USER'S GUIDE FOR TABLES OF ODD UNIMODULAR LATTICES WITHOUT ROOTS

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1. Introduction

This user's guide is devoted to using *PARI-GP* tables of odd unimodular lattices without roots (i.e., of minimum $m \geq 3$) in dimensions $n \leq 28$. These lattices are discrete subgroups of rank n in an n-dimensional Euclidean space E, which may be identified to $\mathbb{R}^n$ once we have chosen an orthonormal basis for E. The tables rely on Bacher-Venkov's work [B-V], where lattices in dimensions $n = 27$ and $n = 28$ are described as *Kneser-neighbours* of $\mathbb{Z}^n \subset \mathbb{R}^n$ relative to a pair (v, p) of a vector $v \in Z^n$ and a prime p.

Recall that given a lattice L, a vector $v \in L$ and an integer $p \geq 2$, the *neighbour of* L *for* (v, p) is

$$L_p^v = \langle L_{p,v}, \tfrac{v}{p} \rangle$$

where

$$L_{p,v} = \{x \in L \mid x \cdot v\} \equiv 0 \mod p\} .$$

It was known before Bacher-Venkov's paper ([C-S], Chapters 16 and 17; [Bo]) that lattices of dimension n and minimum $m \geq 3$ do not exist for $n \leq 22$ and that there exists exactly one such odd lattice for $n = 23$ (O_{23}, the shorter Leech lattice), for $n = 24$ (O_{24}, the odd Leech lattice), and $n = 26$ (let's call it O_{26} — Borcherds's lattice, named T_{26} in [Ne-Sl]), but none for $n = 25$. In [B-V], Bacher and Venkov have classified these lattices for $n = 27$ (3 lattices) and $n = 28$ (38 lattices). It turns out that unimodular lattices of minimum 3 in the range $[24, 31]$ share out among two types, related to their parity vectors; see Definition 2 below.

Key words and phrases. Euclidean unimodular lattices, short vectors, bases
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May 14th, 2010 .

2. Parity vectors.

Recall that a *parity vector for an integral lattice* L is a vector $v \in L$ such that

$$\forall\, x \in L,\; x \cdot x \equiv v \cdot x \mod 2\,.$$

It is easy to see that parity vectors exist, that they constitute a single class modulo $2L$ if $\det(L)$ is odd, and that there norms $v \cdot v$ are well-defined modulo 8. For unimodular lattices, one has the more precise result: *the norms of parity vectors are congruent to* n *modulo* 8. [The result holds for any signature: replacing L by $L \perp \mathbb{Z}^-$, we are reduced to the case of odd "indefinite lattices", and the result is then easy since these lattices are isometric to a direct orthogonal sum $(\mathbb{Z}^+)^p \perp (\mathbb{Z}^+)^q$; see e.g. [Se], Chapter 5. Note that 0 is a parity vector for L if and only if L is even, so that the dimension of an even unimodular lattice is divisible by 8.]

Definition 1. We denote by $\mathcal{P}$ *the set of parity vectors of* L and by N_{par} the smallest possible norm of a parity vector.

It was proved by Elkies in [El1] that we have $N_{par} \le n$, with equality if and only if $L \simeq \mathbb{Z}^n$, so that if $\min L \ge 2$, we have $N_{par} \le n - 8$; and in [El2] that if $\min L \ge 2$ and $n \ge 24$, then $N_{par} \le n - 16$. (For the lattice O_{23}, we have $N_{par} = n - 8$ $(= 15)$.)

In dimension 24, we may have either $N_{par} = n - 24 = 0$, and then L is even, hence isometric to the Leech lattice Λ_{24}, or $N_{par} = n - 16 = 8$, and then L is odd, hence isometric to O_{24}.

In the range $[25, 31]$, N_{par} may *a priori* be equal to $n - 16$ or to $n - 24$.

Definition 2. (Bacher-Venkov.) We say that a unimodular lattice of dimension $n \in [24, 31]$ and minimum 3 is of *general type* (G.T. for short) if $N_{par} = n-16$ and of *exceptional type* (E.T. for short) if $N_{par} = n-24$. We may extend the notion of a G.T. lattice to any n; note ([Ne-V], Corollary 3.4) that n is then bounded from above by 46.

Thus O_{24} and O_{26} (because O_{26} has no vectors of norm 2) are of general type whereas Λ_{24} is of exceptional type.

In dimensions 27 and 28, Bacher and Venkov have proved:

Theorem 3. *Consider unimodular lattices* L *of minimum* 3 *and dimensions* $n = 27, 28$.

(1) *If* $n = 27$*, there are two G.T. lattices and one E.T. lattice.*
(2) *If* $n = 28$*, there are* 36 *G.T. lattices and* 2 *E.T. lattices.*

Notation. In the ordering of [B-V], the G.T. lattices are denoted below by $o_{27a\,i}$, $i = 1, 2$ and $o_{28a\,i}$, $i = 1, \dots, 36$, and the E.T. by o_{27b1}, o_{28b1}, o_{28b2}.

3. Some data.

In this section we restrict ourselves to unimodular lattices L of minimum 3. We denote by L_0 its even sublattice, by S, S_0 and S_0^* the sets of minimal vectors of L, L_0 and L_0^* (the dual of L_0), respectively, and by s, s_0 and s_0^* the corresponding numbers of pairs of minimal vectors.

3.1. **Theta series.** It is proved in [B-V] that the theta series Θ_L of a lattice L as above only depends on its type (general or exceptional). The even coefficients of this series are then those of the even sublattice L_{even} of L, and the theta series also defines that of L_{even}^*.

The proposition below lists the kissing numbers of the three lattices L, L_{even}, and L_{even}^*.

Proposition 4. *The kissing numbers of the lattices above in dimensions* 24 *to* 28 *are as follows:*
- $n = 24$ *(G.T.)* $s = 2048$, $s_0 = 49128$, $s_0^* = 24$ $(S_0^* \sim S(\mathbb{Z}^{24}))$;
- $n = 26$ *(G.T.)* $s = 1560$, $s_0 = 51090$, $s_0^* = 312$;
- $n = 27$ *(G.T.)* $s = 1332$, $s_0 = 50571$, $s_0^* = 864$;
- $n = 27$ *(E.T.)* $s = 820$, $s_0 = 59787$, $s_0^* = 1$;
- $n = 28$ *(G.T.)* $s = 1120$, $s_0 = 49140$, $s_0^* = 3360$;
- $n = 28$ *(E.T.)* $s = 864$, $s_0 = 53236$, $s_0^* = 1$.

For the sake of completion, we give below the data for $n = 23$:
- $n = 23$ $(N_{par} = 15)$ $s = 2300$, $s_0 = 46575$, $s_0^* = 2300$ $(S(L_{even}^*) \sim S(L))$.

3.2. **Strong eutaxy.** Recall (Venkov, [Ve]) that a lattice is *strongly eutactic* if the set of its minimal vectors is a spherical 3-design. This amounts to the fact that the sum of the orthogonal projections to the lines which support its minimal vectors is proportional to the identity, a condition which can be easily checked on a computer. The following proposition is proved (though not explicitly stated) in Section 2 of [Ne-V].

Proposition 5. (Nebe–Venkov). *Let L be a unimodular lattice of minimum* 3 *and dimension $n \in [24, 31]$. If L is of general type, then L, L_0 and L_0^* are strongly eutactic.*

If L is one of the three lattices of exceptional type of dimension27 or 28, none of the lattices L, L_0 and L_0^* is strongly eutactic.

3.3. **Perfection (minimum 3).** The basic facts concerning the perfection property can be read in the third chapter of [Ma2]. Recall that the *perfection rank of a lattice* L is the rank $r \in [1, \frac{n(n+1)}{2}]$ in the space of symmetric endomorphisms of E of the set of orthogonal projections

p_x to the minimal vectors x of L. The *co-rank* of L is $\frac{n(n+1)}{2} - r$. A lattice is *perfect* if its perfection rank is maximal.

In [Ve], Venkov has defined the notion of a *strongly perfect lattice*, a lattice the minimal vectors of which constitute a 5-design, and proved that such lattices are indeed extreme, hence in particular perfect. He has also classified the integral, strongly perfect lattices of minimum 3, proving that there are exactly *five* such lattices, one in each of the dimensions 1, 7, 16, 22, and 23 — in dimension 23, this is O_{23}, even a 7-design. For $n \geq 24$, our lattices L are never strongly perfect, and indeed no general perfection rules show up. We list below the status of the lattices L with respect to perfection.

Proposition 6. *Among the unimodular lattices of minimum* 3 *and dimension* $n \in [24, 28]$*, there are* 29 *perfect lattices, all of general type, namely* o_{27a1} *(*$r = 378$*), and* 28 *lattices* $o_{28a\,i}$ *(*$r = 406$*), those with* $i = 1, \ldots, 25, 28, 30, 32$*. The remaining* 13 *lattices are listed below according to their perfection co-ranks:*
co-rank 1*:* o_{26}*,* o_{28a29} *; co-rank* 2*:* o_{28a26}*,* o_{28a27}*,* o_{28b37} *;*
co-rank 6*:* o_{27a2}*,* o_{28a31} *; co-rank* 14*:* o_{28a33} *; co-rank* 23*:* o_{24} *;*
co-rank 26*:* o_{27b1} *; co-rank* 37*:* o_{28b38} *; co-rank* 42*:* o_{28a34} *;*
co-rank 70*:* o_{28a35} *; co-rank* 105*:* o_{28a36} *;*
[Note that the perfect lattices above are actually extreme and dual-extreme.]

3.4. **More on parity vectors.** Let Λ be a unimodular lattices of minimum 3, with even sublattice Λ_0, and let v be a parity vector for Λ. We have $\Lambda_0 \subset \Lambda \subset \Lambda^*$, and $\frac{v}{2}$ clearly belongs to Λ^*. The easy congruence $N(v) \equiv n \mod 2$ shows that we have $v \in \Lambda \smallsetminus \Lambda_0$ if n is odd, and $v \in \Lambda_0$ if n is even. The quotient Λ_0/Λ_0^*, of order 4, is cyclic, with representatives $\{0, v, \pm\frac{v}{2}\}$ if n is odd, and elementary, with representatives $\{0, w, \frac{v}{2}, w + \frac{v}{2}\}$ if n is even, where w is any vector in $\Lambda \smallsetminus \Lambda_0$. In both cases, we have $L_0^* \smallsetminus L = \{\frac{v}{2},\, v \in \mathcal{P}\}$. As a consequence, we have $\min L_0^* = \min(\frac{n-16}{4}, 9)$ ($\frac{n-16}{4}$ for $n = 24, 26, 27$; 9 for $n = 28, 29, 30, 31$). Moreover, except for $n = 28$, $v \mapsto \frac{v}{2}$ induces a one-to-one correspondence between the set of shortest vectors of $\mathcal{P}$ and S_0^*. (For $n = 28$, S_0^* also contains S.)

3.5. **Perfection (minimum** 4**).** All the even sublattices L_0 of the 44 unimodular lattices L of minimum 3 are perfect, so that those for which L is of general type are extreme and dual-extreme (the remaining three lattices have not been tested for eutaxy). Among the lattices L_0^*, those for which $L = \mathrm{O}_{23}$ or L is of general type and dimension 28 are perfect (note that $S(L_o^*)$ then contains $S(L)$), the others are not. These 37 perfect lattices are of course extreme and dual-extreme.

3.6. **Density.** The even lattices L_0 all have minimum 4 and determinant 4. Their Hermite invariants are smaller than those of the laminated lattices for $n = 24$, 26, 30, 31, and are equal to them for for $n = 27$, 28, 29. However they are not the densest known lattices in these dimensions; see just above in this WEB-page the lattices constructed by Roland Bacher.

4. The tables.

The file *unimod23to28.gp* is a PARI-GP-file containing LLL-reduced Gram matrices for the 44 unimodular lattices of minimum 3 which exist in dimensions $n \in [23, 28]$. This can be downloaded in a PARI-GP session (strike $\backslash r$ *unimod23to28.gp*).

Gram matrices are named as above, i.e., o23, o24, o26, 027a1, o27a2, o28a1,...,o28a36 for the G.T. lattices, o27b1, o28b1,o28b2 for the E.T. lattices. In dimensions 27 and 28, one can load them as vectors $o27$ of length 3 and $o28$ of length 38. The G.T. lattices are $o27[i]$, $i = 1, 2$ and $o28[i]$, $i = 1, \dots, 36$ where the subscript i is the same as in o27ai and o28ai, and the E.T. lattices are similarly $o27[3]$, $o28[37]$, and $o28[38]$.

The file also contains Gram matrices o29 and o31 taken from Nebe=Sloanes catalogue, examples of general type in dimensions 29 and 31, respectively,
and two little gp-programs:

- *esl(a)* outputs an LLL-reduced Gram matrix for the even sublattice of the lattice with Gram matrix a;
- *vpar(a)* outputs a parity vector for the lattice with Gram matrix a.

Acknowledgements. Merci, Christine!

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