First edition: November 29th, 2004
Updated (J. M.) January 8th, 2010

WEB PAGES ON THE VORONOI ALGORITHM AND MINIMAL CLASSES

C. BATUT, J. MARTINET (*)

ABSTRACT. The first 2 *PARI-GP* files below give the sets of vertices and edges of the Voronoi graph first in dimensions 2 to 6, then in dimension 7. The numerical data are extracted from Jaquet's thesis [Ja]. The third file, based on Chapters 9 and 14 of [Mar], is devoted to minimal classes in dimensions 2 to 4. We present below a short account of Voronoi's theory and minimal classes.

1. The perfection rank

Let $S \subset \mathbb{R}^n$ $(n \geq 2)$ be a set of non-zero vectors viewed as column-matrices, invariant under the symmetry $x \mapsto -x$. Unless otherwise stated, we assume that S is finite, and we set $s = \frac{1}{2}\,|S|$. With S we associate the set of the s matrices $M = X\,{}^tX$ for $X \in S$ (note that X and $-X$ define the same matrix M) and its span V_S in the space $\mathrm{Sym}_n(\mathbb{R})$ of real, $n \times n$ symmetric matrices
[The symmetric matrices $X\,{}^tX$ may be viewed as *projection matrices*: indeed, let E be an n-dimensional Euclidean vector space equipped with a basis $\mathcal{B}$; if X is the set of components on $\mathcal{B}$ of some vector $x \in E$, then $X\,{}^tX$ is the matrix in the pair of bases $(\mathcal{B}^*, \mathcal{B})$ of the orthogonal projection $p_x \in \mathrm{End}^s(E)$ to x.]

Definition 1.1. The rank in $\mathrm{Sym}_n(\mathbb{R})$ of the set $\{X\,{}^tX\}_{X \in S}$ is called the *perfection rank of* S and denoted by $\mathrm{perf}\,S$ The *perfection rank of a lattice* Λ or a *positive definite quadratic form* Q *on* $\mathbb{R}^n$ (or the symmetric matrix A such that $Q(X) = {}^tXAX$) is the perfection rank of its set of minimal vectors.

We say that Λ or Q (or A) is *perfect* if its perfection rank attains its maximal possible value, namely $\dim \mathrm{Sym}_n(\mathbb{R}) = \frac{n(n+1)}{2}$.

Key words and phrases. Perfect Lattices, Voronoï Graph, Minimal Classes.
(*) Université de Bordeaux, I.M.B, (UMR 5251), équipe A2X..

2. The facets

In what follows, the space $\mathrm{Sym}_n(\mathbb{R})$ is equipped with the *Voronoi scalar product* $\langle M, N\rangle = \mathrm{Tr}(MN)$. We consider a set S as above and we set $r = \operatorname{perf} S$.

Definition 2.1. We say that a subset T of S is *admissible* if it is maximal among the subsets T' of S with $\operatorname{perf} T' = \operatorname{perf} T$.

We denote by W the orthogonal of V_T in V_S:

$$W = \{M \in V_S,\ \forall X \in T,\ \langle M, X^tX\rangle\,(= {}^tXMX)\ = 0\,.$$

We say that $F \in W$ is a *face vector for T in S* the scalar products $\langle F, X^tX\rangle$ are strictly positive for all $X \in S\smallsetminus T$.

The set $\mathcal{C}_S(T)$ of face vectors for T in S is a cone, called the *positive cone of T in S*.

In what follows, we restrict ourselves to the important case where V_T has codimension 1 in V_S. In this case, the cone $\mathcal{C}_S(T)$ is a half-line or is empty.

Definition 2.2. Suppose that $\mathcal{C}_T$ is not empty. An element of $\mathcal{C}_T$ is called a *facet for T in S*.

Note that any T such that $s(T) = s(S) - 1$ (and $\operatorname{perf} T < \operatorname{perf} S \Leftrightarrow \operatorname{perf} T = \operatorname{perf} S - 1$) is a facet. However, this special case is far from being general.

3. Contiguity

This time, we start with a positive definite quadratic form Q, with matrix A and minimum m. (If we are dealing with a lattice Λ, we chose a Gram matrix A for Λ and denote by Q the corresponding quadratic form.) We take for S the set $S(Q)$ of minimal vectors of Q and we chose T as above.

Given $F \in V_S$ and $t \in \mathbb{R}$, let R be the quadratic form with matrix F and let $Q_t = Q + t * R$. For $y \in \mathbb{Z}^n$, the function $t \mapsto Q_t(y)$ is increasing, constant or decreasing according to whether $R(y)$ is positive, zero, or negative.

Chose $F \in V_T^{\perp}$. Then we have $Q_t(x) = m$ for all $x \in T$. If $\mathcal{C}_T = \emptyset$, then for every $t \neq 0$, there exists $y \in S$ such that $Q_t(y) < m$. Otherwise, for $F \in \mathcal{C}_T$ and $t > 0$ and small enough, we have $\min Q_t = m$ (and $S(Q_t) = T$).

Let $\theta \in (0, +\infty)$ be the least upper bound of t for which $\min Q_t = m$. If θ is finite, we can consider the form Q_θ.

Definition 3.1. The form Q_θ above is the *contiguous (or neighbour) form of* Q *relative to* T (or *along* F).

Then $S(Q_\theta)$ contains strictly T. Since T is maximal among the subsets of S of perfection rank $r-1$, we have perf $S(Q_\theta) \geq r$. The form R is proportional to the difference $Q - Q_\theta$. Denoting by B the matrix of Q_θ and rescaling the parameter t, we may put the the matrices of Q_t in the interval $0 \leq t \leq \theta$ in the form

$$A_{t'} = B + t'\,(B - A)\,,\ 0 \leq t' \leq 1$$

to be used in the *PARI-GP* files.

So far the theory has been established (by Voronoi) only for perfect forms. In this case, the contiguous form exists for all facets and is again perfect. Also, and this is the fundamental result of Voronoi, the contiguity graph is connected. The proofs can be read in [Mar], Chapter 7, Sections 1, 2, 3.

4. Voronoi Graphs: the Results

The references are those of the bibliography of [Mar], which can be found on this WEB page.

Perfect lattices were classified up to dimension 5 by Korkine and Zolotareff in [K-Z3] (1877). Their results were recovered by Voronoi ([Vo1], 1907), using the construction of the contiguity graph.

Dimension 6 was dealt with 50 years later by Barnes ([Bar4], 1957), who determined directly the contiguity graph. A direct classification of perfect lattices was produced later by Baranovskii and Ryshkov ([Br-R], 1985). However, they did not published the details of their proof.

Finally the results for dimension 7 were first published by Stacey ([Sta1], 1975), but her proof was not considered as definitive, in particular because the lattices she found were not tested for isometry. The first recognized proof was produced by Jaquet in his Neuchâtel thesis ([Ja2], 1991; see also [Ja5] for a published proof). The reference [Ja2] contains the complete data for dimensions 2 to 7.

In their Bordeaux PhD theses, M. Laïhem (1992), J.-L. Baril and H. Napias (19967) obtained a list of 10770 perfect lattices, which was completed a few months later by C. Batut to a list of 10916 perfect lattices; see [Mar] and the file "Perfection: An introductory paper about perfect lattices." in this WEB page. To obtain the complete classification of 8-dimensional perfect lattices looked out of scope of any existing method. Nevertheless, M. Dutour Sikirić, A. Schürmann and F. Vallentin ([D-S-V]) were able to prove in 2005 that the list above is complete, by constructing the Voronoi graph. This Voronoi graph can be obtained on request from the authors.

The 48 perfect lattices in dimensions n≤7.

Dimension	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Perfect	1	1	1	2	3	7	33
Edges		1	1	2	4	18	357

5. Voronoi Graphs: the Files

In the two *PARI-GP* files, devoted to dimensions 2–6 and 7 respectively, Gram matrices for the 47 perfect lattices in dimensions n from 2 to 7 are given as vectors $p\mathrm{n}[i]$.

For $n = 2, 3, 4, 5, 6$, the contiguous forms are displayed as a vector $V\mathrm{n}[j]$ with one index j for each edge of the contiguity graph with endpoint some $p\mathrm{n}[k]$ with $k \geq i$.

For $n = 7$, the contiguous forms are displayed as a vector $V7[i][j]$ for $n = 7$, where the index i is that of a perfect form whose neighbour is *equivalent* to some $p7[k]$ with $k \geq i$, and there is one index j for each edge starting form $p7[i]$. **However**, the components of the vectors $V7[i]$ are such that $j \geq j_0$, the least index for which the contiguous form is a $p7[k]$ for some $k \geq i$.

Finally, for $n \leq 6$, the path $A + t * (B - A)$ defined by $V\mathrm{n}[j]$ is directly available using the command $VR\mathrm{n}[j]$.

As usual, the files contain other informations which can be read under an editor, for instance *emacs*.

6. Minimal Classes and (weak) eutaxy

We follow [Mar], Chapters 9 (for the theory) and 14 (for the numerical data, and [Bt] for the (not yet available) data for dimension 5.

In terms of lattices, *minimal classes* are the classes for the equivalence relation on the set of lattices in a given n-dimensional Euclidean space:

$$\Lambda \sim \Lambda' \iff \exists u \in \mathrm{GL}(E),\ u(\Lambda) = \Lambda' \text{ and } \ u(S(\Lambda)) = S(\Lambda')\,.$$

We then define on these classes an ordering relation by

$$\mathcal{C}' \prec \mathcal{C} \iff \exists\, \Lambda \in \mathcal{C},\, \exists\, \Lambda' \in \mathcal{C}',\ S(\Lambda') \subset S(\Lambda)\,.$$

In terms of quadratic forms, we consider the finite subsets $S \in \mathbb{Z}^n$ which are such that the set $\mathcal{Q}(S)$ of positive definite quadratic forms q

such that $S = S(q)$ is not empty. Then $\mathcal{Q}(S)$ is a convex open polyhedron. We obtain this way a cell decomposition for the set of positive definite quadratic forms. Minimal classes correspond to equivalence classes of cells under the action of $\mathrm{GL}_n(\mathbb{Z})$.

Recall that a lattice Λ (or a positive definite quadratic form) q is *well-rounded* if it contains $n = \dim\Lambda$ independent minimal vectors, and that a class is *well-rounded* if all its lattices are. In the sequel, we restrict ourselves to well rounded classes. This is no important restriction: classes whose minimal vectors have rank $n' < n$ are in one-to-one correspondence with n'-dimensional classes.

The perfection rank r (and the cardinality s) of S are invariants of the class $\mathcal{C}$ defined by S. The dimension of a cell is its *perfection corank*, namely $\frac{n(n+1)}{2} - r$. The 0-cells are perfect forms and 1-cells are Voronoi paths connecting two perfect forms. Thus minimal classes are objects which generalize both perfect forms (or lattices) and Voronoi paths. They are related to a construction we considered in Section 2: given S, $S' \subset S$ defines a class $\mathcal{C}'$ (with necessarily $\mathcal{C}' \prec \mathcal{C}$) if and only if it is admissible in the sense of Definition 2.1 and satifies moreover sign conditions with respect to faces. The topological closure $\overline{\mathcal{C}}$ of a class $\mathcal{C}$ is the union of the classes $\mathcal{D} \succ \mathcal{C}$ (we say that $\mathcal{D}$ lies above $\mathcal{C}$).

For $x \in E \smallsetminus \{0\}$, denote by $p_x \in \mathrm{End}^s(E)$ the orthogonal projection to $\mathbb{R}x$. Given a lattice Λ (resp. a positive definite quadratic form q with matrix A), a *eutaxy relation for* Λ (resp. *for* q) is an equality

$$\mathrm{Id} = \sum_{x \in S(\Lambda)} \rho_x \, p_x \quad (\text{resp. } A^{-1} = \sum_{X \in S(q)} \rho'_X \, X^t X)\,.$$

[These definitions are compatible with the dictionary *lattices* $\longleftrightarrow$ *quadratic forms*: if $\mathcal{B}$ is a basis for Λ with Gram matrix A, then $A = \mathrm{Mat}(Id, \mathcal{B}, \mathcal{B}^*)$ and $X^t X = \mathrm{Mat}(p_x, \mathcal{B}^*, \mathcal{B})$ — note the exchange $\mathcal{B} \longleftrightarrow \mathcal{B}^*$.]

Definition 6.1. We say that a lattice or a form is *weakly eutactic* if there exists a eutaxy relation between its minimal vectors. We say that it is *semi-eutactic* (resp. *eutactic*) if moreover the eutaxy coefficients can be chosen to be non-negative (resp. strictly positive). We say that it is *strongly eutactic* if there exist a eutaxy relation with equal coefficients. (It is also useful to consider *strongly semi-eutactic* lattices and forms, those for which there exists a eutaxy relation with equal non-zero eutaxy coefficients.)
[Strongly eutactic lattices are the lattices whose set of minimal vectors is a spherical 2-design (or 3-design, this amounts to the same by central symmetry). Similarly, strongly semi-eutactic lattices are the lattices whose set of minimal vectors having *non-zero* eutaxy coefficients is a spherical 2-design.]

On the closure of a given class $\mathcal{C}$, the Hermite invariant γ (defined by $\gamma(\Lambda) = \dfrac{\min\Lambda}{\det(\Lambda)^{1/n}}$) attains a minimum. By a theorem of A.-M. Bergé and J. Martinet, this minimum is attained on a unique lattice (up to similarity), which is also the unique weakly eutactic lattice (up to similarity) in its class. This class is some class $\mathcal{C}' \subset \overline{\mathcal{C}}$ (whence a canonical map $\mathcal{C} \mapsto \mathcal{C}' \succ \mathcal{C}$).

7. Minimal Classes: the Results

The classification of minimal classes together with the corresponding classification of weakly eutactic lattices is known up to dimension $n = 5$.

For $n \leq 4$, the first results were found by Štogrin and completed by Bergé-Martinet (references [St] and [B-M5] in [Mar]); the classification of minimal classes used an *ascending algorithm*, starting with the unique class with $s = r = n$ and enlarging successively the possible sets of minimal vectors; this algorithm gives the classification of perfect lattice as a byproduct,

In dimension 5, such a procedure would be lengthy, and Batut used a descending algorithm, searching for admissible subsets of one of three sets $S(\Lambda)$ for Λ one of the perfect lattices $P_5^1 = \mathbb{D}_5$, $P_5^2 = \mathbb{A}_5^3$, and $P_5^3 = \mathbb{A}_5$, These results have been partially extended to dimensions 6 and 7 by P. Elbaz-Vincent, H. Gangl and C. Soulé ([E-G-S])

The following table displays for dimensions $n = 2, 3, 4, 5, 6$ the number of minimal classes, then (at least for $n \leq 5$) the number of these classes which contain a weakly eutactic, then a semi-eutactic, then a strongly semi-eutactic, then a eutactic, and finally a strongly eutactic lattice.

Minimal classes in dimensions 1 to 6.

Dimension	**1**	**2**	**3**	**4**	**5**	**6**
Number of classes	1	2	5	18	136	5634
Weakly eutactic	0	0	0	0	4	?
Semi-eutactic	0	0	0	0	4	?
Strongly semi-eutactic	0	0	0	1	1	6
Eutactic	0	0	2	10	109	?
Strongly eutactic	1	2	3	6	9	21

In the table above, "eutactic" implies "non-strongly eutactic", "semi-eutactic" implies "non-eutactic", and "weakly eutactic" implies "non-semi-eutactic".
Hence, to obtain, for instance, the total number of classes containing a weakly eutactic lattice. one must add the numbers of the lines 2, 3, 4, 5, 6. More details on stronly (semi-)eutactic lattices can be found in the file "Lattices and spherical designs." below on this home page.

At the time [E-G-S] was written, about 1 500 000 minimal classes had been detected in dimension 7.

8. Minimal Classes: the Files

For the while, there is only *one* file (*Sdim2to4.gp*) devoted to dimensions 2, 3 and 4. The notation is essentially that of [Mar]: minimal classes are denoted $Snx\mathrm{r}$ (instead of merely $x\mathrm{r}$) where $n \in \{2, 3, 4\}$ is the dimension, $r \in [n, \frac{n(n+1)}{2}$ is the perfection rank, and the letter $x \in \{a, b, c, d\}$ characterizes the class among all classes with the same values of n and r. We have $s = r$ except for $S4a10$ (the class of $\mathbb{D}_4$) for which $r = 10$ and $s = r + 2$. The set $Snx\mathrm{r}$ is the set of minimal vectors on a basis, chosen as in [Mar].

To each such class corresponds a family of Gram matrices depending affinely on $\frac{n(n+1)}{2} - r$ (the perfection co-rank of the class) parameters, denoted by $Mnx\mathrm{r}$. Finally, the weakly eutactic matrix in the class (if any) is denoted by $Mnx\mathrm{r}E$.

The reference list below contains only five items. For the other papers cited in the text, see the updated reference list of [Mar] on this home page.

References

[Bt] C. Batut, *Classification of quintic eutactic forms*, Math. Comp. **70** (2001), 395–417.

[D-S-V] M. Dutour Sikirić, A. Schürmann, F. Vallentin, *Classification of eight dimensional perfect forms*, Electron. Res. Announc. Amer. Math. Soc. **13** (2007), 21–32.

[E-G-S] P. Elbaz-Vincent, H. Gangl, C. Soulé, *Perfect Lattices, Homology of Modular Groups and Algebraic K-theory*, Oberwolfach Reports 1/2005 (2005), 36–39.

[Ja] D-O. Jaquet, *Énumération complète des classes de formes parfaites en dimension* 7, Thèse, Neuchâtel (1991).
[This is reference [Ja2] of Section 4.]

[Mar] J. Martinet, *Perfect Lattices in Euclidean Spaces*, Grundlehren **327**, Springer-Verlag, Heidelberg (2003).

Other items are given in Section 4 above.