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Some Contributions to Scattering Theory in General Relativity

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Publications (hors thèse de doctorat) :

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Prépublication :

[10] Jean-François Bony, Dietrich Häfner, *The semilinear wave equation on asymptotically euclidean manifolds*, arXiv:0810.0464.

Ce document se divise en deux parties. La première partie est consacrée à la théorie de la diffusion en relativité générale. La deuxième partie contient deux résultats sur l'équation de Schrödinger.

Contents

Part I

Some contributions to scattering theory in general relativity

Chapter 1

Introduction

One of the fundamental problems of theoretical physics consists of describing the behavior of a material field which evolves in an electromagnetic field which evolves itself in a gravitational field. This is done by studying the following system of equations :

$$G_{ab} = 8\pi(T^{\Psi}_{ab} + T^{Fel}_{ab}), G_{ab} := R_{ab} + \frac{1}{2}Rg_{ab} + \Lambda g_{ab}, \tag{1.0.1}$$

$$\nabla^a F^{el}_{ab} = -4\pi J^{\Psi}_b, \quad \nabla_{[a} F^{el}_{b\,c]} = 0, \tag{1.0.2}$$

$$\mathcal{F}(g_{ab}, \nabla^a, F^{el}_{ab}, \Psi) = 0. \tag{1.0.3}$$

Equation (1.0.1) is the Einstein equation, where Λ, R, R_{ab}, g_{ab}, G_{ab}, T^{Ψ}_{ab}, T^{Fel}_{ab} are the cosmological constant, the scalar curvature, the Ricci tensor, the metric tensor, the Einstein tensor, the energy momentum tensor associated to the field Ψ and the energy momentum tensor associated to the electromagnetic field. (1.0.2) is the Maxwell equation, where J^{Ψ}_b is the current and F^{el}_{ab} is the electromagnetic tensor. Eventually (1.0.3) is an equation for a material field. To the equations (1.0.1)-(1.0.3) have to be added initial data. For the Einstein equations the initial data must fulfil the constraint equations. The system (1.0.1)-(1.0.3) is extremely complicated and we do not see how to analyze this system directly without doing any simplifications. In addition there is today no satisfying quantum theory of gravitation. Nevertheless quantum effects play an important role in particular near the event horizons of black holes. The most efficient way to analyze the above system is the following : First we look for explicit solutions of (1.0.1), (1.0.2) with $T^{\Psi}_{ab} = 0$, $J^{\Psi}_b = 0$. Once g_{ab} and F^{el}_{ab} are fixed, one solves and quantizes (1.0.3). In order to assure that this procedure makes sense one has to prove that the explicit solution of (1.0.1) and (1.0.2) is stable under small perturbations.

The Cauchy problem for the Einstein equations was solved by Choquet-Bruhat in 1952 (see [49]). Global nonlinear stability for Minkowski space-time was shown in the work of Christodoulou and Klainerman [28]. We also refer to the work of Rodnianski and Linblad [77] whose result is less precise than that of Christodoulou and Klainerman, but their proof is simpler. For black hole type space-times the question of nonlinear stability remains open.

Spectacular asymptotic properties of material fields have been observed in black hole type space-times. The most general solution of (1.0.1), (1.0.2) (with cosmological constant $\Lambda = 0$) describing a black hole is the Kerr-Newman space-time which describes a rotating charged black-hole. It is characterized by its mass M, its charge Q and its angular momentum per unit mass a. The particular solutions with $Q = 0$, $a = 0$, $Q = a = 0$, $Q = a = M = 0$ are called resp. Kerr, Reissner-Nordström, Schwarzschild and Minkowski solutions. The most interesting effects for the material fields evolving in this kind of geometries are the following :

- The Hawking effect

  In his paper [64], S. W. Hawking considers the collapse of a star forming a black-hole. An observer far away from the black hole observes at the last moments of the collapse in his proper time the emergence of a thermal state coming from the future event horizon.

- Superradiance

  The exterior of the Kerr black hole is not stationary in the sense that there is no global timelike Killing vector field. One of the consequences of this is that the field equations of entire spin do not have a conserved positive quantity. Therefore the energy of a material field can increase during a scattering process.

Although both effects have been known for more than thirty years, mathematically rigorous studies about them are relatively rare. Their understanding demands a precise analysis of the asymptotic behavior of material fields in black hole type space-times. On the one hand this analysis is the key to a mathematically precise description of these effects. On the other hand it contributes to an understanding of the stability of the space-times itself. Indeed linearization of gravity around a certain fixed background leads to linear equations on this fixed background which are similar to the wave equation. That is why the study of local energy and related properties of the wave equation on a fixed background is believed to be a prerequisite for the study of the nonlinear stability problem.

The biggest part of the present work concerns classical and quantum scattering in the Kerr(-Newman) space-time. The scattering properties of classical and quantum fields outside a Schwarzschild black hole have been thoroughly studied. The first results on the subject were obtained by J. Dimock in 1985 [39] and by J. Dimock and B. Kay in 1986 and 1987 [41], [40], [42] for classical and quantum scalar fields. This work was pushed further by A. Bachelot in the 1990's ; his series of papers starts with scattering theories for classical fields, Maxwell in 1991 [6] and Klein-Gordon in 1994 [7], then tackles quantum fields in 1997 [8] and culminates with a rigorous mathematical description of the Hawking effect for a spherical symmetric gravitational collapse in 1999 [10] and 2000 [11]. Meanwhile, other authors contributed to the subject, such as J.-P. Nicolas in 1995 with a scattering theory for classical massless Dirac fields [86], W.M. Jin in 1998 with a construction of wave operators in the massive case [66] and F. Melnyk in 2003 who obtained a complete scattering for massive charged Dirac fields [80] and the Hawking effect for charged, massive spin 1/2 fields [81]. Note that in [80], [81] and [86], the cases of Reissner-Nordstrøm (charged) and de Sitter (with a cosmological horizon) black holes are also treated ; these geometries do not fundamentally change the analytic difficulties in the construction of classical or quantum scattering theories. All these works use trace class perturbation methods and therefore cannot be extended to the Kerr case because of the lack of symmetry of the geometry (see below). One paper using different techniques appeared in 1992, due to S. De Bièvre, P. Hislop and I.M. Sigal [34] : by means of a Mourre estimate, they study the wave equation on non compact Riemannian manifolds ; possible applications are therefore static situations, such as the Schwarzschild case, which they treat, but the Kerr geometry is not even stationary and the results cannot be applied. Scattering theory in a superradiant situation has been studied in the one dimensional case by Bachelot (see [12]). In the Kerr case superradiance has been studied by Finster, Kamran, Smoller and Yau (see [47]). In the framework of Kerr black holes, the analysis of the scattering properties of fields is faced with several fundamental difficulties not present in the Schwarzschild framework.

i) Lack of symmetry. The Kerr solutions possess only two commuting Killing vector fields. In the Boyer-Lindquist coordinate system (t, r, θ, φ), based on these Killing vector fields, they are interpreted as the time coordinate vector field $\partial/\partial t$ and the longitude coordinate vector field $\partial/\partial\varphi$. Kerr space-time therefore has cylindrical, but not spherical, spatial symmetry. This prevents a straightforward decomposition in spin-weighted spherical harmonics, that reduces the problem to the study of a $1+1$-dimensional evolution system with potential. The trace-class perturbation methods used in the Schwarzschild case are in consequence not applicable. Another effect of the lack of spherical symmetry is the presence of artificial long-range terms at infinity in the field equations. To get rid of these terms it is necessary to have a deeper understanding of the geometry, and of the dynamics, than what is required in the Schwarzschild case.

ii) The point of view of scattering theory is that of an observer static at infinity. Such an observer perceives the propagation of a field outside the black hole as an evolution on a cylindrical manifold $\Sigma \simeq \mathbb{R}\times\mathbb{S}^2$, with one asymptotically euclidean end corresponding to infinity and one asymptotically hyperbolic end representing the horizon. In the absence of spherical symmetry, the asymptotically hyperbolic end is awkward for scattering theory, more particularly for the choice of a conjugate operator in the framework of Mourre theory. The generator of dilations, that is the usual conjugate operator, cannot be used here.

iii) Kerr space-time is not stationary ; there exists no globally defined timelike Killing vector field outside the black hole. In particular, the vector $\partial/\partial t$ is spacelike in a toroidal region, called the ergosphere, surrounding the horizon. As already mentioned for field equations of integral spin, such as the wave equation, Klein-Gordon or Maxwell, this means that no positive definite conserved energy exists, which allows fields to extract energy from the ergosphere. For field equations of half integral spin (Weyl, Dirac or Rarita-Schwinger), we have a conserved L^2 norm, there is no superradiance and the lack of stationarity is not in itself a serious difficulty. This conserved L^2 norm is usually interpreted as a conserved charge. It is the good conserved quantity to work with.

iv) The Hawking effect is characterized by a regime which obliges us to consider large times and high frequencies simultaneously. As no decomposition into spherical harmonics is possible in the Kerr case, the difference between the full operator and a comparison operator with constant coefficients is always a differential operator of order at least one. As a consequence these operators are in general not close to each other in the high energy regime we are interested in.

v) In the Schwarzschild case perturbation arguments very often work out quite well. As in the Kerr case the perturbation is itself a differential operator of maximal order, this procedure is in this case quite limited.

My contribution is the following. I first considered the wave equation in a class of stationary asymptotically flat space-times (see [60]). This model can be considered as an intermediate step between the Schwarzschild and the Kerr case. I then showed asymptotic completeness for the non superradiant modes of the Klein-Gordon field in the Kerr geometry (see [61]). A complete scattering theory for massless Dirac fields in the Kerr geometry was subsequently obtained in collaboration with Jean-Philippe Nicolas (see [63]). This result was later generalized by T. Daudé to the massive charged case in the Kerr-Newman geometry (see [33]). Eventually I was able to give a mathematically precise description of the Hawking effect for fermions in the setting of the collapse of a rotating charged star (see [58]). The study of scattering properties for linear fields was largely motivated by the Hawking effect and the proof in [58] uses the scattering results obtained in [63], [33].

We now turn to the question of stability of black hole type space-times. As explained above one of the prerequisites of a possible proof of a global nonlinear stability result is believed to be decay estimates for the solution of the wave equation. This question has been treated using different methods. Dafermos and Rodnianski use vector field methods to first give a mathematically rigorous treatment of Price's law and then to give a precise description of the red shift effect (see [30], [29]). They also succeed in showing that the solution of the wave equation in slowly rotating Kerr backgrounds is uniformly bounded (see [32]). Blue, Soffer and Blue Sterbenz mix this vector field approach with a spectral approach based on positive commutator estimates (see [18], [19]). Finster, Kamran, Smoller and Yau obtain pointwise decay for the Dirac equation (with a rate) and for the wave equation (without a rate) on the Kerr metric (see [45], [46]) using spectral methods. Note that the point of view of Finster, Kamran, Smoller, Yau is slightly different from the point of view of Dafermos, Rodnianski, in particular their result on the wave equation does not imply [32].

In collaboration with Jean-François Bony we have used our knowledge about the localization of resonances for the wave equation in the De Sitter-Schwarzschild case to analyze the behavior of the local energy in this case (see [22]). Resonances have been studied by Bachelot, Motet-Bachelot in the Schwarzschild case and by Sá Barreto-Zworski in the De Sitter-Schwarzschild case (see [13], [93]). The study of Sá Barreto-Zworski is essentially a high energy analysis. In [22] we have found in addition a zero resonance which prevents decay of the local energy. For initial data in the complement of a one dimensional space we obtain exponential decay of the local energy. Resonances are also important because they correspond to the frequencies and rates of dumping of signals emitted by the black hole in the presence of perturbations. Today it is theoretically possible to measure the corresponding gravitational waves and to detect in this way the presence of a black hole. For the wave equation on the De Sitter-Schwarzschild case, see also the work of Dafermos, Rodnianski (see [31]), who obtain weaker decay but can permit that the support of the solution contains the sphere of bifurcation.

Another interesting problem in this context is the study of the semilinear wave equation on a given space-time. The semilinear wave equation in Minkowski space has been thoroughly studied. Global existence is known in dimension $d \geq 4$ for small initial data (see the article of S. Klainerman and G. Ponce [75]). Almost global existence in dimension $d = 3$ for small data was shown by F. John and S. Klainerman in [68]. Almost global means that the life time of a solution is at least $e^{1/\delta}$, where δ is the size of the initial data in some Sobolev space. Note that, in dimension $d = 3$, T. Sideris [94] has proved that global existence does not hold in general (see also F. John [67]).

In [71], M. Keel, H. Smith and C. Sogge give a new proof of the almost global existence result for quadratic nonlinearities in dimension 3 using estimates of the form

$$(\ln(2+T))^{-1/2}\|\langle x\rangle^{-1/2}v'\|_{L^2([0,T]\times\mathbb{R}^3)} \lesssim \|v'(0,.)\|_{L^2(\mathbb{R}^3)} + \int_0^T \|G(s,.)\|_{L^2(\mathbb{R}^3)}ds, \quad (1.0.4)$$

and a certain Sobolev type estimate due to S. Klainerman (see [74]). Here v solves the wave equation $\Box v = G$ in $\mathbb{R}_+ \times \mathbb{R}^3$ and $v' = (\partial_t v, \partial_x v)$. They also treat the non trapping obstacle case. In [72] similar results are obtained for the corresponding quasilinear equation. The obstacle case, in which the trapped trajectories are of hyperbolic type, is treated in the article of J. Metcalfe and C. Sogge [82].

S. Alinhac shows an estimate similar to (1.0.4) on a curved background. The metric is depending on and decaying in time (see [2], [3]). The results of J. Metcalfe and D. Tataru [84] imply estimates similar to (1.0.4) for a variable coefficients wave equation outside a star shaped obstacle (see also [83]). Outside the obstacle, their wave operator is a small perturbation of the wave operator in Minkowski space.

In collaboration with J.-F. Bony we consider the quadratically semilinear wave equation on asymptotically euclidean riemannian manifolds (see [23]). If the metric is non trapping we show for small data global existence in dimension $d \geq 4$ and long time existence in dimension $d = 3$. More precisely if the data is of size δ in some appropriate Sobolev norm, then we obtain existence on $[0, T_\delta]$ with $T_\delta = \delta^{-n}$ for all $n > 0$.

In contrast to the papers cited above which all use vector field methods, in this paper we use a somewhat different approach. We will show how estimates of type (1.0.4) follow from a Mourre estimate and the Kato theory of H-smoothness. This method will permit us to consider non trapping riemannian metrics which are asymptotically euclidean without requiring that they are everywhere a small perturbation of the euclidean metric. We will suppose for simplicity that the metric is C^∞, but a C^k approach should in principle be possible. Spectral methods for proving dispersive estimates were previously used by N. Burq. In [25] he obtains global Strichartz estimates for compactly supported non trapping perturbations of the euclidean case. In complicated geometries, conjugate operators are probably not vector fields and we think it is worth trying to mix the classical vector field approach with this Mourre theory approach. In particular we will see in the following chapters that the conjugate operators for the Kerr metric are not vector fields.

This document is organized in the following way :

- Chapter 2 collects some preliminaries for what follows. We give a description of the Kerr-Newman metric and the classical Dirac equation. This chapter also contains a very brief overview of Mourre theory.

- Chapter 3 treats the wave equation on stationary asymptotically flat space-times. We introduce the notion of an asymptotic velocity. Under a long range condition we can show its existence and compute its spectrum. Asymptotic completeness is shown under a short range condition.

- In Chapter 4 we consider the Klein-Gordon equation on the Kerr metric. We start with a general Hilbert space setting. It is shown that a Mourre estimate for an operator $h \geq 0$ entails a Mourre estimate for $\sqrt{h}$. We also show that it entails a Mourre estimate for a matrix problem. It is then shown that the Klein-Gordon field on the Kerr geometry possesses a large number of modes for which we have a conserved quantity. For these modes we obtain an asymptotic completeness result.

- For the massless Dirac equation outside a slow Kerr black hole, we prove asymptotic completeness in Chapter 5. We introduce a new Newman-Penrose tetrad in which the expression of the equation contains no artificial long-range perturbations. The main technique used is then a Mourre estimate. The geometry near the horizon requires us to apply a unitary transformation before we find ourselves in a situation where the generator of dilations is a good conjugate operator. The results are eventually reinterpreted geometrically as providing the solution to a Goursat problem on the Penrose compactified exterior.

- Chapter 6 is devoted to the mathematical study of the Hawking effect for fermions in the setting of the collapse of a rotating charged star. We show that an observer who is located far away from the star and at rest with respect to the Boyer Lindquist coordinates observes the emergence of a thermal state when his proper time goes to infinity. We first introduce a model of the collapse of the star. The assumptions on the asymptotic behavior of the surface of the star are inspired by the asymptotic behavior of certain timelike geodesics in the Kerr-Newman metric. The Dirac equation is then written using coordinates and a Newman-Penrose tetrad which are adapted to the collapse. This coordinate system and tetrad are based on the so called simple null geodesics. The quantization of Dirac fields in a globally hyperbolic space-time is described. We formulate and prove a theorem about the Hawking effect in this setting. The proof of the theorem contains a minimal velocity estimate for Dirac fields that is slightly stronger than the usual ones and an existence and uniqueness result for solutions of a characteristic Cauchy problem for Dirac fields in the Kerr-Newman space-time. The simple null geodesics can be used to construct a Penrose compactification of block I of the Kerr-Newman space-time.

- In Chapter 7 we describe an expansion of the solution of the wave equation on the De Sitter-Schwarzschild metric in terms of resonances. The principal term in the expansion is due to a resonance at 0. The error term decays polynomially if we permit a logarithmic derivative loss in the angular directions and exponentially if we permit an ε derivative loss in the angular directions.

- In Chapter 8 we study the quadratically semilinear wave equation on an asymptotically euclidean riemannian manifold. The metric is supposed to approach the euclidean metric like $\langle x \rangle^{-\rho}$ and to be non trapping. If $\rho \geq 1$ we obtain long time existence for small data in all dimensions $d \geq 3$. If $\rho > 1$ and $d \geq 4$ we obtain global existence for small data.

- In Chapter 9 we discuss the results presented in the previous chapters and state some open problems for future work.

Notations

Let $(\mathcal{M}, g)$ be a smooth 4-manifold equipped with a lorentzian metric g with signature $(+,-,-,-)$. We denote by ∇_a the Levi-Civita connection on $(\mathcal{M}, g)$.

Many of our equations will be expressed using the two-component spinor notations and abstract index formalism of R. Penrose and W. Rindler [91].

Abstract indices are denoted by light face latin letters, capital for spinor indices and lower case for tensor indices. Abstract indices are a notational device for keeping track of the nature of objects in the course of calculations, they do not imply any reference to a coordinate basis, all expressions and calculations involving them are perfectly intrinsic. For example, g_{ab} will refer to the space-time metric as an intrinsic symmetric tensor field of valence $\left[\begin{array}{c} 0 \\ 2 \end{array}\right]$, i.e. a section of $\mathrm{T}^*\mathcal{M} \odot \mathrm{T}^*\mathcal{M}$ and g^{ab} will refer to the inverse metric as an intrinsic symmetric tensor field of valence $\left[\begin{array}{c} 2 \\ 0 \end{array}\right]$, i.e. a section of $\mathrm{T}\mathcal{M} \odot \mathrm{T}\mathcal{M}$ (where $\odot$ denotes the symmetric tensor product, $\mathrm{T}\mathcal{M}$ the tangent bundle to our space-time manifold $\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ its cotangent bundle).

Concrete indices defining components in reference to a basis are represented by bold face latin letters. Concrete spinor indices, denoted by bold face capital latin letters, take their values in $\{0,1\}$ while concrete tensor indices, denoted by bold face lower case latin letters, take their values in $\{0,1,2,3\}$. Consider for example a basis of $\mathrm{T}\mathcal{M}$, that is a family of four smooth vector fields on $\mathcal{M}$: $\mathcal{B} = \{e_0, e_1, e_2, e_3\}$ such that at each point p of $\mathcal{M}$ the four vectors $e_0(p), e_1(p), e_2(p), e_3(p)$ are linearly independent, and the corresponding dual basis of $\mathrm{T}^*\mathcal{M}$: $\mathcal{B}^* = \left\{e^0, e^1, e^2, e^3\right\}$ such that $e^{\mathbf{a}}\left(e_{\mathbf{b}}\right) = \delta^{\mathbf{a}}_{\mathbf{b}}$, $\delta^{\mathbf{a}}_{\mathbf{b}}$ denoting the Kronecker symbol ; $g_{\mathbf{ab}}$ will refer to the components of the metric g_{ab} in the basis $\mathcal{B}$: $g_{\mathbf{ab}} = g(e_{\mathbf{a}}, e_{\mathbf{b}})$ and $g^{\mathbf{ab}}$ will denote the components of the inverse metric g^{ab} in the dual basis $\mathcal{B}^*$, i.e. the 4×4 real symmetric matrices $(g_{\mathbf{ab}})$ and $\left(g^{\mathbf{ab}}\right)$ are the inverse of one another. In the abstract index formalism, the basis vectors $e_{\mathbf{a}}$, $\mathbf{a} = 0,1,2,3$, are denoted $e_{\mathbf{a}}{}^a$ or $g_{\mathbf{a}}{}^a$. In a coordinate basis, the basis vectors $e_{\mathbf{a}}$ are coordinate vector fields and will also be denoted by $\partial_{\mathbf{a}}$ or $\frac{\partial}{\partial x^{\mathbf{a}}}$; the dual basis covectors $e^{\mathbf{a}}$ are coordinate 1-forms and will be denoted by $\mathrm{d}x^{\mathbf{a}}$.

We adopt Einstein's convention for the same index appearing twice, once up, once down, in the same term. For concrete indices, the sum is taken over all the values of the index. In the case of abstract indices, this signifies the contraction of the index, i.e. $f_a V^a$ denotes the action of the 1-form f_a on the vector field V^a.

For a manifold Y we denote by $C_b^\infty(Y)$ the set of all C^∞ functions on Y, that are bounded together with all their derivatives. We denote by $C_\infty(Y)$ the set of all continuous functions tending to zero at infinity.

We will use the pseudodifferential calculus and in particular the Weyl quantization. We will note :

$$\begin{aligned} d \in S^{k,m} \quad &\text{iff} \quad \forall \alpha,\, \beta \in \mathbb{N}^n \, |\partial_\xi^\alpha \partial_x^\beta d(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{k-|\alpha|} \langle x \rangle^{m-|\beta|}, \\ d^w \in \Psi^{m,k} \quad &\text{iff} \quad d \in S^{k,m}. \end{aligned}$$

Chapter 2

Preliminaries

2.1 The Kerr-Newman metric

We give a brief description of the Kerr-Newman metric, which describes an eternal rotating charged black-hole. A detailed description can be found e.g. in [99].

Boyer-Lindquist coordinates

In Boyer-Lindquist coordinates, a Kerr-Newman black-hole is described by a smooth 4-dimensional lorentzian manifold $\mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega$, whose space-time metric g and electromagnetic vector potential Φ_a are given by :

$$\begin{aligned} g &= \left(1+\frac{Q^2-2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{2a\sin^2\theta(2Mr-Q^2)}{\rho^2}\mathrm{d}t\mathrm{d}\varphi - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 \\ &- \frac{\sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\varphi^2, \end{aligned} \tag{2.1.1}$$

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta\,,\ \Delta = r^2 - 2Mr + a^2 + Q^2, \\ \sigma^2 &= \left(r^2+a^2\right)\rho^2 + (2Mr - Q^2)a^2\sin^2\theta = (r^2+a^2)^2 - a^2\Delta\sin^2\theta, \\ \Phi_a dx^a &= -\frac{Qr}{\rho^2}(dt - a\sin^2\theta d\varphi). \end{aligned}$$

Here M is the mass of the black hole, a its angular momentum per unit mass and Q the charge of the black-hole. If $Q = 0$, g reduces to the Kerr metric, and if $Q = a = 0$ we recover the Schwarzschild metric. The expression (2.1.1) of the Kerr metric has two types of singularities. While the set of points $\{\rho^2 = 0\}$ (the equatorial ring $\{r = 0\,,\ \theta = \pi/2\}$ of the $\{r = 0\}$ sphere) is a true curvature singularity, the spheres where Δ vanishes, called horizons, are mere coordinate singularities. We will consider in this work subextremal Kerr-Newman space-times, that is we suppose $Q^2 + a^2 < M^2$. In this case Δ has two real roots:

$$r_\pm = M \pm \sqrt{M^2 - (a^2+Q^2)}. \tag{2.1.2}$$

The spheres $\{r = r_-\}$ and $\{r = r_+\}$ are called event horizons. The two horizons separate $\mathcal{M}_{BH}$ into three connected components called Boyer-Lindquist blocks : $B_I, B_{II}, B_{III}(r_+ < r, r_- < r < r_+, r < r_-)$. No Boyer-Lindquist block is stationary, that is, there exists no globally defined timelike Killing vector field on any given block. In particular, block I contains a toroidal region, called the ergosphere, surrounding the horizon,

$$\mathcal{E} = \left\{(t,r,\theta,\varphi)\,;\ r_+ < r < M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}\right\}, \tag{2.1.3}$$

where the vector $\partial/\partial t$ is spacelike.

An important feature of the Kerr-Newman space-time is that it has Petrov type D (see e.g. [88]). This means that the Weyl tensor has two double roots at each point. These roots, referred to as the principal null directions of the Weyl tensor, are given by the two vector fields

$$V^{\pm} = \frac{r^2 + a^2}{\Delta}\partial_t \pm \partial_r + \frac{a}{\Delta}\partial_\varphi.$$

Since V^+ and V^- are twice repeated null directions of the Weyl tensor, by the Goldberg-Sachs theorem (see for example [88, Theorem 5.10.1]) their integral curves define shear-free null geodesic congruences. We shall refer to the integral curves of V^+ (respectively V^-) as the outgoing (respectively incoming) principal null geodesics and from this point on we shall write PNG for principal null geodesic. The plane determined at each point by the two prinipal null directions is called the principal plane.

Instead of r, we will often use a Regge-Wheeler type coordinate r_* in B_I (see e.g. [27]), which is given by

$$r_* = r + \frac{1}{2\kappa_+}\ln|r - r_+| - \frac{1}{2\kappa_-}\ln|r - r_-| + R_0, \tag{2.1.4}$$

where R_0 is any constant of integration and

$$\kappa_\pm = \frac{r_+ - r_-}{2(r_\pm^2 + a^2)} \tag{2.1.5}$$

are the surface gravities at the outer and inner horizons. The variable r_* satisfies :

$$\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}. \tag{2.1.6}$$

When r varies from r_+ to ∞, r_* varies from $-\infty$ to ∞. We put:

$$\Sigma := \mathbb{R}_{r_*} \times S^2. \tag{2.1.7}$$

2.2 The Dirac equation

This section contains a discussion about spin structures and Dirac fields which is valid in general globally hyperbolic space-times.

2.2.1 Spin structures

Let $(\mathcal{M}, g)$ be a smooth 4-manifold with a lorentzian metric g with signature $(+,-,-,-)$ which is assumed to be oriented, time oriented and globally hyperbolic. Global hyperbolicity implies :

i) $(\mathcal{M}, g)$ admits a spin structure (see R.P. Geroch [55, 56, 57] and E. Stiefel [96]). We will choose one of them. We denote by $\mathbb{S}$ (or $\mathbb{S}^A$ in the abstract index formalism) the spin bundle over $\mathcal{M}$ and by $\bar{\mathbb{S}}$ (or $\mathbb{S}^{A'}$) the same bundle with the complex structure replaced by its opposite. The dual bundles $\mathbb{S}^*$ and $\bar{\mathbb{S}}^*$ will be denoted respectively $\mathbb{S}_A$ and $\mathbb{S}_{A'}$. The complexified tangent bundle to $\mathcal{M}$ is recovered as the tensor product of $\mathbb{S}$ and $\bar{\mathbb{S}}$, i.e.

$$T\mathcal{M} \otimes \mathbb{C} = \mathbb{S} \otimes \bar{\mathbb{S}} \text{ or } T^a\mathcal{M} \otimes \mathbb{C} = \mathbb{S}^A \otimes \mathbb{S}^{A'}$$

and similarly

$$T^*\mathcal{M} \otimes \mathbb{C} = \mathbb{S}^* \otimes \bar{\mathbb{S}}^* \text{ or } T_a\mathcal{M} \otimes \mathbb{C} = \mathbb{S}_A \otimes \mathbb{S}_{A'} .$$

An abstract tensor index a is thus understood as an unprimed spinor index A and a primed spinor index A' clumped together : $a = AA'$. The symplectic forms on $\mathbb{S}$ and $\bar{\mathbb{S}}$ are denoted ϵ_{AB}, $\epsilon_{A'B'}$ and are referred to as the Levi-Civita symbols. ϵ_{AB} can be seen as an isomorphism from $\mathbb{S}$ to $\mathbb{S}^*$ which to κ^A associates $\kappa_A = \kappa^B\epsilon_{BA}$. Similarly, $\epsilon_{A'B'}$ and the corresponding $\epsilon^{A'B'}$ can be regarded as lowering and raising devices for primed indices. The metric g is expressed in terms of the Levi-Civita symbols as $g_{ab} = \epsilon_{AB}\epsilon_{A'B'}$.

ii) There exists a global time function t on $\mathcal{M}$. The level surfaces Σ_t, $t \in \mathbb{R}$, of the function t define a foliation of $\mathcal{M}$, all Σ_t being Cauchy surfaces and homemorphic to a given smooth $3-$manifold Σ (see Geroch [56]). Geroch's theorem does not say anything about the regularity of the leaves Σ_t; the time function is only proved to be continuous and they are thus simply understood as topological submanifolds of $\mathcal{M}$. A regularization procedure for the time function can be found in [16], [17]. In the concrete cases which we will consider the time function is smooth and all the leaves are diffeomorphic to Σ. The function t is then a smooth time coordinate on $\mathcal{M}$ and it is increasing along any non space-like future oriented curve. Its gradient $\nabla^a t$ is everywhere orthogonal to the level surfaces Σ_t of t and it is therefore everywhere timelike; it is also future oriented. We identify $\mathcal{M}$ with the smooth manifold $\mathbb{R} \times \Sigma$ and consider g as a tensor valued function on $\mathbb{R} \times \Sigma$.

Let T^a be the future-pointing timelike vector field normal to Σ_t, normalized for later convenience to satisfy :

$$T^aT_a = 2,$$

i.e.

$$T^a = \frac{\sqrt{2}}{|\nabla t|}\nabla^a t, \text{ where } |\nabla t| = (g_{ab}\nabla^a t\nabla^b t)^{1/2}.$$

2.2.2 The Dirac equation and the Newman-Penrose formalism

In terms of two component spinors (sections of the bundles $\mathbb{S}^A$, $\mathbb{S}_A$, $\mathbb{S}^{A'}$ or $\mathbb{S}_{A'}$), the charged Dirac equation takes the form (see [90], page 418):

$$\left\{ \begin{array}{l} (\nabla^A_{A'} - iq\Phi^A_{A'})\phi_A = \mu\chi_{A'}, \\ (\nabla^{A'}_A - iq\Phi^{A'}_A)\chi_{A'} = \mu\phi_A\,,\ \mu = \frac{m}{\sqrt{2}}\,, \end{array} \right. \tag{2.2.1}$$

where $m \geq 0$ is the mass of the field. In the massless case, equation (2.2.1) reduces to the Weyl anti-neutrino equation

$$\nabla^{AA'}\phi_A = 0\,, \tag{2.2.2}$$

since the equation on χ (the Weyl neutrino equation),

$$\nabla^{AA'}\chi_{A'} = 0\,,$$

is the complex conjugate of the anti-neutrino equation

$$\nabla^{AA'}\bar{\chi}_A = 0\,.$$

We shall refer to equation (2.2.2) as the Weyl equation. The Dirac equation (2.2.1) possesses a conserved current (see for example [87]) on general curved space-times, defined by the future oriented non-spacelike vector field, sum of two future oriented null vector fields :

$$V^a = \phi^A\bar{\phi}^{A'} + \bar{\chi}^A\chi^{A'}\,.$$

The vector field V^a is divergence free, i.e. $\nabla^a V_a = 0$. Consequently the 3-form $\omega = *V_a dx^a$ is closed. Let Σ be a spacelike or characteristic hypersurface, $d\Omega$ the volume form on $\mathcal{M}$ induced by the metric ($d\Omega = \rho^2 dt \wedge dr \wedge d\omega$ for the Kerr-Newman metric), $\mathcal{N}^a$ the (future pointing) normal to Σ and $\mathcal{L}^a$ transverse to Σ with $\mathcal{N}^a\mathcal{L}_a = 1$. Then

$$\begin{aligned} &\int_\Sigma *(\phi_A\bar{\phi}_{A'}dx^{AA'} + \bar{\chi}_A\chi_{A'}dx^{AA'}) \\ &= \int_\Sigma \mathcal{N}_{BB'}\mathcal{L}^{BB'} * (\phi_A\bar{\phi}_{A'}dx^{AA'} + \bar{\chi}_A\chi_{A'}dx^{AA'}) \\ &= \int_\Sigma \mathcal{L}^{CC'}\partial_{CC'} \lrcorner\, (\mathcal{N}_{BB'}dx^{BB'} \wedge *(\phi_A\bar{\phi}_{A'}dx^{AA'} + \bar{\chi}_A\chi_{A'}dx^{AA'})) \\ &= \int_\Sigma \mathcal{N}^{AA'}(\phi_A\bar{\phi}_{A'} + \bar{\chi}_A\chi_{A'})(\mathcal{L}^{BB'}\partial_{BB'}) \lrcorner\, d\Omega. \end{aligned}$$

If Σ is spacelike we can take $\mathcal{L}^{AA'} = \mathcal{N}^{AA'}$ and the integral defines a norm and by this norm the space $L^2(\Sigma; \mathbb{S}_A \oplus \mathbb{S}^{A'})$ as completion of $C^\infty_0(\Sigma; \mathbb{S}_A \oplus \mathbb{S}^{A'})$. Note that if Σ is characteristic $\int_\Sigma *\phi_A\bar{\phi}_{A'}dx^{AA'} = 0$ does not entail $\phi_A = 0$ on Σ (see Remark 2.2.1). If Σ_t are the level surfaces of t, then we see by Stokes' theorem that the total charge

$$C(t) = \frac{1}{\sqrt{2}}\int_{\Sigma_t} V_a T^a d\sigma_{\Sigma_t} \tag{2.2.3}$$

is constant throughout time. Here $d\sigma_{\Sigma_t} = \frac{1}{\sqrt{2}} T^a \lrcorner\, d\Omega$.

Using the Newman-Penrose formalism, equation (2.2.1) can be expressed as a system of partial differential equations with respect to a coordinate basis. This formalism is based on the choice of a null tetrad, i.e. a set of four vector fields l^a, n^a, m^a and $\bar{m}^a$, the first two being real and future oriented, $\bar{m}^a$ being the complex conjugate of m^a, such that all four vector fields are null and m^a is orthogonal to l^a and n^a, that is to say

$$l_a l^a = n_a n^a = m_a m^a = l_a m^a = n_a m^a = 0\,. \tag{2.2.4}$$

The tetrad is said to be normalized if in addition

$$l_a n^a = 1\,,\ m_a \bar{m}^a = -1\,. \tag{2.2.5}$$

The vectors l^a and n^a usually describe "dynamic" or scattering directions, i.e. directions along which light rays may escape towards infinity (or more generally asymptotic regions corresponding to scattering channels). The vector m^a tends to have, at least spatially, bounded integral curves, typically m^a and $\bar{m}^a$ generate rotations. The principle of the Newman-Penrose formalism is to decompose the covariant derivative into directional covariant derivatives along the frame vectors. We introduce a spin-frame $\{o^A, \iota^A\}$, defined uniquely up to an overall sign factor by the requirements that

$$o^A \bar{o}^{A'} = l^a\,,\ \iota^A \bar{\iota}^{A'} = n^a\,,\ o^A \bar{\iota}^{A'} = m^a\,,\ \iota^A \bar{o}^{A'} = \bar{m}^a\,,\ o_A \iota^A = 1\,. \tag{2.2.6}$$

We will also denote the spin frame by $\{\epsilon_0{}^A, \epsilon_1{}^A\}$. The dual basis of $\mathbb{S}_A$ is $\{\epsilon_A{}^0, \epsilon_A{}^1\}$, where $\epsilon_A{}^0 = -\iota_A$, $\epsilon_A{}^1 = o_A$. Let ϕ_0 and ϕ_1 be the components of ϕ_A in $\{o^A, \iota^A\}$, and $\chi_{0'}$ and $\chi_{1'}$ the components of $\chi_{A'}$ in $(\bar{o}^{A'}, \bar{\iota}^{A'})$:

$$\phi_0 = \phi_A o^A \ ,\quad \phi_1 = \phi_A \iota^A \ ,\quad \chi_{0'} = \chi_{A'} \bar{o}^{A'} \ ,\quad \chi_{1'} = \chi_{A'} \bar{\iota}^{A'}\,.$$

Dirac's equation then takes the form (see for example [27])

$$\left.\begin{array}{l}
n^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\phi_0 - m^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\phi_1 + (\mu-\gamma)\phi_0 + (\tau-\beta)\phi_1 = \frac{m}{\sqrt{2}}\chi_{1'}\,,\\[2ex]
l^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\phi_1 - \bar{m}^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\phi_0 + (\alpha-\pi)\phi_0 + (\varepsilon-\tilde{\rho})\phi_1 = -\frac{m}{\sqrt{2}}\chi_{0'}\,,\\[2ex]
n^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\chi_{0'} - \bar{m}^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\chi_{1'} + (\bar{\mu}-\bar{\gamma})\chi_{0'} + (\bar{\tau}-\bar{\beta})\chi_{1'} = \frac{m}{\sqrt{2}}\phi_1\,,\\[2ex]
l^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\chi_{1'} - m^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\chi_{0'} + (\bar{\alpha}-\bar{\pi})\chi_{0'} + (\bar{\varepsilon}-\bar{\tilde{\rho}})\chi_{1'} = -\frac{m}{\sqrt{2}}\phi_0
\end{array}\right\} \tag{2.2.7}$$

The μ, γ etc. are the so called spin coefficients, for example $\mu = -\bar{m}^a \delta n_a, \delta = m^a \nabla_a$. For the formulas of the spin coefficients and details about the Newman-Penrose formalism see e.g. [90].

It is often useful to allow simultaneous consideration of bases of $T^a\mathcal{M}$ and $\mathbb{S}^A$, which are completely unrelated to one another. Let $\{e_0, e_1, e_2, e_3\}$ be such a basis of $T^a\mathcal{M}$, which is not related to the Newman-Penrose tetrad.

We define the Infeld-Van der Waerden symbols as the spinor components of the frame vectors in the spin frame $\{\epsilon_0{}^A, \epsilon_1{}^A\}$:

$$g_{\mathbf{a}}{}^{\mathbf{AA'}} = e_{\mathbf{a}}{}^{\mathbf{AA'}} = g_{\mathbf{a}}{}^a \epsilon_A{}^{\mathbf{A}} \epsilon_{A'}{}^{\mathbf{A'}} = \begin{pmatrix} n_{\mathbf{a}} & -\bar{m}_{\mathbf{a}} \\ -m_{\mathbf{a}} & l_{\mathbf{a}} \end{pmatrix}$$

(recall that $g_{\mathbf{a}}^a = e_{\mathbf{a}}^a$ denotes the vector field $e_{\mathbf{a}}$). We use these quantities to express (2.2.1) in terms of spinor components :

$$\left.\begin{array}{rcl} -i\epsilon_{A'}^{\mathbf{A'}}(\nabla^{AA'} - iq\Phi^{AA'})\phi_A & = & -ig^{\mathbf{aAA'}}\epsilon_{\mathbf{A}}^A(\nabla_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\phi_A = -i\mu\chi^{\mathbf{A'}}, \\ -i\epsilon_{\mathbf{A}}^A(\nabla_{AA'} - iq\Phi_{AA'})\chi^{A'} & = & -ig^{\mathbf{a}}{}_{\mathbf{AA'}}\epsilon_{A'}^{\mathbf{A'}}(\nabla_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\chi^{A'} = i\mu\phi_{\mathbf{A}}, \end{array}\right\} \tag{2.2.8}$$

where $\nabla_{\mathbf{a}}$ denotes $\nabla_{e_{\mathbf{a}}}$. For $\mathbf{a} = 0, 1, 2, 3$, we introduce the 2×2 matrices

$$A^{\mathbf{a}} = {}^t g^{\mathbf{aAA'}},\ B^{\mathbf{a}} = g^{\mathbf{a}}{}_{\mathbf{AA'}},$$

and the 4×4 matrices

$$\gamma^{\mathbf{a}} = \begin{pmatrix} 0 & i\sqrt{2}B^{\mathbf{a}} \\ -i\sqrt{2}A^{\mathbf{a}} & 0 \end{pmatrix}. \tag{2.2.9}$$

We find :

$$\gamma^{\mathbf{a}} = i\sqrt{2}\begin{pmatrix} 0 & 0 & l^{\mathbf{a}} & m^{\mathbf{a}} \\ 0 & 0 & \bar{m}^{\mathbf{a}} & n^{\mathbf{a}} \\ -n^{\mathbf{a}} & m^{\mathbf{a}} & 0 & 0 \\ \bar{m}^{\mathbf{a}} & -l^{\mathbf{a}} & 0 & 0 \end{pmatrix}. \tag{2.2.10}$$

Putting $\Psi = \phi_A \oplus \chi^{A'}$, the components of Ψ in the spin frame are $\mathbf{\Psi} =^t (\phi_0, \phi_1, \chi^{0'}, \chi^{1'})$ and (2.2.8) becomes

$$\sum_{\mathbf{a}=\mathbf{0}}^{3} \gamma^{\mathbf{a}}\mathcal{P}(\nabla_{e_{\mathbf{a}}} - iq\Phi_{\mathbf{a}})\Psi + im\mathbf{\Psi} = 0, \tag{2.2.11}$$

where $\mathcal{P}$ is the mapping that to a Dirac spinor associates its components in the spin frame :

$$\Psi = \phi_A \oplus \chi^{A'} \mapsto \mathbf{\Psi} = \phi_{\mathbf{A}} \oplus \chi^{\mathbf{A'}}.$$

Remark 2.2.1. *For a vector field X^a we have :*

$$X^{AA'}(\phi_A\bar{\phi}_{A'} + \bar{\chi}_A\chi_{A'}) = \langle \mathbf{X}\Psi, \Psi\rangle_{\mathbb{C}^4}, \tag{2.2.12}$$

$$\mathbf{X} = \begin{pmatrix} n_aX^a & -m_aX^a & 0 & 0 \\ -\bar{m}_aX^a & l_aX^a & 0 & 0 \\ 0 & 0 & l_aX^a & m_aX^a \\ 0 & 0 & \bar{m}_aX^a & n_aX^a \end{pmatrix}.$$

If Σ is a characteristic hypersurface with conormal n_a, then

$$\int_\Sigma *((\phi_A\bar{\phi}_{A'} + \bar{\chi}_A\chi_{A'})dx^{AA'}) = \int_\Sigma n^a(\phi_A\bar{\phi}_{A'} + \bar{\chi}_A\chi_{A'})d\sigma_\Sigma = \int_\Sigma (|\Psi_2|^2 + |\Psi_3|^2)d\sigma_\Sigma,$$

where $d\sigma_\Sigma = (l^a\partial_a) \lrcorner\, d\Omega$.

2.3 Mourre Theory

In this section we recall some elements of Mourre theory. We consider the commutator $[H, iA]$ between the Hamiltonian H and another selfadjoint operator A, called the conjugate operator. We say that the pair (H, A) satisfies a Mourre estimate on some open bounded energy interval Δ (included in the spectrum of H), if

$$(ME) \quad \mathbf{1}_\Delta(H)[iH, A]\mathbf{1}_\Delta(H) \geq \delta \mathbf{1}_\Delta(H)$$

for some $\delta > 0$. As both operators H and A are unbounded we have to be careful to define correctly the commutator. We say that the pair (H, A) satisfies the Mourre conditions (see [85]) iff

(M1') $D(A) \cap D(H)$ is dense in $D(H)$,

(M2') e^{isA} preserves $D(H), \sup_{|s|\leq 1} ||He^{isA}u|| < \infty, \quad \forall u \in D(H)$,

(M3') $[iH, A]$ which is defined as a quadratic form on $D(H) \cap D(A)$ is semi-bounded, closable and can be extended to a bounded operator from $D(H)$ to $\mathcal{H}$:

$$|[iH, A](u, v)| \leq C||Hu||||v||, \quad \forall u, v \in D(H) \cap D(A).$$

It has been remarked in [51] that the Virial theorem remains valid under the following conditions :

(M1) e^{isA} preserves $D(H)$,

(M2) $[iH, A]$ defined as a quadratic form on $D(H) \cap D(A)$ can be extended to a bounded operator from $D(H)$ to $\mathcal{H}$:

$$|[iH, A](u, v)| \leq C||Hu||||v||, \quad \forall u, v \in D(H) \cap D(A).$$

In fact, (M1')+(M2') is even equivalent to (M1). Note also that even in Mourre's original work [85], the assumption that $[iH, A]$ is semi-bounded is not necessary.

In our opinion the simplest and most useful condition for the Mourre estimate is the following (see [4]) :

A bounded operator C is of class $C^k(A; \mathcal{H})$ iff

$$\mathbb{R} \ni s \mapsto e^{isA}Ce^{-isA} \quad \text{is} \quad C^k \quad \text{for the strong topology of} \quad \mathcal{B}(\mathcal{H}).$$

$H \in C^k(A)$ if there exists $z \in \mathbb{C} \setminus \sigma(H)$ such that $(z - H)^{-1} \in C^k(A; \mathcal{H})$. (M1)-(M2) implies $H \in C^1(A)$ and the Virial theorem is valid as long as $H \in C^1(A)$ (see [4]).

If I is an interval, then we set

$$I^\pm = \{z \in \mathbb{C}; \operatorname{Re} z \in I; \pm \operatorname{Im} z > 0\}.$$

The following limiting absorption principle holds :

Theorem 2.3.1 (Mourre). *Assume that $H \in C^2(A)$ and that (ME) holds. Then*

$$\sup_{z \in I^{\pm}} ||\langle A \rangle^{-s} (H - z)^{-1} \langle A \rangle^{-s}|| < \infty,$$

for all closed intervals $I \subset \Delta$ and $s > \frac{1}{2}$.

In the original work of Mourre $s = 1$ is required, also the technical assumptions are slightly stronger. We refer to [4] for a detailed discussion of the weakest possible assumptions. The theorem in the form presented here can be found e.g. in [52], where energy methods are used for the proof. As observed by Kato (see [69]) the limiting absorption principle entails $(s > 1/2)$:

$$\int_0^\infty ||\langle A \rangle^{-s} e^{-itH} \mathbf{1}_I(H) \Phi||^2 dt \leq C ||\Phi||^2. \tag{2.3.1}$$

Inequality (2.3.1) means that $\langle A \rangle^{-s} \mathbf{1}_I(H)$ is *H-smooth*, a property which is discussed in detail in [92]. The above estimate can be used to obtain asymptotic completeness results. In Chapter 8 we will show that it can also be used to obtain estimates which are useful for nonlinear problems. From the Mourre estimate also follow propagation estimates (see [95]). We do not state the abstract results here but refer to Chapter 3 for an application to the wave equation.

Chapter 3

Asymptotic completeness for the wave equation in a class of stationary and asymptotically flat space-times

In this chapter, which summarizes the article [60], we show asymptotic completeness for the wave equation in a class of stationary and asymptotically flat space-times. We introduce the asymptotic velocity observable and we describe its spectrum (under hypotheses weaker than for the asymptotic completeness).

3.1 A class of stationary space-times

In this section we describe the space-time and the associated wave equation.

3.1.1 The space-time

We consider a lorentzian manifold $(\mathcal{M}, g)$ of dimension $n+1$, $n \geq 2$ [1], orientable and orientable in time, globally hyperbolic, stationary and asymptotically flat. We choose the signature $(+, -, ..., -)$ for the lorentzian metric g. By a theorem due to Geroch a globally hyperbolic space-time is homeomorphic to $\mathbb{R} \times \Sigma$, where Σ is a Cauchy surface (see [56]). If the space-time is also stationary, then there exists a globally timelike Killing vector field X and the hoemomorphism can be described in the folowing way :

$$\begin{array}{ccc} \mathbb{R} \times \Sigma & \rightarrow & \mathcal{M} \\ (t, p) & \mapsto & \Phi_t(p), \end{array} \tag{3.1.1}$$

[1] The method also works in dimension $n = 1$, but in this case we have to change the energy space, because the elements of the usual energy space are not distributions in dimension 1.

where $\Phi_t(p)$ is the flow of the vector field X. We consider the same representation of this space-time as in [70]:

Introducing local coordinates x^i on Σ, we obtain coordinates on $\mathcal{M}$ via the homeomorphism (3.1.1). As we use the Killing parameter, the metric in these coordinates is independent of t. Note also that in these coordinates the vector field X equals $\frac{\partial}{\partial t}$. Let ${}^n g$ be the riemannian metric induced by g on Σ. Let $N(\Sigma)$ be the future directed normal vector to Σ. On Σ the Killing vector field takes the form :

$$\left(\frac{\partial}{\partial t} \equiv\right) X = \alpha N(\Sigma) + \beta$$

with α a scalar function and β a vector field on Σ. As X is timelike we have :

$$\alpha > 0, \quad \alpha^2 -^n g(\beta, \beta) > 0.$$

In the coordinates (t, x^i), the metric can be written :

$$(g_{\mu\nu}) = \begin{pmatrix} \alpha^2 - \beta^i\beta_i & -\beta_j \\ -\beta_i & -({}^n g)_{ij} \end{pmatrix}.$$

Here and in the rest of this chapter latin indices go from 1 to n and greek indices from 0 to n. The Minkowski metric is denoted $\eta_{\mu\nu}$ and upper indices are used for the dual metric. We consider the case $\Sigma = \mathbb{R}^n$ and we introduce the following conditions :

General condition (asymptotically flat)

There exist coordinates x on $\Sigma = \mathbb{R}^n$ s.t., if we denote x=(t, x), $g = \sum_{\mu,\nu=0}^{n} g_{\mu\nu}(\mathrm{x})d\mathrm{x}^\mu d\mathrm{x}^\nu$ we have :

$$g_{\mu\nu} \quad \text{is independent of} \quad t, \tag{3.1.2}$$

$$\forall x \in \mathbb{R}^n, \quad g_{00}(x) > 0, \tag{3.1.3}$$

$$\forall x \in \mathbb{R}^n, \quad \forall \xi \in \mathbb{R}^n, \quad -\sum_{ij} g_{ij}(x)\xi_i\xi_j > 0, \tag{3.1.4}$$

$$\exists \epsilon > 0, \quad \forall \alpha \in \mathbb{N}^n, \quad \partial_x^\alpha(g_{\mu\nu} - \eta_{\mu\nu}) \in \mathcal{O}(\langle x \rangle^{-\epsilon-|\alpha|}). \tag{3.1.5}$$

These conditions are also satisfied by the dual metric and the estimates are uniform.

3.1.2 The associated wave equation

Let $|g| = |\mathrm{det} g|$. The d'Alembertian associated to g is given by :

$$\Box_g := |g|^{-\frac{1}{2}} \sum_{\mu,\nu} \partial_\mu |g|^{\frac{1}{2}} g^{\mu\nu} \partial_\nu. \tag{3.1.6}$$

After a unitary transformation we can write the wave equation

$$\left.\begin{array}{rcl} \Box_g u &=& 0, \\ u|_{t=0} &=& u_0, \\ \partial_t u|_{t=0} &=& u_1 \end{array}\right\} \tag{3.1.7}$$

in the following form

$$\left.\begin{array}{rcl}\partial_t^2 u - ib\partial_t u + au & = & 0,\\ u|_{t=0} & = & u_0,\\ \partial_t u|_{t=0} & = & u_1\end{array}\right\} \quad (3.1.8)$$

with

$$a = \frac{1}{\sqrt{g^{00}}|g|^{\frac{1}{4}}}\sum_{jk}\partial_j|g|^{\frac{1}{2}}g^{jk}\partial_k\frac{1}{\sqrt{g^{00}}|g|^{\frac{1}{4}}}, \quad (3.1.9)$$

$$b = i\sum_k \partial_k\frac{g^{0k}}{g^{00}} + \frac{g^{0k}}{g^{00}}\partial_k. \quad (3.1.10)$$

3.1.3 The abstract setting

We consider the following abstract wave equation on a Hilbert space $\mathcal{H}$.

$$\left.\begin{array}{rcl}\partial_t^2 u - ib\partial_t u + au & = & 0,\\ u|_{t=0} & = & u_0,\\ \partial_t u|_{t=0} & = & u_1.\end{array}\right\} \quad (3.1.11)$$

We suppose in the following that :

$$a, b \quad \text{selfadjoint}, \quad (3.1.12)$$

$$a \geq 0, \quad 0 \notin \sigma_{\text{pp}}(a), \quad (3.1.13)$$

$$D(b) \supset D(a^{\frac{1}{2}}) \quad \text{and} \quad \forall u \in D(a^{\frac{1}{2}}), \quad ||bu|| \leq C(||a^{\frac{1}{2}}u|| + ||u||). \quad (3.1.14)$$

Let $\mathcal{H}^k$ be the scale of Sobolev spaces associated to a. On $\mathcal{H}^1$ we introduce the norm $||u||_1^2 := (au, u)$ (as $0 \notin \sigma_{\text{pp}}(a)$ this is indeed a norm). In an analogous way we introduce the norm $||u||_2^2 := (au, u) + ||au||^2$ on $\mathcal{H}^2$. Let $\mathcal{H}_c^k$, $k = 1, 2$ the completion of $\mathcal{H}^k$ in these norms. For $f = (f_0, f_1) \in \mathcal{H}_c^1 \oplus \mathcal{H}$, we put

$$||f||_{\mathcal{E}}^2 = ||f_1||^2 + (af_0, f_0) \quad \text{(energy norm)}$$

and define the energy space $\mathcal{E} = \mathcal{H}_c^1 \oplus \mathcal{H}$. We rewrite the wave equation as a first order system :

$$\left.\begin{array}{rcl}i\partial_t f & = & Rf\\ f|_{t=0} & = & (u_0, u_1),\end{array}\right\} \quad R = \begin{pmatrix} 0 & i \\ -ia & -b\end{pmatrix}. \quad (3.1.15)$$

The operator R is selfadjoint on $\mathcal{E}$ with domain $D(R) = \mathcal{H}_c^2 \oplus \mathcal{H}^1$. By a unitary transformation $U(a)$ we can transform the operator R into :

$$L := U(a)RU^{-1}(a) = \begin{pmatrix} a^{\frac{1}{2}} - \frac{b}{2} & \frac{b}{2} \\ \frac{b}{2} & -a^{\frac{1}{2}} - \frac{b}{2}\end{pmatrix}.$$

The operator L is selfadjoint on $\mathcal{H} \oplus \mathcal{H}$ with domain $D(L) = \mathcal{H}^1 \oplus \mathcal{H}^1$. The evolution is therefore described by a unitary group e^{-itL}.

It is easy to check that the operators introduced in Subsection 3.1.2 satisfy :

$$a(x,\xi) - \xi^2 \in S^{2,-\epsilon}, \tag{3.1.16}$$
$$b(x,\xi) \in S^{1,-\epsilon}. \tag{3.1.17}$$

The operators a^w, b^w satisfy the conditions (3.1.12)-(3.1.14). In particular a is selfadjoint on $\mathcal{H} := L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$ and b^w is essentially selfadjoint with domain $H^1(\mathbb{R}^n)$. In the following we note

$$L := \begin{pmatrix} (a^w)^{\frac{1}{2}} - \frac{b^w}{2} & \frac{b^w}{2} \\ \frac{b^w}{2} & -(a^w)^{\frac{1}{2}} - \frac{b^w}{2} \end{pmatrix}, \; D(L) = H^1 \oplus H^1.$$

3.1.4 Absence of eigenvalues

For the rest of this chapter we suppose the following stronger condition on the metric :

$$\exists \delta > \frac{1}{2}, \quad \forall \alpha \in \mathbb{N}^n, \quad \partial_x^\alpha g^{0k} \in \mathcal{O}(\langle x \rangle^{-\delta-|\alpha|}). \tag{3.1.18}$$

Theorem 3.1.1. *Under the supplementary condition (3.1.18), L has no eigenvalues.*

3.2 The Mourre estimate

In this section we show a Mourre estimate for L. We proceed by first showing a Mourre estimate for a^w, then for $(a^w)^{1/2}$ and eventually for L. We give here only the result for L. Let

$$c := \frac{1}{2}(\langle x, D\rangle + \langle D, x\rangle) \quad \left(D = \frac{1}{i}\nabla\right) \quad \text{and}$$
$$D(c) = \{\Phi \in L^2(\mathbb{R}^n) | c\Phi \in L^2(\mathbb{R}^n)\}.$$

For $\chi \in C_0^\infty(]0,\infty[)$, we always consider the function $\mathbb{R}^+ \ni \lambda \to \chi(\lambda^{\frac{1}{2}})$ to be extended by 0 on $\mathbb{R}^-$. In this way the expression $\chi(a(x,\xi)^{\frac{1}{2}})$ is always well defined, even if $a(x,\xi)$ is not necessarily positive.

Let $\chi, \tilde{\chi} \in C_0^\infty(]0,\infty[)$ or $\chi, \tilde{\chi} \in C_0^\infty(]-\infty, 0[), \chi\tilde{\chi} = \chi$. From the regularity properties of a^w we deduce that the operators $\tilde{\chi}((a^w)^{\frac{1}{2}})c\tilde{\chi}((a^w)^{\frac{1}{2}})$ resp. $\tilde{\chi}(-(a^w)^{\frac{1}{2}})c\tilde{\chi}(-(a^w)^{\frac{1}{2}})$ are well defined on $D(c)$. The closure, denoted $c_{\tilde{\chi}}$, is selfadjoint. For $\chi \in C_0^\infty(]0,\infty[)$ or $\chi \in C_0^\infty(]-\infty, 0[)$, we put :

$$C_\chi = \begin{pmatrix} c_\chi & 0 \\ 0 & c_\chi \end{pmatrix}, \quad D(C_\chi) = D(c_\chi) \oplus D(c_\chi).$$

Theorem 3.2.1.

(*i*) *For all* $\chi \in C_0^\infty(]0,\infty[) \cup C_0^\infty(]-\infty,0[)$, *we have :* $L \in C^2(C_\chi)$.

(*ii*) *For all* $\chi, \tilde{\chi} \in C_0^\infty(]0,\infty[) \cup C_0^\infty(]-\infty,0[)$ *with* $\chi\tilde{\chi} = \chi$, *we have :*

$$\chi(L)[iL, C_{\tilde{\chi}}]\chi(L) = \chi(L)L\chi(L) + K \quad \text{with} \quad K \quad \text{compact.} \tag{3.2.1}$$

(*iii*) *For all* $\lambda > 0$ *and* $\delta > 0$, *we can find an open neighborhood* Δ *of* λ *s.t. for all* $\tilde{\chi} \in C_0^\infty(]0,\infty[), \tilde{\chi} = 1$ *on* Δ:

$$\mathbb{1}_\Delta(L)[iL, C_{\tilde{\chi}}]\mathbb{1}_\Delta(L) \geq (\lambda - \delta)\mathbb{1}_\Delta(L). \tag{3.2.2}$$

We have an analogous property for $\lambda < 0$.

3.3 Propagation estimates

In this section we collect several propagation estimates which are important for the construction of the asymptotic velocity and the proof of the asymptotic completeness result. For $\Phi \in \mathcal{H} \oplus \mathcal{H}$ we put :

$$\Phi_t := e^{-itL}\Phi.$$

Proposition 3.3.1 (Maximal velocity estimate)**.** (*i*) *Let* $1 < \theta_1 < \theta_2, \chi \in C_0^\infty(]0,\infty[)$ *or* $\chi \in C_0^\infty(]-\infty,0[)$ *and* $\Phi \in \mathcal{H} \oplus \mathcal{H}$. *Then*

$$\int_1^\infty ||\mathbb{1}_{[\theta_1,\theta_2]}\left(\frac{|x|}{t}\right)\chi(L)\Phi_t||^2\frac{dt}{t} \leq C||\Phi||^2. \tag{3.3.1}$$

(*ii*) *Let* $F \in C^\infty(\mathbb{R}), F' \in C_0^\infty(\mathbb{R})$ *and* $\mathrm{supp}F \subset]1,\infty[$. *Then :*

$$s - \lim_{t\to\infty} F\left(\frac{|x|}{t}\right)e^{-itL} = 0. \tag{3.3.2}$$

Let $\chi, \tilde{\chi} \in C_0^\infty(]0,\infty[)$ or $\chi, \tilde{\chi} \in C_0^\infty(]-\infty,0[)$ s.t. $\chi\tilde{\chi} = \chi, \tilde{g}(\lambda) = \tilde{\chi}(\sqrt{\lambda})\sqrt{\lambda}\tilde{\chi}(\sqrt{\lambda})$ (resp. $\tilde{g}(\lambda) = -\tilde{\chi}(-\sqrt{\lambda})\sqrt{\lambda}\tilde{\chi}(-\sqrt{\lambda})$). The operator v^w for

$$v(x,\xi) := \tilde{g}'(a(x,\xi))\nabla_\xi a(x,\xi)$$

is called the *local velocity*. Note that $v \in S^{0,0}$ and that v^w is a bounded operator. We obtain :

Proposition 3.3.2 (Microlocal velocity estimate)**.** (*i*) *Let* $0 < \theta_1 < \theta_2, \Phi \in \mathcal{H} \oplus \mathcal{H}$. *Then :*

$$\int_1^\infty ||\mathbb{1}_{[\theta_1,\theta_2]}\left(\frac{|x|}{t}\right)\left(\frac{x}{t} - v^w\right)\chi(L)\Phi_t||^2\frac{dt}{t} \leq C||\Phi||^2. \tag{3.3.3}$$

(*ii*) *Let* $0 < \theta_1 < \theta_2$. *Then :*

$$s - \lim_{t\to\infty} \mathbb{1}_{[\theta_1,\theta_2]}\left(\frac{|x|}{t}\right)\left(\frac{x}{t} - v^w\right)\chi(L)e^{-itL} = 0.$$

Proposition 3.3.3 (Minimal velocity estimate)**.** *Let* $\chi \in C_0^\infty(]0,\infty[)$ *or* $\chi \in C_0^\infty(]-\infty,0[), \Phi \in \mathcal{H} \oplus \mathcal{H}$ *and* $0 < \theta_0 < 1$. *Then we have :*

$$i) \qquad \int_1^\infty ||\mathbb{1}_{[0,\theta_0]}\left(\frac{|x|}{t}\right)\chi(L)\Phi_t||^2\frac{dt}{t} \leq C||\Phi||^2, \tag{3.3.4}$$

$$ii) \qquad s - \lim_{t\to\infty} \mathbb{1}_{[0,\theta_0]}\left(\frac{|x|}{t}\right)e^{-itL} = 0. \tag{3.3.5}$$

3.4 Asymptotic velocity

In this section, we construct the asymptotic velocity and we describe its fundamental properties. In the following we will always identify (as algebras) $C_\infty(\mathbb{R}^n, \mathbb{C}^2)$ and $C_\infty(\mathbb{R}^n \times \{+,-\})$, resp. $C_\infty(\mathbb{R}^n \times \{+,-\}) \otimes C_\infty(\mathbb{R})$ and $C_\infty(\mathbb{R}^n \times \{+,-\} \times \mathbb{R})$.

Theorem 3.4.1. *(i) For all $f = (f_1, f_2) \in C_\infty(\mathbb{R}^n, \mathbb{C}^2)$ there exists*

$$s - \lim_{t \to +\infty} e^{itL} \begin{pmatrix} f_1(\frac{x}{t}) & 0 \\ 0 & f_2(\frac{x}{t}) \end{pmatrix} e^{-itL} =: \gamma^+(f) \in B(\mathcal{H} \oplus \mathcal{H}).$$

$\gamma^+ : C_\infty(\mathbb{R}^n, \mathbb{C}^2) \to B(\mathcal{H} \oplus \mathcal{H})$ is a $*$-morphism of C^*-algebras. We have : $[L, \gamma^+(f)] = 0$ for all $f \in C_\infty(\mathbb{R}^n, \mathbb{C}^2)$. We put $\mathcal{U}^+ := \mathrm{Im}\gamma^+$.

(ii) We have :

$$\gamma^+(f) = s - \lim_{t \to \infty} e^{itL} \begin{pmatrix} f_1(\frac{D}{|D|}) & 0 \\ 0 & f_2(-\frac{D}{|D|}) \end{pmatrix} e^{-itL}.$$

(iii) We have :

$$I := \mathrm{Ker}(\gamma^+) = \{f \in C_\infty(\mathbb{R}^n \times \{+,-\}) | \quad f|_{\mathrm{S}^{n-1} \times \{+,-\}} = 0\},$$

which gives the following isomorphisms :

$$\begin{array}{ccc} C_\infty(\mathbb{R}^n \times \{+,-\})/I & \xrightarrow[\gamma^+]{\simeq} & \mathcal{U}^+ \\ & \searrow^{\simeq} \quad \nearrow_{\simeq} & \\ & C(\mathrm{S}^{n-1} \times \{+,-\}) & \end{array}$$

, in particular $\sigma(\mathcal{U}^+) = \mathrm{S}^{n-1} \times \{+,-\}$.

(iv) We consider the $*$-morphism of C^*- algebras

$$\Gamma^+ : \begin{array}{ll} C_\infty(\mathbb{R}^n \times \{+,-\}) \otimes C_\infty(\mathbb{R}) & \to B(\mathcal{H} \oplus \mathcal{H}) \\ g \otimes h & \mapsto \gamma^+(g) h(L). \end{array}$$

We put :

$$\mathcal{W}^+ := \mathrm{Im}\Gamma^+, \mathcal{N} := \mathrm{S}^{n-1} \times \{+\} \times \mathbb{R}^+ \cup \mathrm{S}^{n-1} \times \{-\} \times \mathbb{R}^-.$$

We have :

$$J := \mathrm{Ker}(\Gamma^+) = \{f \in C_\infty(\mathbb{R}^n \times \{+,-\} \times \mathbb{R}) \quad | \quad f|_{\mathcal{N}} = 0\},$$

which gives the following isomorphisms :

$$\begin{array}{ccc} C_\infty(\mathbb{R}^n \times \{+,-\} \times \mathbb{R})/J & \xrightarrow[\Gamma^+]{\simeq} & \mathcal{W}^+ \\ & \searrow^{\simeq} \quad {}^{\simeq}\nearrow & \\ & C_\infty(\mathcal{N}) & \end{array}$$

, in particular $\sigma(\mathcal{W}^+) = \mathcal{N}$.

Remark

1) We can associate to the algebra $\mathcal{U}^+$ an observable in one of the following two manners :

a) For $f \in C_\infty(\mathbb{R}^n)$, we have :

$$\gamma^+(f,f) = s - \lim_{t\to\infty} e^{itL} \begin{pmatrix} f(\frac{x}{t}) & 0 \\ 0 & f(\frac{x}{t}) \end{pmatrix} e^{-itL}.$$

If in addition $f(0) = 1$, we have :

$$s - \lim_{R\to\infty} \left(s - \lim_{t\to\infty} e^{itL} \begin{pmatrix} f(\frac{x}{Rt}) & 0 \\ 0 & f(\frac{x}{Rt}) \end{pmatrix} e^{-itL} \right) = \mathbb{1}$$

by Proposition 3.3.1. By [36, Proposition B.2.1.], there is a vector of selfadjoint operators commuting with P^+s.t.

$$\gamma^+(f,f) = f(P^+), \quad \forall f \in C_\infty(\mathbb{R}^n).$$

The operator P^+ commutes with L.

b) By (ii), the limit

$$V^+ \ := s - \lim_{t\to\infty} e^{itL} \begin{pmatrix} \frac{D}{|D|} & 0 \\ 0 & -\frac{D}{|D|} \end{pmatrix} e^{-itL}$$

exists. Clearly :

$$f(V^+) = f(P^+), \quad \forall f \in C_\infty(\mathbb{R}^n), \quad \text{i.e.} \quad V^+ = P^+,$$

in particular P^+ is bounded. We call $V^+ = P^+$ the *asymptotic velocity*. Let us consider :

$$\kappa \ : \begin{array}{ll} C_\infty(\mathbb{R}^n) \to & \mathcal{U}^+ \\ f & \mapsto \gamma^+(f,f). \end{array}$$

By (iii), we have

$$\mathrm{Ker}(\kappa) = \{f \in C_\infty(\mathbb{R}^n) | \quad f|_{\mathrm{S}^{n-1}} = 0\},$$

i.e. $\sigma(P^+) = \mathrm{S}^{n-1}$. Using (iv), we can compute the joint spectrum :

$$\sigma(P^+, L) = \mathrm{S}^{n-1} \times \mathbb{R}.$$

If we put

$$V_1^+ \ := s - \lim_{t\to\infty} e^{itL} \begin{pmatrix} \frac{D}{|D|} & 0 \\ 0 & 0 \end{pmatrix} e^{-itL}, \quad V_2^+ \ := s - \lim_{t\to\infty} e^{itL} \begin{pmatrix} 0 & 0 \\ 0 & -\frac{D}{|D|} \end{pmatrix} e^{-itL},$$

then :

$$\sigma(V_1^+, L) = \mathrm{S}^{n-1} \times \mathbb{R}^+, \quad \sigma(V_2^+, L) = \mathrm{S}^{n-1} \times \mathbb{R}^-.$$

2) We can see $(iii), (iv)$ as a very weak version of asymptotic completeness. For $(iii), (iv)$, we only need a decay $\mathcal{O}(\langle x\rangle^{-\epsilon})$ at infinity whereas we need $\mathcal{O}(\langle x\rangle^{-1-\epsilon})$ for asymptotic completeness (short range).

3.5 Asymptotic completeness

Let

$$\tilde{L} = \begin{pmatrix} |D| & 0 \\ 0 & -|D| \end{pmatrix}, \; D(\tilde{L}) = H^1 \oplus H^1.$$

We will compare e^{-itL} and $e^{-it\tilde{L}}$. As already announced, we have to strengthen our conditions. Instead of (3.1.16),(3.1.17) we require the existence of $\epsilon > 0$ s.t. :

$$a(x,\xi) - \xi^2 \in S^{2,-1-\epsilon}, \tag{3.5.1}$$

$$b(x,\xi) \in S^{1,-1-\epsilon}. \tag{3.5.2}$$

Note that this is a condition on the operators and not directly on the metric. In particular the wave equation on the Schwarzschild metric enters in this setting after diagonalization with respect to the eigenvalues of the Laplacian on the sphere. Note however that in the Schwarzschild case we have to change the energy space (compare Chapter 7).

Theorem 3.5.1. *We have the existence of*

$$s - \lim_{t\to+\infty} e^{itL} e^{-it\tilde{L}}, \tag{3.5.3}$$

$$s - \lim_{t\to+\infty} e^{it\tilde{L}} e^{-itL}. \tag{3.5.4}$$

If (3.5.3) equals Ω_{sr}^+*, then (3.5.4) equals* $\Omega_{\mathrm{sr}}^{+*}$ *and we have* $\Omega_{\mathrm{sr}}^{+*}\Omega_{\mathrm{sr}}^+ = \mathbb{1} = \Omega_{\mathrm{sr}}^+\Omega_{\mathrm{sr}}^{+*}$*. In addition :*

$$L = \Omega_{\mathrm{sr}}^+ \tilde{L} \Omega_{\mathrm{sr}}^{+*}, \tag{3.5.5}$$

$$P^+ = \Omega_{\mathrm{sr}}^+ \begin{pmatrix} \frac{D}{|D|} & 0 \\ 0 & -\frac{D}{|D|} \end{pmatrix} \Omega_{\mathrm{sr}}^{+*}, \tag{3.5.6}$$

in particular $\sigma_{\mathrm{sc}}(L) = \emptyset$ *and* $(P^+)^2 = \mathbb{1}$.

Chapter 4

On scattering theory for the Klein-Gordon equation on the Kerr metric

4.1 Introduction

In this chapter, which summarizes the article [61], we show asymptotic completeness for the Klein-Gordon equation on the Kerr metric within certain spaces of positive energy solutions. We compare the Klein-Gordon dynamics with the free dynamics near the horizon of the black-hole and with a modified dynamics of Dollard type at infinity.

Recall that the Kerr metric is not stationary, i.e. there is no global timelike Killing vector field. In particular there exists a region called the ergosphere where the vector field ∂_t becomes spacelike. The consequence for the Klein-Gordon equation is the following. If we write it in the form

$$\left.\begin{aligned} (\partial_t^2 - 2ik\partial_t + h)u = 0, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1 \end{aligned}\right\} \tag{4.1.1}$$

with h, k selfadjoint operators on a certain Hilbert space, then h is not a positive operator. Thus the energy $E(u) = ||\partial_t u||^2 + (hu, u)$, which is conserved along the evolution, is not positive. More generally there exists no positive conserved energy for the Klein-Gordon equation on the Kerr metric. We will construct Hilbert subspaces where h is positive. For the existence of these subspaces we need that the mass of the particle is strictly positive. Our asymptotic completeness result is valid after restriction to these subspaces. The results presented in this and the following chapter are contained in [61] and [63]. In these works a general geometric setting is described in which we formulate and prove the results. In particular the results in [61] and [63] can be applied to the Klein-Gordon (wave) and Dirac equations on a riemannian manifold with euclidean and hyperbolic ends. In [61] and [63] they are applied to the Kerr case. In the present work, we only summarize the applications to the Kerr case and refer to [61], [63] for the more general setting.

The proof uses a Mourre estimate. In [34], De Bièvre, Hislop and Sigal observed that a Mourre estimate for a non negative operator h entails a Mourre estimate for the square root of h. We will give here a somewhat different formulation and proof of this result. This is sufficient to treat static situations. As the space-time is not static we also need an additional argument to deduce a Mourre estimate for a matrix problem. The operator h itself is in our case similar to a Laplacian on a riemannian manifold with two ends, one asymptotically euclidean and the other asymptotically hyperbolic. For this kind of Laplacian a Mourre estimate is established in the article of Froese and Hislop [50]. In [61] we give a somewhat different formulation and proof of this result. In the next chapter we will give a new argument that also works for the Dirac operator (whereas the argument of Froese, Hislop only works for the Laplacian). Therefore we will not deal with the Mourre estimate for h in this chapter.

The present chapter is organized as follows. Section 4.2 is devoted to the Hilbert space setting. We deduce from a Mourre estimate for h a Mourre estimate for the square root of h and a Mourre estimate for a matrix problem. In Section 4.3 we collect the main results of this chapter. We construct the subspaces on which the energy is positive. We consider the energy associated to the Killing vector field (using Boyer-Lindquist coordinates and denoting Ω_H the angular velocity of the horizon with respect to infinity) $\partial_t + \Omega_H$. This Killing vector field is timelike close to the horizon and spacelike at infinity. The axial symmetry of the Kerr solution gives the conservation of the angular momentum for the solutions of the Klein-Gordon equation. This permits us to impose in the following a restriction on the angular momentum of the initial data. The positive mass of the particle makes the operator positive. For neutral mesons we can permit at least 10^{16} modes.

4.2 The Hilbert space setting

4.2.1 An abstract wave equation

We consider the same abstract wave equation as in Chapter 3 :

$$\left.\begin{array}{r} \partial_t^2 u - 2ik\partial_t u + hu = 0, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1. \end{array}\right\} \tag{4.2.1}$$

We first suppose that

$$h, k \quad \text{selfadjoint}, \tag{4.2.2}$$

$$h \geq 0, \quad 0 \notin \sigma_{\text{pp}}(h), \tag{4.2.3}$$

$$D(k) \supset D(h^{1/2}), \quad \forall u \in D(h^{1/2}),\ ||ku|| \leq C(||h^{1/2}u|| + ||u||). \tag{4.2.4}$$

Let $\mathcal{H}^k$ be the scale of Sobolev spaces associated to h and $\mathcal{H}^k_c$ the completion of $\mathcal{H}^k$ in the norm :

$$\|u\|^2_{\mathcal{H}^k} = \sum_{j=1}^{k} \|P^{j/2}u\|^2.$$

The energy space will be denoted in this chapter by $\mathcal{R} := \mathcal{H}^1_c \oplus \mathcal{H}$. We rewrite the wave equation as a first order system :

$$\left.\begin{array}{r} i\partial_t f = Rf \\ f|_{t=0} = (u_0, u_1) \end{array}\right\}, \quad R = \begin{pmatrix} 0 & i \\ -ih & -2k \end{pmatrix}. \tag{4.2.5}$$

Recall from Chapter 3 that R is selfadjoint on $\mathcal{R}$ with domain $D(R) = \mathcal{H}^2_c \oplus \mathcal{H}^1$. Let $\mathcal{L} := \mathcal{H} \oplus \mathcal{H}$. We introduce the following unitary transformation

$$U : \mathcal{R} \to \mathcal{L},\, U = \frac{1}{\sqrt{2}} \begin{pmatrix} h^{1/2} & i \\ h^{1/2} & -i \end{pmatrix},\, L := URU^{-1} = \begin{pmatrix} h^{1/2} - k & k \\ k & -h^{1/2} - k \end{pmatrix}.$$

Then, again by the results of Chapter 3, L is selfadjoint with $D(L) = \mathcal{H}^1 \oplus \mathcal{H}^1$. The evolution is therefore described by a unitary group e^{-itL}.

We now want to solve (4.2.1) without supposing $h \geq 0$. To avoid confusion we introduce new symbols $\tilde{h}, \tilde{k}$. For simplicity we will suppose that $\tilde{h}$ (resp. $\tilde{k}$) is a differential operator of order 2 (resp. 1) acting on $C_0^\infty(\tilde{\mathcal{M}})$, $\tilde{\mathcal{M}} = \mathbb{R}_s \times S^2_\omega$. We consider the following wave equation :

$$\left.\begin{array}{r} (\partial_t^2 - 2i\tilde{k}\partial_t + \tilde{h})u = 0, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1. \end{array}\right\} \tag{4.2.6}$$

Our hypotheses will be :

$$\tilde{h} + \tilde{k}^2 \geq 0 \quad \text{in the sense of quadratic forms on } C_0^\infty(\tilde{\mathcal{M}}). \tag{4.2.7}$$

(We note the Friedrichs extension $(\tilde{h} + \tilde{k}^2, D(\tilde{h} + \tilde{k}^2))$.)

$$\pm[\tilde{h}, i\tilde{k}] \leq C(\tilde{h} + \tilde{k}^2) \text{ in the sense of quadratic forms on } C_0^\infty(\tilde{\mathcal{M}}). \tag{4.2.8}$$

$$0 \notin \sigma_{\mathrm{pp}}(\tilde{h} + \tilde{k}^2). \tag{4.2.9}$$

(4.2.7) is the condition that (4.2.6) is hyperbolic. (4.2.8) will assure a-priori estimates. These two conditions are geometric conditions. (4.2.9) is necessary to define an energy norm which coincides with the energy norm we have defined before in the case $\tilde{k} = 0$. We define the scale of energy spaces in the following way. We put :

$$E := \begin{pmatrix} 0 & 1 \\ -\tilde{h} & 2i\tilde{k} \end{pmatrix}.$$

For $(u_0, u_1) \in C_0^\infty(\tilde{\mathcal{M}}) \times C_0^\infty(\tilde{\mathcal{M}})$, we put :

$$||(u_0, u_1)||^2_{\mathcal{E}} = ||u_1 - i\tilde{k}u_0||^2 + \langle (\tilde{h} + \tilde{k}^2)u_0, u_0 \rangle.$$

The space $\mathcal{E}^k$ is defined as the completion of $C_0^\infty(\tilde{\mathcal{M}}) \times C_0^\infty(\tilde{\mathcal{M}})$ in the norm :

$$\|(u_0, u_1)\|^2_{\mathcal{E}^k} = \sum_{i=0}^{k} \left\| E^i \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|^2_{\mathcal{E}}.$$

The condition (4.2.9) assures that $||.||_{\mathcal{E}^k}$ defines a norm. We will often write $\mathcal{E}$ in place of $\mathcal{E}^0$ and we note $\mathcal{E}^{-k}$ the dual of $\mathcal{E}^k$.

If u is a solution of (4.2.6), then $v = e^{-i\tilde{k}t}u$ is a solution of

$$(\partial_t^2 + h(t))v = 0 \tag{4.2.10}$$

with $h(t) = e^{-i\tilde{k}t}(\tilde{h} + \tilde{k}^2)e^{it\tilde{k}t} \geq 0$. The natural energy associated to (4.2.10) is

$$||v||_{\mathcal{T}}^2 = ||\partial_t v||^2 + \langle h(t)v, v\rangle.$$

We obtain the energy $\mathcal{E}$ by rewriting the energy $\mathcal{T}$ for u. We put :

$$P := \partial_t - E.$$

We have the following a-priori estimate :

Lemma 4.2.1. *Let $u \in C([0,T]\ ;\mathcal{E}^{i+1}) \cap C^1([0,T]\ ;\mathcal{E}^i)$. Then we have :*

$$\sup_{0\leq t\leq T} ||u(t)||_{\mathcal{E}^i}^2 \leq C(||u(0)||_{\mathcal{E}^i}^2 + \int_0^T ||Pu||_{\mathcal{E}^i}^2 dt). \tag{4.2.11}$$

The Lemma gives the following existence and uniqueness result using a Hahn-Banach type argument :

Theorem 4.2.1. *Let $f \in L^2([0,T]\ ;\mathcal{E}^i)$ and $g \in \mathcal{E}^i$. Then there exists a unique solution $u \in C([0,T]\ ;\mathcal{E}^i)$ of*

$$\left.\begin{aligned}(\partial_t - E)u &= f,\\ u|_{t=0} &= g\end{aligned}\right\} \tag{4.2.12}$$

and $u(t)$ satisfies (4.2.11).

4.2.2 The Mourre estimate

In this subsection we suppose $h \geq 0$. Let b be another selfadjoint operator s.t. $h \in C^1(b)$. We suppose that the couple (h, b) satisfies a Mourre estimate on an interval $I^2 = [\alpha^2, \beta^2]$ $(\beta > \alpha > 0)$:

$$1_{I^2}(h)[ih, b]1_{I^2}(h) \geq \nu 1_{I^2}(h) + k \quad \text{with } \nu > 0 \text{ and } k \text{ compact.} \tag{4.2.13}$$

We will construct an operator b_χ, depending on a cut-off function χ, s.t. $h^{1/2} \in C^1(b_\chi)$ and such that the couple $(h^{1/2}, b_\chi)$ satisfies a Mourre estimate on each interval $\tilde{I} \subset I = [\alpha, \beta]$ with $\text{dist}(\tilde{I}, \mathbb{R} \setminus I) > 0$. For this purpose we choose a function $\chi \in C_0^\infty(]0,\infty[)$ with $\chi|_{I^2} = 1$. The regularity of h with respect to b entails that $\chi(h)b\chi(h)$ is well defined on $D(b)$ and closable. The closure, which we denote b_χ, is selfadjoint.

Theorem 4.2.2. *(i) For all $\chi \in C_0^\infty(]0,\infty[)$, the couple $(h^{1/2}, b_\chi)$ satisfies (M1), (M2) and $[h^{1/2}, b_\chi]$ possesses an extension to a bounded operator.*

(ii) For all $\tilde{I} \subset I$ with $\text{dist}(\tilde{I}, \mathbb{R} \setminus I) > 0$ and $\chi \in C_0^\infty(]0,\infty[)$, $\chi|_{I^2} = 1$, we have :

$$1_{\tilde{I}}(h^{1/2})[ih^{1/2}, b_\chi]1_{\tilde{I}}(h^{1/2}) \geq \mu 1_{\tilde{I}}(h^{1/2}) + \tilde{k} \quad \textit{with } \mu > 0 \textit{ and } \tilde{k} \textit{ compact.}$$

Note that from a purely formal point of view the theorem follows rather easily from an integral representation of the square root. Indeed let $\hat{\chi} \in C_0^\infty(I^2)$ with $\hat{\chi}|_{\tilde{I}^2} = 1$, in particular $\hat{\chi}\chi = \hat{\chi}$. Using the formula :

$$\hat{\chi}^2(h)h^{1/2} = \pi^{-1} \int_0^\infty s^{-1/2}\hat{\chi}^2(h)h(s+h)^{-1}ds$$

we find :

$$\hat{\chi}(h)[ih^{1/2}, b_\chi]\hat{\chi}(h) = \pi^{-1} \int_0^\infty s^{1/2}(s+h)^{-1}\hat{\chi}(h)[ih, b]\hat{\chi}(h)(s+h)^{-1}ds,$$

which allows us to use the Mourre estimate for h.

We now establish the Mourre estimate for a matrix problem. Let $\chi \in C_0^\infty(]0,\infty[)$ and $\mathcal{H}, h \geq 0, b, b_\chi$ as before, in particular $h \in C^1(b)$. We consider the following selfadjoint operators on $\mathcal{L} := \mathcal{H} \oplus \mathcal{H}$:

$$\begin{aligned} B_\chi &:= \begin{pmatrix} b_\chi & 0 \\ 0 & b_\chi \end{pmatrix}, \quad D(B_\chi) = D(b_\chi) \oplus D(b_\chi) \quad \text{and} \\ L &:= \begin{pmatrix} h^{1/2} & 0 \\ 0 & -h^{1/2} \end{pmatrix} + c \begin{pmatrix} -\mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix} \end{aligned}$$

with $c \in \mathbb{R}^+$ and $D(L) = \mathcal{H}^1 \oplus \mathcal{H}^1$. Let $S_h \subset \mathbb{R}$ be a discrete subset s.t. $0 \in S_h$ and :

$$\forall \lambda \in \mathbb{R} \setminus S_h, \quad \exists I \quad \text{neighborhood of } \lambda \text{ with} \quad 1_I(h)[ih, b]1_I(h) \geq \nu 1_I(h) + k$$

with $\nu > 0$ and k compact. We call S_h the threshold set of h. We put $S_L := \{-c \pm \sqrt{\lambda^2 + c^2}, \lambda \in S_h\}$. We obtain :

Theorem 4.2.3.

$$\begin{aligned} &(i) \quad \forall \chi \in C_0^\infty(]0,\infty[) \quad (L, B_\chi) \quad \textit{satisfies (M1),(M2).} \\ &(ii) \quad \forall \lambda \in \mathbb{R} \setminus S_L, \quad \exists \chi \in C_0^\infty(]0,\infty[), \quad I \quad \textit{neighborhood of } \lambda \textit{ with :} \\ &\quad 1_I(L)[iL, B_\chi]1_I(L) \geq \mu 1_I(L) + K_1 \quad \textit{if} \quad \lambda > 0, \\ &\quad 1_I(L)[iL, -B_\chi]1_I(L) \geq \mu 1_I(L) + K_2 \quad \textit{if} \quad \lambda < 0, \end{aligned}$$

with $\mu > 0$ and K_i $(i = 1, 2)$ compact.

4.3 Asymptotic completeness

4.3.1 The Klein-Gordon equation and the asymptotic dynamics

Let f be a solution of the wave equation on the Kerr space-times and $u = \frac{\sigma}{\sqrt{r^2+a^2}}f$. Then u satisfies

$$\left(\partial_t^2 + \frac{4aMr}{\sigma^2}\partial_\phi\partial_t - \frac{\sqrt{r^2+a^2}}{\sigma}\partial_s(r^2+a^2)\partial_s\frac{\sqrt{r^2+a^2}}{\sigma} - \frac{\sqrt{\Delta}}{\sigma}\Delta_{S^2}\frac{\sqrt{\Delta}}{\sigma} + \frac{a^2}{\sigma^2}\partial_\phi^2 + \frac{m^2\rho^2\Delta}{\sigma^2}\right)u = 0\,. \tag{KG}$$

Here s is the Regge-Wheeler coordinate. Our Hilbert space will be $\mathcal{H} := L^2(\mathbb{R}\times S^2_\omega, dsd\omega)$. Let $c_\pm = \frac{1}{2\kappa_\pm}$, $T>0$, $\theta_0 \in C_b^\infty(\mathbb{R})$ with supp$\theta_0 \subset\,]-\infty,-\frac{T}{2}[\,\cup\,]\frac{T}{2},\infty[$ and $\theta_0 = 1$ on $]-\infty,-T[\,\cup\,]T,\infty[$. Let $0<\theta_+\in \mathbf{S}^{0,-1}$ and $\theta_+(r)=1$ for $r\ge 1$. We put

$$g_-(s) := \frac{(r_+-r_-)^{\frac{c_-}{c_+}+1}e^{-2\kappa_+(\theta_0(s)|s|+r_+)}}{(r_+^2+a^2)^2},\ f_{(-,1)}(s) := \frac{(r_+-r_-)^{\frac{c_-}{c_+}+1}e^{-2\kappa_+(|s|\theta_0(s)+r_+)}m^2r_+^2}{4M^2r_+^2}.$$

We will consider the following equations :

$$\left.\begin{aligned}(\partial_t^2+\tfrac{2a}{r_+^2+a^2}\partial_\phi\partial_t-\partial_s^2-g_-(s)\Delta_{S^2}+f_{(-,1)}(s)+\tfrac{a^2}{(r_+^2+a^2)^2}\partial_\phi^2)u_-=0,\\ u_-|_{t=0}=u_{-,0},\\ \partial_tu_-|_{t=0}=u_{-,1},\end{aligned}\right\}\ (KG_{(-,1)})$$

$$\left.\begin{aligned}(\partial_t^2+\tfrac{2a}{r_+^2+a^2}\partial_\phi\partial_t-\partial_s^2+\tfrac{a^2}{(r_+^2+a^2)^2}\partial_\phi^2)u_-=0,\\ u_-|_{t=0}=u_{-,0},\\ \partial_tu_-|_{t=0}=u_{-,1},\end{aligned}\right\}\ (KG_{(-,2)})$$

$$\left.\begin{aligned}(\partial_t^2-\partial_s^2-\theta_0(s)\tfrac{1}{|s|^2}\Delta_{S^2}+\theta_+(s)(\tfrac{m^2}{2}-\tfrac{2Mm^2}{|s|}\theta_0(s))+\tfrac{m^2}{2}\\ +(\theta_+(s)-1)\tfrac{a^2}{2(r_+^2+a^2)^2}\partial_\phi^2)u_+=0,\\ u_+|_{t=0}=u_{+,0},\\ \partial_tu_+|_{t=0}=u_{+,1},\end{aligned}\right\}\ (KG_{(+,1)})$$

$$\left.\begin{aligned}(\partial_t^2-\partial_s^2+m^2-\tfrac{2Mm^2}{|s|}\theta_0(s))u_+=0,\\ u_+|_{t=0}=u_{+,0},\\ \partial_tu_+|_{t=0}=u_{+,1},\end{aligned}\right\}\ (KG_{(+,2)})$$

$$\left.\begin{aligned}(\partial_t^2+\tfrac{4aMr}{(r^2+a^2)^2}\partial_\phi\partial_t-\partial_s^2-\tfrac{\Delta}{(r^2+a^2)^2}\Delta_{S^2}+\tfrac{a^2}{(r^2+a^2)^2}\partial_\phi^2\\ +\tfrac{m^2r^2\Delta}{(r^2+a^2)^2})v=0,\\ v|_{t=0}=v_0,\\ \partial_tv|_{t=0}=v_1,\end{aligned}\right\}\ (KG_0)$$

$$\left.\begin{aligned}(\partial_t^2+\tfrac{4aMr}{\sigma^2}\partial_\phi\partial_t-\tfrac{\sqrt{r^2+a^2}}{\sigma}\partial_s(r^2+a^2)\partial_s\tfrac{\sqrt{r^2+a^2}}{\sigma}-\tfrac{\sqrt{\Delta}}{\sigma}\Delta_{S^2}\tfrac{\sqrt{\Delta}}{\sigma}\\ +\tfrac{a^2}{\sigma^2}\partial_\phi^2+\tfrac{m^2\rho^2\Delta}{\sigma^2})u=0,\\ u|_{t=0}=u_0,\\ \partial_tu|_{t=0}=u_1,\end{aligned}\right\}\ (KG)$$

The equations $(KG_{(-,1)})$ and $(KG_{(-,2)})$ describe the dynamics near the horizon, the equations $(KG_{(+,1)})$ and $(KG_{(+,2)})$ describe the dynamics near infinity. (KG_0) describes a spherically symmetric intermediate dynamics. It is needed for the proof. Its introduction avoids the necessity of dealing with artificial long range terms. (KG) is the Klein-Gordon equation. The cut-off θ_+ in $(KG_{(+,1)})$ avoids the creation of bounded states. Close to infinity this equation becomes :

$$(\partial_t^2-\partial_s^2-\theta_0(s)\frac{1}{|s|^2}\Delta_{S^2}+m^2-\frac{2Mm^2}{|s|}\theta_0(s))u_+=0.$$

All the above equations fit in the abstract setting developed in Section 4.2. We put in the following $\mathcal{N}=\{0,(-,i),(+,i);\ i=1,2\}$. For $\nu\in\mathcal{N}$, the corresponding energy space is denoted $\mathcal{E}_\nu$.

4.3.2 Restricting the space of solutions

We start with the following change of coordinates :

$$\left\{\begin{array}{c}\tilde{\phi}=\phi-\frac{a}{r_+^2+a^2}t,\\ \tilde{t}=t,\\ \tilde{\theta}=\theta\\ \tilde{s}=s\end{array}\right\}\Rightarrow\left\{\begin{array}{c}\partial_{\tilde{\phi}}=\partial_\phi,\\ \partial_{\tilde{t}}=\partial_t+\frac{a}{r_+^2+a^2}\partial_\phi,\\ \partial_{\tilde{\theta}}=\partial_\theta\\ \partial_{\tilde{s}}=\partial_s\end{array}\right\}$$

In these new coordinates, which we denote again t, s, ϕ, θ, the Klein-Gordon equation becomes :

$$\begin{aligned}&\Bigg(\partial_t^2+\left(\frac{4aMr}{\sigma^2}-\frac{2a}{r_+^2+a^2}\right)\partial_\phi\partial_t+\left(\frac{a^2}{(r_+^2+a^2)^2}-\frac{4a^2Mr}{\sigma^2(r_+^2+a^2)}+\frac{a^2}{\sigma^2}\right)\partial_\phi^2\\&-\frac{\sqrt{r^2+a^2}}{\sigma}\partial_s(r^2+a^2)\partial_s\frac{\sqrt{r^2+a^2}}{\sigma}-\frac{\sqrt{\Delta}}{\sigma}\Delta_{S^2}\frac{\sqrt{\Delta}}{\sigma}+\frac{m^2\rho^2\Delta}{\sigma^2}\Bigg)v=0.\end{aligned}\tag{4.3.1}$$

The coefficients are independent of ϕ and so we can fix the angular momentum $\partial_\phi = in$ in the above equation :

$$\begin{aligned}&\Bigg(\partial_t^2+i\left(\frac{4aMr}{\sigma^2}-\frac{2a}{r_+^2+a^2}\right)n\partial_t-\left(\frac{a^2}{(r_+^2+a^2)^2}-\frac{4a^2Mr}{\sigma^2(r_+^2+a^2)}+\frac{a^2}{\sigma^2}\right)n^2\\&-\frac{\sqrt{r^2+a^2}}{\sigma}\partial_s(r^2+a^2)\partial_s\frac{\sqrt{r^2+a^2}}{\sigma}-\frac{\sqrt{\Delta}}{\sigma}\Delta_{S^2}\frac{\sqrt{\Delta}}{\sigma}+\frac{m^2\rho^2\Delta}{\sigma^2}\Bigg)v=0.\end{aligned}\tag{4.3.2}$$

The above equation has the form (4.2.1) and the operator h is positive if the angular momentum is not too large with respect to the particle. More precisely h is positive if

$$m^2\geq\frac{n^2a^2}{r_+^3}\left(\frac{1}{M}+\frac{1}{r_+}\right).\tag{4.3.3}$$

We will suppose (4.3.3) in the following. An explicit calculation shows that for neutral mesons we can permit at least 2.82×10^{16} modes. We define the operators h_n, k_n using equation (4.3.2). We denote $\mathcal{H}_n^i, L_n, \mathcal{R}_n, R_n$ the spaces and operators constructed starting from (h_n, k_n) as described in Section 4.2. A similar procedure is applied to the comparison dynamics. The corresponding spaces and operators have an additional index ν. Recall that $(-\Delta_{S^2}, H^2(S^2))$ is a selfadjoint operator with compact resolvent. Its eigenvalues are of the form $l(l+1)$ with multiplicity $2l+1$. We can decompose $\mathcal{H}$ into spherical harmonics :

$$L^2(\mathbb{R}\times S^2_\omega)=\oplus_{l=0}^{\infty}\oplus_{m=-l}^{m=l}L^2(\mathbb{R})\otimes Y_{m,l},\ \mathcal{H}_{m,l}\ :=L^2(\mathbb{R})\otimes Y_{m,l},\quad \mathcal{L}_{m,l}\ :=\mathcal{H}_{m,l}\oplus\mathcal{H}_{m,l}.$$

The spherically symmetric operators can be restricted to the spaces with fixed spherical harmonics, the corresponding operators and spaces are indexed by both m and l. We now introduce the spaces on which R_n describes the real dynamics of the system :

$$\begin{aligned}\mathcal{H}_n^\phi\ &:=\{u(r,\theta,\phi)=e^{in\phi}v,\ v\in L^2(\mathbb{R}\times[0,\pi],\sin\theta dsd\theta)\}\subset\mathcal{H},\ \mathcal{H}_n^{1,\phi}:=\mathcal{H}_n^1\cap\mathcal{H}_n^\phi,\\ \mathcal{H}_{c,n}^{1,\phi}\ &:=\text{completion of }\mathcal{H}_n^{1,\phi}\text{ in the norm }(h_nu,u),\ \mathcal{R}_n^\phi:=\mathcal{H}_{c,n}^{1,\phi}\oplus\mathcal{H}_n^\phi.\end{aligned}$$

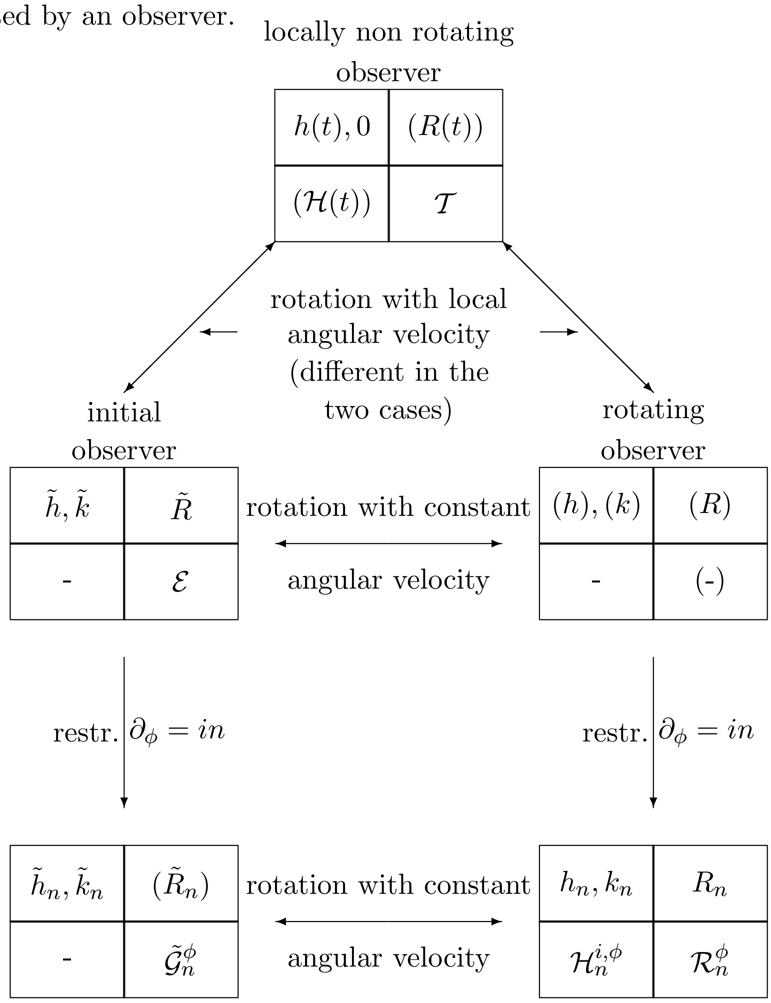
The spaces $\mathcal{R}^{\phi}_{\nu,n}$ are defined in the same manner. Note that $e^{\pm itR_n}$ sends $\mathcal{R}^{\phi}_n$ into itself and that $e^{\pm itR_{\nu,n}}$ sends $\mathcal{R}^{\phi}_{\nu,n}$ into itself. We put :

$$\tilde{\mathcal{G}}^{\phi}_n = \begin{pmatrix} \mathbf{1} & 0 \\ -ic_{1,n} & \mathbf{1} \end{pmatrix} \mathcal{R}^{\phi}_n, \quad \|(u_0,u_1)\|_{\tilde{\mathcal{G}}^{\phi}_n} = \left\| \begin{pmatrix} \mathbf{1} & 0 \\ ic_{1,n} & \mathbf{1} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\mathcal{R}^{\phi}_n},$$

where $c_{1,n} = \frac{an}{r_+^2+a^2}$. The spaces $\tilde{\mathcal{G}}^{\phi}_{\nu,n}$ are constructed in the same manner. We also define $\tilde{\mathcal{G}}_n$:

$$\tilde{\mathcal{G}}_n = \begin{pmatrix} \mathbf{1} & 0 \\ -ic_{1,n} & \mathbf{1} \end{pmatrix} \mathcal{R}_n, \quad ||(u_0,u_1)||_{\tilde{\mathcal{G}}_n} = \left\| \begin{pmatrix} \mathbf{1} & 0 \\ ic_{1,n} & \mathbf{1} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\mathcal{R}_n}.$$

The spaces $\tilde{\mathcal{G}}^{\phi}_{\nu,n}$, $\tilde{\mathcal{G}}_{\nu,n}$ are constructed in an analogous way. Considering the Klein Gordon equations (KG), (KG_ν) we construct the operators $\tilde{h}$, $\tilde{k}$ etc. and the energy spaces $\mathcal{E}$, $\mathcal{E}_\nu$ as explained in Section 4.2. The operators $\tilde{h}_n$, $\tilde{k}_n$ are obtained by restriction $D_\varphi = n$. The following figure summarizes the notations we need. Each bloc contains the Sobolev spaces, energy spaces, operators describing the KG equation and the dynamics which are naturally associated to a given coordinate system. All coordinates can be understood as coordinates naturally attributed by an observer.



We check that

$$\tilde{\mathcal{G}}_n^{\phi} \hookrightarrow \mathcal{E}, \quad \tilde{\mathcal{G}}_{\nu,n}^{\phi} \hookrightarrow \mathcal{E}_{\nu}, \, \mathcal{E}_{(+,2)} \hookrightarrow \tilde{\mathcal{G}}_{(+,2),n}$$

with continuous embeddings. On $\tilde{\mathcal{G}}_n^{\phi}$ the norms $||.||_{\mathcal{E}}$ and $||.||_{\tilde{\mathcal{G}}_n^{\phi}}$ are equivalent and for all $\nu \in \mathcal{N}$ the norms $||.||_{\mathcal{E}_{\nu}}$ and $||.||_{\tilde{\mathcal{G}}_{\nu,n}^{\phi}}$ are equivalent on $\tilde{\mathcal{G}}_{\nu,n}^{\phi}$. We will also need the spaces

$$\mathcal{R}_n^{\phi,f} = \{u \in \mathcal{R}_n^{\phi} \; ; \quad u = \sum_{finite} u_{ml}, \, u_{ml} \in \mathcal{L}_{m,l}\}, \, \tilde{\mathcal{G}}_n^{\phi,f} = \begin{pmatrix} \mathbf{1} & 0 \\ -ic_{1,n} & \mathbf{1} \end{pmatrix} \mathcal{R}_n^{\phi,f}.$$

The spaces $\mathcal{R}_{\nu,n}^{\phi,f}, \tilde{\mathcal{G}}_{\nu,n}^{\phi,f}$ are defined in the same way. We note that $\tilde{\mathcal{G}}_{(+,2),n}^{\phi,f} \subset \tilde{\mathcal{G}}_n^{\phi}$.

We now fix $N > 0$ s.t. $m^2 \geq \frac{N^2 a^2}{r_+^3}(\frac{1}{M} + \frac{1}{r_+})$. We define :

$$\mathcal{G}_N^{\phi} \; := \bigoplus_{|n|\leq N} \tilde{\mathcal{G}}_n^{\phi}, \quad \mathcal{G}_N^{\phi,f} \; := \bigoplus_{|n|\leq N} \tilde{\mathcal{G}}_n^{\phi,f}.$$

The spaces $\mathcal{G}_{\nu,N}^{\phi}$ and $\mathcal{G}_{\nu,N}^{\phi,f}$ are defined in the same manner. We have $\mathcal{G}_N^{\phi} \hookrightarrow \mathcal{E}$ and $\mathcal{G}_{\nu,N}^{\phi} \hookrightarrow \mathcal{E}_{\nu}$ with continuous embedding.

4.3.3 Asymptotic completeness

The first result in this subsection is that R_n, h_n and $h_{\nu,n}$, $R_{\nu,n}\, \nu \neq (+,2)$ do not have eigenvalues. Let $j_{\pm} \in C_b^{\infty}(\mathbb{R})$ with $\mathrm{supp} j_- \subset]-\infty, 1[, \quad \mathrm{supp} j_+ \subset]-1, \infty[$ and $j_+^2 + j_-^2 = 1$. Let $j_{(\delta,i)} \; := j_{\delta}$ for $\delta \in \{+,-\}$, $i \in \{1,2\}$ and $j_0 = 1$. We put :

$$\tilde{\mathcal{G}}_{\mathrm{c},\nu,n}^{\phi} \; := \begin{pmatrix} \mathbf{1} & 0 \\ -ic_{1,n} & \mathbf{1} \end{pmatrix} \mathrm{Im}\mathbf{1}^{\mathrm{c}}(R_{\nu,n}) \cap \tilde{\mathcal{G}}_{\nu,n}^{\phi}, \quad \mathcal{G}_{\mathrm{c},\nu,N}^{\phi} \; := \bigoplus_{|n|\leq N} \tilde{\mathcal{G}}_{\mathrm{c},\nu,n}^{\phi}.$$

In this subsection we compare the solution of the Klein-Gordon equation to the solutions of the other equations. We recall that a solution of (KG) (resp. (KG_{ν})) is a couple $(f(t), \partial_t f(t))$ such that $f(t)$ satisfies (KG) (resp. (KG_{ν})).

Theorem 4.3.1. *Let* $\nu \in \{0, (\pm, 1)\}$.

(i) Let $(v_{(\nu,0)}, v_{(\nu,1)}) \in \mathcal{G}_{\nu,N}^{\phi}$ *and* $v_{\nu}(t)$ *be the solution of* (KG_{ν}) *with initial data* $(v_{(\nu,0)}, v_{(\nu,1)})$. *Then there exists a solution* $u_{\nu}^+ \in C([0,\infty[\; ; \mathcal{G}_N^{\phi})$ *of* (KG) *such that*

$$||u_{\nu}^+(t) - j_{\nu} v_{\nu}(t)||_{\mathcal{E}} \to 0 \quad (t \to +\infty). \tag{4.3.4}$$

(ii) Let $(u_0, u_1) \in \mathcal{G}_N^{\phi}$ *and* $u(t)$ *be the solution of* (KG) *with initial data* (u_0, u_1). *Then there exists a solution* $v_{\nu}^+ \in C([0,\infty[\; ; \mathcal{G}_{\nu,N}^{\phi})$ *of* (KG_{ν}) *such that*

$$||v_{\nu}^+(t) - j_{\nu} u(t)||_{\mathcal{E}_{\nu}} \to 0 \quad (t \to +\infty). \tag{4.3.5}$$

Theorem 4.3.2. *Let* $\nu \in \{(\pm, 2)\}$.

(i) Let $(v_{(\nu,0)}, v_{(\nu,1)}) \in \mathcal{G}^{\phi,f}_{\nu,N} \cap \mathcal{G}^{\phi}_{c,\nu,N}$ *and* $v_\nu(t)$ *be the solution of* (KG_ν) *with initial data* $(v_{(\nu,0)}, v_{(\nu,1)})$. *Then there exists a solution* $u^+_\nu \in C([0,\infty[\,; \mathcal{G}^{\phi}_N)$ *of* (KG) *such that*

$$||u^+_\nu(t) - j_\nu v_\nu(t)||_{\mathcal{E}} \to 0 \quad (t \to +\infty). \tag{4.3.6}$$

(ii) Let $(u_0, u_1) \in \mathcal{G}^{\phi}_N$ *and* $u(t)$ *be the solution of* (KG) *with initial data* (u_0, u_1). *Then there exists a solution* $v^+_\nu \in C([0,\infty[\,; \mathcal{G}^{\phi}_{\nu,N})$ *of* (KG_ν) *such that*

$$||v^+_\nu(t) - j_\nu u(t)||_{\mathcal{E}_\nu} \to 0 \quad (t \to +\infty). \tag{4.3.7}$$

Remark 4.3.1. *We can replace the norm* $||.||_{\mathcal{E}}$ *resp.* $||.||_{\mathcal{E}_\nu}$ *in (4.3.4)-(4.3.7) by the norm* $||.||_{\mathcal{G}^{\phi}_N}$ *resp.* $||.||_{\mathcal{G}^{\phi}_{\nu,N}}$.

4.3.4 Profiles

In this subsection we will describe the asymptotic behavior of the solution in terms of profiles. Let

$$\begin{aligned}&\mathcal{U}_D(t)\\ &= \exp\left(-it\begin{pmatrix} 0 & i \\ -i(D_s^2+m^2) & 0\end{pmatrix} - i\ln t\begin{pmatrix} 0 & -i\frac{m^2M}{|D_s|}(D_s^2+m^2)^{-1/2} \\ i\frac{m^2M}{|D_s|}(D_s^2+m^2)^{1/2} & 0\end{pmatrix}\right).\end{aligned}$$

Concerning the situation near the horizon we note that $(KG_{(-,2)})$ possesses an explicit solution. Let $c_1 := \frac{a}{r_+^2+a^2}$. For $u_0, u_1 \in C_0^\infty(\mathbb{R} \times S^2_\omega)$ the solution $(u(t), \partial_t u(t))$ of $(KG_{(-,2)})$ is given by :

$$\begin{aligned} u(t) &= \frac{1}{2}(u_0(s+t,\theta,\phi - c_1 t) + u_0(s-t,\theta,\phi-c_1 t)) \\ &+ \frac{1}{2}\int_{s-t}^{s+t} u_1(\tau,\theta,\phi-c_1\tau) + c_1\partial_\phi u_0(\tau,\theta,\phi-c_1\tau)d\tau. \end{aligned} \tag{4.3.8}$$

For $(v_{-,0}, v_{-,1}) \in \mathcal{E}_{(-,2)}$ we note $\mathcal{U}_{(-,2)}(t)(v_{-,0}, v_{-,1})$ the solution of $(KG_{(-,2)})$ with initial data $(v_{-,0}, v_{-,1})$. We have the

Theorem 4.3.3. *(i) For all* $(v_{(-,0)}, v_{(-,1)}) \in \mathcal{G}^{\phi,f}_{(-,2),N}$ *and all* $(v_{(+,0)}, v_{(+,1)}) \in \mathcal{G}^{\phi,f}_{(+,2),N}$, *there exist solutions* $u^+_-, u^+_+ \in C([0,\infty[\,; \mathcal{G}^{\phi}_N)$ *of* (KG) *s.t.*

$$||u^+_-(t) - j_-\mathcal{U}_{(-,2)}(t)(v_{(-,0)}, v_{(-,1)})||_{\mathcal{E}} \to 0 \quad (t \to +\infty) \quad \textit{and} \tag{4.3.9}$$

$$||u^+_+(t) - j_+\mathcal{U}_D(t)(v_{(+,0)}, v_{(+,1)})||_{\mathcal{E}} \to 0 \quad (t \to +\infty). \tag{4.3.10}$$

(ii) Let $(u_0, u_1) \in \mathcal{G}^{\phi}_N$ *and* $u(t)$ *be the solution of* (KG) *with initial data* (u_0, u_1). *Then there exist* $(v^+_{(-,0)}, v^+_{(-,1)}) \in \mathcal{G}^{\phi}_{(-,2),N}$ *and* $(v^+_{(+,0)}, v^+_{(+,1)}) \in \mathcal{G}^{\phi}_{(+,2),N}$ *such that*

$$||\mathcal{U}_{(-,2)}(t)(v^+_{(-,0)}, v^+_{(-,1)}) - j_-u(t)||_{\mathcal{E}_{(-,2)}} \to 0 \quad (t \to +\infty), \tag{4.3.11}$$

$$||\mathcal{U}_D(t)(v^+_{(+,0)}, v^+_{(+,1)}) - j_+u(t)||_{\mathcal{E}_{(+,2)}} \to 0 \quad (t \to +\infty). \tag{4.3.12}$$

Chapter 5

Scattering of massless Dirac fields by a Kerr black hole

5.1 Introduction

In this chapter we show asymptotic completeness for massless Dirac fields in the Kerr geometry. We use two types of comparison dynamics; asymptotic profiles and Dirac type comparison dynamics. We obtain the existence of an asymptotic velocity and describe its spectrum. There are two main ingredients of the proof : a new Newman-Penrose tetrad which is particularly well-adapted to the time dependent scattering problem and a Mourre estimate. The geometry near the horizon requires us to apply a unitary transformation before we find ourselves in a situation where the generator of dilations is a good conjugate operator. The chapter is organized as follows:

- In Section 5.2 we introduce the new Newman-Penrose tetrad.
- Section 5.3 summarizes the main results of this chapter.
- In Section 5.4 we give some elements of the proof.
- The results are reinterpreted geometrically in Section 5.5

The present chapter summarizes results obtained in collaboration with Jean-Philippe Nicolas (see [62], [63]).

5.2 A new Newman-Penrose tetrad

The tetrad which is the most frequently used is Kinnersley's tetrad (see [73]). The choice of this tetrad comes from the type D structure of the Kerr space-time. The two real vectors are chosen along the two principal null directions V^+ and V^-:

$$l^a \frac{\partial}{\partial x^a} = \lambda V^+ \, , \; n^a \frac{\partial}{\partial x^a} = \mu V^- \tag{5.2.1}$$

and the normalization condition $l_a n^a = 1$ gives

$$\lambda \, \mu \, g \left(V^+, V^- \right) = 1 \, ,$$

thus,

$$\lambda \, \mu \, \frac{2\rho^2}{\Delta} = 1 \, .$$

Kinnersley's choice was to take $\lambda = 1$. Once the directions l^a and n^a are chosen, the complex vector field m^a is uniquely determined, modulo a phase factor $e^{i\theta}$. We will take here $\lambda = \mu$ rather $\lambda = 1$ in order to have an equivalent behavior of the vectors l^a, n^a close to the horizon. We obtain :

$$l^a \frac{\partial}{\partial x^a} = \frac{1}{\sqrt{2\Delta\rho^2}} \left[\left(r^2 + a^2 \right) \frac{\partial}{\partial t} + \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right] , \tag{5.2.2}$$

$$n^a \frac{\partial}{\partial x^a} = \frac{1}{\sqrt{2\Delta\rho^2}} \left[\left(r^2 + a^2 \right) \frac{\partial}{\partial t} - \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right] , \tag{5.2.3}$$

$$m^a \frac{\partial}{\partial x^a} = \frac{1}{p\sqrt{2}} \left[ia \sin\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi} \right] , \tag{5.2.4}$$

$$\bar{m}^a \frac{\partial}{\partial x^a} = \frac{1}{\bar{p}\sqrt{2}} \left[-ia \sin\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} - \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi} \right] , \tag{5.2.5}$$

where $p = r + ia\cos\theta$. We will use the Regge-Wheeler type coordinate and define the density spinor :

$$\tilde{\phi}_A = \left(\frac{\Delta \, \sigma^2 \rho^2}{\left(r^2 + a^2 \right)^2} \right)^{1/4} \phi_A \, . \tag{5.2.6}$$

Then the Weyl equation in the Kerr metric can be written :

$$\partial_t \tilde{\Phi} = i \not{D}_K^K \tilde{\Phi}, \tag{5.2.7}$$

$$\not{D}_K^K = M_r D_r + M_{S^2} \not{D}_{S^2} + M_\varphi D_\varphi + M_P . \tag{5.2.8}$$

Here $\tilde{\Phi}$ is the vector of components of the spinor $\tilde{\phi}_A$ in the spin frame corresponding to the above tetrad. We will specify only the term which is awkward for the scattering theory.

$$M_{S^2} = \begin{pmatrix} \frac{(r^2+a^2)\sqrt{\Delta\rho^2}}{p\sigma^2} & \frac{ia\Delta\sin\theta}{\sigma^2} \\ -\frac{ia\Delta\sin\theta}{\sigma^2} & \frac{(r^2+a^2)\sqrt{\Delta\rho^2}}{\bar{p}\sigma^2} \end{pmatrix} \simeq \frac{1}{r} \, \mathrm{Id}_2, \; r \to +\infty.$$

At positive infinity ($r \to +\infty$) we have to compare $M_{S^2} \not{D}_{S^2}$ to $\frac{1}{r} \not{D}_{S^2}$. Note that the short range condition is :

$$(M_{S^2} - \frac{1}{r}) \not{D}_{S^2} \in O(r^{-1-\epsilon} (\frac{1}{r} \not{D}_{S^2})), \; \epsilon > 0, \quad \text{i.e.} \quad M_{S^2} - \frac{1}{r} \in O(r^{-2-\epsilon}),$$

which is not satisfied. More generally, there is no spherically symmetric matrix M_0 such that :

$$(M_{S^2} - M_0) \in \mathcal{O}(r^{-2-\epsilon}).$$

Note that this is not linked to the choice of λ, μ in (5.2.1).

To solve this problem we will choose another tetrad.

Once the Newman-Penrose tetrad chosen, the vector field $l^a + n^a$ is a future pointing timelike vector field. By the choice of the foliation Σ_t we fix a future pointing timelike vector field, which is the future normal to Σ_t. The idea is to put

$$\mathbf{l}^a + \mathbf{n}^a = T^a.$$

We will call such tetrads adapted to the foliation (see [87]). We single out a pair of null vectors that are not accelerated in the angular directions; i.e. $\mathbf{l}^a$ and $\mathbf{n}^a$ are in the plane spanned by T^a, ∂_r. Requiring that $\mathbf{l}^a$ should be outgoing, $\mathbf{n}^a$ incoming, and a similar behavior of the two vectors near the horizon, we obtain (the choice of $\mathbf{m}^a$ is now imposed, except for the freedom of a complex factor of modulus 1) :

$$\begin{aligned}
\mathbf{l}^a \frac{\partial}{\partial x^a} &= \frac{1}{2} T^a \frac{\partial}{\partial x^a} + \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r} = \frac{\sigma}{\sqrt{2\Delta\rho^2}} \left(\frac{\partial}{\partial t} + \frac{2aMr}{\sigma^2} \frac{\partial}{\partial \varphi} \right) + \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r}, \\
\mathbf{n}^a \frac{\partial}{\partial x^a} &= \frac{1}{2} T^a \frac{\partial}{\partial x^a} - \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r} = \frac{\sigma}{\sqrt{2\Delta\rho^2}} \left(\frac{\partial}{\partial t} + \frac{2aMr}{\sigma^2} \frac{\partial}{\partial \varphi} \right) - \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r}. \\
\mathbf{m}^a \frac{\partial}{\partial x^a} &= \frac{1}{\sqrt{2\rho^2}} \left(\frac{\partial}{\partial \theta} + \frac{\rho^2}{\sigma} \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi} \right), \\
\bar{\mathbf{m}}^a \frac{\partial}{\partial x^a} &= \frac{1}{\sqrt{2\rho^2}} \left(\frac{\partial}{\partial \theta} - \frac{\rho^2}{\sigma} \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi} \right).
\end{aligned}$$

A first test to decide if this is a good choice of the Newman-Penrose tetrad consists of computing the conserved quantity. We find :

$$T^{\mathbf{AA'}} = \begin{pmatrix} \mathbf{n}_a T^a & -\bar{\mathbf{m}}_a T^a \\ -\mathbf{m}_a T^a & \mathbf{l}_a T^a \end{pmatrix} = I,$$

which is indeed the simplest possible expression. We compute the expression of the Weyl equation using the new tetrad :

$$\begin{aligned}
\partial_t \Psi &= i \not{D}_K \Psi, \\
\not{D}_K &= h \not{D}_0 h + V_\varphi D_\varphi + V, \\
\not{D}_0 &= \gamma D_{r_*} + g(r_*) \not{D}_{S^2} + f(r_*) D_\varphi, \\
g(r_*) &= \frac{\sqrt{\Delta}}{r^2 + a^2}, \; f(r_*) = -\frac{2Mra}{(r^2+a^2)^2}, \\
h &= \sqrt{\frac{r^2 + a^2}{\sigma}}, \; \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{5.2.9}$$

The potentials V_φ, V are short range potentials and we have :

$$h^2 - 1 \in \begin{cases} \mathcal{O}(\langle r_* \rangle^{-2}), \; r_* \to +\infty, \\ \mathcal{O}(e^{2\kappa_+ r_*}), \; r_* \to -\infty. \end{cases}$$

The exact expression for the potential can be found in [63]. In a first step, we will compare $\not{D}_K$ to $\not{D}_0$, the difference is short range after diagonalization in $D_\varphi = n$. In a second step, we compare $\not{D}_0$ to asymptotic hamiltonians. In this second step, the angular part can be treated as a potential, thanks to the spherical symmetry of the operators. Thus the problem that existed in the expression of the Weyl equation in Kinnersley's tetrad no longer exists in this new tetrad. The vector Ψ of the components of the spinor $\tilde{\phi}_A$ in the spin frame corresponding to the new tetrad is linked to the vector $\tilde{\Phi}$ by a unitary transformation :

$$\Psi = \mathbf{U}\tilde{\Phi}, \quad \mathbf{U} = \sqrt{\frac{p}{2\sigma\rho}} \begin{pmatrix} \sqrt{\sigma_+} & -\frac{\bar{p}}{\rho}\frac{ia\sin\theta\sqrt{\Delta}}{\sqrt{\sigma_+}} \\ \frac{ia\sin\theta\sqrt{\Delta}}{\sqrt{\sigma_+}} & \frac{\bar{p}}{\rho}\sqrt{\sigma_+} \end{pmatrix},$$

where $\sigma_+ = \sigma + r^2 + a^2$.

5.3 Principal results

In this section we present the principal results of this chapter. Let $\theta_0 \in C^\infty(\mathbb{R})$ be zero in a neighborhood of 0 and 1 away from the origin, $\theta_1(x) = \mathbf{1}_{\mathbb{R}^+}(x)\theta_0(x)$. We define :

$$\begin{aligned} \mathcal{H} &= L^2((\mathbb{R}\times S^2); dr_* d\omega); \mathbb{C}^2), \\ \mathbb{D}_H &= \gamma D_{r_*} - \frac{a}{r_+^2 + a^2} D_\varphi, \\ \mathbb{D}_\infty &= \gamma D_{r_*}, \\ \not{D}_H &= \gamma D_{r_*} + e^{-\kappa_+ |r_*| \theta_0(r_*)} \not{D}_{S^2} - \frac{a}{r_+^2 + a^2} D_\varphi, \\ \not{D}_\infty &= \gamma D_{r_*} + \frac{\theta_1(r_*)}{|r_*|} \not{D}_{S^2}. \end{aligned}$$

$\mathbb{D}_H$ and $\mathbb{D}_\infty$ have the same spaces of incoming (resp. outgoing) solutions :

$$\mathcal{H}^- = \{(\psi_0, 0) \in \mathcal{H}\} \, (\text{resp.}\, \mathcal{H}^+ = \{(0, \psi_1) \in \mathcal{H}\}).$$

$\not{D}_K, \not{D}_H, \not{D}_\infty$ are selfadjoint and their spectra are purely absolutely continuous. The absence of eigenvalues for $\not{D}_K$ follows from a result about separation of variables (see [97]). The first result concerns the asymptotic velocity :

Theorem 5.3.1 (Asymptotic velocity)**.** *There exist bounded operators $P^\pm$, $P_H^\pm$, $P_\infty^\pm$ such that, for all $J \in \mathcal{C}_\infty(\mathbb{R})$:*

$$\begin{aligned} J(P^\pm) &= s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} J\left(\frac{r_*}{t}\right) e^{it\not{D}_K}, \\ J(P_H^\pm) &= s - \lim_{t\to\pm\infty} e^{-it\not{D}_H} J\left(\frac{r_*}{t}\right) e^{it\not{D}_H}, \\ J(P_\infty^\pm) &= s - \lim_{t\to\pm\infty} e^{-it\not{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\not{D}_\infty}, \\ J(\mp\gamma) &= s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_H} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_H} = s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty}. \end{aligned}$$

In addition we have :

$$\sigma(P^+) = \sigma(P_H^+) = \sigma(P_\infty^+) = \{-1, 1\}.$$

Theorem 5.3.2 (Asymptotic profiles)**.** *i) The classical wave operators defined by the strong limits*

$$\mathfrak{W}_H^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\mathbb{D}_H} P_{\mathcal{H}^{\mp}} \,, \tag{5.3.1}$$

$$\mathfrak{W}_\infty^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\mathbb{D}_\infty} P_{\mathcal{H}^{\pm}} \,, \tag{5.3.2}$$

$$\tilde{\mathfrak{W}}_H^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_H} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^-}(P^{\pm}) \,, \tag{5.3.3}$$

$$\tilde{\mathfrak{W}}_\infty^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_\infty} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^+}(P^{\pm}) \tag{5.3.4}$$

exist and satisfy :

$$\tilde{\mathfrak{W}}_H^{\pm} = \left(\mathfrak{W}_H^{\pm}\right)^* \,,\ \tilde{\mathfrak{W}}_\infty^{\pm} = \left(\mathfrak{W}_\infty^{\pm}\right)^* \,,$$
$$\tilde{\mathfrak{W}}_H^{\pm}\mathfrak{W}_H^{\pm} + \tilde{\mathfrak{W}}_\infty^{\pm}\mathfrak{W}_\infty^{\pm} = \mathfrak{W}_H^{\pm}\tilde{\mathfrak{W}}_H^{\pm} + \mathfrak{W}_\infty^{\pm}\tilde{\mathfrak{W}}_\infty^{\pm} = \mathrm{Id}_{\mathcal{H}} \,,$$
$$\mathrm{Ker}\left(\mathfrak{W}_H^{\pm}\right) = \mathcal{H}^{\pm} \,,\ \ \mathrm{Ker}\left(\mathfrak{W}_\infty^{\pm}\right) = \mathcal{H}^{\mp} \,,\ \ \mathrm{ran}\left(\tilde{\mathfrak{W}}_H^{\pm}\right) = \mathcal{H}^{\mp} \,,\ \ \mathrm{ran}\left(\tilde{\mathfrak{W}}_\infty^{\pm}\right) = \mathcal{H}^{\pm} \,.$$

ii) If we put :

$$\begin{aligned} W^+ : \quad & \mathcal{H}^- \oplus \mathcal{H}^+ \longrightarrow \mathcal{H} \,, \\ & ((\psi_0, 0), (0, \psi_1)) \longmapsto \mathfrak{W}_H^{+}(\psi_0, 0) + \mathfrak{W}_\infty^{+}(0, \psi_1) \,, \end{aligned} \tag{5.3.5}$$

$$\begin{aligned} W^- : \quad & \mathcal{H}^+ \oplus \mathcal{H}^- \longrightarrow \mathcal{H} \\ & ((0, \psi_1), (\psi_0, 0)) \longmapsto \mathfrak{W}_H^{-}(0, \psi_1) + \mathfrak{W}_\infty^{-}(\psi_0, 0) \,, \end{aligned} \tag{5.3.6}$$

$$\tilde{W}^+ : \mathcal{H} \longrightarrow \mathcal{H}^- \oplus \mathcal{H}^+ \,,\ \tilde{W}^+\Psi = \left(\tilde{\mathfrak{W}}_H^{+}\Psi \,,\ \tilde{\mathfrak{W}}_\infty^{+}\Psi\right) \,, \tag{5.3.7}$$

$$\tilde{W}^- : \mathcal{H} \longrightarrow \mathcal{H}^+ \oplus \mathcal{H}^- \,,\ \tilde{W}^-\Psi = \left(\tilde{\mathfrak{W}}_H^{-}\Psi \,,\ \tilde{\mathfrak{W}}_\infty^{-}\Psi\right) \,, \tag{5.3.8}$$

then $W^{\pm}$ are isometries and satisfy :

$$\tilde{W}^+W^+ = \mathrm{Id}_{\mathcal{H}^-\oplus\mathcal{H}^+} \,,\ \tilde{W}^-W^- = \mathrm{Id}_{\mathcal{H}^+\oplus\mathcal{H}^-} \,,\ W^+\tilde{W}^+ = W^-\tilde{W}^- = \mathrm{Id}_{\mathcal{H}} \,.$$

Theorem 5.3.3 (Dirac-type comparison dynamics)**.** *The classical wave operators defined by the strong limits :*

$$\Omega_H^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\not{D}_H} \mathbf{1}_{\mathbb{R}^-}(P_H^{\pm}) \,, \tag{5.3.9}$$

$$\Omega_\infty^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\not{D}_\infty} \mathbf{1}_{\mathbb{R}^+}(P_\infty^{\pm}) \,, \tag{5.3.10}$$

$$\tilde{\Omega}_H^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_H} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^-}(P^{\pm}) \,, \tag{5.3.11}$$

$$\tilde{\Omega}_\infty^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_\infty} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^+}(P^{\pm}) \,, \tag{5.3.12}$$

exist and satisfy :

$$\tilde{\Omega}_H^{\pm} = \left(\Omega_H^{\pm}\right)^* \,,\ \tilde{\Omega}_\infty^{\pm} = \left(\Omega_\infty^{\pm}\right)^* \,,$$
$$\tilde{\Omega}_H^{\pm}\Omega_H^{\pm} + \tilde{\Omega}_\infty^{\pm}\Omega_\infty^{\pm} = \Omega_H^{\pm}\tilde{\Omega}_H^{\pm} + \Omega_\infty^{\pm}\tilde{\Omega}_\infty^{\pm} = \mathrm{Id}_{\mathcal{H}} \,.$$

5.4 Elements of the proof

In this section we will explain some ideas of the proof. We start with proving a version of Theorem 5.3.3 using cut-off functions, more precisely we will show the existence of the limits

$$s-\lim_{t\to\infty} e^{-it\not{D}_H} j_- e^{it\not{D}_K} \quad \text{etc.,}$$

where j_- is a smooth function equal to 1 for $r_* \leq -1$ and zero for $r_* \geq 1$. Suppose for a moment that this result is already established. We deduce a version of Theorem 5.3.2 with cut-off functions using also [86]. This gives us Theorem 5.3.1. Indeed the proof is elementary for the dynamics $\mathbb{D}_H$ and $\mathbb{D}_\infty$. For the other dynamics we use the asymptotic completeness results already established. Theorems 5.3.2 and 5.3.3 follow.

That means that the key point is to prove a version of Theorem 5.3.3 with cut-off functions. We first compare $\not{D}_K$ with the intermediate spherically symmetric operator $\not{D}_0$. In a second step we compare $\not{D}_0$ with $\not{D}_H, \not{D}_\infty$. This second step is easy. After diagonalization we are concerned with a one dimensional Dirac operator with potential. We therefore concentrate on the first step. The proof uses a Mourre estimate. The evolution can be described as the evolution on a riemannian manifold with two ends. We will only consider the end corresponding to the horizon of the black hole, where the construction of the conjugate operator is the most difficult. This end is asymptotically hyperbolic. The toy model is the following :

$$H = \gamma D_r + e^{\eta r}\not{D}_{S^2} + C \quad \mathbb{R}_- \times S^2;\ \eta > 0, C \in \mathbb{R}.$$

We first treat the case $C = 0$. The conjugate operator which is the most used is the generator of dilations $A = \frac{1}{2}(rD_r + D_r r)$. We try

$$[iH, A] = \gamma D_r - \eta r e^{\eta r}\not{D}_{S^2}$$

and see that almost none of the conditions appearing in Mourre theory are fulfilled ! Thanks to the spherical symmetry we can diagonalize and use spin weighted spherical harmonics :

$$H^{nl} = \gamma D_r + e^{\eta r}(l + \frac{1}{2})\tau,$$

where τ is a constant matrix. Now we can establish a Mourre estimate because $e^{\eta r}\tau$ is a potential which is "very short range". We can use e.g. that $e^{\eta r}\tau(l + \frac{1}{2})\chi(H^{nl})$ is compact for $\chi \in C_0^\infty(\mathbb{R})$. On the one hand, in cases like the Kerr metric, where the hamiltonian is not spherically symmetric, we cannot proceed in this manner. On the other hand, we will show that the Mourre estimate that one can establish for the intermediate spherically symmetric hamiltonian is in fact uniform in n, l. To do so, we introduce the following unitary transformation :

$$\begin{aligned} U &:= e^{\eta^{-1} i D_r \ln|\not{D}_{S^2}|}, \\ \hat{H} &= U^* H U = \gamma D_r + e^{\eta r}\frac{\not{D}_{S^2}}{|\not{D}_{S^2}|}, \\ \hat{H}^{nl} &= \gamma D_r + e^{\eta r}\tau. \end{aligned}$$

By the preceding arguments $A^{nl} = \frac{1}{2}(rD_r + D_r r)$ is a good conjugate operator for $\hat{H}^{nl}$. The Mourre estimate that we obtain for $(\hat{H}^{nl}, A^{nl})$ is uniform in n, l because the two operators do not depend on n, l ! We therefore choose $\hat{A} = UAU^*$ as conjugate operator. The case where C is different from zero is on each spherical harmonics similar to the case of the charged Dirac equation in the Reissner-Nordstroem metric. In [80] the conjugate operator A is replaced by $A + \gamma Cr$ and we modify A here in the same manner. We then conjugate with the unitary transform to obtain the good conjugate operator.

Remark 5.4.1. *$\hat{A}$ is similar to an operator introduced by Froese, Hislop for the Laplacian on riemannian manifolds with an asymptotically hyperbolic end (see [50]). They write the Laplacian as a sum of two operators, one part in r and one part on the sphere. They then explicitly use the positivity of each part. Thus we cannot apply their argument to the Dirac case.*

5.5 Geometric interpretation

5.5.1 Theorem 5.3.2 in terms of principal null geodesics

Let $P_N = \gamma D_{r_*} - \frac{a}{r^2+a^2} D_\varphi$. The operator P_N is linked to the null vector fields $v^\pm$, which are generators of the principal null geodesics, normalized so that the flow preserves the foliation $\{\Sigma_t\}_t$:

$$v^\pm = \frac{\Delta}{r^2+a^2} V^\pm = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial r_*} + \frac{a}{r^2+a^2} \frac{\partial}{\partial \varphi}.$$

We have :

Theorem 5.5.1. *The strong limits :*

$$\mathfrak{W}^\pm_{H,pn} := s - \lim_{t \to \pm\infty} e^{-it\not{D}_K} e^{it\mathbf{P_N}} P_{\mathcal{H}^\mp} , \tag{5.5.1}$$

$$\mathfrak{W}^\pm_{\infty,pn} := s - \lim_{t \to \pm\infty} e^{-it\not{D}_K} e^{it\mathbf{P_N}} P_{\mathcal{H}^\pm} , \tag{5.5.2}$$

$$\tilde{\mathfrak{W}}^\pm_{H,pn} := s - \lim_{t \to \pm\infty} e^{-it\mathbf{P_N}} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^-}(P^\pm) , \tag{5.5.3}$$

$$\tilde{\mathfrak{W}}^\pm_{\infty,pn} := s - \lim_{t \to \pm\infty} e^{-it\mathbf{P_N}} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^+}(P^\pm) \tag{5.5.4}$$

$$\tag{5.5.5}$$

exist and satisfy the same properties as the wave operators of Theorem 5.3.2. The corresponding global wave operators are :

$$\begin{aligned} W^+_{pn} : \quad & \mathcal{H}^- \oplus \mathcal{H}^+ \longrightarrow \mathcal{H} , \\ & ((\psi_0, 0), (0, \psi_1)) \longmapsto \mathfrak{W}^+_{H,pn} (\psi_0, 0) + \mathfrak{W}^+_{\infty,pn} (0, \psi_1) , \end{aligned} \tag{5.5.6}$$

$$\begin{aligned} W^-_{pn} : \quad & \mathcal{H}^+ \oplus \mathcal{H}^- \longrightarrow \mathcal{H} \\ & ((0, \psi_1), (\psi_0, 0)) \longmapsto \mathfrak{W}^-_{H,pn} (0, \psi_1) + \mathfrak{W}^-_{\infty,pn} (\psi_0, 0) . \end{aligned} \tag{5.5.7}$$

$$\tilde{W}^+_{pn} : \mathcal{H} \longrightarrow \mathcal{H}^- \oplus \mathcal{H}^+ , \; \tilde{W}^+_{pn} \Psi = \left(\tilde{\mathfrak{W}}^+_{H,pn} \Psi , \; \tilde{\mathfrak{W}}^+_{\infty,pn} \Psi \right) , \tag{5.5.8}$$

$$\tilde{W}^-_{pn} : \mathcal{H} \longrightarrow \mathcal{H}^+ \oplus \mathcal{H}^- , \; \tilde{W}^-_{pn} \Psi = \left(\tilde{\mathfrak{W}}^-_{H,pn} \Psi , \; \tilde{\mathfrak{W}}^-_{\infty,pn} \Psi \right) . \tag{5.5.9}$$

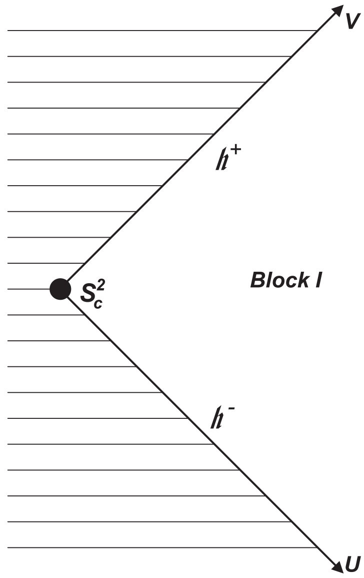


Figure 5.1: $B_I^{KBL} = [0, \infty[_U \times [0, \infty[_V \times S^2_{\theta,\varphi^\sharp}$.

5.5.2 Inverse wave operators at the horizon as trace operators

As already mentioned in the introduction, the singularities for $\Delta = 0$ are coordinate singularities. To see this, we introduce coordinates $(t^*, r, \theta, \varphi^*)$ which are called Kerr-* coordinates. They are chosen so that the principal incoming null coordinates are given by :

$$\dot{r} = -1, \dot{t^*} = \dot{\varphi^*} = \dot{\theta} = 0.$$

An explicit calculation shows that g can be extended in a smooth way to $\{r = r_+\}$ and we define the future event horizon :

$$\mathfrak{H}^+ := \mathbb{R}_{t^*} \times \{r = r_+\} \times S^2_{\theta,\varphi^*}.$$

This is a null hypersurface. The *-Kerr coordinates $({}^*t, r, \theta, {}^*\varphi)$ are constructed in the same manner using the outgoing principal null geodesics. This gives an analogous definition of the past event horizon ($\mathfrak{H}^-$). In order to put the two coordinate systems together, we define Kruskal-Boyer-Lindquist coordinates : $U = e^{-\kappa_+ {}^*t}$, $V = e^{\kappa_+ t^*}$, $\varphi^\sharp = \varphi - \frac{a}{r_+^2 + a^2} t$. The Kruskal-Boyer-Lindquist coordinates give us a global description of the horizon $\mathfrak{H} := ([0, \infty[_U \times \{0\}_V \times S^2_{\theta,\varphi^\sharp}) \cup (\{0\}_U \times [0, \infty[_V \times S^2_{\theta,\varphi^\sharp})$ (see Figure 5.1). All these coordinate systems are explained in detail in [88].

We can apply Leray's theorem to see that the spinor Φ_A solution of (2.2.2) for the Kerr metric has a trace on $\mathfrak{H}^\pm$. By explicit calculation we show that the limit $\lim_{r \to r_+} \Psi_0(\gamma^-_{V,\theta,\varphi^\sharp}(r)) =: \Psi_0|_{\mathfrak{H}^+}(0, V, \theta, \varphi^\sharp)$ exists and that

$$\lim_{r \to r_+} \Psi_1(\gamma^-_{V,\theta,\varphi^\sharp}(r)) = 0,$$

where Ψ is solution of (5.2.9) and $\gamma^-_{V,\theta,\varphi^\sharp}$ denotes the principal incoming null geodesic that encounters $\mathfrak{H}^+$ at the point $(0, V, \theta, \varphi^\sharp)$. This permits to define a trace operator :

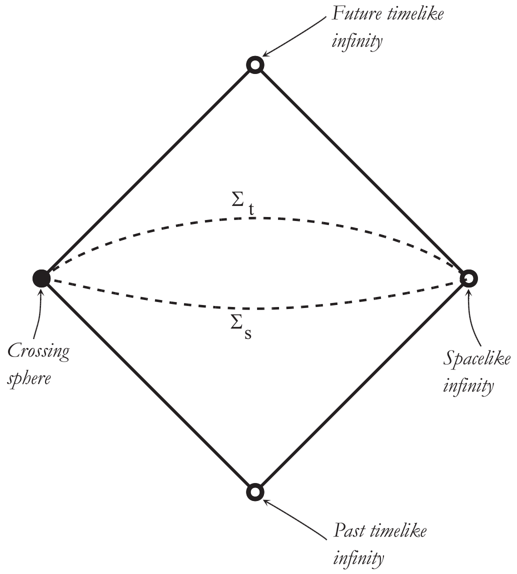


Figure 5.2: Penrose compactification of block I

Definition 5.5.1.

$$\mathcal{T}_{\mathfrak{H}}^{+} : \begin{array}{ccc} C_0^\infty(\Sigma_0, \mathbb{C}^2) & \to & C^\infty(\mathfrak{H}^+, \mathbb{C}) \\ \Psi_{\Sigma_0} & \mapsto & \Psi_0|_{\mathfrak{H}^+}. \end{array}$$

$\mathcal{T}_{\mathfrak{H}}^{-}$ is defined in an analogous manner using outgoing geodesics. We also define a diffeomorphism $\mathfrak{F}_{\mathfrak{H}}^{\pm} : \mathfrak{H}^{\pm} \to \Sigma_0$ identifying points along incoming (resp. outgoing) principal null geodesics and $\mathcal{H}_{\mathfrak{H}^\pm} := L^2(\mathfrak{H}^\pm, dVol_{\mathfrak{H}^\pm})$, where $dVol_{\mathfrak{H}^\pm}$ is the pull back of $dr_* d\omega$ by $\mathfrak{F}_{\mathfrak{H}}^{\pm}$.

Theorem 5.5.2. *$\mathcal{T}_{\mathfrak{H}}^{\pm}$ can be extended to a bounded operator from $\mathcal{H}$ to $\mathcal{H}_{\mathfrak{H}^\pm}$ and this extension is the pull-back of the inverse wave operator $\tilde{\mathfrak{W}}_{H,pn}^{\pm}$ by the diffeomorphism $\mathfrak{F}_{\mathfrak{H}}^{\pm}$:*

$$\mathcal{T}_{\mathfrak{H}}^{\pm} = (\mathfrak{F}_{\mathfrak{H}}^{\pm})^* \tilde{\mathfrak{W}}_{H,pn}^{\pm}.$$

5.5.3 Inverse wave operators at infinity as trace operators

We define $\Omega := \frac{1}{r}$. We have : $\mathcal{B}_I = \mathbb{R}_{t^*} \times]0, \frac{1}{r_+}[_\Omega \times S^2_{\theta,\varphi^*}$ and we put : $\hat{g} := \Omega^2 g$. An explicit calculation shows that $\hat{g}$ can be extended smoothly to $\mathbb{R}_{t^*} \times [0, \frac{1}{r_+}]_\Omega \times S^2_{\theta,\varphi^*}$ and we define past null infinity as $\mathcal{I}^- := \mathbb{R}_{t^*} \times \{\Omega = 0\} \times S^2_{\theta,\varphi^*}$. This is a smooth null hypersurface for $\hat{g}$. We define future null infinity $\mathcal{I}^+$ in the same manner using *-Kerr coordinates. The Penrose compactification of block I is defined as

$$(\bar{\mathcal{B}}_I, \hat{g}),\ \bar{\mathcal{B}}_I = \mathcal{B}_I \cup \mathfrak{H}^+ \cup \mathfrak{H}^- \cup S^2_c \cup \mathcal{I}^- \cup \mathcal{I}^+.$$

In spite of the terminology used, the compactified space-time is not compact. There are three "points" of the boundary that are missing : future and past timelike infinity as well as spacelike infinity (see Figure 5.2).

Remark 5.5.1. *The Weyl equation is conformally invariant :*

$$\hat{\nabla}^{AA'}\hat{\phi}_A = 0, \text{ where } \quad \hat{\phi}_A = \Omega^{-1}\phi_A$$

and $\hat{\nabla}$ is the covariant derivative associated with the rescaled metric $\hat{g}$.

We now follow the same procedure as at the horizon to define $\mathcal{T}_{\mathfrak{I}}^{\pm}, \mathfrak{F}_{\mathfrak{I}}^{\pm}$ and $\mathcal{H}_{\mathfrak{I}^{\pm}}$. We obtain :

Theorem 5.5.3. *The operator $\mathcal{T}_{\mathfrak{I}}^{\pm}$ can be extended to a bounded operator from $\mathcal{H}$ to $\mathcal{H}_{\mathfrak{I}^{\pm}}$ and we have : $\mathcal{T}_{\mathfrak{I}}^{\pm} = (\mathfrak{F}_{\mathfrak{I}}^{\pm})^{*}\tilde{\mathfrak{W}}_{\infty,pn}^{\pm}$.*

5.5.4 The Goursat problem

We can now solve the Goursat problem in the compactification of block I. We define :

$$\Pi_F : \begin{array}{ll} \mathcal{H} & \to \mathcal{H}_{\mathfrak{H}^+} \oplus \mathcal{H}_{\mathfrak{I}^+} =: \mathcal{H}_F \\ \Psi_{\Sigma_0} & \mapsto (\mathcal{T}_{\mathfrak{H}}^{+}\Psi_{\Sigma_0}, \mathcal{T}_{\mathfrak{I}}^{+}\Psi_{\Sigma_0}). \end{array}$$

Theorem 5.5.4 (Goursat problem)**.** *Π_F is an isomorphism, i.e. for all $\Phi \in \mathcal{H}_F$ there exists a unique $\Psi \in C(\mathbb{R}_t, \mathcal{H})$ solution of (5.2.9) such that $\Phi = \Pi_F\Psi(0)$.*

Chapter 6

Creation of fermions by rotating charged black-holes

6.1 Introduction

It was in 1975 that S. W. Hawking published his famous paper about the creation of particles by black holes (see [64]). Later this effect was analyzed by other authors in more detail (see e.g. [98]) and we can say that the effect was well understood from a physical point of view at the end of the 1970's. From a mathematical point of view, however, fundamental questions linked to the Hawking radiation such as scattering theory for field equations on black-hole space-times were not addressed at that time.

The aim of the present chapter, which summarizes the article [58], is to give a mathematically precise description of the Hawking effect for spin 1/2 fields in the setting of the collapse of a rotating charged star. We show that an observer, who is located far away from the black hole and at rest with respect to the Boyer-Lindquist coordinates, observes the emergence of a thermal state when his proper time t goes to infinity. In the proof we use the results of Chapter 5 as well as their generalizations to the Kerr-Newman case in [33].

Let us give an idea of the theorem describing the effect. Let r_* be the Regge-Wheeler coordinate. We suppose that the boundary of the star is described by $r_* = z(t,\theta)$. The space-time is then given by

$$\mathcal{M}_{col} = \bigcup_t \Sigma_t^{col},\ \Sigma_t^{col} = \{(t, r_*, \omega) \in \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2;\ r_* \geq z(t,\theta)\}.$$

The typical asymptotic behavior of $z(t,\theta)$ is $(A(\theta) > 0,\ \kappa_+ > 0)$:

$$z(t,\theta) = -t - A(\theta)e^{-2\kappa_+ t} + B(\theta) + \mathcal{O}(e^{-4\kappa_+ t}),\ t \to \infty.$$

Let $\mathcal{H}_t = L^2((\Sigma_t^{col}, \mathrm{dVol}); \mathbb{C}^4)$. The Dirac equation can be written as

$$\partial_t \Psi = i\not{D}_t \Psi + \quad \text{boundary condition.} \tag{6.1.1}$$

We will put a MIT boundary condition on the surface of the star. The evolution of the Dirac field is then described by an isometric propagator $U(s,t) : \mathcal{H}_s \to \mathcal{H}_t$. The Dirac equation on the whole exterior Kerr-Newman space-time $\mathcal{M}_{BH}$ will be written as

$$\partial_t \Psi = i\not{D}\Psi.$$

Here $\not{D}$ is a selfadjoint operator on $\mathcal{H} = L^2((\mathbb{R}_{r_*} \times S^2, dr_* d\omega); \mathbb{C}^4)$. There exists an asymptotic velocity operator $P^{\pm}$ s.t. for all continuous functions J with $\lim_{|x|\to\infty} J(x) = 0$ we have

$$J(P^{\pm}) = s - \lim_{t\to\pm\infty} e^{-it\not{D}} J\left(\frac{r_*}{t}\right) e^{it\not{D}}.$$

Let $\mathcal{U}_{col}(\mathcal{M}_{col})$ (resp. $\mathcal{U}_{BH}(\mathcal{M}_{BH})$) be the algebras of observables outside the collapsing body (resp. on the space-time describing the eternal black-hole) generated by $\Psi^*_{col}(\Phi_1)\Psi_{col}(\Phi_2)$ (resp. $\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\Phi_2)$). Here $\Psi_{col}(\Phi)$ (resp. $\Psi_{BH}(\Phi)$) are the quantum spin fields on $\mathcal{M}_{col}$ (resp. $\mathcal{M}_{BH}$). Let ω_{col} be a vacuum state on $\mathcal{U}_{col}(\mathcal{M}_{col})$; ω_{vac} a vacuum state on $\mathcal{U}_{BH}(\mathcal{M}_{BH})$ and $\omega^{\eta,\sigma}_{Haw}$ be a KMS-state on $\mathcal{U}_{BH}(\mathcal{M}_{BH})$ with inverse temperature $\sigma > 0$ and chemical potential $\mu = e^{\sigma\eta}$ (see Section 6.5 for details). For a function $\Phi \in C_0^\infty(\mathcal{M}_{BH})$ we define :

$$\Phi^T(t, r_*, \omega) = \Phi(t - T, r_*, \omega).$$

The theorem about the Hawking effect is the following :

Theorem 6.1.1 (Hawking effect)**.** *Let*

$$\Phi_j \in (C_0^\infty(\mathcal{M}_{col}))^4,\ j = 1, 2.$$

Then we have

$$\begin{aligned}\lim_{T\to\infty} \omega_{col}(\Psi^*_{col}(\Phi_1^T)\Psi_{col}(\Phi_2^T)) &= \omega^{\eta,\sigma}_{Haw}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_2)) \\ &\quad + \omega_{vac}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_2)), \quad (6.1.2)\\ T_{Haw} = 1/\sigma = \kappa_+/2\pi, \quad \mu = e^{\sigma\eta},\ \eta &= \frac{qQr_+}{r_+^2 + a^2} + \frac{aD_\varphi}{r_+^2 + a^2}.\end{aligned}$$

Here q is the charge of the field, Q the charge of the black-hole, a is the angular momentum per unit mass of the black-hole, $r_+ = M + \sqrt{M^2 - (a^2 + Q^2)}$ defines the outer event horizon and κ_+ is the surface gravity of this horizon. The interpretation of (6.1.2) is the following. We start with a vacuum state which we evolve in the proper time of an observer at rest with respect to the Boyer Lindquist coordinates. The limit when the proper time of this observer goes to infinity is a thermal state coming from the event horizon in formation and a vacuum state coming from infinity as expressed on the R.H.S of (6.1.2). The Hawking effect comes from an infinite Doppler effect and the mixing of positive and negative frequencies. To explain this a little bit more, we describe the analytic problem behind the effect. Let $f(r_*, \omega) \in C_0^\infty(\mathbb{R} \times S^2)$. The key result about the Hawking effect is :

$$\begin{aligned}\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(\not{D}_0)U(0,T)f||_0^2 &= \langle \mathbf{1}_{\mathbb{R}^+}(P^-)f, \mu e^{\sigma\not{D}}(1 + \mu e^{\sigma\not{D}})^{-1}\mathbf{1}_{\mathbb{R}^+}(P^-)f\rangle \\ &\quad + ||\mathbf{1}_{[0,\infty)}(\not{D})\mathbf{1}_{\mathbb{R}^-}(P^-)f||^2, \qquad (6.1.3)\end{aligned}$$

where μ, η, σ are as in the above theorem. Equation (6.1.3) implies (6.1.2).

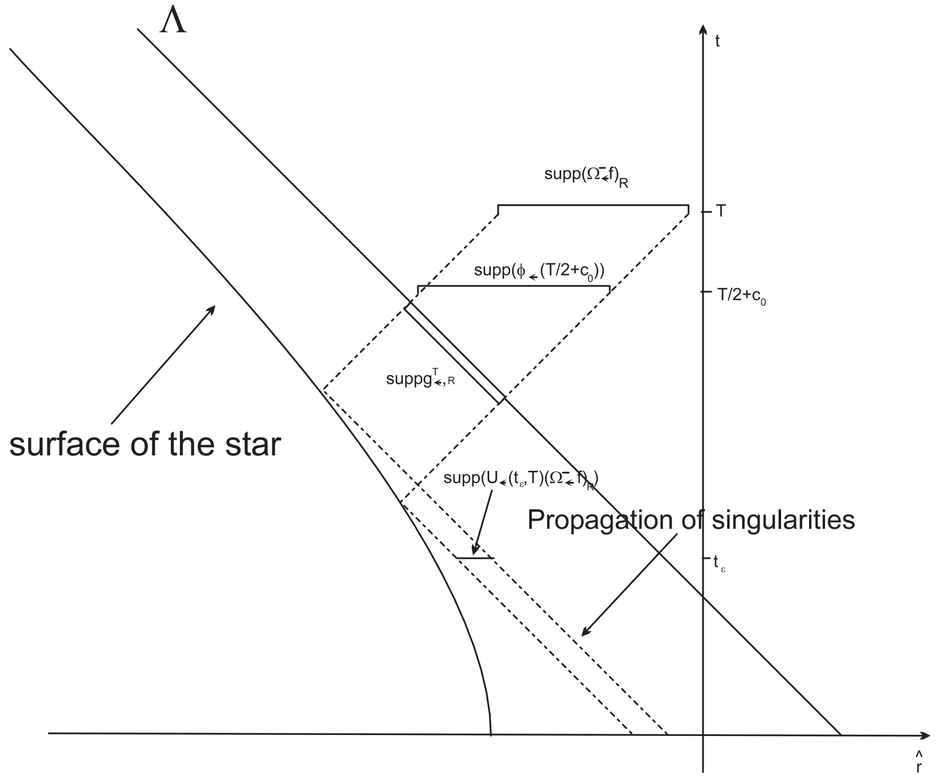


Figure 6.1: The collapse of the star

The term on the L.H.S. comes from the vacuum state we consider. We have to project on the positive frequency solutions (see Section 6.5 for details). Note that in (6.1.3) we consider the time reversed evolution. This comes from the quantization procedure. When time becomes large the solution hits the surface of the star at a point closer and closer to the future event horizon. Figure 6.1 shows the situation for an asymptotic comparison dynamics, which satisfies Huygens' principle. For this asymptotic comparison dynamics the support of the solution concentrates more and more when time becomes large, which means that the frequency increases. The consequence of the change in frequency is that the system does not stay in the vacuum state.

6.2 The analytic problem

Let us consider a model, where the eternal black-hole is described by a static space-time (although the Kerr-Newman space-time is not even stationary, the problem will be essentially reduced to this kind of situation). Then the problem can be described as follows. Consider a riemannian manifold Σ_0 with one asymptotically euclidean end and a boundary. The boundary will move when t becomes large asymptotically with the speed of light. The manifold at time t is denoted Σ_t. The "limit" manifold Σ is a manifold with two ends, one asymptotically euclidean and the other asymptotically hyperbolic (see Figure 6.2). The problem consists in evaluating the limit

$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(\not{D}_0)U(0,T)f||_0,$$

where $U(0,T)$ is the isometric propagator for the Dirac equation on the manifold with moving boundary and suitable boundary conditions and $\not{D}_0$ is the Dirac hamiltonian at time $t=0$. It is worth noting that the underlying scattering theory is not the scattering theory for the problem with moving boundary but the scattering theory on the "limit" manifold. It is largely believed that the result does not depend on the boundary condition. We will show in this chapter that it does not depend on the chiral angle in the MIT boundary condition. Note also that the boundary viewed in $\bigcup_t \{t\} \times \Sigma_t$ is only weakly timelike, a problem that has been rarely considered (but see [9]).

One of the problems for the description of the Hawking effect is to derive a reasonable model for the collapse of the star. We will suppose that the metric outside the collapsing star is always given by the Kerr-Newman metric. Whereas this is a genuine assumption in the rotational case, in the spherically symmetric case Birkhoffs theorem assures that the metric outside the star is the Reissner-Nordström metric. We will suppose that a point on the surface of the star will move along a curve which behaves asymptotically like a timelike geodesic with $L=\mathcal{Q}=\tilde{E}=0$, where L is the angular momentum, $\tilde{E}$ the rotational energy and $\mathcal{Q}$ the Carter constant. The choice of geodesics is justified by the fact that the collapse creates the space-time, i.e. angular momenta and rotational energy should be zero with respect to the space-time. We will need an additional asymptotic condition on the collapse. It turns out that there is a natural coordinate system $(t,\hat{r},\omega)$ associated to the collapse. In this coordinate system the surface of the star is described by $\hat{r}=\hat{z}(t,\theta)$. We need to assume the existence of a constant C s.t.

$$|\hat{z}(t,\theta)+t+C| \to 0,\ t\to\infty. \tag{6.2.1}$$

It can be checked that this asymptotic condition is fulfilled if we use the above geodesics for some appropriate initial condition. On the one hand we are not able to compute this initial condition explicitly, on the other hand it seems more natural to impose a (symmetric) asymptotic condition than an initial condition. If we would allow in (6.2.1) a function $C(\theta)$ rather than a constant, the problem would become more difficult. Indeed one of the problems for treating the Hawking radiation in the rotational case is the high frequencies of the solution. In contrast with the spherically symmetric case, the difference between the Dirac operator and an operator with constant coefficients is near the horizon always a differential operator of order one [1]. This explains that in the high energy regime we are interested in, the Dirac operator is not close to a constant coefficient operator. Our method for proving (6.1.3) is to use scattering arguments to reduce the problem to a problem with a constant coefficient operator, for which we can compute the radiation explicitly. If we do not impose a condition of type (6.2.1), then in all coordinate systems the solution has high frequencies, in the radial as well as in the angular directions. With condition (6.2.1) these high frequencies only occur in the radial direction. Our asymptotic comparison dynamics will differ from the real dynamics only by derivatives in angular directions and by potentials.

[1]In the spherically symmetric case we can diagonalize the operator. After diagonalization the difference is just a potential.

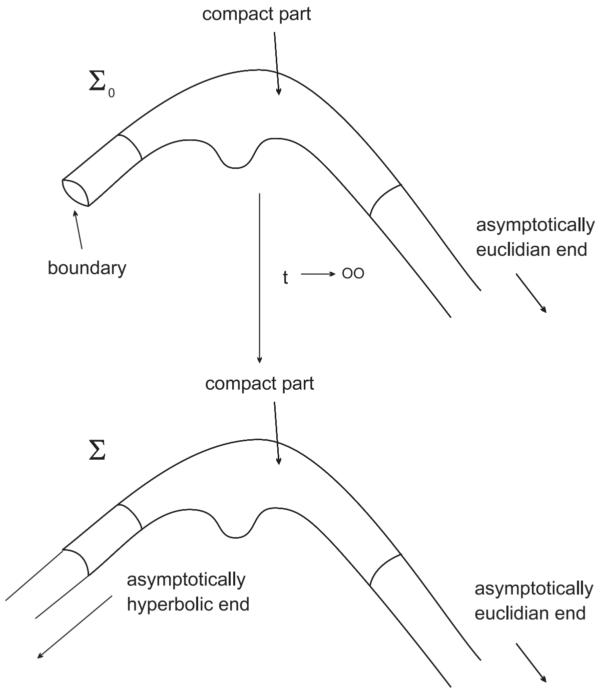


Figure 6.2: The manifold at time $t = 0$ Σ_0 and the limit manifold Σ.

Let us now give some ideas of the proof of (6.1.3). We want to reduce the problem to the evaluation of a limit that can be explicitly computed. To do so, we use the asymptotic completeness results obtained in [63] and [33]. There exists a constant coefficient operator $\not{D}_{\leftarrow}$ s.t. the following limits exist :

$$W^{\pm}_{\leftarrow} := s - \lim_{t\to\pm\infty} e^{-it\not{D}} e^{it\not{D}_{\leftarrow}} \mathbf{1}_{\mathbb{R}^{\mp}}(P^{\pm}_{\leftarrow}),$$
$$\Omega^{\pm}_{\leftarrow} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_{\leftarrow}} e^{it\not{D}} \mathbf{1}_{\mathbb{R}^{\mp}}(P^{\pm}).$$

Here $P^{\pm}_{\leftarrow}$ is the asymptotic velocity operator associated to the dynamics $e^{it\not{D}_{\leftarrow}}$. Then the R.H.S. of (6.1.3) equals :

$$||\mathbf{1}_{[0,\infty)}(\not{D})\mathbf{1}_{\mathbb{R}^-}(P^-)f||^2 + \langle \Omega^-_{\leftarrow} f, \mu e^{\sigma\not{D}_{\leftarrow}}(1+\mu e^{\sigma\not{D}_{\leftarrow}})^{-1}\Omega^-_{\leftarrow} f\rangle.$$

The aim is to show that the incoming part is :

$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(D_{\leftarrow,0})U_{\leftarrow}(0,T)\Omega^-_{\leftarrow} f||^2_0 = \langle \Omega^-_{\leftarrow} f, \mu e^{\sigma\not{D}_{\leftarrow}}(1+\mu e^{\sigma\not{D}_{\leftarrow}})^{-1}\Omega^-_{\leftarrow} f\rangle,$$

where the equality can be shown by explicit calculation. Here $\not{D}_{\leftarrow,t}$ and $U_{\leftarrow}(s,t)$ are the asymptotic operator with boundary condition and the associated propagator. The outgoing part is easy to treat.

As already mentioned, we have to consider the solution in a high frequency regime. Using the Regge-Wheeler variable as a position variable and, say, the Newman-Penrose tetrad used in [63] we find that the modulus of the local velocity

$$[ir_*, \not{D}] = h^2(r_*, \omega)\Gamma^1$$

is not equal to 1, whereas the asymptotic dynamics must have constant local velocity. Here h is a continuous function and Γ^1 a constant matrix. Whereas the (r_*, ω) coordinate system and the tetrad used in [63] were well adapted to the time dependent scattering theory developed in [63], they are no longer well adapted when we consider large times and high frequencies. We are therefore looking for a variable $\hat{r}$ s.t.

$$\{(t, \hat{r}, \omega); \hat{r} \pm t = const.\}$$

are characteristic surfaces. By a separation of variables Ansatz we find a family of such variables and we choose the one which is well adapted to the collapse of the star in the sense that along an incoming null geodesic with $L_z = \mathcal{Q} = 0$ we have :

$$\frac{\partial \hat{r}}{\partial t} = -1.$$

This variable turns out to be a generalized Bondi-Sachs variable. The null geodesics with $L_z = \mathcal{Q} = 0$ are generated by null vector fields $N^{\pm}$ that we choose to be l and n in the Newman-Penrose tetrad. If we write down the hamiltonian for the Dirac equation with this choice of coordinates and tetrad we find that the local velocity now has modulus 1 everywhere and our initial problem disappears. Note that (6.1.3) is of course independent of the choice of the coordinate system and the tetrad, i.e. both sides of (6.1.3) are independent of these choices.

The chapter is organized as follows :

- In Section 6.3 we present the model of the collapsing star. We first analyze geodesics in the Kerr-Newman space-time and explain how the Carter constant can be understood in terms of the hamiltonian flow. We construct the variable $\hat{r}$ and show that

  $$\frac{\partial \hat{r}}{\partial t} = \pm 1 \text{ along null geodesics with } L_z = \mathcal{Q} = 0.$$

  We then show that in the $(t, \hat{r}, \omega)$ coordinate system the surface of the collapsing star is given by

  $$\hat{r} = \hat{z}(t, \theta) = -t - \hat{A}(\theta) e^{-2\kappa_+ t} + \hat{B}(\theta) + \mathcal{O}(e^{-4\kappa_+ t})$$

  with $\hat{A}(\theta) > 0$. Our technical hypothesis will be $\hat{B}(\theta) = const.$

- In Section 6.4 we describe classical Dirac fields. We introduce a new Newman-Penrose tetrad and compute the new expression of the equation. New asymptotic hamiltonians are introduced and classical scattering results are obtained from scattering results in [63] and [33]. The MIT boundary condition is discussed.

- Dirac quantum fields are discussed in Section 6.5. We describe the quantization of Dirac fields in a globally hyperbolic space-time. The theorem about the Hawking effect is formulated and discussed in Subsection 6.5.2.

- In Section 6.6 we give the main ideas of the proof. The remaining sections collect results needed for the proof.

- In Section 6.7 we show additional scattering results. A minimal velocity estimate slightly stronger than the usual ones is established.
- In Section 6.8 we solve the characteristic problem for the Dirac equation. We approximate the characteristic surface by smooth spacelike hypersurfaces and recover the solution in the limit. This method is close to that used by Hörmander in [65] for the wave equation.
- Section 6.9 is devoted to the comparison of the dynamics on an interval $[t_\epsilon, T]$.
- In Section 6.10 we study the propagation of singularities for the Dirac equation in the Kerr-Newman metric. We show that "outgoing" singularities located in
$$\{(\hat{r},\omega,\xi,q);\, \hat{r}\ge -t_\epsilon - C^{-1},\, |\xi|\ge C|q|\}$$
stay away from the surface of the star for C large.

6.3 The model of the collapsing star

The purpose of this section is to describe the model of the collapsing star. We will suppose that the metric outside the star is given by the Kerr-Newman metric. Geodesics are discussed in Subsection 6.3.1. We give a description of the Carter constant in terms of the associated hamiltonian flow. A new position variable is introduced. In Subsection 6.3.2 we give the precise asymptotic behavior of the boundary of the star using this new position variable. We require that a point on the surface behaves asymptotically like incoming timelike geodesics with $L = \mathcal{Q} = \tilde{E} = 0$, which are studied in Subsection 6.3.2. The precise assumptions are also given in Subsection 6.3.2. For the whole chapter $\mathcal{M}_{BH}$ denotes block I of the Kerr-Newman space-time.

6.3.1 Some remarks about geodesics in the Kerr-Newman space-time

It is one of the most remarkable facts about the Kerr-Newman metric that there exist four first integrals for the geodesic equations. If γ is a geodesic in the Kerr-Newman space-time, then $p := \langle\gamma',\gamma'\rangle$ is conserved. The two Killing vector fields ∂_t, ∂_φ give two first integrals, the energy $E := \langle\gamma',\partial_t\rangle$ and the angular momentum $L := -\langle\gamma',\partial_\varphi\rangle$. There exists a fourth constant of motion, the so-called Carter constant $\mathcal{K}$ (see e.g. [26]). We will also use the Carter constant $\mathcal{Q} = \mathcal{K} - (L - aE)^2$, which has a somewhat more geometrical meaning, but gives in general more complicated formulas. Let

$$\mathbf{P} := (r^2 + a^2)E - aL,\, \mathbf{D} := L - aE\sin^2\theta. \tag{6.3.1}$$

Let $\Box_g$ be the d'Alembertian associated to the Kerr-Newman metric. We will consider the hamiltonian flow of the principal symbol of $\frac{1}{2}\Box_g$ and then use the fact that a geodesic can be understood as the projection of the hamiltonian flow on $\mathcal{M}_{BH}$. The principal symbol of $\frac{1}{2}\Box_g$ is :

$$P := \frac{1}{2\rho^2}\left(\frac{\sigma^2}{\Delta}\tau^2 - \frac{2a(Q^2-2Mr)}{\Delta}q_\varphi\tau - \frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta}q_\varphi^2 - \Delta|\xi|^2 - q_\theta^2\right). \tag{6.3.2}$$

Let

$$\mathcal{C}_p := \left\{(t,r,\theta,\varphi;\tau,\xi,q_\theta,q_\varphi); P(t,r,\theta,\varphi;\tau,\xi,q_\theta,q_\varphi) = \frac{1}{2}p\right\}.$$

Here $(\tau,\xi,q_\theta,q_\varphi)$ is dual to (t,r,θ,φ). We have the following :

Theorem 6.3.1. *(i) Let $x_0 = (t_0,r_0,\varphi_0,\theta_0,\tau_0,\xi_0,q_{\theta_0},q_{\varphi_0}) \in \mathcal{C}_p$ and $x(s) = (t(s),r(s),\theta(s),\varphi(s);\tau(s),\xi(s),q_\theta(s),q_\varphi(s))$ be the associated hamiltonian flow line. Then we have the following constants of motion :*

$$p = 2P, E = \tau,\ L = -q_\varphi,\ \mathcal{K} = q_\theta^2 + \frac{\mathbf{D}^2}{\sin^2\theta} + pa^2\cos^2\theta = \frac{\mathbf{P}^2}{\Delta} - \Delta|\xi|^2 - pr^2, \qquad (6.3.3)$$

where $\mathbf{D}$, $\mathbf{P}$ *are defined in (6.3.1).*

The case $L = 0$ is of particular interest. Let γ be a null geodesic with energy $E > 0$, Carter constant $\mathcal{K}$, angular momentum $L = 0$ and given signs of r_0', θ_0'. We can associate a hamiltonian flow line using (6.3.3) to define the initial data $\tau_0,\xi_0,q_{\theta_0},q_{\varphi_0}$ given $t_0,r_0,\theta_0,\varphi_0$. The signs of q_{θ_0} and ξ_0 are fixed by $sign q_{\theta_0} = -sign\theta_0'$, $sign\xi_0 = -sign r_0'$. From (6.3.3) we infer conditions under which ξ, q_θ do not change their signs.

$$\mathcal{K} < \min_{r\in(r_+,\infty)} \frac{(r^2+a^2)^2E^2}{\Delta} \Rightarrow \xi \text{ does not change its sign,}$$

$$\mathcal{Q} \geq 0 \Rightarrow q_\theta \text{ does not change its sign.}$$

Note that in the case $\mathcal{Q} = 0$ γ is either in the equatorial plane or it does not cross it. Under the above conditions ξ (resp. q_θ) can be understood as a function of r (resp. θ) alone. In this case let $k_{\mathcal{K},E}$ and $l_{\mathcal{K},E}$ s.t.

$$\frac{dk_{\mathcal{K},E}(r)}{dr} = \frac{\xi(r)}{E}, \quad l_{\mathcal{K},E}' = \frac{q_\theta(\theta)}{E}, \quad \hat{r}_{\mathcal{K},E} := k_{\mathcal{K},E}(r) + l_{\mathcal{K},E}(\theta). \qquad (6.3.4)$$

It is easy to check that $(t,\hat{r}_{\mathcal{K},E},\omega)$ is a coordinate system on block I.

Lemma 6.3.1. *We have :*

$$\frac{\partial \hat{r}_{\mathcal{K},E}}{\partial t} = -1 \quad along \quad \gamma, \qquad (6.3.5)$$

where t is the Boyer-Lindquist time.

We will suppose from now on $r_0' < 0$, i.e. our construction is based on incoming null geodesics.

We will often use the r_* variable and its dual variable ξ^*. In this case we have to replace $\xi(r)$ by $\frac{r^2+a^2}{\Delta}\xi^*(r_*)$. The function $k_{\mathcal{K},E}$ is then a function of r_* satisfying :

$$k_{\mathcal{K},E}'(r_*) = \frac{\xi^*}{E}, \qquad (6.3.6)$$

where the prime denotes derivation with respect to r_*. Using the explicit form of the Carter constant in Theorem 6.3.1 we find :

$$(k_{\mathcal{K},E}')^2 = 1 - \frac{\Delta\mathcal{K}}{(r^2+a^2)^2E^2}, \qquad (6.3.7)$$

$$(l_{\mathcal{K},E}')^2 = \frac{\mathcal{K}}{E^2} - a^2\sin^2\theta. \qquad (6.3.8)$$

In particular we have :

$$(k'_{\mathcal{K},E})^2\frac{(r^2+a^2)^2}{\sigma^2}+(l'_{\mathcal{K},E})^2\frac{\Delta}{\sigma^2}=1. \tag{6.3.9}$$

We will often consider the case $\mathcal{Q}=0$ and write in this case simply k,l instead of $k_{a^2E^2,E}$, $l_{a^2E^2,E}$.

Corollary 6.3.1. *For given Carter constant $\mathcal{K}$, energy $E>0$ and sign of θ'_0 the surfaces*

$$\mathcal{C}^{c,\pm}_{\mathcal{K},E}=\{(t,r_*,\theta,\varphi);\ \pm t=\hat{r}_{\mathcal{K},E}(r_*,\theta)+c\}$$

are characteristic.

Remark 6.3.1. (i) *The variable $\hat{r}_{\mathcal{K},E}$ is a Bondi-Sachs type coordinate. This coordinate system is discussed in some detail in [48]. As in [48] we will call the null geodesics with $L=\mathcal{Q}=0$ simple null geodesics (SNG's).*

(ii) *A natural way of finding the variable $\hat{r}_{\mathcal{K},E}$ is to start with Corollary 6.3.1. Look for functions $k_{\mathcal{K},E}(r_*)$ and $l_{\mathcal{K},E}(\theta)$ such that $\mathcal{C}^{c,\pm}_{\mathcal{K},E}=\{\pm t=k_{\mathcal{K},E}(r_*)+l_{\mathcal{K},E}(\theta)\}$ is characteristic. The condition that the normal is null is equivalent to (6.3.9). The curve generated by the normal lies entirely in $\mathcal{C}^{c,\pm}_{\mathcal{K},E}$.*

Remark 6.3.2. *From the explicit form of the Carter constant in Theorem 6.3.1 follows :*

$$q_\theta^2+(p-E^2)a^2\cos^2\theta+\frac{q_\varphi^2}{\sin^2\theta}=\mathcal{Q}. \tag{6.3.10}$$

This is the equation of the θ motion and it is interpreted as conservation of the mechanical energy with $V(\theta)=(p-E^2)a^2\cos^2\theta+\frac{q_\varphi^2}{\sin^2\theta}$ as potential energy and q_θ^2 in the role of kinetic energy. The quantity $\tilde{E}=(E^2-p)a^2$ is usually called the rotational energy.

6.3.2 The model of the collapsing star

Let $\mathcal{S}_0$ be the surface of the star at time $t=0$. We suppose that elements $x_0\in\mathcal{S}_0$ will move along curves which behave asymptotically like certain incoming timelike geodesics γ_p. All these geodesics should have the same energy E, angular momentum L, Carter constant $\mathcal{K}$ (resp. $\mathcal{Q}=\mathcal{K}-(L-aE)^2$) and "mass" $p:=\langle\gamma'_p,\gamma'_p\rangle$. We will suppose :

(A) The angular momentum L vanishes : $L=0$.

(B) The rotational energy vanishes : $\tilde{E}=a^2(E^2-p)=0$.

(C) The total angular momentum about the axis of symmetry vanishes : $\mathcal{Q}=0$.

The conditions (A)-(C) are imposed by the fact that the collapse itself creates the space-time, so that momenta and rotational energy should be zero with respect to the space-time.

Timelike geodesics with $L=\mathcal{Q}=\tilde{E}=0$

Next, we will study the above family of geodesics. The starting point of the geodesic is denoted $(0,r_0,\theta_0,\varphi_0)$. Given a point in the space-time, the conditions (A)-(C) define a unique cotangent vector provided you add the condition that the corresponding tangent vector is incoming. The choice of p is irrelevant because it just corresponds to a normalization of the proper time.

Lemma 6.3.2. *Along the geodesic γ_p we have :*

$$\frac{\partial\theta}{\partial t} = 0, \tag{6.3.11}$$

$$\frac{\partial\varphi}{\partial t} = \frac{a(2Mr - Q^2)}{\sigma^2}, \tag{6.3.12}$$

where t is the Boyer-Lindquist time.

The function $\frac{\partial\varphi}{\partial t} = \frac{a(2Mr-Q^2)}{\sigma^2}$ is usually called the *local angular velocity of the space-time.* Our next aim is to adapt our coordinate system to the collapse of the star. The most natural way of doing this is to choose an incoming null geodesic γ with $L = Q = 0$ and then use the Bondi-Sachs type coordinate as in the previous subsection. In addition we want that $k(r_*)$ behaves like r_* when $r_* \to -\infty$. We therefore put :

$$k(r_*) = r_* + \int_{-\infty}^{r_*} \left(\sqrt{1 - \frac{a^2\Delta(s)}{(r(s)^2 + a^2)^2}} - 1 \right) ds, \tag{6.3.13}$$

$$l(\theta) = a \sin\theta. \tag{6.3.14}$$

The choice of the sign of l' is not important, the opposite sign would have been possible. Recall that $\cos\theta$ does not change its sign along a null geodesic with $L = \mathcal{Q} = 0$. We fix the notation for the null vector fields generating γ and the corresponding outgoing vector field :

$$N^{\pm,a}\partial_a = \frac{E\sigma^2}{\rho^2\Delta}\left(\partial_t \pm \frac{(r^2+a^2)^2}{\sigma^2}k'(r_*)\partial_{r_*} \pm \frac{\Delta}{\sigma^2}a\cos\theta\partial_\theta + \frac{a(2Mr-Q^2)}{\sigma^2}\partial_\varphi\right). \tag{6.3.15}$$

These vector fields will be important for the construction of the Newman-Penrose tetrad. We put :

$$\hat{r} = k(r_*) + l(\theta) \tag{6.3.16}$$

and by Lemma 6.3.1 we have :

$$\frac{\partial\hat{r}}{\partial t} = -1 \quad \text{along}\,\gamma. \tag{6.3.17}$$

In order to describe the model of the collapsing star we have to evaluate $\frac{\partial\hat{r}}{\partial t}$ along γ_p. We start by studying $r(t,\theta)$. Recall that $\theta(t) = \theta_0 = const.$ along γ_p and that $\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)}$ is the surface gravity of the outer horizon. In what follows a dot will denote derivation in t.

Lemma 6.3.3. *There exist smooth functions $\hat{A}(\theta, r_0) > 0$, $\hat{B}(\theta, r_0)$ such that along γ_p we have uniformly in θ, $r_0 \in [r_1, r_2] \subset (r_+, \infty)$:*

$$\hat{r} = -t - \hat{A}(\theta, r_0)e^{-2\kappa_+ t} + \hat{B}(\theta, r_0) + \mathcal{O}(e^{-4\kappa_+ t}),\ t \to \infty. \tag{6.3.18}$$

Furthermore there exists $k_0 > 0$ s.t. for all $t > 0, \theta \in [0,\pi]$ we have :

$$\left(\frac{(r^2+a^2)^2}{\sigma^2}k'^2 - \dot{\hat{r}}^2\right) \geq k_0 e^{-2\kappa_+ t}.$$

Precise assumptions

Let us now make the precise assumptions on the collapse. We will suppose that the surface at time $t = 0$ is given in the $(t, \hat{r}, \theta, \varphi)$ coordinate system by $\mathcal{S}_0 = \{(\hat{r}_0(\theta_0), \theta_0, \varphi_0);\ (\theta_0, \varphi_0) \in S^2\}$, where $\hat{r}_0(\theta_0)$ is a smooth function. As $\hat{r}_0$ does not depend on φ_0, we will suppose that $\hat{z}(t, \theta_0, \varphi_0)$ will be independent of φ_0 : $\hat{z}(t, \theta_0, \varphi_0) = \hat{z}(t, \theta_0) = \hat{z}(t, \theta)$ as this is the case for $\hat{r}(t)$ along timelike geodesics with $L = \mathcal{Q} = 0$. Thus the surface of the star is given by :

$$\mathcal{S} = \{(t, \hat{z}(t, \theta), \omega); t \in \mathbb{R},\ \omega \in S^2\}. \tag{6.3.19}$$

The function $\hat{z}(t, \theta)$ satisfies

$$\forall t \leq 0, \theta \in [0, \pi] \quad \hat{z}(t, \theta) = \hat{z}(0, \theta) < 0, \tag{6.3.20}$$

$$\forall t > 0,\ \theta \in [0, \pi] \quad \dot{\hat{z}}(t, \theta) < 0, \tag{6.3.21}$$

$$\exists k_0 > 0\, \forall t > 0, \theta \in [0, \pi]$$
$$\left(\left(\frac{(r^2+a^2)^2}{\sigma^2} k'^2 \right) (\hat{z}(t, \theta), \theta) - \dot{\hat{z}}^2(t, \theta) \right) \geq k_0 e^{-2\kappa_+ t}, \tag{6.3.22}$$

$$\exists \hat{A} \in C^\infty([0, \pi]),\ \hat{\xi} \in C^\infty(\mathbb{R} \times [0, \pi]) \quad \hat{z}(t, \theta) = -t - \hat{A}(\theta) e^{-2\kappa_+ t} + \hat{\xi}(t, \theta)$$
$$\hat{A}(\theta) > 0\, \forall \theta \in [0, \pi],\ \forall 0 \leq \alpha, \beta \leq 2\, |\partial_t^\alpha \partial_\theta^\beta \hat{\xi}(t, \theta)| \leq C_{\alpha,\beta} e^{-4\kappa_+ t}\, \forall t > 0,\ \theta \in [0, \pi]. \tag{6.3.23}$$

As explained above these assumptions are motivated by the preceding analysis. We do not suppose that a point on the surface moves exactly on a geodesic. Note that (6.3.21), (6.3.22) imply :

$$\forall t > 0, \theta \in [0, \pi] \quad -1 < \dot{\hat{z}}(t, \theta) < 0.$$

Equations (6.3.20)-(6.3.23) summarize our assumptions on the collapse. The space-time of the collapsing star is given by :

$$\mathcal{M}_{col} = \{(t, \hat{r}, \theta, \varphi);\ \hat{r} \geq \hat{z}(t, \theta)\}.$$

We will also note :

$$\Sigma_t^{col} = \{(\hat{r}, \theta, \varphi);\ \hat{r} \geq \hat{z}(t, \theta)\}.$$

Thus :

$$\mathcal{M}_{col} = \bigcup_t \Sigma_t^{col}.$$

Remark 6.3.3. *(i) Let us compare assumptions (6.3.19)-(6.3.23) to the preceding discussion on geodesics. The assumption (6.3.23) contains, with respect to the previous discussion, an additional asymptotic assumption. Comparing to Lemma 6.3.3 this condition can be expressed as $\hat{B}(\theta, r_0(\theta)) = const.\, (r_0(\theta) = r(\hat{r}_0(\theta), \theta))$. Using the freedom of the constant of integration in (2.1.4) we can suppose*

$$\hat{B}(\theta, r_0(\theta)) = 0. \tag{6.3.24}$$

(ii) The Penrose compactification of block I can be constructed based on the SNG's rather than on the principal null geodesics (PNG's), see [58] for details.

We finish this section with a lemma which shows that the asymptotic form (6.3.23) can be achieved by incoming timelike geodesics with $L = \mathcal{Q} = \tilde{E} = 0$.

Lemma 6.3.4. *There exists a smooth function $\hat{r}_0(\theta)$ with the following property. Let γ be a timelike incoming geodesic with $\mathcal{Q} = L = \tilde{E} = 0$ and starting point $(0, \hat{r}_0(\theta_0), \theta_0, \varphi_0)$. Then we have along γ:*

$$\hat{r} + t \to 0, \quad t \to \infty.$$

6.4 Classical Dirac Fields

In this section we describe classical Dirac fields on B_I as well as on $\mathcal{M}_{col}$. We give a result about the existence of a unitary propagator describing the solution of the Dirac equation on $\mathcal{M}_{col}$ with suitable boundary conditions.

6.4.1 A new Newman-Penrose tetrad

If we choose the analogue of the tetrad introduced in Chapter 5 we find the following tetrad for the Kerr-Newman case (see [33]) :

$$\left.\begin{array}{l} L^a = \frac{1}{2}T^a + \sqrt{\frac{\Delta}{2\rho^2}}\partial_r = \sqrt{\frac{\sigma^2}{2\Delta\rho^2}}(\partial_t + \frac{a(2Mr-Q^2)}{\sigma^2}\partial_\varphi) + \sqrt{\frac{\Delta}{2\rho^2}}\partial_r, \\ N^a = \frac{1}{2}T^a - \sqrt{\frac{\Delta}{2\rho^2}}\partial_r = \sqrt{\frac{\sigma^2}{2\Delta\rho^2}}(\partial_t + \frac{a(2Mr-Q^2)}{\sigma^2}\partial_\varphi) - \sqrt{\frac{\Delta}{2\rho^2}}\partial_r, \\ M^a = \frac{1}{\sqrt{2\rho^2}}(\partial_\theta + \frac{\rho^2}{\sqrt{\sigma^2}}\frac{i}{\sin\theta}\partial_\varphi). \end{array}\right\} \tag{6.4.1}$$

The Dirac equation in the Kerr-Newman metric is then described in the following way. Let Φ be the vector of components of the spinor ϕ_A (solution of the Dirac equation) in the spin frame associated to the above tetrad and :

$$\Psi = \sqrt{\frac{\sqrt{\Delta}\sigma\rho}{r^2 + a^2}}\Phi.$$

Then the equation satisfied by Ψ is (see [33]) :

$$\partial_t \Psi = i\not{D}\Psi, \tag{6.4.2}$$

$$\not{D} = h\not{D}_{\mathfrak{s}}h + V_\varphi D_\varphi + V_1, \tag{6.4.3}$$

$$\not{D}_{\mathfrak{s}} = \Gamma^1 D_{r_*} + a_0(r_*)\not{D}_{S^2} + b_0(r_*)\Gamma^4 + c_1(r_*) + c_2^\varphi(r_*)D_\varphi, \tag{6.4.4}$$

$$a_0(r_*) = \frac{\sqrt{\Delta}}{r^2 + a^2},\ b_0(r_*) = \frac{m\sqrt{\Delta}}{\sqrt{r^2 + a^2}},\ c_1(r_*) = -\frac{qQr}{r^2 + a^2},$$

$$c_2^\varphi(r_*) = -\frac{a(2Mr - Q^2)}{(r^2 + a^2)^2},\ h(r_*, \theta) = \sqrt{\frac{r^2 + a^2}{\sigma}}.$$

The potentials V_φ and V_1 are short range potentials [2]. Even if the above tetrad was successfully used for the proof of the asymptotic completeness result, it has a major drawback for the treatment of the Hawking effect. As we have already explained in the introduction, in this representation and using the Regge-Wheeler type coordinate r_* the modulus of local velocity is not equal to 1. The consequence is that in the high frequency regime which is characteristic

[2] This means that the two potentials go to constants $C_\pm$ at least like $\langle r_* \rangle^{-1-\epsilon}$ when $r_* \to \pm\infty$.

of the Hawking effect the full dynamics $\not{D}$ and the free dynamics $\not{D}_H$ or $\not{D}_{\mathfrak{S}}$ are no longer close to each other. Now recall that $\partial_t \hat{r} = -1$ along incoming null geodesics with the correct sign of θ'_0. This means that the observable $\hat{r}$ should increase (resp. decrease) exactly like t along the evolution if we focus on scattering directions in which the variable $\hat{r}$ increases (decreases) in this way. We therefore choose

$$l^a = \lambda N^{+,a},\ n^a = \lambda N^{-,a} \tag{6.4.5}$$

for some normalization constant λ. The choice of m^a is now imposed except for a factor of modulus 1. We find :

$$\left.\begin{array}{rl} l^a = & \sqrt{\frac{\sigma^2}{2\rho^2\Delta}}\left(\partial_t + \partial_{\hat{r}} + \frac{\Delta}{\sigma^2} a\cos\theta\partial_\theta + \frac{a(2Mr-Q^2)}{\sigma^2}\partial_\varphi\right) \\ n^a = & \sqrt{\frac{\sigma^2}{2\rho^2\Delta}}\left(\partial_t - \partial_{\hat{r}} - \frac{\Delta}{\sigma^2} a\cos\theta\partial_\theta + \frac{a(2Mr-Q^2)}{\sigma^2}\partial_\varphi\right) \\ m^a = & \sqrt{\frac{\rho^2}{2\sigma^2}}\left(-ik'\frac{r^2+a^2}{\rho^2}\partial_\theta + \frac{1}{\sin\theta}\partial_\varphi\right). \end{array}\right\} \tag{6.4.6}$$

and that the tetrad l^a, n^a, m^a is adapted to the foliation.

The new expression of the Dirac equation

We now want to use the variable introduced in (6.3.16). Let $\mathcal{H} := L^2((\mathbb{R}\times S^2, d\hat{r}d\omega);\mathbb{C}^4)$. Let H be the Dirac operator acting on $\mathcal{H}$ which we obtain if we use the $(t,\hat{r},\omega)$ coordinate system and the $l^a, n^a, m^a, \bar{m}^a$ tetrad. We obtain :

$$H = \Gamma^1 D_{\hat{r}} + (m_\theta^1\Gamma^1 - m_\theta^2\Gamma^3) - h^2 a_0\Gamma^2\frac{D_\varphi}{\sin\theta} + h^2c_1 + h^2c_2^\varphi D_\varphi + \hat{V}_\varphi D_\varphi + \hat{V}_1, \tag{6.4.7}$$

where

$$\begin{aligned} m_\theta^1 &= \frac{\beta h\sqrt{a_0}}{\sqrt{\alpha+1}}\left(D_\theta + \frac{\cot\theta}{2i}\right)\sqrt{a_0}h\sqrt{\alpha+1} + \sqrt{\alpha+1}h\sqrt{a_0}\left(D_\theta + \frac{\cot\theta}{2i}\right)\frac{\beta h\sqrt{a_0}}{\sqrt{\alpha+1}}, \\ m_\theta^2 &= \frac{\beta h\sqrt{a_0}}{\sqrt{\alpha+1}}\left(D_\theta + \frac{\cot\theta}{2i}\right)\frac{h\beta\sqrt{a_0}}{\sqrt{\alpha+1}} - \sqrt{\alpha+1}h\sqrt{a_0}\left(D_\theta + \frac{\cot\theta}{2i}\right)\sqrt{a_0}h\sqrt{\alpha+1}, \\ \alpha &= k'h^2,\ \beta = \sqrt{\frac{\Delta}{\sigma^2}}a\cos\theta,\ \Gamma^1 = Diag(1,-1,-1,1) \end{aligned}$$

and Γ^2, Γ^3 are constant coefficient matrices. The potentials $\hat{V}_\varphi$ and $\hat{V}_1$ are short range. We put

$$\mathcal{H}^1 := D(H), \quad ||u||^2_{\mathcal{H}^1} = ||Hu||^2 + ||u||^2.$$

Let also

$$P_\omega := (m_\theta^1\Gamma^1 - m_\theta^2\Gamma^3) - h^2a_0\Gamma^2\frac{D_\varphi}{\sin\theta} + h^2c_2^\varphi D_\varphi + \hat{V}_\varphi D_\varphi, \tag{6.4.8}$$

$$W := h^2c_1 + \hat{V}_1. \tag{6.4.9}$$

6.4.2 Scattering results

A complete scattering theory for massless Dirac fields in the Kerr metric was obtained in [63]. This result has been generalized to the case of massive charged Dirac fields in [33]. In both works the comparison dynamics are the dynamics which are natural in the (t, r_*, ω) coordinate system and the L^a, N^a, M^a Newman-Penrose tetrad. We will need comparison dynamics which are natural with respect to the $(t, \hat{r}, \theta, \varphi)$ coordinate system and the l^a, n^a, m^a tetrad. To this purpose we define :

$$H_{\leftarrow} = \Gamma^1 D_{\hat{r}} - \frac{a}{r_+^2 + a^2} D_\varphi - \frac{qQr_+}{r_+^2 + a^2}.$$

Note that

$$H - H_{\leftarrow} = \tilde{P}_\omega + \tilde{W},$$

where $\tilde{P}_\omega$ is a differential operator of order one with derivatives only in the angular directions and $\tilde{W}$ is a short range potential. Therefore the high frequency of the solution in angular directions is no longer a problem for the analysis below. The operator $H_{\leftarrow}$ is selfadjoint on $\mathcal{H}$ with domain $D(H_{\leftarrow}) = \{u \in \mathcal{H}; H_{\leftarrow} u \in \mathcal{H}\}$, $D(H_{\rightarrow}) = \{u \in \mathcal{H}; \mathcal{U}^* \mathcal{V}^* u \in D(\not{D}_{\rightarrow})\}$. Let

$$\mathcal{H}^+ = \{\Psi = (0, \Psi_2, \Psi_3, 0) \in \mathcal{H}\},$$
$$\mathcal{H}^- = \{\Psi = (\Psi_1, 0, 0, \Psi_4) \in \mathcal{H}\}.$$

We note that

$$\mathbf{1}_{\mathbb{R}^\pm}(-\Gamma^1) = P_{\mathcal{H}^\pm},$$

where $P_{\mathcal{H}^\pm}$ is the projection from $\mathcal{H}$ to $\mathcal{H}^\pm$.

The following proposition gives the existence of the asymptotic velocity :

Proposition 6.4.1. *There exist selfadjoint operators $P^\pm$ s.t. for all $g \in C_\infty(\mathbb{R})$:*

$$g(P^\pm) = s - \lim_{t \to \pm\infty} e^{-itH} g\left(\frac{\hat{r}}{t}\right) e^{itH}. \tag{6.4.10}$$

The operators $P^\pm$ commute with H. Furthermore we have :

$$g(P^\pm)\mathbf{1}_{\mathbb{R}^\mp}(P^\pm) = s - \lim_{t \to \pm\infty} e^{-itH} g(-\Gamma^1) e^{itH} \mathbf{1}_{\mathbb{R}^\mp}(P^\pm), \tag{6.4.11}$$
$$\sigma(P^\pm) = \{-1\} \cup [0, 1], \tag{6.4.12}$$
$$\sigma_{pp}(H) = \mathbf{1}_{\{0\}}(P^\pm) = \emptyset. \tag{6.4.13}$$

For limits of the form (6.4.10) we will write the following :

$$P^\pm = s - C_\infty - \lim_{t \to \pm\infty} e^{-itH} \frac{\hat{r}}{t} e^{itH}.$$

Theorem 6.4.1. *The wave operators*

$$W_{\leftarrow}^\pm = s - \lim_{t \to \pm\infty} e^{-itH} e^{itH_{\leftarrow}} P_{\mathcal{H}^\mp}, \tag{6.4.14}$$
$$\Omega_{\leftarrow}^\pm = s - \lim_{t \to \pm\infty} e^{-itH_{\leftarrow}} e^{itH} \mathbf{1}_{\mathbb{R}^\mp}(P^\pm) \tag{6.4.15}$$

exist and satisfy

$$(W_{\leftarrow}^\pm)^* = \Omega_{\leftarrow}^\pm, \; (\Omega_{\leftarrow}^\pm)^* = W_{\leftarrow}^\pm \tag{6.4.16}$$

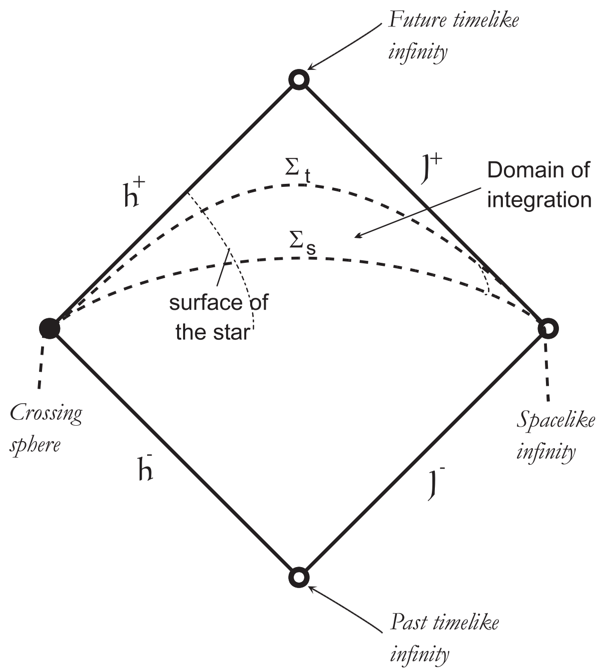


Figure 6.3: The surface of the star and the domain of integration.

6.4.3 The Dirac equation on $\mathcal{M}_{col}$

We want to impose a boundary condition on the surface of the star such that the evolution can be described by a unitary propagator. We will use the conserved current

$$V_a = \phi_A \bar{\phi}_{A'} + \bar{\chi}_A \chi_{A'}.$$

Integrating over the domain indicated in Figure 6.3 and supposing that the field is 0 in a neighborhood of i^0 gives by Stokes' theorem :

$$\int_{\Sigma_s} V_a T^a d\sigma_{\Sigma_s} - \int_{\Sigma_t} V_a T^a d\sigma_{\Sigma_t} + \int_{\mathcal{S}_I} V_a \mathcal{N}^a d\sigma_{\mathcal{S}_I} = 0,$$

where $\mathcal{N}^a$ is the normal to the surface of the star. Therefore the necessary condition for charge conservation outside the collapsing body is

$$V_a \mathcal{N}^a = 0 \quad \text{on} \quad \mathcal{S}. \tag{6.4.17}$$

We will impose :

$$\left.\begin{array}{r} \mathcal{N}^{AA'}\phi_A = \frac{1}{\sqrt{2}} e^{-i\nu}\chi^{A'}, \\ \mathcal{N}^{AA'}\chi_{A'} = \frac{1}{\sqrt{2}} e^{i\nu}\phi^A. \end{array}\right\} \quad \text{on} \quad \mathcal{S}. \tag{6.4.18}$$

Here ν is the so called chiral angle. We note that (6.4.18) implies (6.4.17) and :

$$\mathcal{N}^a \mathcal{N}_a \phi_B = -\phi_B.$$

We therefore impose

$$\mathcal{N}^a\mathcal{N}_a = -1.$$

In any other case the boundary condition would force $(\phi_A, \chi^{A'})$ to equal 0 on $\mathcal{S}$. We will from now on suppose that $\mathcal{N}_a$ is past directed, but the opposite choice would of course be possible. Rewriting condition (6.4.18) using the $(t, \hat{r}, \omega)$ coordinate system and the $(l^a, n^a, m^a, \bar{m}^a)$ tetrad we find :

$$\sum_{\hat{\mu}\in\{t,\hat{r},\theta,\varphi\}} \mathcal{N}_{\hat{\mu}}\hat{\gamma}^{\hat{\mu}}\Psi = -i\mathcal{B}\Psi \quad \text{on} \quad \mathcal{S}, \tag{6.4.19}$$

where $\mathcal{B} = e^{-i\nu\gamma^5}$, $\gamma^5 = Diag(1, 1, -1, -1)$, $\hat{\gamma}^{\hat{\mu}}$ are some suitable Dirac matrices and $\mathcal{N}_{\hat{\mu}}$ are the coordinates of the conormal in the $(t, \hat{r}, \theta, \varphi)$ coordinate system.

We introduce the following Hilbert spaces:

$$\mathcal{H}_t = ((L^2(\Sigma_t^{col}, d\hat{r}d\omega))^4, ||.||_t), \tag{6.4.20}$$

where the norm $||.||_t$ is defined by

$$||\Psi||_t = ||[\Psi]_L||, \quad [\Psi]_L(\hat{r}, \omega) = \begin{cases} \Psi(\hat{r}, \omega) & \hat{r} \geq \hat{z}(t, \theta) \\ 0 & \hat{r} \leq \hat{z}(t, \theta). \end{cases} \tag{6.4.21}$$

Let

$$\mathcal{H}_t^1 = \{u \in \mathcal{H}_t;\ Hu \in \mathcal{H}_t\}, \quad ||u||^2_{\mathcal{H}_t^1} = ||u||_t^2 + ||Hu||_t^2.$$

We also need an extension from $\mathcal{H}_t^1$ to $\mathcal{H}^1$. For this purpose we set for $\phi \in \mathcal{H}_t^1$:

$$[\phi]_H(\hat{r}, \omega) = \begin{cases} \phi(\hat{r}, \omega) & \hat{r} \geq \hat{z}(t, \theta) \\ \phi(2\hat{z}(t, \theta) - \hat{r}, \omega) & \hat{r} \leq \hat{z}(t, \theta) \end{cases}$$

It is easy to check that $[\phi]_H$ is in $\mathcal{H}^1$. On $\mathcal{M}_{col}$ we consider the following mixed problem :

$$\left.\begin{array}{c} \partial_t\Psi = iH_t\Psi \qquad\qquad \hat{z}(t, \theta) < \hat{r}, \\ (\sum_{\hat{\mu}\in\{t,\hat{r},\theta,\varphi\}} \mathcal{N}_{\hat{\mu}}\hat{\gamma}^{\hat{\mu}})\Psi(t, \hat{z}(t, \theta), \omega) = -i\mathcal{B}\Psi(t, \hat{z}(t, \theta), \omega), \\ \Psi(t = s, .) = \Psi_s(.). \end{array}\right\} \tag{6.4.22}$$

Here the operator H_t is given by

$$H_t = H, D(H_t) = \left\{\Psi \in \mathcal{H}_t^1;\ \left(\sum_{\hat{\mu}\in\{t,\hat{r},\theta,\varphi\}} \mathcal{N}_{\hat{\mu}}\hat{\gamma}^{\hat{\mu}}\Psi\right)(t, \hat{z}(t, \theta), \omega) = -i\mathcal{B}\Psi(t, \hat{z}(t, \theta), \omega)\right\}.$$

The problem (6.4.22) is solved by the following proposition.

Proposition 6.4.2. *Let $\Psi_s \in D(H_s)$. Then there exists a unique solution $[\Psi(.)]_H = [U(., s)\Psi_s]_H$ $\in C^1(\mathbb{R}_t; \mathcal{H}) \cap C(\mathbb{R}_t; \mathcal{H}^1)$ of (6.4.22) s.t. for all $t \in \mathbb{R}$ $\Psi(t) \in D(H_t)$. Furthermore we have $||\Psi(t)|| = ||\Psi_s||$ and $U(t, s)$ possesses an extension to an isometric and strongly continuous propagator from $\mathcal{H}_s$ to $\mathcal{H}_t$ s.t. for all $\Psi_s \in D(H_s)$ we have:*

$$\frac{d}{dt}U(t, s)\Psi_s = iH_tU(t, s)\Psi_s.$$

6.5 Dirac Quantum Fields

We adopt the approach of Dirac quantum fields in the spirit of [37] and [38]. This approach is explained in Section 6.5.1. In Section 6.5.2 we present the theorem about the Hawking effect.

6.5.1 Quantization in a globally hyperbolic space-time

Following J. Dimock [38] we construct the local algebra of observables in the space-time outside the collapsing star. This construction does not depend on the choice of the representation of the CAR's, or on the spin structure of the Dirac field, or on the choice of the hypersurface. In particular we can consider the Fermi-Dirac-Fock representation and the following foliation of our space-time (see Subsection 6.3.2):

$$\mathcal{M}_{col} = \bigcup_{t\in\mathbb{R}} \Sigma_t^{col}, \quad \Sigma_t^{col} = \{(t,\hat{r},\theta,\varphi); \hat{r} \geq \hat{z}(t,\theta)\}.$$

We construct the Dirac field Ψ_0 and the C^*-algebra $\mathcal{U}(\mathcal{H}_0)$ in the usual way. We define the operator:

$$S_{col} : \Phi \in (C_0^\infty(\mathcal{M}_{col}))^4 \mapsto S_{col}\Phi := \int_{\mathbb{R}} U(0,t)\Phi(t)dt \in \mathcal{H}_0, \tag{6.5.1}$$

where $U(0,t)$ is the propagator defined in Proposition 6.4.2. The quantum spin field is defined by :

$$\Psi_{col} : \Phi \in (C_0^\infty(\mathcal{M}_{col}))^4 \mapsto \Psi_{col}(\Phi) := \Psi_0(S_{col}\Phi) \in \mathcal{L}(\mathcal{H}_0)$$

and for an arbitrary set $\mathcal{O} \subset \mathcal{M}_{col}$, we introduce $\mathcal{U}_{col}(\mathcal{O})$, the C^*-algebra generated by $\psi^*_{col}(\Phi_1)\Psi_{col}(\Phi_2)$, $\operatorname{supp}\Phi_j \subset \mathcal{O}$, $j=1,2$. Eventually, we have:

$$\mathcal{U}_{col}(\mathcal{M}_{col}) = \overline{\bigcup_{\mathcal{O}\subset\mathcal{M}_{col}} \mathcal{U}_{col}(\mathcal{O})}.$$

Then we define the fundamental state on $\mathcal{U}_{col}(\mathcal{M}_{col})$ as follows:

$$\omega_{col}(\Psi^*_{col}(\Phi_1)\Psi_{col}(\Phi_2)) := \omega_{vac}(\Psi^*_0(S_{col}\Phi_1)\Psi_0(S_{col}\Phi_2)) = \langle \mathbf{1}_{[0,\infty)}(H_0)S_{col}\Phi_1, S_{col}\Phi_2\rangle.$$

Let us now consider the future black-hole. We consider the space-time $\mathcal{M}_{BH}$ with the Dirac hamiltonian H for a field with one particle. For $\Phi \in \mathcal{H}$ we construct the Dirac field $\Psi(\Phi)$ in the usual way. Let

$$S : \Phi \in (C_0^\infty(\mathcal{M}_{BH}))^4 \mapsto S\Phi := \int_{\mathbb{R}} e^{-itH}\Phi(t)dt.$$

We also introduce :

$$\Psi_{BH} : \Phi \in (C_0^\infty(\mathcal{M}_{BH}))^4 \mapsto \Psi_{BH}(\Phi) := \Psi(S\Phi)$$

and the C^*-algebra $\mathcal{U}_{BH}(\mathcal{O})$ generated by $\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\Phi_2)$, $\Phi_1,\Phi_2 \in (C_0^\infty(\mathcal{O}))^4$, $\mathcal{O} \subset \mathcal{M}_{BH}$. As before we put

$$\mathcal{U}_{BH}(\mathcal{M}_{BH}) = \overline{\bigcup_{\mathcal{O}\subset\mathcal{M}_{BH}} \mathcal{U}_{BH}(\mathcal{O})}.$$

We also define the thermal Hawking state:

$$\omega^{\eta,\sigma}_{Haw}(\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\Phi_2)) = \langle \mu e^{\sigma H}(1+\mu e^{\sigma H})^{-1}S\Phi_1, S\Phi_2\rangle_{\mathcal{H}} \tag{6.5.2}$$

$$=: \omega^{\eta,\sigma}_{KMS}(\Psi^*(S\Phi_1)\Psi(S\Phi_2)) \tag{6.5.3}$$

with

$$T_{Haw} = \sigma^{-1},\ \mu = e^{\sigma\eta},\ \sigma > 0,$$

where T_{Haw} is the Hawking temperature and μ is the chemical potential. We will also need a vacuum state which is given by :

$$\omega_{vac}(\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\phi_2)) = \langle \mathbf{1}_{[0,\infty)}(H)S\phi_1, S\phi_2\rangle.$$

6.5.2 The Hawking effect

In this subsection we formulate the main result of this paper. Let $\Phi \in (C_0^\infty(\mathcal{M}_{col}))^4$. We put

$$\Phi^T(t,\hat{r},\omega) = \Phi(t-T,\hat{r},\omega). \tag{6.5.4}$$

Theorem 6.5.1 (Hawking effect)**.** *Let*

$$\Phi_j \in (C_0^\infty(\mathcal{M}_{col}))^4,\ j=1,2.$$

Then we have

$$\lim_{T\to\infty} \omega_{col}(\Psi^*_{col}(\Phi_1^T)\Psi_{col}(\Phi_2^T)) = \omega^{\eta,\sigma}_{Haw}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_2))$$
$$+ \omega_{vac}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_2)), \tag{6.5.5}$$
$$T_{Haw} = 1/\sigma = \kappa_+/2\pi, \quad \mu = e^{\sigma\eta},\ \eta = \frac{qQr_+}{r_+^2+a^2} + \frac{aD_\varphi}{r_+^2+a^2}.$$

In the above theorem $P^\pm$ is the asymptotic velocity introduced in Section 6.4. The projections $\mathbf{1}_{\mathbb{R}^\pm}(P^\pm)$ separate outgoing and incoming solutions.

Remark 6.5.1. *The result is independent of the choices of coordinate system and tetrad, i.e. both sides of (6.5.5) are independent of these choices. Indeed a change of coordinate system or a change of tetrad is equivalent to a conjugation of the operators by a unitary transformation. We also note that the result is independent of the chiral angle ν in the boundary condition.*

6.6 Strategy of the proof

The radiation can be explicitly computed for the asymptotic dynamics near the horizon. For $f = (0, f_2, f_3, 0)$ and T large the time reversed solution of the mixed problem for the asymptotic dynamics is well approximated by the so called geometric optics approximation :

$$F^T(\hat{r},\omega) := \frac{1}{\sqrt{-\kappa_+\hat{r}}}(f_3,0,0,-f_2)\left(T + \frac{1}{\kappa_+}\ln(-\hat{r}) - \frac{1}{\kappa_+}\ln\hat{A}(\theta),\omega\right).$$

For this approximation the radiation can be computed explicitly :

Lemma 6.6.1. *We have :*

$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_{\leftarrow})F^T||^2 = \langle f, e^{\frac{2\pi}{\kappa_+}H_{\leftarrow}} \left(1+e^{\frac{2\pi}{\kappa_+}H_{\leftarrow}}\right)^{-1} f\rangle.$$

The strategy of the proof is now the following :

i) We decouple the problem at infinity from the problem near the horizon by cut-off functions. The problem at infinity is easy to treat.

ii) We consider $U(t,T)f$ on a characteristic hypersurface Λ. The resulting characteristic data is denoted g^T. We will approximate $\Omega_{\leftarrow}^{-}f$ by a function $(\Omega_{\leftarrow}^{-}f)_R$ with compact support and higher regularity in the angular derivatives. Let $U_{\leftarrow}(s,t)$ be the isometric propagator associated to the asymptotic hamiltonian $H_{\leftarrow}$ with MIT boundary conditions. We also consider $U_{\leftarrow}(t,T)(\Omega_{\leftarrow}^{-}f)_R$ on Λ. The resulting characteristic data is denoted $g^T_{\leftarrow,R}$. The situation for the asymptotic comparison dynamics is shown in Figure 6.1.

iii) We solve a characteristic Cauchy problem for the Dirac equation with data $g^T_{\leftarrow,R}$. The solution at time zero can be written in a region near the boundary as

$$G(g^T_{\leftarrow,R}) = U(0,T/2+c_0)\Phi(T/2+c_0), \tag{6.6.1}$$

where Φ is the solution of a characteristic Cauchy problem in the whole space (without the star). The solutions of the characteristic problems for the asymptotic hamiltonian are written in a similar way and denoted respectively $G_{\leftarrow}(g^T_{\leftarrow,R})$ and $\Phi_{\leftarrow}$.

iv) Using the asymptotic completeness result we show that $g^T - g^T_{\leftarrow,R} \to 0$ when $T,R\to\infty$. By continuous dependence on the characteristic data we see that :

$$G(g^T) - G(g^T_{\leftarrow,R}) \to 0, T, R\to\infty.$$

v) We write

$$\begin{aligned} G(g^T_{\leftarrow,R}) - G_{\leftarrow}(g^T_{\leftarrow,R}) &= U(0,T/2+c_0)(\Phi(T/2+c_0) - \Phi_{\leftarrow}(T/2+c_0)) \\ &\quad + (U(0,T/2+c_0) - U_{\leftarrow}(0,T/2+c_0))\Phi_{\leftarrow}(T/2+c_0). \end{aligned}$$

The first term becomes small near the boundary when T becomes large. We then note that for all $\epsilon>0$ there exists $t_\epsilon>0$ s.t.

$$||(U(t_\epsilon,T/2+c_0) - U_{\leftarrow}(t_\epsilon,T/2+c_0))\Phi_{\leftarrow}(T/2+c_0)|| < \epsilon$$

uniformly in T when T is large. We fix the angular momentum $D_\varphi = n$. The function $U_{\leftarrow}(t_\epsilon,T/2+c_0)\Phi_{\leftarrow}(T/2+c_0)$ will be replaced by a geometric optics approximation $F^T_{t_\epsilon}$ which has the following properties :

$$\text{supp}\, F^T_{t_\epsilon} \subset (-t_\epsilon - |\mathcal{O}(e^{-\kappa_+ T})|, -t_\epsilon), \tag{6.6.2}$$

$$F^T_{t_\epsilon} \rightharpoonup 0,\ T\to\infty, \tag{6.6.3}$$

$$\forall\lambda>0 \quad Op(\chi(\langle\xi\rangle \le \lambda\langle q\rangle))F^T_{t_\epsilon} \to 0,\ T\to\infty. \tag{6.6.4}$$

Here ξ and q are the dual coordinates to $\hat{r},\theta$ respectively.

vi) We show that for λ sufficiently large possible singularities of $Op(\chi(\langle\xi\rangle \geq \lambda\langle q\rangle))F^T_{t_\epsilon}$ are transported by the group $e^{-it_\epsilon H}$ in such a way that they always stay away from the surface of the star.

vii) From the points *i) to v) follows :*

$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_0)j_-U(0,T)f||_0^2 = \lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_0)U(0,t_\epsilon)F^T_{t_\epsilon}||_0^2,$$

where j_- is a smooth cut-off which equals 1 near the boundary and 0 at infinity. Let ϕ_δ be a cut-off outside the surface of the star at time 0. If $\phi_\delta = 1$ sufficiently close to the surface of the star at time 0 we see by the previous point that

$$(1-\phi_\delta)e^{-it_\epsilon H}F^T_{t_\epsilon} \to 0,\ T \to \infty. \tag{6.6.5}$$

Using (6.6.5) we show that (modulo a small error term):

$$(U(0,t_\epsilon) - \phi_\delta e^{-it_\epsilon H})F^T_{t_\epsilon} \to 0,\ T \to \infty.$$

Therefore it remains to consider :

$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_0)\phi_\delta e^{-it_\epsilon H}F^T_{t_\epsilon}||_0.$$

viii) We show that we can replace $\mathbf{1}_{[0,\infty)}(H_0)$ by $\mathbf{1}_{[0,\infty)}(H)$. This will essentially allow us to commute the energy cut-off and the group. We then show that we can replace the energy cut-off by $\mathbf{1}_{[0,\infty)}(H_\leftarrow)$. We end up with :

$$\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(H_\leftarrow)e^{-it_\epsilon H_\leftarrow}F^T_{t_\epsilon}||, \tag{6.6.6}$$

which we have already computed.

6.7 Propagation estimates

In this section we state several propagation estimates which are needed for the proof of the main theorem. We start with the maximal velocity estimate :

Lemma 6.7.1. *Let $J \in C_b^\infty(\mathbb{R})$, $\operatorname{supp} J \subset]-\infty,-1-\epsilon] \cup [1+\epsilon,\infty[$ for some $\epsilon > 0$. Then we have :*

$$(i)\ \int_1^\infty ||J\left(\tfrac{\hat{r}}{t}\right)e^{itH}\phi||^2\tfrac{dt}{t} \lesssim ||\phi||^2,$$
$$(ii)\ \ s-\lim_{t\to\pm\infty} J\left(\tfrac{\hat{r}}{t}\right)e^{itH} = 0.$$

The minimal velocity estimate is given by the following lemma :

Lemma 6.7.2. *Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\chi \subset \mathbb{R}\setminus\{-m,m\}$. Then there exists a strictly positive constant ϵ_χ such we have :*

$$\int_1^\infty ||\mathbf{1}_{[0,\epsilon_\chi]}\left(\frac{|\hat{r}|}{t}\right)e^{itH}\chi(H)\Phi||^2\frac{dt}{t} \leq C_\chi||\Phi||^2.$$

Furthermore

$$s-\lim_{t\to\infty}\mathbf{1}_{[0,\epsilon_\chi]}\left(\frac{|\hat{r}|}{t}\right)e^{itH}\chi(H) = 0.$$

It turns out that we need a stronger minimal velocity estimate near the horizon :

Lemma 6.7.3. *Let* $m \in C(\mathbb{R}^+)$, $0 < \epsilon < 1$, $\lim_{t\to\infty} m(t) = \infty$, $0 \le m(|t|) \le \epsilon|t|$ *for all* $|t| \ge 1$. *Then*

$$s - \lim_{t\to\pm\infty} \mathbf{1}_{[0,1]}\left(\frac{|\hat{r}|}{|t| - m(|t|)}\right) e^{itH}\mathbf{1}_{\mathbb{R}^\mp}(P^\pm) = 0. \tag{6.7.1}$$

An analogous result holds if we replace $e^{itH}\mathbf{1}_{\mathbb{R}^\mp}(P^\pm)$ *by* $e^{itH_\leftarrow}$.

6.8 The characteristic Cauchy problem

In this section we study a characteristic Cauchy problem for the Dirac equation in the whole exterior Kerr-Newman space-time. These results can be used to solve the characteristic Cauchy problem near the collapsing star and to write the solution in the form (6.6.1) (see [58] for details). Our strategy is similar to that of Hörmander in [65] for the characteristic Cauchy problem for the wave equation. Let (see Figure 6.4) :

$$\begin{aligned}
\Lambda_T^\pm &:= \{(\pm\hat{r}, \hat{r}, \omega);\ 0 \le \pm\hat{r} \le T,\ \omega \in S^2\}, \Lambda_T := \Lambda_T^+ \cup \Lambda_T^-,\\
K_T &:= \{(t, \hat{r}, \omega);\ |\hat{r}| \le T,\ |\hat{r}| \le t \le T,\ \omega \in S^2\},\\
\Sigma_T &:= \{(T, \hat{r}, \omega);\ |\hat{r}| \le T,\ \omega \in S^2\}.
\end{aligned}$$

We need the spaces :

$$\begin{aligned}
\mathcal{H}_T &:= L^2(([-T, T] \times S^2, d\hat{r}d\omega); \mathbb{C}^4),\\
\mathcal{H}_T^1 &:= \{u \in \mathcal{H}_T;\ Hu \in \mathcal{H}_T\},\ ||u||^2_{\mathcal{H}_T^1} = ||u||^2_{\mathcal{H}_T} + ||Hu||^2_{\mathcal{H}_T},\\
L^2_{T,-} &:= L^2(([-T, 0] \times S^2, d\hat{r}d\omega); \mathbb{C}^2),\\
L^2_{T,+} &:= L^2(([0, T] \times S^2, d\hat{r}d\omega); \mathbb{C}^2).
\end{aligned}$$

Let $\Phi_T \in C^\infty(\Sigma_T; \mathbb{S}_A \oplus \mathbb{S}^{A'})$. By the usual theorems for hyperbolic equations we can associate to Φ_T a smooth solution $\phi_A \oplus \chi^{A'} \in C^\infty(K_T; \mathbb{S}_A \oplus \mathbb{S}^{A'})$. We use the l^a, n^a, m^a tetrad and the $(t, \hat{r}, \omega)$ coordinate system. Let $\Psi = \sqrt[4]{\frac{\rho^2\Delta\sigma^2}{(r^2+a^2)^2 k'^2}}(\phi_0, \phi_1, \chi_{1'}, -\chi_{0'})$ be the associated density spinor. The spinor fields o^A and ι^A are smooth and non vanishing on $\Lambda_T^\pm$, therefore we can associate to this solution the smooth trace of Ψ :

$$\mathcal{T} :\ \Phi_T \mapsto (\Psi_2, \Psi_3)(-\hat{r}, \hat{r}, \omega) \oplus (\Psi_1, \Psi_4)(\hat{r}, \hat{r}, \omega) \in C^\infty([-T, 0] \times S^2; \mathbb{C}^2) \oplus C^\infty([0, T] \times S^2; \mathbb{C}^2).$$

Using the conserved current we obtain by Stokes' theorem :

$$\begin{aligned}
\int_{\Sigma_T} *(\phi_A\bar{\phi}_{A'}dx^{AA'} + \bar{\chi}_A\chi_{A'}dx^{AA'}) &= \sqrt{2}\int_{-T}^0\int_{S^2}(|\Psi_2|^2 + |\Psi_3|^2)(-\hat{r}, \hat{r}, \omega)d\hat{r}d\omega\\
&+ \sqrt{2}\int_0^T\int_{S^2}(|\Psi_1|^2 + |\Psi_4|^2)(\hat{r}, \hat{r}, \omega)d\hat{r}d\omega. \qquad (6.8.1)
\end{aligned}$$

Therefore the operator $\mathcal{T}$ possesses an extension to a bounded operator

$$\mathcal{T} \in \mathcal{L}(L^2(\Sigma_T; \mathbb{S}_A \oplus \mathbb{S}^{A'}); L^2([-T, 0] \times S^2; \mathbb{C}^2) \oplus L^2([0, T] \times S^2; \mathbb{C}^2)).$$

Our first result is

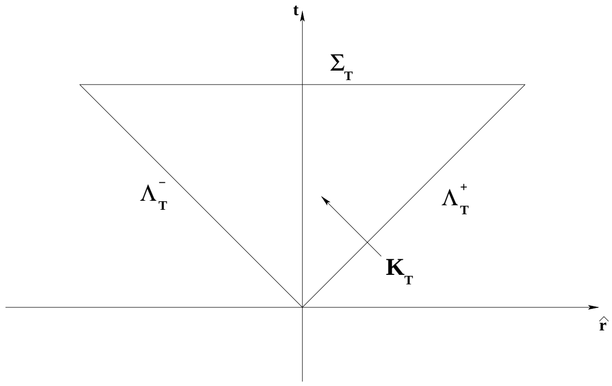


Figure 6.4: The characteristic Cauchy problem

Theorem 6.8.1. $\frac{1}{\sqrt[4]{2}}\mathcal{T}$ *is an isometry.*

The characteristic data $\mathcal{T}(\Phi_T)$ contains information only about $\phi_1, \chi_{1'}$ on Λ_T^- and $\phi_0, \chi_{0'}$ on Λ_T^+. The functions $\phi_0, \chi_{0'}$ on Λ_T^- resp. $\phi_1, \chi_{1'}$ on Λ_T^+ are obtained from the given data by restriction of the equation to $\Lambda_T^{\pm}$. On Λ_T^- we have (see (2.2.7)) :

$$\left.\begin{array}{r} n^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\phi_0 - m^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\phi_1 + (\mu - \gamma)\phi_0 + (\tau - \beta)\phi_1 = \frac{m}{\sqrt{2}}\chi_{1'}, \\ n^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\chi_{0'} - \bar{m}^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\chi_{1'} + (\bar{\mu} - \bar{\gamma})\chi_{0'} + (\bar{\tau} - \bar{\beta})\chi_{1'} = \frac{m}{\sqrt{2}}\phi_1, \end{array}\right\}$$

where $\phi_1, \chi_{1'}$ have to be considered as source terms. Putting $g(\hat{r}, \omega) = \Psi(-\hat{r}, \hat{r}, \omega)$, $\hat{g}(\hat{r}, \omega) = \Psi(\hat{r}, \hat{r}, \omega)$ we find :

$$\left.\begin{array}{r} -D_{\hat{r}} g_{1,4} = ((P_\omega + W)g)_{1,4}, \\ g_{1,4}(0, \omega) = \hat{g}_{1,4}(0, \omega), \end{array}\right\}, \tag{6.8.2}$$

$$\left.\begin{array}{r} D_{\hat{r}} \hat{g}_{2,3} = ((P_\omega + W)\hat{g})_{2,3}, \\ \hat{g}_{2,3}(0, \omega) = g_{2,3}(0, \omega). \end{array}\right\} \tag{6.8.3}$$

Here P_ω, W are defined in (6.4.8), (6.4.9). We understand $\hat{r}$ as a time parameter which goes from $-T$ to 0 for (6.8.2) and from 0 to T for (6.8.3). We write (6.8.2) as

$$\left.\begin{array}{r} \partial_{\hat{r}} g_{1,4} = iA(\hat{r})g_{1,4} + S(\hat{r}), \\ g_{1,4}(0, \omega) = \hat{g}_{1,4}(0, \omega), \end{array}\right\} \tag{6.8.4}$$

where $A(\hat{r})$ is a family of linear operators and $S(\hat{r})$ a source term. For some suitable Y, (6.8.4) possesses a unique Y-valued solution given by :

$$g_{1,4}(\hat{r}, \omega) = V(\hat{r}, 0)\hat{g}_{1,4}(0, \omega) + \int_0^{\hat{r}} V(\hat{r}, \hat{r}')S(\hat{r}')d\hat{r}'. \tag{6.8.5}$$

For given $g_{2,3}$, $\hat{g}_{1,4}$ we define $g_{1,4}$, $\hat{g}_{2,3}$ as the solutions of the partial differential equations (6.8.2), (6.8.3) and put :

$$g_{2,3}^H(\hat{r}, \omega) := \frac{1}{2}(-D_{\hat{r}} g_{2,3} + ((P_\omega + W)g)_{2,3}), \tag{6.8.6}$$

$$\hat{g}_{1,4}^H(\hat{r}, \omega) := \frac{1}{2}(D_{\hat{r}} \hat{g}_{1,4} + ((P_\omega + W)\hat{g})_{1,4}). \tag{6.8.7}$$

As $g_{1,4}$ is a Y-valued solution we have (the argument for $\hat{g}_{2,3}$ is analogous):

$$g^H_{2,3} \in L^2(([-T,0]\times S^2, d\hat{r}d\omega);\mathbb{C}^2),\ \hat{g}^H_{1,4} \in L^2(([0,T]\times S^2, d\hat{r}d\omega);\mathbb{C}^2).$$

We define $\tilde{H}^1$ as the completion of $C^\infty([-T,0]\times S^2;\mathbb{C}^2)\oplus C^\infty([0,T]\times S^2;\mathbb{C}^2)$ in the norm :

$$||(g_{2,3},\hat{g}_{1,4})||^2_{\tilde{H}^1} = 2(||g_{2,3}||^2_{L^2_{T,-}} + ||\hat{g}_{1,4}||^2_{L^2_{T,+}} + ||g^H_{2,3}||^2_{L^2_{T,-}} + ||\hat{g}^H_{1,4}||^2_{L^2_{T,+}}). \tag{6.8.8}$$

We now start with $\Psi_T \in C^\infty(\Sigma_T;\mathbb{C}^4)$. Then we can associate a classical solution $\Psi \in C^\infty(K_T;\mathbb{C}^4)$ and the traces $g_{2,3}(\hat{r},\omega) = \Psi_{2,3}(-\hat{r},\hat{r},\omega)$, $-T\le\hat{r}\le 0$ and $\hat{g}_{1,4}(\hat{r},\omega) = \Psi_{1,4}(\hat{r},\hat{r},\omega)$, $0\le\hat{r}\le T$ are well defined. By the previous discussion $g_{1,4}(\hat{r},\omega) = \Psi_{1,4}(-\hat{r},\hat{r},\omega)$, $-T\le\hat{r}\le 0$ and $\hat{g}_{2,3}(\hat{r},\omega) = \Psi_{2,3}(\hat{r},\hat{r},\omega)$, $0\le\hat{r}\le T$ are solutions of equations (6.8.2), (6.8.3). As Ψ is a solution of the Dirac equation, so is $H\Psi \in C^\infty(K_T;\mathbb{C}^4)$. We compute $H\Psi|_{\Lambda^\pm_T}$ in terms of $g_{2,3}$, $\hat{g}_{1,4}$ and find :

$$\begin{aligned}(H\Psi)_{2,3}(-\hat{r},\hat{r},\omega) &= \frac{1}{2}(-D_{\hat{r}}g_{2,3} + ((P_\omega+W)g)_{2,3}),\\ (H\Psi)_{1,4}(\hat{r},\hat{r},\omega) &= \frac{1}{2}(D_{\hat{r}}\hat{g}_{1,4} + ((P_\omega+W)\hat{g})_{1,4}).\end{aligned}$$

Using the identity (6.8.1) we find :

$$||H\Psi_T||^2_{\mathcal{H}_T} = 2(||g^H_{2,3}||^2_{L^2_{T,-}} + ||\hat{g}^H_{1,4}||^2_{L^2_{T,+}})$$

and therefore :

$$||\Psi_T||^2_{\mathcal{H}^1_T} = ||(g_{2,3},\hat{g}_{1,4})||^2_{\tilde{H}^1}. \tag{6.8.9}$$

This means that the trace operator

$$\mathcal{T}\ :\ \begin{array}{ccc} C^\infty(\Sigma_T;\mathbb{C}^4) & \to & C^\infty([-T,0]\times S^2;\mathbb{C}^2)\oplus C^\infty([0,T]\times S^2;\mathbb{C}^2)\\ \Psi_T & \mapsto & (\Psi_{2,3}(-\hat{r},\hat{r},\omega),\Psi_{1,4}(\hat{r},\hat{r},\omega))\end{array}$$

extends to a bounded operator

$$\mathcal{T}_H \in \mathcal{L}(\mathcal{H}^1_T;\tilde{H}^1).$$

Our second result is

Theorem 6.8.2. *$\mathcal{T}_H$ is an isometry.*

6.9 Comparison of the dynamics

In this section we give two fundamental estimates which are used in step *iv)* and step *v)* of the proof of the theorem. Let $\mathcal{J}\in C^\infty_b(\mathbb{R})$, $0<a<b<1$ and

$$\mathcal{J}(\hat{r}) = \begin{cases} 1 & \hat{r}\le a\\ 0 & \hat{r}>b\end{cases}$$

The first estimate concerns the comparison of the characteristic data. Let

$$\begin{aligned} g^T(t,\omega) &:= (P_{2,3}U(t,T)f)(-t+1,\omega),\\ g^T_{\leftarrow}(t,\omega) &:= (P_{2,3}U_{\leftarrow}(t,T)\Omega^-_{\leftarrow}f)(-t+1,\omega).\end{aligned}$$

Lemma 6.9.1. *We have :*

$$\int_0^\infty \int_{S^2} |g^T(t,\omega) - g^T_{\leftarrow}(t,\omega)|^2 dt d\omega \to 0,\ T \to \infty.$$

The proof of the lemma uses the asymptotic completeness result. Note that the time interval would be finite if both dynamics satisfied Huygens' principle and $\Omega^-_{\leftarrow} f$ had compact support. In this case we could just use the Sobolev embedding $H^1 \hookrightarrow L^\infty$ in dimension one and show strong convergence of the inverse wave operators in $H^1(\mathbb{R}; (L^2(S^2))^4)$. Instead of Huygens' principle which is not satisfied we use Lemma 6.7.3. This argument is also sufficient to treat the difficulty linked to the fact that the support of $\Omega^-_{\leftarrow} f$ is in general not compact. The function $m(t)$ will be adapted to the convergence rate of the inverse wave operators.

The second estimate concerns the comparison with the asymptotic dynamics.

Lemma 6.9.2. *We have uniformly in $t_\epsilon \geq 0$:*

$$||\mathcal{J}(\hat{r}+t_\epsilon)U(t_\epsilon, T/2+c_0)(\Phi^{R,N}(T/2+c_0,.) - \Phi^{R,N}_{\leftarrow}(T/2+c_0,.))||_{\mathcal{H}_{t_\epsilon}} \to 0,\ T \to \infty.$$

Here $\Phi^{R,N}$ and $\Phi^{R,N}_{\leftarrow}$ are some approximations of the solution of the characteristic Cauchy problem in the whole space. The proof follows ideas similar to those in the proof of Lemma 6.9.1.

6.10 Study of the hamiltonian flow

In this subsection we study the hamiltonian flow of

$$P = h^2\sqrt{k'^2\xi^2 + a_0^2(l'\xi+q)^2}. \tag{6.10.1}$$

We obtain this principal symbol if we fix $\partial_\varphi = in$ in the expression of H and then diagonalize it, ξ is dual to $\hat{r}$ and q is dual to θ. Let for $L > 0$

$$\mathcal{E}_L := \left\{(\hat{r},\theta;\xi,q);\ \frac{\xi}{|q|} \geq L\right\},$$
$$\mathcal{I}_L^{t_0} := \left\{(\hat{r},\theta;\xi,q);\ \hat{r} \geq -t_0 - L^{-1}\right\}.$$

Lemma 6.10.1. *For $t_0 > 0$ sufficiently large there exist $\delta > 0,\ L_0 > 0$ s.t. for all $L \geq L_0$ we have:*

$$\forall 0 \leq s \leq t_0 \quad \phi_s(\mathcal{I}_L^{t_0} \cap \mathcal{E}_L) \subset \{(\hat{r},\theta;\xi,q);\ \hat{r} \geq \hat{z}(t_0-s,\theta)+\delta\}.$$

Because of the fixed angular momentum we are in a situation with angular momentum 0 for the hamiltonian flow of the principal symbol. Thus we can associate a variable $\hat{r}_{\mathcal{K}}$ with $\partial_t \hat{r}_{\mathcal{K}} = 1$. We find

$$\frac{\mathcal{K}}{E^2} = a^2 + \mathcal{O}(L^{-1}).$$

Without the error term we have therefore $\hat{r}_{\mathcal{K}} = \hat{r}$ and the lemma is obvious. The proof of the lemma is then essentially an argument of continuous dependence on the initial conditions.

Chapter 7

Decay and non-decay of the local energy for the wave equation on the De Sitter–Schwarzschild metric

7.1 Introduction

In this chapter, which summarizes results obtained in collaboration with J.-F. Bony (see [22]), we describe an expansion of the solution of the wave equation on the De Sitter–Schwarzschild metric in terms of resonances. Resonances correspond to the frequencies and rates of dumping of signals emitted by the black hole in the presence of perturbations (see [27, Chapter 4.35]). On the one hand these resonances are today an important hope of effectively detecting the presence of a black hole as we are theoretically able to measure the corresponding gravitational waves. On the other hand, the distance of the resonances to the real axis reflects the stability of the system under the perturbation: larger distances correspond to more stability. In particular the knowledge of the localization of resonances gives precise information about the decay of the local energy and its rate. The aim of the present chapter is to show how this method applies to the simplest model of a black hole: the De Sitter–Schwarzschild black hole.

Thanks to the work of Sá Barreto and Zworski ([93]) we have a very good knowledge of the localization of resonances for the wave equation on the De Sitter–Schwarzschild metric. Using their results we can describe an expansion of the solution of the wave equation on the De Sitter–Schwarzschild metric in terms of resonances. The principal term in the expansion is due to a resonance at 0. The error term decays polynomially if we permit a logarithmic derivative loss in the angular directions and exponentially if we permit an ε derivative loss in the angular directions. For initial data in the complement of a one-dimensional space the local energy is integrable if we permit a $(\ln\langle -\Delta_\omega \rangle)^\alpha$ derivative loss with $\alpha > 1$. This estimate is almost optimal in the sense that it becomes false for $\alpha < \frac{1}{2}$.

The method presented in this chapter does not directly apply to the Schwarzschild case. This is not linked to the difficulty of the photon sphere which we treat in this chapter, but to the possible accumulation of resonances at the origin in the Schwarzschild case.

7.2 Results

The exterior of the De Sitter–Schwarzschild black hole is given by

$$(\mathcal{M}, g), \quad \mathcal{M} = \mathbb{R}_t \times X \text{ with } X =]r_-, r_+[_r \times \mathbb{S}^2_\omega \tag{7.2.1}$$

$$g = \alpha^2 dt^2 - \alpha^{-2} dr^2 - r^2 d\omega^2, \quad \alpha = \Big(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\Big)^{1/2}, \tag{7.2.2}$$

where $M > 0$ is the mass of the black holes and $\Lambda > 0$ with $9M^2\Lambda < 1$ is the cosmological constant. r_- and r_+ are the two positive roots of $\alpha = 0$. We also denoted by $d\omega^2$ the standard metric on $\mathbb{S}^2$.

The corresponding d'Alembertian is

$$\square_g = \alpha^{-2}\big(D_t^2 - \alpha^2 r^{-2} D_r(r^2\alpha^2) D_r + \alpha^2 r^{-2}\Delta_\omega\big), \tag{7.2.3}$$

where $D_\bullet = \frac{1}{i}\partial_\bullet$ and $-\Delta_\omega$ is the positive Laplacian on $\mathbb{S}^2$. We also denote

$$\widehat{P} = \alpha^2 r^{-2} D_r(r^2\alpha^2) D_r - \alpha^2 r^{-2}\Delta_\omega,$$

the operator on X which governs the situation on $L^2(X, r^2\alpha^{-2} dr\, d\omega)$. We define

$$P = r\widehat{P}r^{-1},$$

on $L^2(X, \alpha^{-2} dr\, d\omega)$, and, in the coordinates (r, ω), we have

$$P = \alpha^2 D_r(\alpha^2 D_r) - \alpha^2 r^{-2}\Delta_\omega + r^{-1}\alpha^2(\partial_r \alpha^2).$$

We introduce the Regge–Wheeler coordinate given by

$$x'(r) = \alpha^{-2}. \tag{7.2.4}$$

In the coordinates (x, ω), the operator P is given by

$$P = D_x^2 - \alpha^2 r^{-2}\Delta_\omega + \alpha^2 r^{-1}(\partial_r\alpha^2), \tag{7.2.5}$$

on $L^2(\mathbb{R} \times \mathbb{S}^2, dx\, d\omega)$. Let $V = \alpha^2 r^{-2}$ and $W = \alpha^2 r^{-1}(\partial_r \alpha^2)$ be the potentials appearing in P. The potential V is exponentially decaying at $\pm\infty$. It is of the following form :

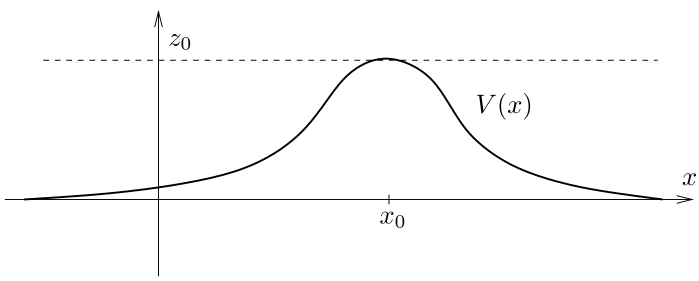


Figure 7.1: The potential $V(x)$.

In particular, V admits at x_0 a non-degenerate maximum at energy $z_0 > 0$. As stated in Proposition 2.1 of [93], the work of Mazzeo–Melrose [79] implies that for $\chi \in C_0^\infty(\mathbb{R})$

$$R_\chi(\lambda) = \chi(x)(P - \lambda^2)^{-1}\chi(x)$$

has a meromorphic extension from the upper half plane to $\mathbb{C}$, whose poles λ are called resonances. The set of the resonances is denoted by $\operatorname{Res} P$. We recall the main result of [93]:

Theorem 7.2.1 (Sá Barreto–Zworski). *There exist $K > 0$ and $\theta > 0$ such that for any $C > 0$ there exists an injective map, $\tilde{b}$, from the set of pseudo-poles*

$$\frac{(1 - 9\Lambda M^2)^{\frac{1}{2}}}{3^{\frac{3}{2}} M} \left(\pm\mathbb{N} \pm \frac{1}{2} - i\frac{1}{2}\left(\mathbb{N}_0 + \frac{1}{2}\right)\right),$$

into the set of poles of the meromorphic continuation of $(P - \lambda^2)^{-1} : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ such that all the poles in

$$\Omega_C = \{\lambda : \operatorname{Im}\lambda > -C,\ |\lambda| > K,\ \operatorname{Im}\lambda > -\theta|\operatorname{Re}\lambda|\},$$

are in the image of $\tilde{b}$ and for $\tilde{b}(\mu) \in \Omega_C$,

$$\tilde{b}(\mu) - \mu \to 0 \quad \text{as} \quad |\mu| \to \infty.$$

If $\mu = \mu^{\pm}_{\ell,j} = 3^{-\frac{3}{2}} M^{-1}(1-9\Lambda M^2)^{\frac{1}{2}}\big((\pm\ell\pm\frac{1}{2})-i\frac{1}{2}(j+\frac{1}{2})\big)$, $\ell \in \mathbb{N}$, $j \in \mathbb{N}_0$, then the corresponding pole, $\tilde{b}(\mu)$, has multiplicity $2\ell + 1$.

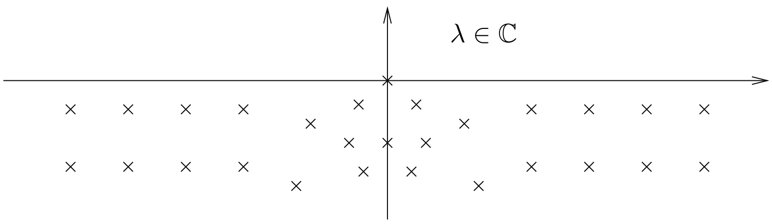


Figure 7.2: The resonances of P near the real axis.

The natural energy space $\mathcal{E}$ for the wave equation is given by the completion of $C_0^\infty(\mathbb{R} \times \mathbb{S}^2) \times C_0^\infty(\mathbb{R} \times \mathbb{S}^2)$ in the norm

$$\|(u_0, u_1)\|^2_{\mathcal{E}} = \|u_1\|^2 + \langle P u_0, u_0\rangle. \tag{7.2.6}$$

It turns out that this is not a space of distributions. The problem is very similar to the problem for the wave equation in dimension 1. We therefore introduce another energy space $\mathcal{E}^{\mathrm{mod}}_{a,b}$ $(-\infty < a < b < \infty)$ defined as the completion of $C_0^\infty(\mathbb{R}\times\mathbb{S}^2)\times C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$ in the norm

$$\|(u_0, u_1)\|^2_{\mathcal{E}^{\mathrm{mod}}_{a,b}} = \|u_1\|^2 + \langle P u_0, u_0\rangle + \int_a^b \int_{\mathbb{S}^2} |u_0(s,\omega)|^2 ds\, d\omega.$$

Note that for any $-\infty < a < b < \infty$ the norms $\mathcal{E}^{\mathrm{mod}}_{a,b}$ and $\mathcal{E}^{\mathrm{mod}}_{0,1}$ are equivalent. We will therefore only work with the space $\mathcal{E}^{\mathrm{mod}}_{0,1}$ in the future and note it from now on $\mathcal{E}^{\mathrm{mod}}$. Let us write the wave equation $\Box_g u = 0$ as a first order system in the following way:

$$\begin{cases} i\partial_t v = Lv \\ v(0) = v_0 \end{cases} \quad \text{with} \quad L = \begin{pmatrix} 0 & i \\ -iP & 0 \end{pmatrix}.$$

Let $\mathcal{H}^k$ be the scale of Sobolev spaces associated to P. We denote $\mathcal{H}^2_c$ the completion of $\mathcal{H}^2$ in the norm $\|u\|_2^2 := \langle Pu, u\rangle + \|Pu\|^2$. Then $(L, D(L) = \mathcal{H}^2_c \oplus \mathcal{H}^1)$ is selfadjoint on $\mathcal{E}$. We denote $\mathcal{E}^k$ the scale of Sobolev spaces associated to L. Note that because of

$$(L-\lambda)^{-1} = (P-\lambda^2)^{-1}\begin{pmatrix} \lambda & i \\ -iP & \lambda \end{pmatrix}, \tag{7.2.7}$$

the meromorphic extension of the cut-off resolvent of P entails a meromorphic extension of the cut-off resolvent of L and the resonances of L coincide with the resonances of P.

Recall that $(-\Delta_\omega, H^2(\mathbb{S}^2))$ is a selfadjoint operator with compact resolvent. Its eigenvalues are $\ell(\ell+1)$, $\ell \geq 0$ with multiplicity $2\ell+1$. We denote by

$$P_\ell = r^{-1}D_x r^2 D_x r^{-1} + \alpha^2 r^{-2}\ell(\ell+1) \tag{7.2.8}$$

the operator P restricted to $\mathcal{H}_\ell = L^2(\mathbb{R}) \times Y_\ell$ where Y_ℓ is the eigenspace of the eigenvalue $\ell(\ell+1)$. In the following, P_ℓ will be identified with the operator on $L^2(\mathbb{R})$ given by (7.2.8). The operators L_ℓ and spaces $\mathcal{E}_\ell$, $\mathcal{E}_\ell^{\mathrm{mod}}$, $\mathcal{E}_\ell^k$ are defined in a similar manner to the operator L and the spaces $\mathcal{E}$, $\mathcal{E}^{\mathrm{mod}}$, $\mathcal{E}^k$. Let Π_ℓ be the orthogonal projector on $\mathcal{E}_\ell^{\mathrm{mod}}$. For $\ell \geq 1$, the space $\mathcal{E}_\ell^{\mathrm{mod}}$ and $\mathcal{E}_\ell$ are the same and the norms are uniformly equivalent with respect to ℓ.

Using Proposition II.2 of Bachelot and Motet-Bachelot [14], the group e^{-itL} preserves the space $\mathcal{E}^{\mathrm{mod}}$ and there exist $C, k > 0$ such that

$$\|e^{-itL}u\|_{\mathcal{E}^{\mathrm{mod}}} \leq Ce^{k|t|}\|u\|_{\mathcal{E}^{\mathrm{mod}}}.$$

From the previous discussion, the same estimate holds with $k = 0$ uniformly in $\ell \geq 1$. In particular, $(L-z)^{-1}$ is bounded on $\mathcal{E}^{\mathrm{mod}}$ for $\operatorname{Im} z > k$, and we denote $\mathcal{E}^{\mathrm{mod},-\mathrm{j}} = (L-z)^j\mathcal{E}^{\mathrm{mod}} \subset \mathcal{D}'(\mathbb{R}\times\mathbb{S}^2)\times\mathcal{D}'(\mathbb{R}\times\mathbb{S}^2)$ for $j \in \mathbb{N}_0$.

We first need a result on P:

Proposition 7.2.1. *For $\ell \geq 1$, the operator P_ℓ has no resonance and no eigenvalue on the real axis.*

For $\ell = 0$, P_0 has no eigenvalue in $\mathbb{R}$ and no resonance in $\mathbb{R}\setminus\{0\}$. But, 0 is a simple resonance of P_0, and, for z close to 0, we have

$$(P_0 - z^2)^{-1} = \frac{i\gamma}{z} r\langle r|\ \cdot\ \rangle + H(z), \tag{7.2.9}$$

where $\gamma \in]0, +\infty[$ and $H(z)$ is a holomorphic (bounded) operator near 0. Equation (7.2.9) is an equality between operators from L^2_{comp} to L^2_{loc}.

For $\chi \in C_0^\infty(\mathbb{R})$ we denote henceforth:

$$\widehat{R}_\chi(\lambda) = \chi(x)(L-\lambda)^{-1}\chi(x).$$

For a resonance λ_j we define $m(\lambda_j)$ by the Laurent expansion of the cut-off resolvent near λ_j :

$$\widehat{R}_\chi(\lambda) = \sum_{k=-(m(\lambda_j)+1)}^{\infty} A_k(\lambda-\lambda_j)^k.$$

We also define $\pi^{\chi}_{j,k}$ by

$$\pi^{\chi}_{j,k} = \frac{-1}{2\pi i} \oint \frac{(-i)^k}{k!} \widehat{R}_{\chi}(\lambda)(\lambda - \lambda_j)^k d\lambda. \tag{7.2.10}$$

The main result of this chapter is the following:

Theorem 7.2.2. *Let $\chi \in C_0^\infty(\mathbb{R})$.*

(i) Let $0 < \mu \notin \frac{(1-9\Lambda M^2)^{1/2}}{3^{1/2}M}\frac{1}{2}\left(\mathbb{N}_0 + \frac{1}{2}\right)$ such that there is no resonance with $\operatorname{Im} z = -\mu$. Then there exists $M > 0$ with the following property. Let $u \in \mathcal{E}^{\mathrm{mod}}$ such that $\langle -\Delta_\omega\rangle^M u \in \mathcal{E}^{\mathrm{mod}}$. Then we have:

$$\chi e^{-itL}\chi u = \sum_{\substack{\lambda_j \in \operatorname{Res} P \\ \operatorname{Im}\lambda_j > -\mu}} \sum_{k=0}^{m(\lambda_j)} e^{-i\lambda_j t} t^k \pi^{\chi}_{j,k} u + E_1(t)u, \tag{7.2.11}$$

with

$$\|E_1(t)u\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim e^{-\mu t}\|\langle -\Delta_\omega\rangle^M u\|_{\mathcal{E}^{\mathrm{mod}}}, \tag{7.2.12}$$

and the sum is absolutely convergent in the sense that

$$\sum_{\substack{\lambda_j \in \operatorname{Res} P \\ \operatorname{Im}\lambda_j > -\mu}} \sum_{k=1}^{m(\lambda_j)} \|\pi^{\chi}_{j,k}\langle -\Delta_\omega\rangle^{-M}\|_{\mathcal{L}(\mathcal{E}^{\mathrm{mod}})} \lesssim 1. \tag{7.2.13}$$

(ii) There exists $\varepsilon > 0$ with the following property. Let $g \in C([0, +\infty[)$, $\lim_{|x|\to\infty} g(x) = 0$, positive, strictly decreasing with $x^{-1} \le g(x)$ for x large. Let $u = (u_1, u_2) \in \mathcal{E}^{\mathrm{mod}}$ be such that $\left(g(-\Delta_\omega)\right)^{-1} u \in \mathcal{E}^{\mathrm{mod}}$. Then we have

$$\chi e^{-itL}\chi u = \gamma \begin{pmatrix} r\chi\langle r, \chi u_2\rangle \\ 0 \end{pmatrix} + E_2(t)u, \tag{7.2.14}$$

with

$$\|E_2(t)u\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim g(e^{\varepsilon t})\left\|\left(g(-\Delta_\omega)\right)^{-1} u\right\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{7.2.15}$$

Remark 7.2.1. a) By the results of Sá Barreto and Zworski we know that there exists $\mu > 0$ such that 0 is the only resonance in $\operatorname{Im} z > -\mu$. Choosing this μ in (i) the sum on the right hand side contains a single element which is

$$\gamma \begin{pmatrix} r\chi\langle r, \chi u_2\rangle \\ 0 \end{pmatrix}.$$

b) Again by the paper of Sá Barreto and Zworski we know that $\lambda_j = \widetilde{b}(\mu^{\varepsilon}_{\ell,\widetilde{j}})$ for all the λ_j's outside a compact set (see Theorem 7.2.1). For such λ_j, we have $m_j(\lambda_j) = 0$ and $\pi^{\chi}_{j,k} = \Pi_\ell \pi^{\chi}_{j,k} = \pi^{\chi}_{j,k}\Pi_\ell$ is an operator of rank $2\ell + 1$.

c) Let $\mathcal{E}^{\mathrm{mod},\perp} = \{u \in \mathcal{E}^{\mathrm{mod}};\ \langle r, \chi u_2\rangle = 0\}$. By part (ii) of the theorem, for $u \in \mathcal{E}^{\mathrm{mod},\perp}$, the local energy is integrable if $(\ln\langle -\Delta_\omega\rangle)^\alpha u \in \mathcal{E}^{\mathrm{mod}}$, for some $\alpha > 1$, and decays exponentially if $\langle -\Delta_\omega\rangle^\varepsilon u \in \mathcal{E}^{\mathrm{mod}}$ for some $\varepsilon > 0$.

d) In fact, we can replace $\langle -\Delta_\omega\rangle^M$ by $\langle P\rangle^{2M}$ in the first part of the theorem. And, by an interpolation argument, we can obtain the following estimate: for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\chi e^{-itL}\chi u = \gamma \begin{pmatrix} r\chi\langle r, \chi u_2\rangle \\ 0 \end{pmatrix} + E_3(t)u, \tag{7.2.16}$$

with

$$\|E_3(t)u\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim e^{-\delta t}\big\|\langle P\rangle^\varepsilon u\big\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{7.2.17}$$

Remark 7.2.2. In the Schwarzschild case ($\Lambda = 0$) the potential $V(x)$ is only polynomially decreasing at infinity and we cannot apply the result of Mazzeo–Melrose. Therefore we cannot exclude a possible accumulation of resonances at 0. This difficulty has nothing to do with the presence of the photon sphere which is treated by the method presented in this chapter.

Remark 7.2.3. Let $u \in \mathcal{E}^{\mathrm{mod},\perp}$ be such that $(\ln\langle -\Delta_\omega\rangle)^\alpha u \in \mathcal{E}^{\mathrm{mod}}$ for some $\alpha > 1$. Then we have from part (ii) of the theorem, for $\lambda \in \mathbb{R}$,

$$\Big\|\int_0^\infty \chi e^{-it(L-\lambda)}\chi u\, dt\Big\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim \|(\ln\langle -\Delta_\omega\rangle)^\alpha u\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{7.2.18}$$

This estimate is almost optimal since it becomes false for $\alpha < \frac{1}{2}$. Indeed we have ($\lambda \in \mathbb{R}$):

$$\widehat{R}_\chi(\lambda)u = i\int_0^\infty \chi e^{-it(L-\lambda)}\chi u\, dt.$$

Thus from (7.2.18) we obtain the resolvent estimate

$$\|\widehat{R}_\chi(\lambda)(\ln\langle -\Delta_\omega\rangle)^{-\alpha}\|_{\mathcal{L}(\mathcal{E}^{\mathrm{mod},\perp},\mathcal{E}^{\mathrm{mod}})} \lesssim 1.$$

It is easy to see that this entails the resolvent estimate

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi(\ln\langle \ell(\ell+1)\rangle)^{-\alpha}\| \lesssim \frac{1}{|\lambda|},$$

for $\ell \geq 1$. We introduce the semi-classical parameter $h^2 = (\ell(\ell+1))^{-1}$ and $\widetilde{P} = h^2D_x^2 + V(x) + h^2W(x)$. Then, for $R > 0$, the above estimate gives the semi-classical estimate:

$$\|\chi(\widetilde{P} - z)^{-1}\chi\| \lesssim \frac{|\ln h|^\alpha}{h},$$

for $1/R \leq z \leq R$. Such an estimate is known to be false for $\alpha < \frac{1}{2}$ and $z = z_0$, the maximum value of the potential $V(x)$ (see [1, Proposition 2.2]).

Remark 7.2.4. Let $\mathcal{P}_1$ be the projection on the first variable, $\mathcal{P}_1(u_1, u_2) = u_1$. If $u \in \mathcal{E}^{\mathrm{mod}}$ is such that $\big(g(-\Delta_\omega)\big)^{-1}(L+i)u \in \mathcal{E}^{\mathrm{mod}}$, then $\mathcal{P}_1\chi e^{-itL}\chi u \in C^0(\mathbb{R}\times\mathbb{S}^2)$ and the remainder term in (7.2.14) satisfies

$$\|\mathcal{P}_1E_2(t)u\|_{L^\infty(\mathbb{R}\times\mathbb{S}^2)} \lesssim g(e^{\varepsilon t})\big\|\big(g(-\Delta_\omega)\big)^{-1}(L+i)u\big\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{7.2.19}$$

Moreover, if $u \in \mathcal{E}^{\mathrm{mod}}$ is such that $\big(g(-\Delta_\omega)\big)^{-1}(L+i)^2u \in \mathcal{E}^{\mathrm{mod}}$, then $\chi e^{-itL}\chi u \in C^0((\mathbb{R}\times\mathbb{S}^2)\times(\mathbb{R}\times\mathbb{S}^2))$ and the remainder term in (7.2.14) satisfies

$$\|E_2(t)u\|_{L^\infty((\mathbb{R}\times\mathbb{S}^2)\times(\mathbb{R}\times\mathbb{S}^2))} \lesssim g(e^{\varepsilon t})\big\|\big(g(-\Delta_\omega)\big)^{-1}(L+i)^2u\big\|_{\mathcal{E}^{\mathrm{mod}}}. \tag{7.2.20}$$

7.3 Elements of the proof

The proof of the theorem is based on resolvent estimates. Once these estimates are established we use the Cauchy theorem and contour deformations in the complex plane. Using (7.2.7) we see that it is sufficient to prove resolvent estimates for $\chi(P_\ell - \lambda^2)^{-1}\chi$. We will use the description of the resonances given in Sá Barreto–Zworski [93]. Recall that

$$R_\chi(\lambda) = \chi(P - \lambda^2)^{-1}\chi, \tag{7.3.1}$$

has a meromorphic extension from the upper half plane to $\mathbb{C}$. The resonances of P are defined as the poles of this extension. We treat only the case $\operatorname{Re}\lambda > -1$ since we can obtain the same type of estimates for $\operatorname{Re}\lambda < 1$ using $\big(R_\chi(-\overline{\lambda})\big)^* = R_\chi(\lambda)$.

Theorem 7.3.1. *Let $C_0 > 0$ be fixed. The operators $\chi(P_\ell - \lambda^2)^{-1}\chi$ satisfy the following estimates uniformly in ℓ :*

i) For all $R > 0$, the number of resonances of P is bounded in $B(0, R)$. Moreover, there exists $C > 0$ such that

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi\| \leq \|\chi(P - \lambda^2)^{-1}\chi\| \leq C \prod_{\substack{\lambda_j \in \operatorname{Res} P \\ |\lambda_j| < 2R}} \frac{1}{|\lambda - \lambda_j|} \tag{7.3.2}$$

for all $\lambda \in B(0, R)$. As usual, the resonances are counted with their multiplicity.

ii) For R large enough, P_ℓ has no resonance in $[R, \ell/R] + i[-C_0, 0]$. Moreover, there exists $C > 0$ such that

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi\| \leq \frac{C}{\langle\lambda\rangle^2}, \tag{7.3.3}$$

for $\lambda \in [R, \ell/R] + i[-C_0, C_0]$.

iii) Let R be fixed. For ℓ large enough, the resonances of P_ℓ in $[\ell/R, R\ell] + i[-C_0, 0]$ are the $\widetilde{b}(\mu^+_{\ell,j})$ given in Theorem 7.2.1 (in particular their number is bounded uniformly in ℓ). Moreover, there exists $C > 0$ such that

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi\| \leq C\langle\lambda\rangle^C \prod_{\substack{\lambda_j \in \operatorname{Res} P_\ell \\ |\lambda - \lambda_j| < 1}} \frac{1}{|\lambda - \lambda_j|}, \tag{7.3.4}$$

for $\lambda \in [\ell/R, R\ell] + i[-C_0, C_0]$.

Furthermore, P_ℓ has no resonance in $[\ell/R, R\ell] + i[-\varepsilon, 0]$, for some $\varepsilon > 0$, and we have

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi\| \leq C\frac{\ln\langle\lambda\rangle}{\langle\lambda\rangle} e^{C|\operatorname{Im}\lambda|\ln\langle\lambda\rangle}, \tag{7.3.5}$$

for $\lambda \in [\ell/R, R\ell] + i[-\varepsilon, 0]$.

iv) Let $C_1 > 0$ be fixed. For R large enough, P_ℓ has no resonance in $\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > R\ell,\text{ and } 0 \geq \operatorname{Im}\lambda \geq -C_0 - C_1\ln\langle\lambda\rangle\}$. Moreover, there exists $C > 0$ such that

$$\|\chi(P_\ell - \lambda^2)^{-1}\chi\| \leq \frac{C}{\langle\lambda\rangle} e^{C|\operatorname{Im}\lambda|}, \tag{7.3.6}$$

for $\operatorname{Re}\lambda > R\ell$ and $C_0 \geq \operatorname{Im}\lambda \geq -C_0 - C_1\ln\langle\lambda\rangle$.

The results concerning the localization of the resonances in this theorem are proved in [14] and [93], Figure 7.3 summarizes the different estimates of the resolvent.

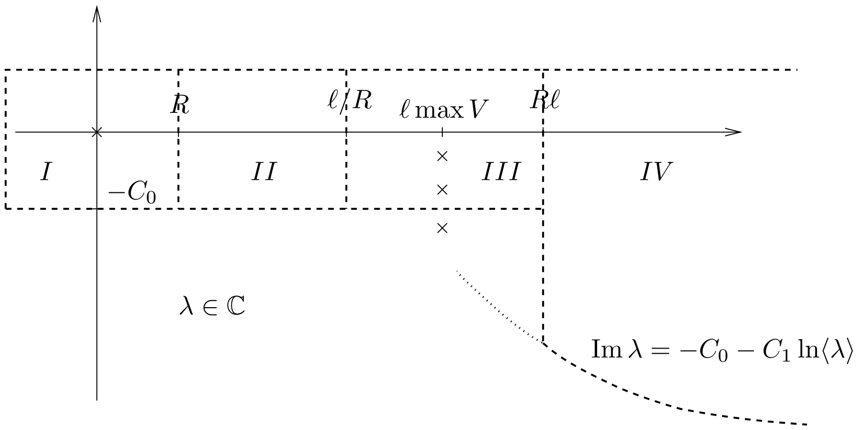


Figure 7.3: The different zones in Theorem 7.3.1.

In zone *I* which is compact, the result of Mazzeo–Melrose [79] gives a bound uniform with respect to ℓ (away from the possible resonances). In particular, part *i)* of Theorem 7.3.1 is a direct consequence of this work.

In zone *II*, the result of Zworski [100] gives us a good (uniform with respect to ℓ) estimate of the resolvent. Here, we use the exponential decay of the potential at $+\infty$ and $-\infty$. By comparison, the corresponding potential for the Schwarzschild metric does not decay exponentially, and our present work cannot be extended to this setting. Note that this problem concerns only zones *I* and *II*, but zones *III* and *IV* can be treated in the same way.

In zone *III*, we have to deal with the so called "photon sphere". The estimate (7.3.4) follows from a general polynomial bound of the resolvent in dimension 1 (see [24]).

In zone *IV*, the potentials $\ell(\ell+1)V$ and W are very small in comparison to λ^2. So they do not play any role, and we obtain the same estimate as in the free case of $-\Delta$ (or as for non trapping geometries).

Chapter 8

The semilinear wave equation on asymptotically euclidean manifolds

In this chapter, which summarizes work in collaboration with J.-F. Bony (see [23]), we study the quadratically semilinear wave equation on an asymptotically euclidean manifold.

8.1 Main results

The manifold that we consider is given by $(\mathbb{R}^d, \mathfrak{g})$, $d \geq 3$, $\mathfrak{g} = \sum_{ij} g_{ij} dx^i dx^j$. We suppose $g_{i,j}(x) \in C^\infty(\mathbb{R}^d)$ and, for some $\rho > 0$,

$$\forall \alpha \in \mathbb{N}^d \qquad \partial_x^\alpha (g_{i,j} - \delta_{i,j}) = \mathcal{O}(\langle x \rangle^{-|\alpha|-\rho}). \tag{H1}$$

We also assume that

$$\mathfrak{g} \text{ is non-trapping.} \tag{H2}$$

Let $g = (\det(\mathfrak{g}))^{1/4}$. The Laplace-Beltrami operator associated to $\mathfrak{g}$ is given by :

$$\Delta_{\mathfrak{g}} = \sum_{ij} \frac{1}{g^2} \partial_i g^{ij} \partial_j,$$

where g^{ij} denotes the inverse metric. Let us consider the following unitary transform :

$$\mathcal{V} : \begin{cases} L^2(\mathbb{R}^d, g^2 dx) & \to L^2(\mathbb{R}^d, dx) \\ \quad v & \mapsto \quad gv \end{cases}$$

The transformation $\mathcal{V}$ sends $-\Delta_{\mathfrak{g}}$ to

$$P = -\mathcal{V} \Delta_{\mathfrak{g}} \mathcal{V}^* = -\sum_{ij} \frac{1}{g} \partial_i g^{ij} \partial_j \frac{1}{g},$$

which is the operator we are interested in. Let $\tilde{\partial}_j := \partial_j \frac{1}{g}$ and $\Omega = \Omega^{k,l} := x_k \partial_l - x_l \partial_k$ be the rotational vector fields. For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer greater than or equal to

x. We consider the following semilinear wave equation :

$$\left.\begin{aligned} \Box_{\mathfrak{g}} u &= Q(u'), \\ (u|_{t=0}, \partial_t u|_{t=0}) &= (u_0, u_1). \end{aligned}\right\} \tag{8.1.1}$$

Here $\Box_{\mathfrak{g}} = \partial_t^2 + P$ and $Q(u')$ is a quadratic form in $u' = (\partial_t u, \tilde{\partial}_x u)$. Our main result is the following

Theorem 8.1.1. *Assume hypotheses* (H1) *and* (H2)*. Suppose* $u_0, u_1 \in C_0^\infty(\mathbb{R}^d)$ *and that, for* $M = 2\left(\left\lceil \frac{d-1}{2} \right\rceil + 1\right)$*, we have*

$$\sum_{|\alpha|+j \le M+1} \left\| \partial_x^j \Omega^\alpha u_0 \right\| + \sum_{|\alpha|+j \le M} \left\| \partial_x^j \Omega^\alpha u_1 \right\| \le \delta. \tag{8.1.2}$$

i) Assume $d \geq 3$ *and* $\rho \geq 1$*. For all* $n > 0$*, there exists a constant* $\delta_n > 0$ *such that, for* $\delta \le \delta_n$*, the problem* (8.1.1) *has a unique solution* $u \in C^\infty([0, T] \times \mathbb{R}^d)$ *with*

$$T = \delta^{-n}.$$

ii) Assume $d \geq 4$ *and* $\rho > 1$*. For* δ *small enough, the problem* (8.1.1) *has a unique global solution* $u \in C^\infty([0, +\infty[\times \mathbb{R}^d)$*.*

Remark 8.1.1. *One may consider more general nonlinearities. For example, the previous result holds for quadratic nonlinearities of the form* $Q(x)(\langle x \rangle^{-\mu} u, u')$ *with* $\mu > 1$ *and* $\|\partial_x^\alpha Q(x)\| = \mathcal{O}(\langle x \rangle^{-|\alpha|})$*. In particular, one can replace* $Q(u')$ *by* $Q(\partial_t u, \partial_x u)$ *or work with the wave equation before the transformation by* $\mathcal{V}$*.*

The main ingredient of the proof are estimates of type (1.0.4). Let us therefore consider the corresponding linear equation :

$$\left.\begin{aligned} \Box_{\mathfrak{g}} v &= G, \\ (v|_{t=0}, \partial_t v|_{t=0}) &= (v_0, v_1). \end{aligned}\right\} \tag{8.1.3}$$

With the notation

$$F_\mu^\varepsilon(T) = \begin{cases} T^{1-2\mu+\varepsilon} & \mu \le 1/2, \\ 1 & \mu > 1/2 \end{cases}$$

we have the following estimate :

Theorem 8.1.2. *Assume that* (H1) *and* (H2) *hold with* $\rho > 0$ *and let* $0 < \mu \le 1$*. For all* $\varepsilon > 0$*, the solution of* (8.1.3) *satisfies*

$$\left\| \langle x \rangle^{-\mu} v' \right\|_{L^2([0,T] \times \mathbb{R}^d)} \lesssim \langle F_\mu^\varepsilon(T) \rangle^{1/2} \left(\|v'(0, \cdot)\|_{L^2(\mathbb{R}^d)} + \int_0^T \|G(s, \cdot)\|_{L^2(\mathbb{R}^d)} ds \right). \tag{8.1.4}$$

It will be important to have higher order estimates. To this purpose let us put $\tilde{\Omega}^{k,\ell} = x_k \tilde{\partial}_\ell - x_\ell \tilde{\partial}_k$, $Z = \{\partial_t, \tilde{\partial}_x, \tilde{\Omega}\}$, $Y = \{\tilde{\partial}_x, \tilde{\Omega}\}$, $X = \{\tilde{\partial}_x\}$, where $\{\tilde{\Omega}\}$ (resp. $\{\tilde{\partial}_x\}$) are the collections of rotational vector fields (resp. partial derivatives with respect to space variables).

Theorem 8.1.3. *Assume that* (H1) *and* (H2) *hold with* $\rho > 1$ *and let* $N > 0$ *and* $1/2 \leq \mu \leq 1$. *For all* $\varepsilon > 0$, *the solution of* (8.1.3) *satisfies*

$$\sup_{0\leq t\leq T} \sum_{1\leq k+j\leq N+1} \big\| \partial_t^k P^{j/2} v(t,\cdot)\big\|_{L^2(\mathbb{R}^d)} + \sum_{|\alpha|\leq N} \langle F_\mu^\varepsilon(T)\rangle^{-1} \big\| \langle x\rangle^{-\mu} Z^\alpha v'\big\|_{L^2([0,T]\times\mathbb{R}^d)}$$
$$\lesssim \sum_{|\alpha|\leq N} \Big(\big\| (Z^\alpha v)'(0,\cdot)\big\|_{L^2(\mathbb{R}^d)} + \int_0^T \big\| Z^\alpha G(s,\cdot)\big\|_{L^2(\mathbb{R}^d)} ds \Big). \tag{8.1.5}$$

Moreover, for $\rho = 1$, *the same inequality holds with* $\langle F_\mu^\varepsilon(T)\rangle^{-1}$ *replaced by* $\langle T\rangle^{-\varepsilon}$.

Note that in Theorems 8.1.1, 8.1.3 $\rho \geq 1$ is required whereas in Theorem 8.1.2 $\rho > 0$ is sufficient.

8.2 The abstract setting

In this section we will show some abstract results for a couple (H, A) of selfadjoint operators on a Hilbert space $\mathcal{H}$ satisfying a Mourre estimate. We will suppose for the whole section $H \in C^2(A)$, that the commutators $ad_A^j H$, $j = 1, 2$ can be extended to bounded operators and that (H, A) satisfies a Mourre estimate on an open interval I :

$$\mathbb{1}_I(H) i[H, A] \mathbb{1}_I(H) \geq \delta \mathbb{1}_I(H), \tag{8.2.1}$$

for some $\delta > 0$. The limiting absorption principle, a result by Kato (see [69]) and an interpolation argument give :

Theorem 8.2.1. *For all closed intervals* $J \subset I$, $\mu > 0$, $0 < \varepsilon < 2\mu$ *and* $u \in \mathcal{H}$ *we have :*

$$\int_0^T \big\| \langle A\rangle^{-\mu} e^{-itH} \mathbb{1}_J(H) u\big\|^2 dt \leq C F_\mu^\varepsilon(T) \|u\|^2.$$

The constant C *only depends on* J, μ, ε, δ *and the norms of the two commutators* $ad_A^j H$, $j = 1, 2$.

We now study the non-homogeneous equation. Let $G(t) \in L^1_{\text{loc}}(\mathbb{R}_t; \mathcal{H})$ be such that $\operatorname{supp} G \subset [0, +\infty[$. We consider the solution u of

$$\left.\begin{aligned} (i\partial_t - H) u(t) &= \varphi(H) G(t) \\ u_{|t=0} &= 0 \end{aligned}\right\} \tag{8.2.2}$$

with $\varphi \in L^\infty(\mathbb{R})$ and $\operatorname{supp} \varphi \subset J$. This means that

$$u(t) = -i \int_0^t e^{-i(t-s)H} \varphi(H) G(s)\, ds, \tag{8.2.3}$$

and then $u \in C^0(\mathbb{R}_t; \mathcal{H}) \cap \mathcal{S}'(\mathbb{R}_t; \mathcal{H})$.

Theorem 8.2.2. *Assume that* $\varphi \in C^1(\mathbb{R})$ *satisfies* $\|\varphi\|_\infty \leq 1$, $\|\varphi'\|_\infty \leq C_1 < \infty$ *and* $\operatorname{supp}\varphi \subset J$, $J \subset I$ *closed. Then, for all* $\mu > 0$, $0 < \varepsilon < 2\mu$ *and* $G(t) \in L^2(\mathbb{R}_t; D(\langle A\rangle^\mu))$ *with* $\operatorname{supp} G \subset [0, +\infty[$,

$$\int_0^T \left\|\langle A\rangle^{-\mu} u(t)\right\|^2 dt \leq C(F_\mu^\varepsilon(T))^2 \int_0^T \left\|\langle A\rangle^{\mu} G(t)\right\|^2 dt.$$

The constant C *only depends on* J, μ, ε, δ, C_1 *and the norms of the two commutators* $\operatorname{ad}_A^j H$, $j = 1, 2$.

We will construct conjugate operators A which we can compare to x. Thus the estimate in Theorem 8.2.1 is very close to the estimate we want to obtain, the principal difference being the energy cut-off.

8.3 The Mourre estimate

As already explained in previous chapters, in order to show a Mourre estimate for the wave equation, we have to show a Mourre estimate for $P^{1/2}$. We divide this section into the study of the low, intermediate and the high frequency part.

8.3.1 Low frequency Mourre estimate

For low frequencies, we will make a dyadic decomposition and use a conjugate operator specific to each part of the decomposition. In this section, we will obtain a Mourre estimate for each part. For $\lambda \geq 1$, we set

$$\mathcal{A}_\lambda = \varphi(\lambda P) A_0 \varphi(\lambda P), \tag{8.3.1}$$

where

$$A_0 = \frac{1}{2}(xD + Dx), \quad D(A_0) = \{u \in L^2(\mathbb{R}^d);\ A_0 u \in L^2(\mathbb{R}^d)\},$$

is the generator of dilations and $\varphi \in C_0^\infty(]0, +\infty[; [0, +\infty[)$ satisfies $\varphi(x) > \delta > 0$ on some open bounded interval $I \subset]0, +\infty[$.

It is well known that $P \in C^1(A_0)$. In particular, $\varphi(\lambda P) : D(A_0) \longrightarrow D(A_0)$ and $\mathcal{A}_\lambda$ is well defined on $D(A_0)$. Its closure, again denoted $\mathcal{A}_\lambda$, is selfadjoint (see [4, Theorem 6.2.5, Lemma 7.2.15]).

Proposition 8.3.1. *i) We have* $(\lambda P)^{1/2} \in C^2(\mathcal{A}_\lambda)$. *The commutators* $\operatorname{ad}^j_{\mathcal{A}_\lambda}(\lambda P)^{1/2}$, $j = 1, 2$, *can be extended to bounded operators and we have, uniformly in* λ,

$$\left\|[\mathcal{A}_\lambda, (\lambda P)^{1/2}]\right\| \lesssim 1, \tag{8.3.2}$$

$$\left\|\operatorname{ad}^2_{\mathcal{A}_\lambda}(\lambda P)^{1/2}\right\| \lesssim \begin{cases} 1 & \rho > 1, \\ \lambda^\varepsilon & \rho \leq 1, \end{cases} \tag{8.3.3}$$

where $\varepsilon > 0$ *can be chosen arbitrary small.*

ii) For λ large enough, we have the following Mourre estimate:

$$\mathbb{1}_I(\lambda P)\big[i(\lambda P)^{1/2}, \mathcal{A}_\lambda\big]\mathbb{1}_I(\lambda P) \geq \frac{\delta^2\sqrt{\inf I}}{2}\mathbb{1}_I(\lambda P). \tag{8.3.4}$$

iii) For $0 \leq \mu \leq 1$ and $\psi \in C_0^\infty(]0, +\infty[)$, we have

$$\big\| |\mathcal{A}_\lambda|^\mu \langle x\rangle^{-\mu}\big\| \lesssim \lambda^{-\mu/2+\varepsilon}, \tag{8.3.5}$$

$$\big\| \langle\mathcal{A}_\lambda\rangle^\mu \psi(\lambda P)\langle x\rangle^{-\mu}\big\| \lesssim \lambda^{-\mu/2+\varepsilon}, \tag{8.3.6}$$

for all $\varepsilon > 0$.

8.3.2 Intermediate frequency Mourre estimate

Here, we obtain a Mourre estimate for frequencies inside the compact interval $[1/C, C]$. For that, we will use a standard argument in scattering theory. Mimicking Section 8.3.1, we set $\mathcal{A} = \varphi(P)A_0\varphi(P)$, where $\varphi \in C_0^\infty(]0, +\infty[; [0, +\infty[)$ with $\varphi = 1$ near $[1/C, C]$. As before, $\mathcal{A}$ is essentially self-adjoint with domain $D(A_0)$ and we denote its closure again by $\mathcal{A}$.

Proposition 8.3.2. *i) We have $P^{1/2} \in C^2(\mathcal{A})$. The commutators $\operatorname{ad}^j_{\mathcal{A}} P^{1/2}$, $j = 1, 2$, can be extended to bounded operators.*

ii) For each $\sigma \in [1/C, C]$, there exists $\delta > 0$ such that

$$\mathbb{1}_{[\sigma-\delta,\sigma+\delta]}(P)\big[iP^{1/2}, \mathcal{A}\big]\mathbb{1}_{[\sigma-\delta,\sigma+\delta]}(P) \geq \frac{1}{2\sqrt{C}}\mathbb{1}_{[\sigma-\delta,\sigma+\delta]}(P). \tag{8.3.7}$$

iii) For $0 \leq \mu \leq 1$, we have

$$\big\|\langle\mathcal{A}\rangle^\mu\langle x\rangle^{-\mu}\big\| \lesssim 1. \tag{8.3.8}$$

8.3.3 High frequency Mourre estimate

For the high frequencies we will use pseudodifferential calculus. Let $p(x, \xi) \in S^{2,0}$ be the symbol of P, and

$$p_0(x, \xi) = \sum_{j,k} g^{j,k}(x)\xi_j\xi_k \in S^{2,0},$$

be its principal part. We have $p - p_0 \in S^{0,0}$. Let

$$\mathrm{H}_{p_0} = \begin{pmatrix} \partial_\xi p_0 \\ -\partial_x p_0 \end{pmatrix},$$

be the Hamiltonian of p_0. Since the metric $\mathfrak{g}$ is non-trapping by assumption, the energy $\{p_0 = 1\}$ is non-trapping for the Hamiltonian flow of p_0. Then, using a result of C. Gérard and Martinez [53], one can construct a function $b(x, \xi) \in S^{1,1}$ such that $b = x \cdot \xi$ for x large enough, and

$$H_{p_0}b \geq \delta, \tag{8.3.9}$$

for some $\delta > 0$ and all $(x,\xi) \in p_0^{-1}([1-\varepsilon, 1+\varepsilon])$, $\varepsilon > 0$. We set $A = \mathrm{Op}(a)$ with

$$a(x,\xi) = b\big(x, (p_0+1)^{-1/2}\xi\big) \in S^{0,1}.$$

Let $f \in C^\infty(\mathbb{R};\mathbb{R})$ be such that $f = 1$ on $[2,+\infty[$ and $f = 0$ on $]-\infty, 1]$. As conjugate operator at high frequency, we choose

$$A_\infty = f(P)Af(P). \tag{8.3.10}$$

Proposition 8.3.3. *i) The operator A_∞ is essentially self-adjoint on $D(\langle x\rangle)$ and*

$$\|A_\infty u\| \lesssim \|\langle x\rangle u\|,$$

for all $u \in D(\langle x\rangle)$.

ii) We have $P^{1/2} \in C^2(A_\infty)$. The commutators $[P^{1/2}, A_\infty]$ and $[[P^{1/2}, A_\infty], A_\infty]$ are in $\Psi^{0,0}$ and can be extended as bounded operator on $L^2(\mathbb{R}^d)$.

iii) For $C > 0$ large enough,

$$\mathbb{1}_{[C,+\infty[}(P) i[P^{1/2}, A_\infty]\mathbb{1}_{[C,+\infty[}(P) \geq \frac{\delta}{8}\mathbb{1}_{[C,+\infty[}(P).$$

8.4 Elements of the proof

Theorem 8.1.2 is a direct consequence of the abstract results of Section 8.2 and the Mourre estimate established in Section 8.3. Let us consider Theorem 8.1.3. The term $\|\partial_t^k P^{j/2}u\|$, $1 \leq k+j \leq N+1$ is controlled using the usual energy estimate and the fact that ∂_t, P commute with the equation. However, the other vector fields do not commute with the equation. In particular we need an additional argument for the vector field $\tilde{\Omega}$. If u satisfies the linear equation, then $\tilde{\Omega}u$ satisfies an equation with an additional $[P,\tilde{\Omega}]u$ on the R.H.S. Let us therefore consider a function v satisfying :

$$\left.\begin{aligned} (\partial_t^2 + P)v &= [P,\tilde{\Omega}]u, \\ (v|_{t=0}, \partial_t v|_{t=0}) &= 0. \end{aligned}\right\}$$

Applying the abstract result Theorem 8.2.2 and the fact that the conjugate operators can be compared to x we find for $0 < \mu \leq 1$:

$$(F_\mu^\epsilon(T))^{-2}\|\langle x\rangle^{-\mu}v'\|^2_{L^2([0,T]\times\mathbb{R}^d)} \lesssim \int_0^T \|\langle x\rangle^\mu [P,\tilde{\Omega}]u\|^2 dt.$$

We only explain the case $\rho > 1$, the case $\rho = 1$ is slightly more technical. We clearly can suppose μ small enough to ensure that $\rho - \mu > 1/2$. Then $\|\langle x\rangle^\mu[P,\tilde{\Omega}]u\|$ is controlled by $\|\langle x\rangle^{-\nu}Pu\|$ with $\nu > 1/2$. But,

$$\int_0^T \|\langle x\rangle^{-\nu}Pu\|^2 dt$$

is estimated by Theorem 8.1.2 and the fact that P commutes with the equation. Proceeding in this way we can prove Theorem 8.1.3 inductively. Once Theorems 8.1.2 and 8.1.3 are established, Theorem 8.1.1 follows from the arguments of Keel, Smith and Sogge (see [71]) in a standard way.

Chapter 9

Discussion

We have developed scattering theory for field equations on a fixed background. The main example was the Kerr-Newman metric, which describes rotating black-holes. The scattering theory is under control for spin 1/2 fields. For entire spin fields superradiance is a major obstacle to developing scattering theory and the scattering results for the Klein-Gordon equation remain partial. This problem should probably be discussed first for simpler examples, see e.g. [12]. Asymptotic completeness results, existence of the asymptotic velocity etc. are of course interesting by themselves, but we also see these results as a starting point to study other questions such as
1) Geometric interpretation of scattering theory,
2) A mathematically precise description of important physical effects such as the Hawking effect,
3) Gravitational waves,
4) Nonlinear problems.
Let us recall what the results were and what we hope for the future :
1) As explained in Chapter 5, the asymptotic completeness result for the Dirac equation can be used to solve the Goursat problem in the Penrose compactification of block I. It is in principle possible to exploit the inverse way. In [78] J.-P. Nicolas and L. Mason show that in asymptotically simple space-times with smooth timelike infinity existence and uniqueness of the solution of the Goursat problem give an asymptotic completeness result. However, in general space-times timelike infinity is a singularity, and in this case the question remains open.
2) In Chapter 6 we give a mathematically precise description of the Hawking effect for fermions in the setting of the collapse of a rotating charged star. Because of superradiance, an analogous result for the boson case seems for the moment out of reach. But even in the fermion case, the model that we have considered, is idealized. In particular the following points should be addressed :
2.1 In Chapter 6 we consider a mirror star for which the boundary is totally reflecting. Even in the Schwarzschild case there is no rigorous result about the Hawking effect where the interior of the star is considered.
2.2 The proof of the Hawking effect in Chapter 6 uses the special structure of the Kerr metric, in particular the separation of variables. Because of the back reaction of the field on the metric, it would be important to have a proof which is independent of this special structure. More

precisely we would like to understand if the effect only depends on the asymptotic structure of the space-time or if the local structure influences the effect.
2.3 The effect is considered in the framework of free quantum field theory. It would be very interesting to take interacting quantum field theory into account. This leads of course to a very large research programme concerning interacting quantum field theory on a curved background. As a preliminary step one could try to couple the system to a detector. Such a model has already been considered for the Unruh effect by De Bièvre, Merkli (see [35]).
3) The expansion of the solution of the wave equation in [22] is important for the understanding of gravitational waves coming from a De Sitter Schwarzschild black hole. It would be interesting to have an analogous result for the De Sitter Kerr metric. One could use the separation of variables for field equations in the De Sitter Kerr metric.
4) As is shown in Chapter 7, the theory of resonances is a very powerful tool to show decay of the local energy. Such decay estimates are of course useful for nonlinear problems. As long as the space-time is fixed, the method is well adapted for the treatment of nonlinear problems. For the full nonlinear stability problem however, it has the drawback that it requires analyticity of the metric. It seems therefore that, for this problem, vector field methods are better adapted. Concerning the Kerr metric however, as we have seen in the previous chapters, a conjugate operator probably needs to be pseudodifferential and can therefore not be represented as a vector field. This is one of the reasons why in our collaboration with J.-F. Bony we wanted to check if nonlinear problems on a curved space-time could be solved using a Mourre method, which is a C^k rather than an analytic method. In Chapter 8 it is shown that this is indeed the case.

Part II

Two remarks on the Schrödinger equation

Chapter 10

Scattering for the Schrödinger equation with a repulsive potential

10.1 Introduction

In this chapter, which summarizes work in collaboration with J.-F. Bony, R. Carles and L. Michel (see [20], [21]), we study scattering theory for the linear Schrödinger equation when the reference Hamiltonian contains a repulsive potential. We consider the family of hamiltonians :

$$\mathbb{H}_{\alpha,0} = -\Delta - \langle x\rangle^{\alpha} \quad \text{with } 0<\alpha\le 2, \text{ and the perturbed hamiltonians} \quad \mathbb{H}_{\alpha} = \mathbb{H}_{\alpha,0} + V_{\alpha},$$

where $\langle x\rangle = \sqrt{1+x^2}$. As we will see, the potential $-\langle x\rangle^{\alpha}$ is a source of acceleration. The case $\alpha = 2$ is critical : if $\alpha > 2$ classical particles can reach infinity in finite time and $(\mathbb{H}_{0,\alpha}, C_0^{\infty}(\mathbb{R}))$ is not essentially selfadjoint (see e.g. [44]).

The principal consequence of the acceleration is that the usual position variable $\langle x\rangle$ increases more rapidly than t along the evolution. This is why we can hope that the usual short range condition

$$|V_0(x)| \lesssim \langle x\rangle^{-1-\varepsilon} \quad (\varepsilon > 0) \tag{10.1.1}$$

can be weakened for $\mathbb{H}_{\alpha}$. This phenomenon has already been observed for Stark type hamiltonians by M. Ben-Artzi [15]. He considers the family of operators

$$\widehat{\mathbb{H}}_{\alpha,0} = -\Delta - \operatorname{sgn}(x_1)|x_1|^{\alpha} \quad 0<\alpha\le 2; \quad \widehat{\mathbb{H}}_{\alpha} = \widehat{\mathbb{H}}_{\alpha,0} + \widehat{V}_{\alpha}(x), \tag{10.1.2}$$

with $x = (x_1, x')$ and proves asymptotic completeness under the condition :

$$|\widehat{V}_{\alpha}(x)| \lesssim M(x').\begin{cases} \langle x_1\rangle^{\alpha-\varepsilon} & \text{for } x_1 \le 0,\\ \langle x_1\rangle^{-1+\alpha/2-\varepsilon} & \text{for } x_1 \ge 0,\end{cases} \tag{10.1.3}$$

with $\varepsilon > 0$ et $M(x') \to 0$ when $|x'| \to \infty$. In dimension 1, we show similar results for $0 < \alpha < 2$ and a weaker short range condition when $\alpha = 2$. In higher dimensions, his methods seem not to apply to our case. For $\alpha = 1$, $\widehat{\mathbb{H}}_{1,0}$ is the Stark hamiltonian associated to a constant electric field in the direction x_1 which was studied in detail (see [5, 89]).

The case $\alpha = 2$ is very instructive. The equations satisfied by the classical particles are very simple and we find $x(t) = x_0 \operatorname{ch}(2t) + \xi_0 \operatorname{sh}(2t)$. Thus, $x(t)$ increases exponentially fast for $\xi_0^2 - x_0^2 \neq 0$, so that a sufficiently short range condition should be :

$$|V_2(x)| \lesssim (1 + \ln\langle x\rangle)^{-1-\varepsilon} \quad (\varepsilon > 0). \tag{10.1.4}$$

More generally we define the new position variable by

$$p_\alpha(x) := \begin{cases} \langle x\rangle^{1-\alpha/2} & \text{if } 0 < \alpha < 2, \\ 1 + \ln\langle x\rangle & \text{if } \alpha = 2. \end{cases}$$

We suppose that $V_\alpha(x) = V_\alpha^1(x) + V_\alpha^2(x)$ where

$$V_\alpha^1(x) \text{ is a measurable function with compact support which is } \Delta\text{–compact,} \tag{10.1.5}$$

and $V_\alpha^2(x) \in L^\infty(\mathbb{R}^n)$ satisfies almost everywhere the short range condition :

$$|V_\alpha^2(x)| \lesssim p_\alpha(x)^{-1-\varepsilon} \quad (\varepsilon > 0). \tag{10.1.6}$$

The operator $\mathbb{H}_\alpha$ is essentially selfadjoint with the domain of the harmonic oscillator and we note again $\mathbb{H}_\alpha$ its selfadjoint extension. $\mathbb{H}_\alpha$ and $\mathbb{H}_{\alpha,0}$ do not have singular continuous spectrum and we note $\mathbf{1}^c(\mathbb{H}_\alpha)$ the projection on the continuous spectrum of $\mathbb{H}_\alpha$. The principal theorem of this chapter is the following :

Theorem 10.1.1. *Under the previous hypotheses, the following limits exist :*

$$\Omega^\pm := s - \lim_{t\to\mp\infty} e^{it\mathbb{H}_\alpha} e^{-it\mathbb{H}_{\alpha,0}}, \tag{10.1.7}$$

$$s - \lim_{t\to\mp\infty} e^{it\mathbb{H}_{\alpha,0}} e^{-it\mathbb{H}_\alpha} \mathbf{1}^c(\mathbb{H}_\alpha). \tag{10.1.8}$$

Furthermore (10.1.8) *equals* $(\Omega^\pm)^*$ *and we have :* $(\Omega^\pm)^*\Omega^\pm = \mathbf{1}, \quad \Omega^\pm(\Omega^\pm)^* = \mathbf{1}^c(\mathbb{H}_\alpha)$.

The operator $\mathbb{H}_{\alpha,0}$ has no point spectrum and the same is true for $\mathbb{H}_\alpha$ if $-\Delta + V_\alpha^1$ satisfies a unique extension theorem.

10.2 Elements of the proof

The proof is based on Mourre theory. The first step is to look for an escape function $a_\alpha(x, \xi)$ satisfying (in an approximate sense)

$$\{\xi^2 - \langle x\rangle^\alpha, a_\alpha(x, \xi)\} = C_0 > 0 \quad \text{when } \xi^2 - \langle x\rangle^\alpha = C_1. \tag{10.2.1}$$

We find

$$a_\alpha(x, \xi) = \begin{cases} x \cdot \xi \langle x\rangle^{-\alpha} & \text{if } 0 < \alpha < 2, \\ \ln\langle x + \xi\rangle - \ln\langle x - \xi\rangle & \text{if } \alpha = 2, \end{cases}$$

and we note that $a_\alpha(x, \xi)$ behaves like $p_\alpha(x)$ on the energy surfaces. In particular, $p_\alpha(x)$ increases like t along the evolution.

Then we prove a Mourre estimate for $\mathbb{H}_\alpha$ with $A_\alpha = \mathrm{Op}(a_\alpha)$. This uses essentially (10.2.1) and the Gårding inequality. Note however that technical difficulties appear because the energy cut-offs are not pseudodifferential operators. Indeed $\chi(\xi^2 - \langle x\rangle^\alpha)$ is not a good symbol (see [21] for details).

From the Mourre estimate we obtain a minimal velocity estimate for A_α and we obtain a minimal velocity estimate for p_α using a theorem due to C. Gérard and F. Nier [54]. To use the theorem we have to modify slightly the conjugate operator by introducing energy cut-offs..

10.3 Asymptotic velocity

Under the same hypotheses it is also possible to construct the asymptotic velocity and to describe its spectrum. More precisely we have the following theorem

Theorem 10.3.1. *There exists a selfadjoint operator P_α^+ commuting with $\mathbb{H}_\alpha$ and such that, for all $g \in C(\mathbb{R})$ going to 0 at infinity,*

$$(i)\ g\big(P_\alpha^+\big) = s - \lim_{t\to\infty} e^{it\mathbb{H}_\alpha} g\Big(\frac{p_\alpha(x)}{t}\Big) e^{-it\mathbb{H}_\alpha}\ ;$$

$$(ii)\ \sigma(P_\alpha^+) = \begin{cases} \{0, \sigma_\alpha\} & \text{if } \sigma_{pp}(\mathbb{H}_\alpha) \neq \emptyset, \\ \{\sigma_\alpha\} & \text{if } \sigma_{pp}(\mathbb{H}_\alpha) = \emptyset, \end{cases} \quad \text{where } \sigma_\alpha = \begin{cases} 2-\alpha & \text{if } 0<\alpha<2, \\ 2 & \text{si } \alpha = 2. \end{cases}$$

Note that if $\sigma_{pp}(\mathbb{H}_\alpha) = \emptyset$, then $p_\alpha(x)$ is equivalent to $\sigma_\alpha t$ along the evolution. In this sense the short range condition (10.1.6) is optimal. Note that for $0 < \alpha < 2$, there exists no explicit formula for $e^{it\mathbb{H}_{\alpha,0}}$. Nevertheless the asymptotic velocity gives a good understanding of the evolution.

10.4 Generalizations

In the case $\alpha = 2$ we can consider a more general hamiltonian. The new reference hamiltonian will be $\mathbb{H}_0 = -\Delta + U(x)$, where U is a real polynomial of degree two. After a change of variables, keeping the Laplacian unchanged, we obtain :

$$U(x) = -\sum_{j=1}^{n_-} \omega_j^2 x_j^2 + \sum_{j=n_-+1}^{n_-+n_+} \omega_j^2 x_j^2 + E x_{n_-+n_++1} + c, \tag{10.4.1}$$

with $\omega_j > 0$ and c, $E \in \mathbb{R}$, where we use the convention that $\sum_{j=p}^{q}(\dots) = 0$ if $p > q$.

We suppose that the perturbative potential V can be written $V = V_1 + V_2 + W$, with :

- $V_j \in L^{p_j}(\mathbb{R}^n; \mathbb{R})$ with $p_j < +\infty$ for $j = 1, 2$;
- $p_j \geq 2$ if $n \leq 3$, $p_j > 2$ if $n = 4$ and $p_j \geq n/2$ if $n \geq 5$;

- $W \in L^\infty(\mathbb{R}^n)$ satisfies

$$|W(x)| \lesssim \prod_{j=1}^{n_-} \big(1+\ln\langle x_j\rangle\big)^{-\beta_j} \langle x_{n_-+n_++1}\rangle^{-\mu\beta_{n_-+n_++1}} \prod_{j=n_-+n_++2}^{n} \langle x_j\rangle^{-\beta_j},$$

  with $\beta_j \geq 0$ and $\sum \beta_j > 1$. $\mu = 1/2$ if $E \neq 0$ and $\mu = 1$ if not.

Under these hypotheses $\mathbb{H} = \mathbb{H}_0 + V$ with domain $C_0^\infty(\mathbb{R}^n)$ is essentially selfadjoint, and we have the following theorem of the existence of wave operators :

Theorem 10.4.1. *(i) If $n_- \geq 1$ and $n_+ \leq 1$, the following strong limits exist in $L^2(\mathbb{R}^n)$:*

$$s - \lim_{t\to\pm\infty} e^{it\mathbb{H}} e^{-it\mathbb{H}_0}.$$

(ii) If V_1 and V_2 are compactly supported and $n_- \geq 1$, then the same conclusions hold.

We also have the following asymptotic completeness theorem :

Theorem 10.4.2. *Suppose that $n_- \geq 1$, $n_- + n_+ = n$, $E = 0$ and $V(x) = V_1(x) + V_2(x)$ where $V_1(x)$ satisfies (10.1.5) and $V_2(x)$ satisfies*

$$|V_2(x)| \lesssim \big(1+\ln\langle (x_1,\dots,x_{n_-})\rangle\big)^{-1-\varepsilon} \quad (\varepsilon > 0).$$

Then the following strong limits exist :

$$s - \lim_{t\to\mp\infty} e^{it\mathbb{H}} e^{-it\mathbb{H}_0}$$
$$s - \lim_{t\to\mp\infty} e^{it\mathbb{H}_0} e^{-it\mathbb{H}} \mathbf{1}^c(\mathbb{H}_\alpha).$$

The proof is similar to that of Theorem 10.1.1. The hypothesis $n_- + n_+ = n$ implies that on the energy surfaces, decay in $(x_1,\dots,x_{n_-})$ gives decay in x.

Chapter 11

Gevrey class regularity for a dissipative Schrödinger-Poisson system

In this chapter, which summarizes the note [59], we are interested in the following dissipative Schrödinger-Poisson system ($\alpha \geq 0$, $\beta \leq 0$, $\gamma \leq 0$, $\epsilon \in \{-1, 1\}$, $\tilde{\epsilon} \in \{0, 1\}$) :

$$i\partial_t \psi_m = -\frac{1}{2}\Delta\psi_m + iH_1\psi_m + V(\Psi)\psi_m, \tag{11.0.1}$$

$$\Delta V = \tilde{\epsilon} \sum \lambda_m ||\psi_m||^2_{L^2} - \epsilon n, \tag{11.0.2}$$

$$H_1 = \alpha\Delta + \beta x^2 + \gamma, \tag{11.0.3}$$

$$n = \sum \lambda_m |\psi_m|^2;\ \lambda_m \geq 0,\ \sum \lambda_m = 1, \tag{11.0.4}$$

$$\Psi = (\psi_m)_{m\in\mathbb{N}}, \tag{11.0.5}$$

$$\Psi(x, 0) = \Phi(x). \tag{11.0.6}$$

We consider this system in dimension 3 either with periodic boundary conditions (where we put $\epsilon = 1$, $\beta = 0$, $\tilde{\epsilon} = 1$), or in the whole space ($\tilde{\epsilon} = 0$). We introduce (see [76]) :

$$X := \{\Gamma = (\gamma_m)_{m\in\mathbb{N}};\ \gamma_m \in L^2 \quad \forall m \in \mathbb{N},\ ||\Gamma||^2_X = \sum \lambda_m ||\gamma_m||^2_{L^2} < \infty\},$$
$$\tilde{X} := \{\Gamma = (\gamma_m)_{m\in\mathbb{N}};\ ||\Gamma||^2_{\tilde{X}} = ||x^2\Gamma||^2_X + ||\Gamma||^2_X < \infty\}.$$

The spaces Y and Z are defined in the same manner starting with H^1 and H^2. Let also $\tilde{Z} := Z \cap \tilde{X}\ (\beta < 0)$ (resp. $\tilde{Z} = Z\ (\beta = 0)$) with the usual norm. For the whole space there exists in the case $\epsilon = 1$ a global existence and uniqueness result in [76]. Our first result concerns global existence and uniqueness in all cases considered here.

Theorem 11.0.3 (Existence and uniqueness)**.** *Let $\Phi \in \tilde{Z}$, $\alpha > 0$, $\beta \leq 0$, $\gamma \leq 0$ or $\alpha = \beta = \gamma = 0$. Then the Schrödinger-Poisson sytem (11.0.1)-(11.0.6) possesses a unique solution (Ψ, n, V) with the following properties :*

$$\Psi \in C([0, \infty); \tilde{Z}) \cap C^1([0, \infty); X);\ n, \Delta V \in C^1([0, \infty); L^1) \cap C([0, \infty); W^{2,1});$$
$$V \in C([0, \infty); L^\infty) \cap L^\infty(\mathbb{R}^3 \times [0, \infty));\ \nabla V \in C([0, \infty); L^2) \cap L^\infty([0, \infty); L^2).$$

Our next result concerns the Gevrey class regularity. Let $B = 1 - \Delta$, $D(B) = Z$. For $T_* > 0$, we put $\varphi(t) = \min\{t, T_*\}$.

Theorem 11.0.4 (Gevrey class regularity)**.** *Let $\alpha > 0$, $\beta \leq 0$, $\gamma \leq 0$ and let $\Psi(t)$ be the solution of the Schrödinger-Poisson system (11.0.1)-(11.0.6).*

i) Let $\Phi \in \tilde{Z}$. Then there exists $T_* > 0$ s.t. $u(t) = e^{\varphi(t)B^{\frac{1}{2}}}\Psi(t) \in \tilde{Z}$ for all $t \geq 0$.

ii) Let $e^{\lambda B^{\frac{1}{2}}}\Phi \in \tilde{Z}$ for some $\lambda > 0$. Then there exists $\mu = \mu(\lambda, ||e^{\lambda B^{\frac{1}{2}}}\Phi||_{\tilde{Z}})$ s.t. $w(t) = e^{\mu B^{\frac{1}{2}}}\Psi(t) \in \tilde{Z}$ for all $t \geq 0$.

Remark 11.0.1. *In the periodic case we can establish, using the Gevrey class regularity, estimates on the convergence velocity for the Galerkin method which are similar to those obtained by Duan, Holmes and Titi for a generalized Ginzburg-Landau equation (see [43]).*

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