatrix}-i\\ i\end{pmatrix}h(t-t',z-z')dz'dt',\end{aligned} \tag{2.32}$$

$$\begin{aligned}n(t,z) =& \int_0^z\!\!\int_0^t \begin{pmatrix}-i, & i\end{pmatrix}\mathcal{E}_1(t',z')F(t-t',z-z')dz'dt' \\ &+ i\int_0^z\!\!\int_0^t \mathcal{E}_2(t',z')h(t-t',z-z')dz'dt'.\end{aligned} \tag{2.33}$$

2.4 Exponentially growing solutions

We show that the rate of amplification e^ρ observed in Propositions 2.5 and 2.6 is sharp. Consider the solution of

$$\begin{cases} i\partial_z u + \Delta_x u - \underline{E} n = 0, \\ i\partial_z v - \Delta_x v + \underline{E} n = 0, \\ (\partial_t^2 - \Delta_x) n - \underline{E}\Delta_x (u+v) = 0. \end{cases} \tag{2.34}$$

with initial-boundary conditions

$$n_{|t=0} = \partial_t n_{|t=0} = 0, \quad u_{|z=0} = \frac{\sin(kt)}{k} e^{i\xi x}, \quad v_{|z=0} = 0. \tag{2.35}$$

This amounts to solve

$$\begin{cases} i\partial_z u - k^2 u - \underline{E} n = 0, \\ i\partial_z v + k^2 v + \underline{E} n = 0, \\ (\partial_t^2 - \Delta_x) n - \underline{E} k^2 (u+v) = 0. \end{cases} \tag{2.36}$$

with $k = |\xi|$, together with initial-boundary conditions

$$n_{|t=0} = \partial_t n_{|t=0} = 0, \quad u_{|z=0} = \frac{\sin(kt)}{k}, \quad v_{|z=0} = 0. \tag{2.37}$$

Because the Fourier-Laplace transform of $1_{\{t>0\}}\frac{\sin(kt)}{k}$ is $\frac{1}{k^2-\zeta^2}$, the solution is

$$U(t,z) = \begin{pmatrix} u(t,z) \\ v(t,z) \end{pmatrix} = \frac{1}{2i\pi} \int_{\mathbb{R}-i\gamma} e^{i\left(t\zeta + zA_k(\zeta)\right)} R \frac{d\zeta}{k^2-\zeta^2}, \tag{2.38}$$

$$n(t,z) = -\frac{1}{2i\pi} \int_{\mathbb{R}-i\gamma} L\, e^{i\left(t\zeta + zA_k(\zeta)\right)} R\, \frac{k^2 E}{(\zeta^2-k^2)^2}\, \frac{d\zeta}{\zeta}, \tag{2.39}$$

where

$$R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 \ , \ 1 \end{pmatrix}.$$

Theorem 2.8. *There are constants $C \ge c > 0$ such that for all $k \ge 2$, $t \in [0,1]$, $z \in [0,1]$:*

$$\begin{aligned} \left|U(t,z)\right| &\le C \frac{1}{k} \frac{e^\rho}{\sqrt{1+\rho}} \\ \left|n(z,t)\right| &\le C\Big(\frac{1}{k} + \frac{t}{1+\rho}\Big) \frac{e^\rho}{\sqrt{1+\rho}}. \end{aligned}$$

Moreover, for $\rho \geq 1$, $t \geq \frac{1}{2}$ and k large enough,

$$\Big|n_k(t,z)\Big| \geq c\rho^{-\frac{5}{2}}e^{\rho}\Big(1+O(\rho^{-1})\Big). \tag{2.40}$$

Proof. One can deform the integration path $\mathbb{R} - i\gamma$ to a contour Γ which is the union of the circles

$$\Gamma_1 = \{|\zeta - k| = r\}, \qquad \Gamma_2 = \{|\zeta + k| = r\}$$

with $r \in [1, \frac{1}{2}k]$. The upper bounds follow from Proposition 2.6.

We now concentrate on the proof of the lower bound for n assuming that $\rho = \sqrt{2|\underline{E}|^2 zt} \geq 2$. We choose the radius $r = \frac{\rho}{2t}$ as in (2.24). By (2.18), for $j = 1, 2$, the integral over Γ_j contributes to

$$\begin{aligned} N_j(t,z) =& \frac{1}{2i\pi}\int_{\Gamma_j} e^{it\zeta+izk\lambda}\frac{\lambda-k}{2\lambda}\,\frac{k^2E}{(\zeta^2-k^2)^2}\,\frac{d\zeta}{\zeta} \\ &+ \frac{1}{2i\pi}\int_{\Gamma_j} e^{it\zeta-izk\lambda}\frac{\lambda+k}{2\lambda}\,\frac{k^2E}{(\zeta^2-k^2)^2}\,\frac{d\zeta}{\zeta}. \end{aligned}$$

Consider first N_1. We evaluate the phases $\varphi = t\zeta - zk\lambda$ and $\psi = t\zeta + zk\lambda$ on Γ_1 as in (2.25) (2.26). For $\zeta = k + re^{i\theta}$ we have

$$\begin{aligned} \varphi(k+re^{i\theta}) &= kt - k^2z + 2i\rho\sin\theta - \sigma_k(r,\theta), \\ \psi(k+re^{i\theta}) &= kt + k^2z + 2\rho\cos\theta + \sigma_k(r,\theta). \end{aligned}$$

where the σ_k are smooth functions of $\theta \in \mathbb{T}$, uniformly bounded as well as their derivatives when $1 \leq r \leq \frac{1}{2}k$.

Note that the imaginary part of ψ is bounded, while by (2.21)

$$\Big|\frac{\lambda-k}{2\lambda}\,\frac{k^2|\underline{E}|^2}{(\zeta^2-k^2)^2}\Big| \leq \frac{C|\underline{E}|^2}{kr^3} \leq \frac{C|\underline{E}|^2}{k} \tag{2.41}$$

on Γ_1. Thus the corresponding term contributes to $O(k^{-1})$ terms in N_1. It remains to evaluate

$$\tilde{N}_1(z,t) = \frac{1}{2i\pi}\int_{\Gamma_1} e^{it\zeta-izk\lambda}\frac{\lambda+k}{2\lambda}\,\frac{k^2E}{(\zeta^2-k^2)^2}\,d\zeta$$

We have the following expansion

$$\frac{\lambda+k}{2\lambda}\,\frac{k^2E}{(\zeta^2-k^2)^2} = \frac{e^{-2i\theta}}{r^2}\,p_k(r,\theta)$$

where the p_k are smooth functions of $\theta \in \mathbb{T}$, uniformly bounded as well as their derivatives when $1 \leq r \leq \frac{1}{2}k$ and such that, uniformly on this domain

$$\lim_{k \to \infty} p_k(r, \theta) = \frac{\underline{E}}{2}$$

Substituting, we see that

$$\tilde{N}_1(z,t) = \frac{e^{i(kt - k^2 z)}}{2\pi r^2} \int_0^{2\pi} e^{-2\rho \sin\theta} \; p_k(r,\theta) e^{-iz\sigma_k(r,\theta) - 2i\theta} d\theta.$$

The stationary phase theorem to implies that

$$\tilde{N}_1(z,t) = e^{i(kt - k^2 z)} \; \frac{e^{2\rho}}{2r^2\sqrt{\pi\rho}} \big(p_k(r, -\pi/2) e^{-i\sigma_k(r,-\pi/2) + i\pi} + O(\rho^{-1}) \big)$$

Therefore, for k large enough,

$$\big| N_1(t,z) \big| \geq \frac{|\underline{E}| t^2}{\sqrt{\pi}\rho^{5/2}} e^{\rho} \Big(1 + O(\rho^{-1}) \Big) + O(1).$$

b) The analysis of the contribution of the integral over Γ_2 is similar, except that the phase with large imaginary part is now ψ. Noticing that the corresponding amplitude is bounded by (2.41), we see that

$$\big| N_2(t,z) \big| \leq \frac{C}{r^2} \frac{e^{\rho}}{\sqrt{\rho}} + O(1)$$

and because $\frac{1}{r} \approx \frac{t}{\rho} \lesssim \frac{1}{\rho}$, the estimate (2.40) follows for k and ρ large enough. $\square$

3 The nonlinear instability

3.1 The method of proof

We fix a constant field $\underline{E} \neq 0$, noticing that $(\underline{E}, 0)$ is a solution of (1.4). We compare it to another solution of the form

$$E = \underline{E} + e^a + \tilde{e}, \quad n = n^a + \tilde{n} \tag{3.1}$$

where (e^a, n^a) is a solution of the homogeneous linearized equation (2.1) (2.2), with small boundary value on $\{z = 0\}$ and such that n^a and n are arbitrarily large at an arbitrarily small distance $z > 0$. Since we consider periodic functions in x, it is sufficient to exhibit an example where *x is one dimensional*, which we now assume. We also assume, without restriction, that *$\underline{E}$ is real.*

We choose (e^a, n^a) to be the solution of

$$
\begin{cases} i\partial_z e^a + \Delta_x e^a - \underline{E} n^a = 0, \\ (\partial_t^2 - \Delta_x) n^a - \Delta_x(\underline{E} e^a + \underline{E}\overline{e^a}) = 0 \end{cases} \tag{3.2}
$$

$$
n^a_{|t=0} = \partial_t n^a_{|t=0} = 0, \quad e^a_{|z=0} = \alpha \frac{\sin(kt)}{k} e^{ikx}, \tag{3.3}
$$

where the frequency k is large and α is a small parameter. Typically, we will choose $\alpha \approx k^{-\sigma}$ so that the boundary data is of order $k^{s-\sigma}$ in H^s. Thus

$$
e^a(t, z, x) = u(t, z) e^{ikx} + \overline{v(t, z)} e^{-ikx}, \tag{3.4}
$$

$$
n^a(t, z, x) = \nu(t, z) e^{ikx} + \overline{\nu(t, z)} e^{-ikx} \tag{3.5}
$$

where (u^a, v^a, ν^a) solve

$$
\begin{cases} i\partial_z u^a - k^2 u^a - \underline{E} \nu^a = 0, \\ i\partial_z v^a + k^2 v^a + \underline{E} \nu^a = 0, \\ (\partial_t^2 + k^2)\nu^a + k^2 \underline{E}(u^a + v^a) = 0, \end{cases} \tag{3.6}
$$

$$
\nu^a_{|t=0} = \partial_t \nu^a_{|t=0} = 0, \quad u^a_{|z=0} = \alpha \frac{\sin(kt)}{k}, \quad v^a_{|z=0} = 0. \tag{3.7}
$$

The solution (e^a, n^a) is amplified as described in Theorem 2.8. We construct correctors $(\tilde{e}, \tilde{n})$ such that (E, n) given by (3.1) is an exact solution of the nonlinear system, on a small domain Ω where $|\tilde{n}| \ll |n^a|$ and such that n^a and thus n is arbitrarily large on a part of the boundary $\partial\Omega$. The equations for the remainders $(\tilde{e}, \tilde{n})$ in (3.1) read

$$
\begin{cases} i\partial_z \tilde{e} + \Delta_x \tilde{e} - \underline{E} \tilde{n} = (e^a + \tilde{e})(n^a + \tilde{n}), \\ (\partial_t^2 - \Delta_x)\tilde{n} - \Delta_x(\overline{\underline{E}} \tilde{e} + \underline{E}\overline{\tilde{e}}) = \Delta_x\big((e^a + \tilde{e})(\overline{e^a} + \overline{\tilde{e}})\big). \end{cases} \tag{3.8}
$$

We add homogeneous initial-boundary data

$$
\tilde{n}_{|t=0} = \partial_t \tilde{n}_{|t=0} = 0, \quad \tilde{e}_{|z=0} = 0. \tag{3.9}
$$

so that E has the same boundary data as $\underline{E} + e^a$.

We solve the nonlinear equation (3.8) by Picard's iterations, in suitable Banach spaces of analytic functions which we now describe.

3.2 Construction of the correctors

The approximate solution (e^a, n^a) has only two frequencies in x, $+k$ and $-k$. The nonlinear interaction will create all the harmonics. This leads to consider functions of the form

$$u = \sum_{p\in\mathbb{Z}} u_p(t,z) e^{ikpx}. \tag{3.10}$$

Definition 3.1. *Given parameters $k \geq 2$, $\delta \in]0,1]$, $b \in]0,1[$ and $s \geq 1$, we denote by $\mathbb{E}$ the space of functions* (3.10) *such that*

$$|u_p(t,z)| \leq C\ \frac{1}{(1+|p|)^s}\ \delta^{\langle p\rangle} e^{\langle p\rangle \rho} \tag{3.11}$$

with

$$\langle p\rangle = \max\{2, |p|\}, \qquad \rho = \sqrt{2k|\underline{E}|^2 zt}, \tag{3.12}$$

on the domain

$$\Omega = \big\{(t,z,x) \in [0,1]\times[0,b]\times\mathbb{T}\ :\ \delta e^{\rho} \leq 1\big\}. \tag{3.13}$$

The best constant in (3.11) *is the norm of u in $\mathbb{E}$.*

Remark 3.2. The factor $\delta e^\rho \leq 1$ present for the frequencies $\pm k$ in the approximate solution is expected to intervene at the power $|p|$ in the harmonic pk by successive nonlinear interaction. Note that the series (3.10) define real analytic functions in x when $(t,z) \in \{\delta e^\rho < 1\}$.

The space $\mathbb{E}$ and the domain Ω depend strongly on the parameters k, b and δ. However, to lighten notations, we do not mention this dependence explicitly.

Lemma 3.3. *Consider the solution (e^a, n^a) of* (3.2) (3.3) *with $\alpha = \gamma\delta$.*

There is a constant C_0 independent of $k \geq 2$, $b \in]0,1]$, $\gamma > 0$ and $\delta \in]0,1]$ such that

i) $f^a = n^a u^a \in \mathbb{E}$ and $h^a = u^a u^a \in \mathbb{E}$ and

$$\|f^a\|_{\mathbb{E}} \leq C_0 \frac{\gamma^2}{k}, \quad \|h^a\|_{\mathbb{E}} \leq C_0 \frac{\gamma^2}{k^2}. \tag{3.14}$$

ii) for all $v \in \mathbb{E}$, $u^a v$ and $n^a v$ belong to $\mathbb{E}$ and

$$\|u^a v\|_{\mathbb{E}} \leq C_0 \frac{\gamma}{k} \|v\|_{\mathbb{E}}, \quad \|n^a v\|_{\mathbb{E}} \leq C_0 \gamma \|v\|_{\mathbb{E}}. \tag{3.15}$$

Lemma 3.4. $\mathbb{E}$ *is a Banach algebra and there is a constant* C_0 *independent of* k, b *and* δ *such that for all* u *and* v *in* $\mathbb{E}$*:*

$$\|uv\|_{\mathbb{E}} \leq C_0 \|u\|_{\mathbb{E}} \ \|v\|_{\mathbb{E}}. \tag{3.16}$$

Corollary 3.5. *Introduce the notations*

$$\Phi(e,n) = (e^a + e)(n^a + n) \tag{3.17}$$

$$\Psi(u,v) := (e^a + u)(\overline{e^a} + \overline{v}). \tag{3.18}$$

Then

$$\|\Phi(e,n)\|_{\mathbb{E}} \leq C_0\Big(\frac{\gamma^2}{k} + \frac{\gamma}{k}\|n\|_{\mathbb{E}} + \gamma\|e\|_{\mathbb{E}}) + \|e\|_{\mathbb{E}} \ \|n\|_{\mathbb{E}}\Big).$$

$$\|\Psi(u,v)\|_{\mathbb{E}} \leq C_0\Big(\frac{\gamma^2}{k^2} + \frac{\gamma}{k}\big(\|u\|_{\mathbb{E}} + \|v\|_{\mathbb{E}}\big) + \|u\|_{\mathbb{E}} \ \|v\|_{\mathbb{E}}\Big).$$

Next, consider the linear problem

$$\begin{cases} i\partial_z e + \Delta_x e - \underline{E} n = f, \\ (\partial_t^2 - \Delta_x) n - \Delta_x(\overline{\underline{E}} e + \underline{E}\overline{e}) = \Delta_x h, \\ n_{|t=0} = \partial_t n_{|t=0} = 0, \quad e_{|z=0} = 0. \end{cases} \tag{3.19}$$

Proposition 3.6. *For* f *and* h *in* $\mathbb{E}$*, the solution* (e,n) *of* (3.19) *belongs to* $\mathbb{E}$ *and*

$$\|e\|_{\mathbb{E}} \leq C_1\, b\, \|f\|_{\mathbb{E}} + C_1 \ln k \|h\|_{\mathbb{E}} \tag{3.20}$$

$$\|n\|_{\mathbb{E}} \leq C_1 \frac{1}{|\underline{E}|^2} \ln(|\ln\delta|) \|f\|_{\mathbb{E}} + C_1 \frac{k}{|\underline{E}|^2} \|h\|_{\mathbb{E}} \tag{3.21}$$

where C_1 *is independent of* k, b *and* δ

Denote by $(e,n) = \mathcal{T}(f,h)$ the solution of (3.19). The equation (3.8) reads

$$(\tilde e, \tilde n) = \mathcal{T}\big(\Phi(\tilde e, \tilde n), \Psi(\tilde e, \tilde e)\big) := \mathcal{F}(\tilde e, \tilde n) := \big(\mathcal{E}(\tilde e, \tilde n), \mathcal{N}(\tilde e, \tilde n)\big). \tag{3.22}$$

Proposition 3.7. *For* $\gamma \geq 1$ *and*

$$4C_0 C_1 \gamma^2 \big(b + \frac{\ln k}{k}\big) \leq 1, \tag{3.23}$$

$$4C_1 C_0 \gamma^2 \big(\frac{\ln(\ln\delta|)}{k} + \frac{1}{k}\big) \leq |\underline{E}|^2, \tag{3.24}$$

$\mathcal{T}$ maps $\mathbb{B}_{\frac{1}{k}} \times \mathbb{B}_1$ into itself, where $\mathbb{B}_R$ denotes the ball of radius R in $\mathbb{E}$, and is a contraction for the norm

$$\|e\|_{\mathbb{E}} + \frac{1}{k}\|n\|_{\mathbb{E}}.$$

if $\gamma > 2$.

Proof. By Corollary 3.5, for $(e, n) \in \mathbb{B}_{\frac{1}{k}} \times \mathbb{B}_1$ and $\gamma \geq 1$, there holds

$$\|\Phi(e,n)\|_{\mathbb{E}} \leq 4C_0\frac{\gamma^2}{k}, \quad \|\Psi(e,e)\|_{\mathbb{E}} \leq 4C_0\frac{\gamma^2}{k^2}.$$

Thus, by Proposition 3.6 and (3.23) (3.24)

$$\|\mathcal{E}(e,n)\|_{\mathbb{E}} \leq 4C_1C_0\big(b\frac{\gamma^2}{k} + \frac{\gamma^2 \ln k}{k^2}\big) \leq \frac{1}{k}$$

and

$$\|\mathcal{N}(e,n)\|_{\mathbb{E}} \leq \frac{4C_1C_0\gamma^2}{|\underline{E}|^2}\big(\frac{\ln(\ln \delta|)}{k} + \frac{1}{k}\big) \leq 1.$$

This shows that $\mathcal{T}$ maps $\mathbb{B}_{\frac{1}{k}} \times \mathbb{B}_1$ into itself.

Consider (e, n) and (e', n') in $\mathbb{B}_{\frac{1}{k}} \times \mathbb{B}_1$. Denote by $\delta e = e - e'$, $\delta n = n - n'$, $\delta\Phi = \Phi(e, n) - \Phi(e', n')$ etc. There holds

$$\begin{aligned}
\|\delta\Phi\|_{\mathbb{E}} &\leq 2C_0\gamma\big(\|\delta e\|_{\mathbb{E}} + \frac{1}{k}\|\delta n\|_{\mathbb{E}}\big),\\
\|\delta\Psi\|_{\mathbb{E}} &\leq \frac{4C_0\gamma}{k}\|\delta e\|_{\mathbb{E}},
\end{aligned}$$

and thus, using Proposition 3.6 and (3.23) (3.24),

$$\begin{aligned}
\|\delta\mathcal{E}\|_{\mathbb{E}} &\leq \frac{1}{\gamma}\big(\|\delta e\|_{\mathbb{E}} + \frac{1}{k}\|\delta n\|_{\mathbb{E}}\big),\\
\frac{1}{k}\|\delta\mathcal{N}\|_{\mathbb{E}} &\leq \frac{1}{\gamma}\big(\|\delta e\|_{\mathbb{E}} + \frac{1}{k}\|\delta n\|_{\mathbb{E}}
\end{aligned}$$

implying that $\mathcal{T}$ is a contraction if $\gamma > 2$. □

Corollary 3.8. *Assume that $\gamma > 2$ and that (3.23) and (3.24) are satisfied. Then the equation (3.8) has a unique solution in $\mathbb{B}_{\frac{1}{k}} \times \mathbb{B}_1$.*

3.3 Choice of parameters, proof of Theorem 1.2

Fix $s = 1$, $\epsilon \in]0, 1[$, $\epsilon' \in]0, \frac{1}{2}\epsilon[$ and $\sigma > 0$. We choose

$$\delta = k^{-\sigma}, \quad b = k^{-\epsilon}, \quad \gamma = k^{\epsilon'}. \tag{3.25}$$

Then the conditions (3.23) and (3.24) are satisfied for k large and there is a solution

$$E = \underline{E} + e^a + \tilde{e}, \quad n = n^a + \tilde{n}$$

of the original system, with

$$\|\tilde{e}\|_{\mathbb{E}} \leq \frac{1}{k}, \qquad \|\tilde{n}\|_{\mathbb{E}} \leq 1. \tag{3.26}$$

Therefore the first Fourier coefficient of n satisfies

$$|n_1(t, z) - n_1^a(t, z)| \leq \delta^2 e^{2\rho} \leq 1 \tag{3.27}$$

while, by Theorem 2.8, for $t \geq \frac{1}{2}$ and ρ large enough

$$|n_1^a(t, z)| \geq c\gamma\rho^{-5/2}\delta e^{\rho}. \tag{3.28}$$

Note that $\delta e^{\rho} = 1$ for $\rho = \sqrt{2k|E|^2 zt} = \sigma \ln k$ thus for $zt = \alpha k^{-1}(\ln k)^2$ for some constant $\alpha > 0$. For k large, $\alpha k^{-1}(\ln k)^2 \ll k^{-\epsilon}$ and therefore

$$\Gamma := \overline{\Omega} \cap \{\delta e^{\rho} = 1\} \neq \emptyset. \tag{3.29}$$

On this set,

$$|n_1^a(t, z)| \geq c' \frac{k^{\epsilon'}}{(\ln k)^{\frac{5}{2}}}. \tag{3.30}$$

which tends to $+\infty$ with k. With (3.27), this finishes the proof of Theorem 1.2.

4 Proofs

4.1 Lemmas

We collect here several elementary estimates which will be used repeatedly in the sequel.

Lemma 4.1. *There is a constant C such that for all $\lambda > 0$, $t \geq 0$ and all $z \geq 0$:*

$$\int_0^z \int_0^t e^{-\sqrt{\lambda z' t'}} dz' dt' \leq C \frac{1}{\lambda} \ln\big(1 + \lambda z t\big), \tag{4.1}$$

$$\int_0^z \int_0^t z' e^{-\sqrt{\lambda z' t'}} dz' dt' \leq \frac{Cz}{\lambda}, \tag{4.2}$$

$$\int_0^z \int_0^t t' e^{-\sqrt{\lambda z' t'}} dz' dt' \leq \frac{Ct}{\lambda}. \tag{4.3}$$

Proof.

$$\int_0^t e^{-\sqrt{\lambda z' t'}} dt' = \frac{1}{\lambda z'} H(\lambda z' t)$$

with

$$H(u) = \int_0^u e^{-\sqrt{u'}} du' \leq C \frac{u}{1+u}.$$

Thus

$$\int_0^z \int_0^t e^{-\sqrt{\lambda z' t'}} dz' dt' \leq C \int_0^z \frac{t}{1 + \lambda z' t}\ dz' = \frac{C}{\lambda} \int_0^{\lambda z t} \frac{ds}{1+s}.$$

and (4.1) follows. The other two estimates are symmetric in t and z. Integrating in t' first as above we see that

$$\int_0^z \int_0^t z' e^{-\sqrt{\lambda z' t'}} dz' dt' \leq C \int_0^z \frac{t z'}{1 + \lambda z' t}\ dz' \leq \frac{C}{\lambda} \int_0^z dz'$$

implying (4.2). $\square$

Lemma 4.2. *For $p \geq 1$ and $\mu \geq 8$,*

$$\frac{1}{p} \ln(1 + \mu p^2) \leq \ln(1 + \mu).$$

Proof. The upper bound of the left hand side for real $p \geq 1$ is equal to $c_0\sqrt{\mu}$ if $\mu \leq y_0^2$ with y_0 being the positive root of $\ln(1 + y_0^2) = \frac{2y_0^2}{1+y_0^2}$, where the function $\frac{\ln(1+y^2)}{y}$ reached its maximum equal to $c_0 = \frac{2y_0}{1+y_0^2}$. When $\mu \geq y_0^2$, the upper bound is attained at $p = 1$ and thus equal to $\ln(1 + \mu)$. Because $y_0^2 \leq 8$, the lemma follows. $\square$

4.2 Linear estimates

In this section we prove the estimates of Proposition 3.6. Expanding the system (3.19) in Fourier series in x and denoting by (e_p, n_p, f_p) and h_p the Fourier coefficients, yields the following system for $U_p := (u_p, v_p) = (e_p, \overline{e_{-p}})$

$$
\tag{4.4}
\begin{cases}
i\partial_z u_p - k^2p^2 u_p - \underline{E} n_p = f_p, \\
i\partial_z v_p + k^2p^2 v_p + \underline{E} n_p = g_p, \\
(\partial_t^2 + k^2) n_p + k^2p^2 \underline{E}(u_p + v_p) = -k^2p^2 h_p,
\end{cases}
$$

with $g_p = -\overline{f_{-p}}$, plus initial boundary boundary conditions:

$$
\tag{4.5}
n_{p|t=0} = 0, \quad \partial_t n_{p|t=0} = 0, \quad u_{p|z=0} = v_{p|z=0} = 0.
$$

Proposition 3.6 is an immediate corollary of the following estimates, where we use the notations

$$
\tag{4.6}
\rho(t, z) = \sqrt{2k|E|^2 zt}\ .
$$

Proposition 4.3. *There is a constant C_1 is independent of $k \geq 2$, $b \leq 1$ and p such that for $F_p := (f_p, g_p)$ and h_p satisfying on Ω*

$$
\tag{4.7}
|F_p(t, z)| \leq A e^{\langle p\rangle \rho(t,z)}, \quad |h_p(t, z)| \leq B e^{\langle p\rangle \rho(t,z)},
$$

the solution (U_p, n_p) of (4.4) (4.5) on Ω satisfies

$$
\tag{4.8}
|U_p(t, z)| \leq C_1 \Big(b\, A + \frac{1}{|\underline{E}|^2} \ln(\ln \delta|)\, B \Big) e^{\langle p\rangle \rho(t,z)},
$$

$$
\tag{4.9}
|n_p(t, z)| \leq C_1 \Big(\frac{1}{|\underline{E}|^2} \ln(|\ln \delta|) A + \frac{1}{|\underline{E}|^2} \big(k + \ln(|\ln \delta|) \big) B \Big) e^{\langle p\rangle \rho(t,z)}.
$$

Proof. We use Proposition 2.7 at the frequency kp.

a) By (2.32), U_p is the sum of three terms. The first one is bounded by

$$
\tag{4.10}
A \int_0^z e^{\langle p\rangle \rho(t,z')} dz' \leq A z e^{\langle p\rangle \rho(t,z)}.
$$

When $p = 0$, only this term is present. When $p \neq 0$, the second term is

$$
U_{p,0} := \mathcal{E}_0 * F_p = \int_0^z\!\!\int_0^t \mathcal{E}_0(t', z') F_p(t - t', z - z') dz' dt'.
$$

Note the following identity for non negative real numbers:

$$
\sqrt{a'b'} + \sqrt{a''b''} \leq \sqrt{(a' + a'')(b' + b'')}.
$$

In particular,

$$\rho(t',z')+\rho(t-t',z-z')\leq\rho(t,z), \tag{4.11}$$

and therefore, by Proposition 2.5 and (4.7):

$$|U_{p,0}(t,z)|\lesssim\alpha e^{\langle p\rangle\rho(t,z)}\int_0^z\!\!\int_0^t e^{-\langle p\rangle\rho(t',z')}\frac{r'_p e^{\rho'_p}}{\sqrt{(1+\rho'_p)}}dz'dt'$$

where $\rho'_p=\sqrt{2kp|E|^2z't'}$ and $r'_p=1+\frac{kp|\underline{E}|^2z'}{1+\rho'_p}\leq 1+kp|\underline{E}|^2z'$.

When $p\geq 2$,

$$p\rho(t',z')-\rho'_p=\sqrt{2(p^2-p)k|E|^2z't'}\geq\sqrt{kp^2|E|^2z't'}\geq\frac{1}{2}p\rho(t',z').$$

When $p=1$,

$$\langle 1\rangle\rho(t',z')-\rho'_1=2\rho(t',z')-\rho(t',z')=\rho(t',z').$$

Thus, in any case,

$$\langle p\rangle\rho(t',z')-\rho'_p\geq\frac{1}{2}p\rho(t',z'). \tag{4.12}$$

Therefore

$$|U_{p,0}(t,z)|\lesssim Ae^{\langle p\rangle\rho(t,z)}\int_0^z\!\!\int_0^t(1+kp|\underline{E}|^2z')e^{-\frac{1}{2}p\rho(t',z')}dz'dt'.$$

Using Lemmas 4.1 with $\lambda=\frac{1}{2}kp^2|\underline{E}|^2$, yields

$$|U_{p,0}(t,z)|\lesssim A\Big(\frac{1}{\lambda}\ln\big(1+\lambda zt\big)+kp|\underline{E}|^2\frac{z}{\lambda}\Big)e^{p\rho}\lesssim Aze^{\langle p\rangle\rho}, \tag{4.13}$$

where we have used that $\frac{1}{\lambda}\ln\big(1+\lambda zt\big)\leq zt$.

Similarly $U_{p,1}=\mathcal{E}_1*h_p$ satisfies

$$\begin{aligned}|U_{p,1}(t,z)|&\lesssim Be^{\langle p\rangle\rho}\int_0^z\!\!\int_0^t e^{-p\rho'}\frac{kp\,e^{\rho'_p}}{\sqrt{(1+\rho'_p)}}dz'dt'\\&\lesssim Be^{\langle p\rangle\rho}\int_0^z\!\!\int_0^t kpe^{-\frac{1}{2}p\rho'}dz'dt'\\&\lesssim B\frac{1}{p|\underline{E}|^2}\ln\big(1+\frac{1}{2}p^2k|E|^2zt\big)\ e^{\langle p\rangle\rho}.\end{aligned}$$

On Ω, $\frac{1}{2}k|E|^2zt\leq\rho^2\leq|\ln\delta|^2$. Thus, for $\delta\geq e^3$, Lemma 4.2 implies that

$$|U_{p,1}(t,z)|\lesssim\frac{1}{|\underline{E}|^2}\ln\big(1+|\ln\delta|^2\big)e^{\langle p\rangle\rho}\lesssim\frac{1}{|\underline{E}|^2}\ln(|\ln\delta|)\ e^{\langle p\rangle\rho}. \tag{4.14}$$

With (4.10) and (4.13), this implies the estimate (4.8).

b) Similarly, when $p = 0$ the estimate for n_0 is immediate. When $p \neq 0$, by (2.33), n_p is the sum of two terms. Up to constant factors, the first one is a convolution of F_p by $\mathcal{E}_1$ (computed at the frequency kp). Using again (4.11) and (4.12) this term satisfies

$$
\begin{aligned}
|n_{p,1}(t,z)| &\lesssim Ae^{p\rho(t,z)} \int_0^z\!\!\int_0^t \frac{kpe^{-\frac{1}{2}p\rho(t',z')}}{\sqrt{(1+\rho'_p)}} dz'dt' \\
&\lesssim Ae^{\langle p\rangle\rho} \int_0^z\!\!\int_0^t kpe^{-\frac{1}{2}p\rho'} dz'dt' \lesssim A\frac{1}{|\underline{E}|^2} \ln(|\ln\delta|)e^{\langle p\rangle\rho}.
\end{aligned} \tag{4.15}
$$

The second term in n_p is the convolution of h_p with $\mathcal{E}_2$ and satisfies

$$
|n_{p,2}(t,z)| \lesssim Be^{\langle p\rangle\rho(t,z)} \int_0^z\!\!\int_0^t e^{-p\rho'} \frac{k^2p^2e^{\rho'_p}}{\tilde{r}'_p\sqrt{(1+\rho'_p)}} dz'dt'
$$

with $\frac{1}{\tilde{r}'_p} \leq \big(t' + \frac{1}{kp}\big)$. Thus, by Lemma 4.1

$$
\begin{aligned}
|n_{p,2}(t,z)| &\lesssim Be^{\langle p\rangle\rho} \int_0^z\!\!\int_0^t k^2p^2\big(\frac{1}{kp} + t'\big)e^{-\frac{1}{2}p\rho(t',z')} dz'dt' \\
&\lesssim B\Big(\frac{kt}{|\underline{E}|^2} + \frac{1}{|\underline{E}|^2} \ln(|\ln\delta|)\Big)e^{p\rho}.
\end{aligned} \tag{4.16}
$$

□

5 Analytic solutions of the Goursat problem

Consider

$$
\begin{cases} i\partial_z E + \Delta_x E = nE, \\ (\partial_t^2 - \Delta_x)n = \Delta_x|E|^2 \end{cases} \tag{5.1}
$$

on $t \in [0,T]$, $z \in [0,Z]$ and $x \in \mathbb{T}^2$, together with initial boundary conditions

$$
n_{|t=0} = 0, \quad \partial_t n_{|t=0} = 0, \quad E_{z=0} = E_0. \tag{5.2}
$$

We prove that this Goursat problem is locally well posed in spaces of analytic functions, following the general approach presented in [11]. One of our objective is to give an explicit lower bound for the domain of existence, depending on the norms of the boundary data (see Remark 5.6 below).

5.1 Spaces

Fix $s >$ and equip $H^s(\mathbb{T})$ with a norm $\|\cdot\|_s$ such that

$$\|uv\|_s \leq \|u\|_s \ \|v\|_s. \tag{5.3}$$

Definition 5.1. *Given a (formal) power series $\phi(\mathrm{x}) = \sum \phi_n \mathrm{x}^n$ in one variable* x*, with nonnegative coefficients, we say that $u(x) \ll \phi$ if for all $\alpha \in \mathbb{Z}^2$, $\|\partial_x^\alpha u\|_s \leq |\alpha|!\, \phi_{|\alpha|}$.*

Lemma 5.2. *i) If $u \ll \phi$, then $\partial_{x_j} u \ll \phi'$.*
i) If $u \ll \phi$ and $v \ll \psi$, then $uv \ll \phi\psi$,

We will apply this definition to functions u and power series which also depend on t and z, seen as parameters. In particular, given a power series $\phi(\mathrm{x})$, we will consider power series

$$\phi_\lambda(z, \mathrm{x}) = \phi(\mathrm{x} + \lambda z), \tag{5.4}$$

and use the notations $u(t, z, x) \ll \phi_\lambda(z, \mathrm{x})$ if $u(t, z, \cdot) \ll \phi_\lambda(z, \cdot)$.

Lemma 5.3. *If $u(t, z, x) \ll \phi'_{R,\lambda}(z, \mathrm{x})$ then*

$$\int_0^z u(t, z', x) dz' \ \ll \ \frac{1}{\lambda} \phi_{R,\lambda}(t, z, \mathrm{x}).$$

In particular, we consider the power series (see [11])

$$\varphi(\mathrm{x}) = c_0 \sum_{n=0}^{\infty} \frac{\mathrm{x}^n}{n^2 + 1}, \quad \varphi_{R,\lambda}(z, \mathrm{x}) = \varphi(R\mathrm{x} + \lambda z). \tag{5.5}$$

where c_0 is such that $\varphi^2 \ll \varphi$. We also introduce the notations

$$\varphi(R\mathrm{x} + \lambda z) = \sum_{n=0}^{\infty} \varphi_n(\lambda z) R^n \mathrm{x}^n. \tag{5.6}$$

Of course the explicit value of φ_n can be deduced from (5.5). We also note that

$$\varphi'_{R,\lambda}(z, \mathrm{x}) = \sum_{n=0}^{\infty} (n+1) \varphi_{n+1}(\lambda z) R^{n+1} \mathrm{x}^n. \tag{5.7}$$

and by integration of $\varphi'(R\mathrm{x} + \lambda z)$ in z, we see that

$$\int_0^z \varphi_{n+1}(\lambda z') dz' = \frac{1}{(n+1)\lambda} \big(\varphi_n(\lambda z) - \varphi_n(0)\big) \leq \frac{1}{(n+1)\lambda} \varphi_n(\lambda z) \tag{5.8}$$

Definition 5.4. *Denote by $\mathbb{E}_{R,\lambda}$ the space of functions u on $[0,T]\times[0,Z]\times\mathbb{T}^2$, with $Z=\frac{1}{\lambda}$, such that*

$$u(t,z,x) \ll C\varphi_{R,\lambda}(z,\mathrm{x}) \tag{5.9}$$

for some constant $C\geq 0$.

Similarly, $\mathbb{F}_{R,\lambda}$ denotes the space of functions u on $[0,T]\times[0,Z]\times\mathbb{T}^2$, such that

$$u(t,z,x) \ll C\varphi'_{R,\lambda}(z,\mathrm{x}) \tag{5.10}$$

for some constant $C\geq 0$.

The norms in $\mathbb{E}$ and $\mathbb{F}$ are the best constant C in (5.9) and (5.10).

For the boundary data $E_0(t,x)$ on $[0,T]\times\mathbb{T}^2$, we consider the spaces $\mathbb{E}_R$ of functions $u(t,x)\ll C\varphi(R\mathrm{x})$.

Theorem 5.5. *Let $E_0\in\mathbb{E}_R$. Then, the problem* (5.1) *has a unique solution $(E,n)\in\mathbb{E}_{R,\lambda}\times\mathbb{F}_{R,\lambda}$ if $\lambda\geq CRT\|E_0\|^2_{\mathbb{E}_R}$ where C is a constant independent of T, R and e_0.*

Remark 5.6. Functions in $\mathbb{E}_{R,\lambda}$ are defined for $R|\mathrm{Im}\,x|+\lambda|z|<1$. Thus R^{-1} measures the width of the complex domain where the boundary data is defined and λ^{-1} is the order of the length of propagation in z. In particular, for boundary data $E_0(t,x)=e(t)e^{ikx}$ then, for $R\approx k$, $\|E_0\|_{\mathbb{E}_R}\lesssim\underline{E}=\|e\|_{L^\infty}$. Theorem 5.5 asserts that the length of stability $Z=\lambda^{-1}$ satisfies

$$ZTk\underline{E}^2\lesssim 1, \tag{5.11}$$

which is very similar to the condition $\rho\lesssim 1$ that was used in Section 3.

5.2 Resolution of (5.1) in $\mathbb{E}$

Consider the wave equation

$$\partial_t^2 n-\Delta_x n=\Delta_x h,\quad n_{|t=0}=\partial_t n_{|t=0}=0. \tag{5.12}$$

Lemma 5.7. *For $h\in\mathbb{E}_{R,\lambda}$, the solution n of* (5.12) *belongs to $\mathbb{F}_{R,\lambda}$ and*

$$\|n\|_{\mathbb{F}_{R,\lambda}}\leq C_0T\|h\|_{\mathbb{E}_{R,\lambda}} \tag{5.13}$$

where C_0 is independent of T, R and λ.

Proof. The mean value of n (the 0-th Fourier coefficient) vanishes and therefore the H^s norm of n is equivalent to the H^{s-1} norm of $\partial_x n$. Thus, by standard energy estimates for the wave equation, there holds

$$\|\partial_x^\alpha n(t,z,\cdot)\|_s \leq C \int_0^t \|\partial_x^\alpha h(t',z,\cdot)\|_{s+1} dt'.$$

Thus, for $|\alpha| = n$,

$$\|\partial_x^\alpha n(t,z,\cdot)\|_s \leq CT \|h\|_{\mathbb{E}_{R,\lambda}} \ (n+1)! \ R^{n+1} \varphi_{n+1}(\lambda z).$$

Therefore

$$n(t,z,x) \ll CT \|h\|_{\mathbb{E}_{R,\lambda}} \ \varphi'_{R,\lambda}(z, \mathrm{x})$$

and the lemma follows. □

We consider next the Schrödinger equation

$$i\partial_t e - \Delta_x e = f, \quad e_{|z=0} = e_0. \tag{5.14}$$

Lemma 5.8. *For $f \in \mathbb{F}_{R,\lambda}$, and $e_0 \in \mathbb{E}_R$, the solution e of* (5.14) *belongs to $\mathbb{E}_{R,\lambda}$ and*

$$\|e\|_{\mathbb{E}_{R,\lambda}} \leq \frac{R}{\lambda} \|f\|_{\mathbb{F}_{R,\lambda}} + \|e_0\|_{\mathbb{E}_R}. \tag{5.15}$$

Proof. Standard energy estimates imply that

$$\|\partial_x^\alpha e(t,z,\cdot)\|_s \leq \int_0^z \|\partial_x^\alpha f(t,z',\cdot)\|_s dz' + \|\partial_x^\alpha e_0(t,\cdot)\|_s$$

Thus, for $|\alpha| = n$,

$$\begin{aligned} \|\partial_x^\alpha e(t,z,\cdot)\|_s \leq \|h\|_{\mathbb{F}_{R,\lambda}} \ (n+1)! R^{n+1} \ &\int_0^z \varphi_{n+1}(\lambda z') dz' \\ &= \|h\|_{\mathbb{F}_{R,\lambda}} \ n! \ \frac{R^{n+1}}{\lambda} \ \varphi_n(\lambda z). \end{aligned}$$

Therefore

$$e(t,z,x) \ll \frac{R}{\lambda} \|f\|_{\mathbb{F}_{R,\lambda}} \ \varphi_{R,\lambda}(z, \mathrm{x})$$

and the lemma follows.

□

Lemma 5.9. *For $e_1 \in \mathbb{E}_{R,\lambda}$, $e_2 \in \mathbb{E}_{R,\lambda}$ and $n \in \mathbb{F}_{R,\lambda}$, there holds $e_1 e_2 \in \mathbb{E}_{R,\lambda}$ and $ne_1 \in \mathbb{F}_{R,\lambda}$ with*

$$\begin{aligned} &\|e_1 e_2\|_{\mathbb{E}_{R,\lambda}} \leq \|e_1\|_{\mathbb{E}_{R,\lambda}} \|e_2\|_{\mathbb{E}_{R,\lambda}}, \\ &\|ne_1\|_{\mathbb{F}_{R,\lambda}} \leq \|e_1\|_{\mathbb{E}_{R,\lambda}} \ \|n\|_{\mathbb{F}_{R,\lambda}}. \end{aligned}$$

5.3 Proof of Theorem 5.5

Denote by $\mathcal{N}(h)$ the solution of (5.12) and by $\mathcal{S}_0(e_0)+\mathcal{S}(f)$ the solution of (5.14). To prove Theorem 5.5 it is sufficient to show that there is $C>0$, such that for $\lambda \geq CRT\|E_0\|^2_{\mathbb{E}_R}$, the equation

$$E = \mathcal{S}(E_0) + \mathcal{T}(E,E,E) \tag{5.16}$$

has a unique solution in $\mathbb{E}_{R,\lambda}$, where $\mathcal{T}$ is the trilinear operator

$$\mathcal{T}(u,v,w) = \mathcal{S}\big(u\mathcal{N}(v\overline{w})\big).$$

The estimates above show that $\mathcal{T}$ maps $(\mathbb{E}_{R,\lambda})^3$ to $\mathbb{E}_{R,\lambda}$ and that

$$\|\mathcal{T}(u,v,w)\|_{\mathbb{E}_{R,\lambda}} \leq \frac{C_0RT}{\lambda}\|u\|_{\mathbb{E}_{R,\lambda}}\|v\|_{\mathbb{E}_{R,\lambda}}\|w\|_{\mathbb{E}_{R,\lambda}}.$$

From here, standard Picard's iterates imply Theorem 5.5.

6 Remarks and comments

6.1 Nonconstant backgrounds

The boundary data we have considered in Sections 2 and 3 are of the form

$$1_{\{t\geq 0\}}\left(\underline{E} + e(t,x)\right), \tag{6.1}$$

thus have a jump at $t=0$, which is not physical. Note that arbitrary functions $\underline{E}(t)$ together with $n=0$ are solutions of (1.4). One could take them as background state. We now briefly sketch how one can start the analysis. Again, there is no restriction in assuming that $\underline{E}$ is real. The linearized systems (2.3) and (2.4) are unchanged, except that $\underline{E}$ is now a function of time. Therefore, we now proceed by using the Fourier-Laplace in z rather than in time: we extend u, v, n by 0 for $z<0$. Note that the extension $\tilde{u}$ satisfies $\partial_z\tilde{u} = \widetilde{\partial_z u} + u_0\delta_{z=0}$. Therefore, the Fourier-Laplace transforms satisfy

$$\begin{cases} -(\zeta+k^2)\hat{u} - \underline{E}\hat{n} = iu_0, \\ (-\zeta+k^2)\hat{v} + \underline{E}\hat{n} = iv_0, \\ (\partial_t^2+k^2)\hat{n} + k^2\underline{E}(\hat{u}+\hat{v}) = 0. \end{cases} \tag{6.2}$$

Hence,

$$\left\{\begin{aligned} \hat{u} &= -\frac{\underline{E}\hat{n}}{\zeta + k^2} - \frac{iu_0}{\zeta + k^2}, \\ \hat{v} &= \frac{\underline{E}\hat{n}}{\zeta - k^2} - \frac{iv_0}{\zeta - k^2}, \end{aligned}\right. \tag{6.3}$$

and

$$\big(\partial_t^2 + k^2(1 + ap(t))\big)\hat{n} = k^2\hat{h}, \tag{6.4}$$

with

$$a = |E|^2\ \frac{2k^2}{\zeta^2 - k^4}, \qquad p(t) = |\underline{E}(t)|^2 \tag{6.5}$$

$$h = iE\big(\frac{u_0}{\zeta - k^2} + \frac{v_0}{\zeta + k^2}\big). \tag{6.6}$$

Denote by $\mathcal{N}(t,s,a)$ the fundamental solution of the second order o.d.e.

$$\big(\partial_t^2 + k^2(1 + ap)\big)\mathcal{N} = 0, \qquad \mathcal{N}_{|t=s} = 0, \quad \partial_t\mathcal{N}_{|t=s} = 1. \tag{6.7}$$

Thus

$$\hat{n}(t) = k^2 \int_0^t U(t,s,a)h(s)ds. \tag{6.8}$$

Performing the inverse Laplace transform, we get

$$n = \int_0^t \mathcal{E}_1(t,s,z)u_0(s)ds + \int_0^t \mathcal{E}_2(t,s,z)v_0(s)ds \tag{6.9}$$

with

$$\mathcal{E}_1(t,s,z) = \frac{i\underline{E}(s)}{2\pi} \int_{\mathbb{R}-i\gamma} e^{iz\zeta} U(t,s,a) \frac{k^2\, d\zeta}{\zeta - k^2}, \tag{6.10}$$

$$\mathcal{E}_2(t,s,z) = \frac{i\underline{E}(s)}{2\pi} \int_{\mathbb{R}-i\gamma} e^{iz\zeta} U(t,s,a) \frac{k^2\, d\zeta}{\zeta + k^2}. \tag{6.11}$$

Lemma 6.1. *$\mathcal{N}$ is an entire function of a. Moreover, for $|a|$small and $0 \leq s \leq t \leq 1$ there holds*

$$\mathcal{N}(t,s,a) = \frac{1}{k}\Big(\alpha(t,s,a)e^{ik\varphi(t,s,a)} - \beta(t,s,a)e^{-ik\varphi(t,s,a)}\Big) \tag{6.12}$$

with

$$\partial_t \varphi = \sqrt{1+ap}, \quad \varphi_{|t=s} = 0, \tag{6.13}$$

and α and β are $O(1)$ as well as their time derivatives.

Because $|a| \to 0$ as $|\zeta| \to \infty$, U is holomorphic in ζ and bounded for large ζ. Thus, one can close the integration path in (6.10) and (6.11): for instance, for $z > 0$:

$$\mathcal{E}_1(t,s,z) = \frac{1}{2\pi}\int_\Gamma e^{iz\zeta} U(t,s,a)\frac{k^2\, d\zeta}{\zeta - k^2} \tag{6.14}$$

where Γ is a large circle in the complex plane.

From here, using the Lemma above, one can repeat most of the computations performed in Section 2, with only technical additional difficulties. The important point is that the amplification is now driven by the kernel e^ρ with

$$\rho := \sqrt{2k(z-z')(P(t)-P(t'))}, \qquad P(t) = \int_0^t |\underline{E}(s)|^2 ds. \tag{6.15}$$

Of course, when $\underline{E}$ is constant we recover the previous results. On an interval where $|\underline{E}| > 0$, there holds $P(t) - P(t') \approx (t-t')$ and all the estimates of Section 2 are unchanged.

In this direction, another interpretation of the constant case $\underline{E}$, different from (6.1) is that the boundary data is

$$\underline{E}(t) + e(t,x)1_{\{t \geq 0\}} \tag{6.16}$$

where $\underline{E}$ vanishes for $t \leq -T_0 < 0$ and is equal to the constant $\underline{E}$ for $t \geq 0$. This means that the perturbation e starts when the background is stabilized at a constant value. This is not physically realistic either, but the expectation is that this analysis is qualitatively relevant for physical interpretations.

6.2 The amplification rate

We come back to the case where $\underline{E}$ is constant. The solution of the homogeneous linearized equations (2.4) is given in Lemma 2.1. It involves convolution by functions defined by contour integrals (2.15). These integrals were estimated in Section 2 using deformations of contours and the saddle point method. According to Lemma 2.4, a model for the integral $\mathcal{E}_1$, is

$$\mathcal{F}_1 = \frac{1}{2i\pi}\int_\Gamma e^{i(t\zeta - zk\lambda)}\frac{d\zeta}{\zeta - k}, \qquad \lambda = k + \frac{|\underline{E}|^2}{2(\zeta - k)} \ . \tag{6.17}$$

Note that

$$\mathcal{F}_1 = e^{i(tk - zk^2)}J(\rho), \qquad \rho = \sqrt{2k|\underline{E}|^2 zt} \tag{6.18}$$

where J is the bessel function

$$J(\rho) = \frac{1}{2i\pi}\int_\Gamma e^{i\frac{1}{2}\rho(\zeta - \frac{1}{\zeta})}\frac{d\zeta}{\zeta} = \int_0^{2\pi} e^{\rho\sin\theta}d\theta = \sum_{n=0}^{\infty}\frac{\rho^n}{n!^2}.$$

This is indeed the core of the analysis performed in Section 2.

Following this path, a model for the solutions of the homogeneous equation (2.4) is

$$u(t,z) = \int_0^t e^{i((t-t')k - zk^2)}J(\sqrt{2k|\underline{E}|^2 z(t-t')})u_0(t')dt'. \tag{6.19}$$

The kernel has an amplification part given by J but it also has an oscillating part given by $e^{i(t-t')k}$. In Theorem 2.8, we have chosen u_0 such that it has itself an oscillation $e^{ikt'}$ to eliminate the oscillations in (6.19) and maximize the amplification. For general u_0 the actual amplification results from a delicate balance between J and the oscillations. Typically, one integration by part in the integral above, wins a factor $k^{-\frac{1}{2}}$ and k^{-1} times a derivative of u_0. Thus the amplification e^ρ is cut down by a factor k^{-m} where m is the smoothness of u_0. This can also be seen on the Fourier Laplace representation of u:

$$u(t,z) = \frac{e^{i(tk - zk^2)}}{2i\pi}\int_\Gamma e^{i(t\zeta - \frac{zk|\underline{E}|^2}{\zeta})}\hat{u}_0(k+\zeta)\frac{d\zeta}{\zeta}. \tag{6.20}$$

In any case, there is one clear conclusion : *the more amplified u_0 are those who have a nontrivial oscillation at frequency k.*

This can also be seen in a different way. Consider the Fourier-Laplace expression of the solution:

$$\hat{U}(z,\zeta) = e^{izA(\zeta)}\hat{U}_0(\zeta). \tag{6.21}$$

For $\zeta = (\tau - i\gamma) \notin \{-k, +k\}$, the eigenvalues of A are

$$\mu(\zeta) = \pm k^2\sqrt{1+\frac{1}{b}}, \qquad b = \frac{\zeta^2 - k^2}{|\underline{E}|^2}. \tag{6.22}$$

Consider *real* time frequencies $\zeta = \tau \notin \{-k, +k\}$. When $1 + \frac{1}{b} > 0$, the eigenvalues are real and there is no amplification. When $1 + \frac{1}{b} < 0$, that is when

$$0 < k^2 - \tau^2 < |\underline{E}|^2, \tag{6.23}$$

the eigenvalue is purely imaginary and this suggests that the *amplification factor for oscillations with frequency* τ is

$$e^{z|\mathrm{Im}\,\mu|}, \qquad |\mathrm{Im}\,\mu| = k^2\,\frac{\sqrt{|\underline{E}|^2 - k^2 + \tau^2}}{\sqrt{k^2 - \tau^2}}. \tag{6.24}$$

In particular, only real frequencies close to k are amplified.

Of course, considering only real frequencies is not consistent with the forward evolution problem under consideration and the condition that U_0 vanishes in the past. $e^{izA(\zeta)}$ has essential singularities at $+k$ and $-k$, and one has to consider integration on complex path as in Section 2, to turn around the singular points. Then the imaginary part of the phase to consider is $t\mathrm{Im}\,\zeta - z\mathrm{Im}\,\mu$. In Section 2 we have chosen optimal contours in order to obtain sharp estimates. In particular, the amplification (6.19) is not correct. But the computation suggests that the amplification is mainly due to frequencies close to k, and this is correct.

6.3 The physical context and physical values

The Zakharov's equations [12] have been introduced at the beginning of the 70's, to describe electronic plasma waves. They couple the slowly varying envelope of the electric field and the low-frequency variation of the density of the ions. When modeling the propagation of a laser beam in a plasma, several phenomena occur. One has to take into account the laser beam itself, the Raman component and the electronic plasma waves (see [2, 3] for example). For laser propagation or for the Raman component, one often uses the paraxial approximation and the Zakharov system that couples the envelop of the vector potential A of the electromagnetic field A to the low-frequency variation of the density δn of the ions reads

$$\left\{\begin{aligned} & i(\partial_t + \frac{k_0 c^2}{\omega_0}\partial_z)A + \frac{c^2}{2\omega_0}\Delta_x A = \frac{\omega_{pe}^2}{2n_0\omega_0}\delta n\, A, \\ & (\partial_t^2 - c_s^2\Delta_x)\delta n = \frac{\omega_{pe}^2}{4\pi m_i c^2}\Delta_x |A|^2, \end{aligned}\right. \tag{6.25}$$

where ω_0 is the frequency of the laser, k_0 its wave number and ω_{pe} the plasma electronic frequency; they are linked by the dispersion relation $\omega_0^2 = \omega_{pe}^2 + k_0^2 c^2$ where c is the speed of light; n_0 the mean density of the plasma, m_i is the mass of the ions and c_s the sound velocity in the plasma. In suitable units for n_0, the plasma frequency ω_{pe} is given by

$$\omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}} \tag{6.26}$$

where $-e$ is the charge of the electron and m_e its mass.

As above, z is the space component in the direction of propagation of the laser beam and x denotes the space components in directions that are transversal to the propagation.

Introducing the dimensionless quantities

$$n = \delta n/n_0, \qquad a = \frac{A}{\underline{A}}, \quad \underline{A} = \frac{m_e c c_s}{e} \tag{6.27}$$

and using (6.26), the system reads

$$\left\{\begin{aligned} & i(\frac{\omega_0}{c^2}\partial_t + k_0\partial_z)a + \frac{1}{2}\Delta_x a = \frac{\omega_{pe}^2}{2c^2}\, n\, a, \\ & (\frac{1}{c_s^2}\partial_t^2 - \Delta_x)n = \frac{m_e}{m_i}\Delta_x |a|^2, \end{aligned}\right. \tag{6.28}$$

In addition to the pulsation ω_0 and the wave length $\lambda_0 = \frac{2\pi}{k_0}$, characteristic quantities for the laser beam are its duration τ_0, its transversal width R_0 (supposed to be smaller than the transversal dimension of the plasma), the expected length of propagation Z_0 and also the characteristic dimension of the transversal variations (speckles) λ which correspond to variations of the spectrum by frequencies $k \approx \frac{2\pi}{\lambda}$. In the application we have in mind $\lambda \ll R_0$ and both for theoretical and computational reasons (spectral methods) it makes sense to *assume periodicity in x* with period X satisfying $\lambda \leq X \ll R_0$.

With this data in mind, rescale the space-time variables introducing the characteristic transversal width X:

$$\tilde{x} = \frac{x}{X}, \quad \tilde{z} = \frac{z}{2k_0 X^2}, \quad \tilde{t} = \frac{c_s t}{X} \tag{6.29}$$

In these variables, the system reads

$$\begin{cases} i(\epsilon\partial_{\tilde{t}} + \partial_{\tilde{z}})a + \Delta_{\tilde{x}} a = \alpha^2 n a, \\ (\partial_{\tilde{t}}^2 - \Delta_{\tilde{x}})n = \dfrac{m_e}{m_i}\Delta_{\tilde{x}}|a|^2, \end{cases} \tag{6.30}$$

with

$$\epsilon = \frac{2\omega_0 c_s X}{c^2}, \qquad \alpha = \frac{\omega_{pe} X}{c}. \tag{6.31}$$

With

$$\tilde{n} = \alpha^2 n, \qquad \tilde{a} = \alpha\sqrt{\frac{m_e}{m_i}}a \tag{6.32}$$

we obtain the dimensionless system:

$$\begin{cases} i(\epsilon\partial_{\tilde{t}} + \partial_{\tilde{z}})\tilde{a} + \Delta_{\tilde{x}}\tilde{a} = \tilde{n}\tilde{a}, \\ (\partial_{\tilde{t}}^2 - \Delta_{\tilde{x}})\tilde{n} = \Delta_{\tilde{x}}|\tilde{a}|^2. \end{cases} \tag{6.33}$$

For this system, the typical *amplification factor* is

$$\rho = \sqrt{2|\tilde{a}|^2\,\tilde{k}\,\tilde{\tau}_0\,\tilde{Z}_0} \tag{6.34}$$

where $\tilde{k}$ is the scaled frequency of speckles, $\tilde{\tau}_0$ the scaled time of propagation under consideration and $\tilde{Z}_0$ the scaled length of propagation. Scaling back to the original variables, there holds

$$\tilde{Z}_0 = \frac{Z_0}{2k_0 X^2}, \quad \tilde{k} = Xk, \quad \tilde{\tau} = \frac{c_s \tau_0}{X}, \tag{6.35}$$

thus

Proposition 6.2. *The dimensioned amplification factor ρ of Section 2 is given by*

$$\rho^2 = 2\frac{|A|^2}{\underline{A}^2}\frac{m_e c_s \omega_{pe}^2}{m_i c^2}\,\frac{k}{k_0}\,\tau_0 Z_0. \tag{6.36}$$

For the physical significance of the analysis it is important to evaluate the various constants. The velocity of light is $c = 3\ 10^8 \mathrm{ms}^{-1}$, the sound velocity of the electrons is of order $c_s \approx 0.005c = 1.5\ 10^6 \mathrm{ms}^{-1}$. With $e = 1.6\ 10^{-19}$C and $m_e = 0.9\ 10^{-30}$Kg, the scaling factor for A is

$$\underline{A} = \frac{m_e c c_s}{e} \approx 2.5\, 10^3\, \mathrm{V}. \tag{6.37}$$

Typical values of ω_{pe} are of order $10^{15} s^{-1}$ and $\frac{m_e}{m_i} \approx 10^{-4}$. Thus

$$2\frac{m_e c_s \omega_{pe}^2}{m_i c^2} \approx 4\ 10^{15} \quad \mathrm{m}^{-1}\mathrm{s}^{-1} \tag{6.38}$$

A typical value of λ_0 is $\lambda_0 = 0.35\mu$m corresponding to a wave number $k_0 = \frac{2\pi}{\lambda_0} = 1.8\ 10^7 \mathrm{m}^{-1}$. The pulsation is given by the dispersion relation $\omega_0^2 = \omega_{pe}^2 + k_0^2 c^2$, yielding $\omega_0 = 5.5\ 10^{15} \mathrm{s}^{-1}$.

By construction, the relative variation $n = \delta n / n_0$ is small compared to 1 and the consistency of (6.28) requires that $\frac{m_e}{m_i}|a|^2$ must also be small compared to 1, that is $|a| \ll 100$ yielding the upper bound for A

$$|A| \ll 10^5\ \mathrm{V}. \tag{6.39}$$

The envelop of the electromagnetic field is linked to A by the polarization relation

$$E = i\frac{\omega_0}{c}A \tag{6.40}$$

yielding the bound

$$|E| \ll 10^{12}\ \mathrm{Vm}^{-1}, \tag{6.41}$$

which allows for very intense fields.

Typical values for the dimension of the speckles is $\lambda \approx 10\mu$m, corresponding to frequencies $k = \frac{2\pi}{\lambda} \approx 6\ 10^5 \mathrm{m}^{-1}$. A constraint for the validity of the paraxial approximation which sustains the Schrödinger envelop equation in (6.25) is that

$$k Z_0 \le k_0 R_0 \tag{6.42}$$

With $k_0/k \approx 30$ as above and a diameter R_0 ranging from 1mm to 1cm, this allows for propagations along distances Z_0 of order 1 to several cm. In this range of k, we finally obtain:

$$\rho = \frac{|A|}{\underline{A}}\sqrt{\beta \tau_0 Z_0} \qquad \text{with} \quad \beta \approx 10^{14} \quad \mathrm{m}^{-1}\mathrm{s}^{-1} \tag{6.43}$$

The duration of the laser impulsion τ_0 may be of order 10^{-8} to 10^{-10}s. Note that for a 1cm long propagation of the laser beam, the time elapsed is of order $3 10^{-11}$s. For Z_0 of order 1cm and τ_0 ranging from 10^{-8} to 10^{-10}s, we obtain that

$$\rho = \gamma \frac{|A|}{\underline{A}} \tag{6.44}$$

with γ ranging from 10 to 100.

Thus, for ρ to be small, one must have

$$|A| \lesssim 10\,\mathrm{V}, \quad \text{or} \quad |E| \lesssim 10^8\,\mathrm{Vm}^{-1}, \tag{6.45}$$

which does nor correspond to high intensity beams.

Note also that with $X \approx 50\ \mu$m, there holds

$$\epsilon \approx 9, \qquad \tilde{\tau} \approx 3. \tag{6.46}$$

In conclusion, we see that the values obtained in (6.45) are compatible with the paraxial approximation and the derivation of the model (6.25). However, they do not allow for high intensity or high energy beams. This suggests that the model could be unadapted to such situations.

References

[1] S.Alinhac, *Le problème de Goursat hyperbolique en dimension deux*, Comm. Partial Differential Equations 1 (1976), no. 3, 231–282.

[2] M.Colin, T.Colin, *On a quasilinear Zakharov system describing laser-plasma interactions*, Differential Integral Equations 17 (2004), no. 3-4, 297–330.

[3] M.Colin, T.Colin, *A numerical model for the Raman amplification for laser-plasma interaction*, J. Comput. Appl. Math. 193 (2006), no. 2, 535–562.

[4] T.Colin, G.Métivier, *Instabilities in Zakharov Equations for laser propagation in a plasma*, in Phase space analysis of partial differential equations, 63–81, Progr. Nonlinear Differential Equations Appl., 69, Birkhäuser Boston, Boston, MA, 2006.

[5] J. Ginibre, Y. Tsutsumi, G. Velo, *On the Cauchy problem for the Zakharov system*. J. Funct. Anal. 151 (1997), no. 2, 384–436.

[6] F.Linares, G.Ponce, J-C.Saut, *On a degenerate Zakharov system.* Bull. Braz. Math. Soc. (N.S.) 36 (2005), no. 1, 1–23.

[7] T. Ozawa, Y. Tsutsumi, *Existence and smoothing effect of solutions for the Zakharov equations.* Publ. Res. Inst. Math. Sci. 28 (1992), no. 3, 329–361.

[8] D.A. Russel, D.F. Dubois and H.A. Rose. *Nonlinear saturation of simulated Raman scattering in laser hot spots.* Physics of Plasmas, Vol. 6 (4), (1999), 1294-1317.

[9] G. Riazuelo. *Etude théorique et numérique de l'influence du lissage optique sur la filamentation des faisceaux lasers dans les plasmas sous-critiques de fusion inertielle.* Thèse de l'Université Paris XI.

[10] C. Sulem and P-L. Sulem. *The nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse.* Applied Mathematical Sciences 139, Springer, (1999).

[11] C. Wagschal,*Le problème de Goursat non linéaire*, J. Math. Pures Appl. (9) 58 (1979), no. 3, 309–337.

[12] V.E. Zakharov, S.L. Musher and A.M. Rubenchik. *Hamiltonian approach to the description of nonlinear plasma phenomena.* Phys. Reports, Vol. 129, (1985), 285-366.