SOME GENERALIZED EUCLIDEAN AND 2-STAGE EUCLIDEAN NUMBER FIELDS THAT ARE NOT NORM-EUCLIDEAN

JEAN-PAUL CERRI

Abstract. We give examples of Generalized Euclidean but not norm-Euclidean number fields of degree greater than 2. In the same way we give examples of 2-stage Euclidean but not norm-Euclidean number fields of degree greater than 2. In both cases, no such examples were known.

1. Introduction

In 1985, Johnson, Queen and Sevilla [9] introduced a generalization of the classical notion of Euclidean number field.

Definition 1.1. A number field $K$ is said to be Generalized Euclidean or simply G.E. if for every $(\alpha, \beta) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$ such that the ideal $(\alpha, \beta)$ is principal, there exists $\Upsilon \in \mathbb{Z}_K$ such that $|N_{K/Q}(\alpha - \Upsilon \beta)| < |N_{K/Q}(\beta)|$.

If $(\alpha, \beta)$ is principal, we thus have at our disposal the Euclidean algorithm to compute a $\gcd$ of $\alpha$ and $\beta$ because it is easy to see that $(\beta, \alpha - \Upsilon \beta)$ is principal again, and so on. Note that if $K$ is norm-Euclidean then $K$ is G.E. and that if $K$ is principal, i.e. has class number 1, then $K$ is G.E. if and only if $K$ is norm-Euclidean.

If we want to illustrate the difference between “G.E.” and “norm-Euclidean”, the interesting case is when $K$ is G.E. but not principal (so not norm-Euclidean). The following result was established by Johnson, Queen and Sevilla in [9].

Theorem 1.1. The quadratic number field $\mathbb{Q}((\sqrt{d}))$ is G.E. but not norm-Euclidean for $d = 10$ and $d = 65$. The quadratic number field $\mathbb{Q}(\sqrt{d})$ is not G.E. for $d = 15, 26, 30, 35, 39, 51, 78, 87, 102, 115, 195$ and $230$.

Furthermore, Johnson, Queen and Sevilla conjectured that $K = \mathbb{Q}(\sqrt{d})$ (with $d > 1$ squarefree) is G.E. if and only if $K$ is norm-Euclidean or $d = 10$ or 65.

Another variation on norm-Euclidean number fields has been introduced by Cooke [7].

Definition 1.2. Let $m$ be a rational integer $\geq 1$. The number field $K$ is $m$-stage Euclidean if and only if for every $\alpha \in \mathbb{Z}_K$ and every $\beta \in \mathbb{Z}_K \setminus \{0\}$ there exists a positive rational integer $k \leq m$ and $k$ pairs $(q_i, r_i)$ ($1 \leq i \leq k$) of elements of $\mathbb{Z}_K$.
such that
\[ \alpha = \beta q_1 + r_1, \]
\[ \beta = r_1 q_2 + r_2, \]
\[ \vdots \]
\[ r_{k-2} = r_{k-1} q_k + r_k, \]
and \(|N_{K/Q}(r_k)| < |N_{K/Q}(\beta)|\).

When it is well defined, let us put
\[ [q_1, q_2, \ldots, q_k] = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \cdots + \frac{1}{q_k}}}, \]
where \(a_k\) and \(b_k\) are given by
\[ a_1 = q_1, \quad b_1 = 1, \]
\[ a_2 = a_1 q_2 + 1, \quad b_2 = q_2, \]
and recursively for \(k \geq 3\) by
\[ a_k = a_{k-1} q_k + a_{k-2}, \quad b_k = q_k b_{k-1} + b_{k-2}. \]

Since
\[ \frac{\alpha}{\beta} = \frac{a_k}{b_k} + (-1)^{k+1} \frac{r_k}{b_k}, \]
this definition is equivalent to the following.

**Definition 1.3.** The number field \(K\) is \(m\)-stage Euclidean if and only if for every \(\xi \in K\), there exists a positive rational integer \(k \leq m\), and \(k\) elements \(q_1, q_2, \ldots, q_k \in \mathbb{Z}_K\) such that
\[ \left| N_{K/Q}(\xi - [q_1, q_2, \ldots, q_k]) \right| < \frac{1}{|N_{K/Q}(b_k)|}. \]

As in the previous case, norm-Euclidean implies \(m\)-stage Euclidean, but contrary to what happens with the G.E. condition, we have the following result [7].

**Theorem 1.2.** A number field \(K\) with unit rank \(r \geq 1\) (i.e. \(r = \text{rank}(\mathbb{Z}_K^*) \geq 1\)) is principal if and only if \(K\) is \(m\)-stage Euclidean for some \(m\).

As a consequence, if we want to study the difference between \(m\)-stage Euclidean and norm-Euclidean, we have to look at number fields with class number 1 and find some example where \(K\) is principal, \(m\)-stage Euclidean but not norm-Euclidean. The following result was established by Cooke [7].

**Theorem 1.3.** For \(d = 14, 22, 23, 31, 38, 43, 46, 53, 61, 69, 89, 93, 97, \mathbb{Q}(\sqrt{d})\) is 2-stage euclidean but not norm-Euclidean.

Furthermore, Cooke and Weinberger [8] proved that, under GRH, every principal number field \(K\) with unit rank \(r \geq 1\) is 4-stage Euclidean, and even 2-stage Euclidean if \(K\) has at least one real place.
For both notions (G.E. and $m$-stage Euclidean), no examples of number fields of degree greater than 2 were known. Our main results are the following.

**Theorem 1.4.** None of the totally real number fields enumerated in Table 1 are principal. They all are G.E. except for the second cubic number field of discriminant 3969, defined by $x^3 - 21x - 35$, which is neither principal nor G.E.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$D_K$</th>
<th>$P(x)$</th>
<th>$h$</th>
<th>$M(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1957</td>
<td>$x^3 - x^2 - 9x + 10$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2597</td>
<td>$x^3 - x^2 - 9x + 8$</td>
<td>3</td>
<td>5/2</td>
</tr>
<tr>
<td>3</td>
<td>2777</td>
<td>$x^3 - x^2 - 14x + 23$</td>
<td>2</td>
<td>5/3</td>
</tr>
<tr>
<td>3</td>
<td>3969</td>
<td>$x^3 - 21x - 28$</td>
<td>3</td>
<td>4/3</td>
</tr>
<tr>
<td>3</td>
<td>3969</td>
<td>$x^3 - 21x - 35$</td>
<td>3</td>
<td>7/3</td>
</tr>
<tr>
<td>3</td>
<td>3981</td>
<td>$x^3 - x^2 - 11x + 12$</td>
<td>2</td>
<td>3/2</td>
</tr>
<tr>
<td>3</td>
<td>4212</td>
<td>$x^3 - 12x - 10$</td>
<td>3</td>
<td>7/2</td>
</tr>
<tr>
<td>3</td>
<td>4312</td>
<td>$x^3 - x^2 - 16x + 8$</td>
<td>3</td>
<td>11/4</td>
</tr>
<tr>
<td>3</td>
<td>5684</td>
<td>$x^3 - 14x - 14$</td>
<td>3</td>
<td>9/2</td>
</tr>
<tr>
<td>4</td>
<td>21025</td>
<td>$x^4 - 17x^2 + 36$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>32625</td>
<td>$x^4 - x^3 - 19x^2 + 4x + 76$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>46400</td>
<td>$x^4 - 22x^2 + 116$</td>
<td>2</td>
<td>5/4</td>
</tr>
<tr>
<td>4</td>
<td>51200</td>
<td>$x^4 - 20x^2 + 50$</td>
<td>2</td>
<td>7/2</td>
</tr>
</tbody>
</table>

**Table 1.** Here, $n$ is the degree of the field $K$, $D_K$ its discriminant, $P(x)$ its equation, $h$ its class number and $M(K)$ its Euclidean minimum.

**Theorem 1.5.** The totally real number fields of degree 3 and of discriminants $< 15000$ which are principal but not norm-Euclidean (82 cases) are 2-stage norm-Euclidean. The same is true for degree 4 and discriminants 18432, 34816, 35152 and for degree 5 and discriminant 390625. In all these cases, the number field is principal, not norm-Euclidean, but 2-stage norm-Euclidean.

Details on the number fields appearing in Theorem 1.5 are available from [6] and are given in the online version of the paper. In Section 2, we recall other definitions and general results. In Section 3 and 4, we study the case of Generalized Euclidean number fields and the case of 2-stage Euclidean number fields, respectively.

2. The algorithm, generalities

Let $K$ be a number field of degree $n$. We have designed an algorithm which allows us to compute the Euclidean minimum of $K$, in particular when $K$ is totally real [5], but also in the general case [3]. According to theoretical results [4], this algorithm can also give the upper part of the Euclidean spectrum of $K$ and this yields new examples of number fields with interesting properties.

---

1In [2] and [10] the Euclidean minimum of the number field with discriminant 3969, defined by $x^3 - 21x - 28$ was erroneously announced to be 1.
From now on, we suppose that $K$ is totally real and that $n > 2$. We denote by $\mathbb{Z}_K$ the ring of its integers and by $N_{K/Q}$ its absolute norm. The Euclidean minimum of an element $\xi \in K$ is

$$m_K(\xi) = \inf_{T \in \mathbb{Z}_K} |N_{K/Q}(\xi - T)|$$

and the Euclidean minimum of $K$ is

$$M(K) = \sup_{\xi \in K} m_K(\xi).$$

The set of values taken by $m_K$ is called the Euclidean spectrum of $K$. We know the following important result [4].

**Theorem 2.1.** The Euclidean spectrum of $K$ is the union of $\{0\}$ and of a strictly decreasing sequence of rationals $(r_i)_{i \geq 0}$ with limit 0. For each $i$, the set of $\xi \in K$ such that $m_K(\xi) = r_i$ is finite modulo $\mathbb{Z}_K$.

In fact, we have a stronger result, which can be formulated in terms of the inhomogeneous spectrum. However, we shall not need this in what follows.

**Corollary 2.2.** The set of $\xi \in K$ such that $m_K(\xi) \geq 1$ is finite modulo $\mathbb{Z}_K$.

Recall now that we have at our disposal an algorithm which can give us all the $\xi \in K$ such that $m_K(\xi) \geq 1$. Without going into details – these can be found in [5] – let us give nevertheless the theorem which justifies the algorithm and the main ideas that are behind it. Let us choose a constant $k > 0$ and a let us embed $K$ into $K \otimes_{\mathbb{Q}} \mathbb{R} =: \tilde{K}$, which we can identify with $\mathbb{R}^n$, in which $\mathbb{Z}_K$ is a lattice. Under this identification an element $\xi$ of $K$ is viewed as $(\sigma_i(\xi))_{1 \leq i \leq n}$, where the $\sigma_i$ are the embeddings of $K$ into $\mathbb{R}$. The map $m_K$ extends to a map $m_{\tilde{K}}$ from $\mathbb{R}^n$ to $\mathbb{R}^+$ in a natural way:

$$m_{\tilde{K}}(x) = \inf_{T \in \mathbb{Z}_K} \prod_{i=1}^n (x_i - \sigma_i(T)).$$

Moreover, the product of two elements of $K$ is extended to the product coordinate by coordinate in $\mathbb{R}^n$. This new product of two elements $x, y \in \mathbb{R}^n$ will be denoted by $x \cdot y$. Let finally $c$ be a non-torsion unit of $\mathbb{Z}_K$.

The main idea is to find in a fundamental domain $\mathcal{F}$ associated to $\mathbb{Z}_K$ in $\mathbb{R}^n$, $s$ distinct bounded sets $\mathcal{T}_i$ ($1 \leq i \leq s$) with the property that for each such $\mathcal{T}_i$ there exists an $X_i \in \mathbb{Z}_K$ and $s_i$ integers $n_{i,1}, \ldots, n_{i,s_i}$ ($s_i > 0$) such that

$$\langle \varepsilon \cdot T_i - X_i \rangle \mathcal{H} \subset \bigcup_{1 \leq l \leq s_i} \mathcal{T}_{n_{i,l}} (i = 1, \ldots, s),$$

where

$$\mathcal{H} = \{ x \in \mathbb{R}^n \text{ such that } m_{\tilde{K}}(x) \leq k \}.$$
associated to $T'_i$. This defines, in a unique way, $j$ elements of $K$, $\xi_0, \ldots , \xi_{j-1}$ by the formula:

$$\xi_r = \frac{\varepsilon^{j-r}X'_r + \varepsilon^{j-2}X'_{r+1} + \cdots + X'_{j-1+r}}{\varepsilon^{j-1}} \quad (r = 0, \ldots , j - 1),$$

the indices being read modulo $j$. In this context, we say that $\xi_0, \ldots , \xi_{j-1}$ are associated to the cycle $c$.

We denote by $E$ the finite set of all elements of $K$ associated to the elements of $C$. The $\xi_i$ associated to a cycle $c$ are in the same orbit modulo $\mathbb{Z}_K$ under the action of $\mathbb{Z}_K^*$ (in fact $\xi_{r+1} = \varepsilon \cdot \xi_r - X'_r$) and satisfy

$$m_\mathcal{P}(\xi_0) = \ldots = m_\mathcal{P}(\xi_{j-1}) = m(c),$$

which is a rational number. Finally, define

$$m(G) = \max_{c \in C} m(c) = \max_{\xi \in E} m_\mathcal{P}(\xi).$$

Let us say that $G$ is convenient if every infinite path of $G$ is ultimately periodic. The essential result is the following.

**Theorem 2.3.** Assume that $G$ is convenient and that there exists $T \in \{T_1, \ldots , T_s\}$ and $x \in \mathbb{R}^n$ such that $m_\mathcal{P}(x) > k$. Then

i) $m_\mathcal{P}(x) \leq m(G)$.

ii) If $x \in K$, there exists $\xi \in E$ such that $x \equiv \xi \mod \mathbb{Z}_K$.

In this situation we know all the potential $\xi \in K$ such that $m_K(\xi) > k$, and since computing $m_K(\xi)$ is possible (again see [5] for more details), we know in fact all the $\xi \in K$ such that $m_K(\xi) > k$. To identify the elements $\xi \in K$ such that $m_K(\xi) \geq 1$, it is sufficient to run the algorithm with $k = 0.999$, for instance.

3. Generalized Euclidean number fields

3.1. Generalities. From the definition of a G.E. number field and the definition of the map $m_K$, we have the following result.

**Proposition 3.1.** The field $K$ is G.E. if and only if for every $\alpha, \beta \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$ such that $m_K(\alpha/\beta) \geq 1$, the ideal $(\alpha, \beta)$ is not principal.

**Remark 1.** Suppose that we have at our disposal the finite set $S$ of all $\xi \in K$ (modulo $\mathbb{Z}_K$) such that $m_K(\xi) \geq 1$, and that for each such $\xi$ we have a representative $u/v$ where $(u, v) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$. Let $\alpha, \beta \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$ such that $m_K(\alpha/\beta) \geq 1$. Then there exists $\xi \equiv u/v$ in $S$ such that $\alpha/\beta = u/v + \gamma$ with $\gamma \in \mathbb{Z}_K$. Since

$$(\alpha, \beta) = (\beta u/v + \gamma/\beta, \beta) = (\beta u/v, \beta) = \beta/(u, v),$$

it is sufficient, for proving that $K$ is G.E., to check that for every $\xi \equiv u/v \in S$, $(u, v)$ is not principal.

3.2. A first example. The purpose of this subsection is to study in detail a particular case. Other results, obtained in another way, will be given in the next subsection. Let $K$ be the normal quartic field generated by any one of the roots of

$$P(X) = X^4 - 20X^2 + 50.$$

The field $K$ is totally real, its discriminant is $51200$, its class number is 2, and a $\mathbb{Z}$-basis of $\mathbb{Z}_K$ is $(e_1, e_2, e_3, e_4)$ with

$$e_1 = 1, \ e_2 = \sqrt{2}, \ e_3 = \sqrt{10 + 5\sqrt{2}}, \ e_4 = \sqrt{10 - 5\sqrt{2}}.$$
Our algorithm shows that
\[ M(K) = \frac{7}{2}, \]
and that there is a unique \( \xi \in K \) (modulo \( \mathbb{Z}_K \)) such that \( m_K(\xi) \geq 1 \). More precisely
\[ \xi \equiv \frac{1}{2}(e_3 + e_4). \]

According to Remark 1, if we want to establish that \( K \) is G.E., we have just to prove that the ideal \((2, e_3 + e_4)\) is not principal.

**Theorem 3.2.** The field \( K \) is not norm-Euclidean but it is G.E.

**Proof.** First of all, we note that \( e_3 + e_4 = e_2 \cdot e_3 \), so that we are reduced to proving that the ideal \((e_2, e_3)\) is not principal. Suppose on the contrary that it is principal so that we have
\[ e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \nu \mathbb{Z}_K, \]
with \( \nu \in \mathbb{Z}_K \). Since \( N_{K/\mathbb{Q}}(e_2) = 4 \) and \( N_{K/\mathbb{Q}}(e_3) = 50 \), we have
\[ N_{K/\mathbb{Q}}(\nu) | 2 = \gcd(4, 50), \]
so that we have two possibilities : either \( \nu \in \mathbb{Z}_K^* \) or \( N_{K/\mathbb{Q}}(\nu) = \pm 2 \).

**First case** : \( \nu \) is a unit and we have in fact \( e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \mathbb{Z}_K \).
In this case, there exist \( u, v \in \mathbb{Z}_K \) such that
\[ 1 = e_2 \cdot u + e_3 \cdot v. \]
Let us write
\[ \begin{cases} u = a + be_2 + ce_3 + de_4 \\ v = a' + b'e_2 + c'e_3 + d'e_4, \end{cases} \]
where \( a, b, c, d, a', b', c', d' \in \mathbb{Z} \).
Since \( e_2 \cdot e_3 = e_3 + e_4, \ e_2 \cdot e_4 = e_3 - e_4 \) and \( e_3 \cdot e_4 = 5e_2 \), if we substitute (3) into (2) we obtain, by identification of the coefficients in our \( \mathbb{Z} \)-basis, that \( 2b + 10c' = 1 \), which is clearly impossible.

**Second case** : \( \nu \) has norm \( \pm 2 \).
If \( \nu = a + be_2 + ce_3 + de_4 \) where \( a, b, c, d \in \mathbb{Z} \), an easy computation leads to
\[ \pm 2 = N_{K/\mathbb{Q}}(\nu) \]
\[ = a^4 + 4b^4 + 50c^4 + 50d^4 - 4a^2b^2 - 20a^2c^2 - 20a^2d^2 - 40b^2c^2 \\
- 40b^2d^2 + 100c^2d^2 + 40abc^2 - 40abd^2 - 20b^2d^2 - 200cd^3 - 200d^3 + 8abcd. \]
This implies that
\[ \pm 2 \equiv (a^2 - 2b^2)^2 \pmod{5}, \]
which is impossible as neither of \( \pm 2 \) are quadratic residues \( \pmod{5} \). \( \square \)
3.3. The Dedekind-Hasse criterion. In this subsection, we study the link between G.E. and a Euclidean-type map that we shall deduce from the Dedekind-Hasse criterion. This will lead us to define an easy test which allows to find new examples, without requiring detailed calculations as above. First of all, recall the Dedekind-Hasse criterion (see for instance [11]).

**Theorem 3.3.** A number field \( K \) has class number 1 if and only if for every \( \alpha, \beta \in \mathbb{Z}_K \setminus \{0\} \) such that \( \beta \nmid \alpha \), there exist \( \gamma, \delta \in \mathbb{Z}_K \) such that

\[
0 < |N_{K/Q}(\alpha \gamma - \beta \delta)| < |N_{K/Q}(\beta)|. 
\]

This leads to the following natural definition.

**Definition 3.1.** For every \( \xi \in K \setminus \mathbb{Z}_K \) we shall denote by \( h_K(\xi) \) the real number defined by

\[
h_K(\xi) = \inf \{ m_K(\Upsilon \xi); \ U \in \mathbb{Z}_K \text{ and } \Upsilon \xi \notin \mathbb{Z}_K \}. 
\]

This map has the following elementary properties, which we give here without proof.

**Proposition 3.4.** For every \( \xi \in K \setminus \mathbb{Z}_K \) we have

1. \( 0 < h_K(\xi) \leq m_K(\xi) \);
2. For every \( \alpha \in \mathbb{Z}_K \), \( h_K(\xi + \alpha) = h_K(\xi) \);
3. For every \( \varepsilon \in \mathbb{Z}_K^* \), \( h_K(\varepsilon \xi) = h_K(\xi) \).

We can now reformulate Dedekind-Hasse criterion as follows.

**Theorem 3.5.** A number field \( K \) has class number 1 if and only if for every \( \xi \in K \setminus \mathbb{Z}_K \) we have \( h_K(\xi) < 1 \).

**Proof.** The norm being multiplicative, (4) can be reformulated as follows: for every \( \xi \in K \setminus \mathbb{Z}_K \) there exist \( \gamma, \delta \in \mathbb{Z}_K \) such that

\[
0 < |N_{K/Q}(\gamma \xi - \delta)| < 1, 
\]

which leads to \( m_K(\xi) < 1 \). Since (5) cannot be true if \( \gamma \xi \in \mathbb{Z}_K \), we have \( h_K(\xi) < 1 \). Conversely, since \( |N_{K/Q}(\gamma \xi - \delta)| = 0 \) implies \( \gamma \xi \in \mathbb{Z}_K \), which is excluded in the definition of \( h_K \), we see that if \( h_K(\xi) < 1 \) then (5) is true. \( \Box \)

Now consider a number field \( K \) and put

\[
S = \{ \xi \in K; m_K(\xi) \geq 1 \}. 
\]

Suppose that \( K \) is not norm-euclidean so that \( S \neq \emptyset \). We have the following result.

**Theorem 3.6.** One of the following three possibilities holds:

1. For every \( \xi \in S \), \( h_K(\xi) < 1 \). Then \( K \) has class number 1 and is not G.E.
2. For every \( \xi \in S \), \( h_K(\xi) \geq 1 \). Then \( K \) is G.E. (and not principal).
3. There exist \( \xi, \mu \in S \) such that \( h_K(\xi) < 1 \) and \( h_K(\mu) \geq 1 \). Then \( K \) is not principal. If in addition, there exists \( \xi = \alpha/\beta \in S \) (with \( \alpha, \beta \in \mathbb{Z}_K \)) with \( h_K(\xi) < 1 \) and such that \( (\alpha, \beta) \) is principal, then \( K \) is not G.E. Otherwise it is G.E.

**Proof.** Clearly we have the three cases.

**Case 1.** The result is a consequence of Theorem 3.5 and of the fact that when the field is principal norm-Euclidean and G.E. are synonymous.
Case 2. Theorem 3.5 indicates that $K$ is not principal. By Proposition 3.1 it is sufficient to prove that for every $\xi = \alpha/\beta \in S$ where $\alpha, \beta \in \mathbb{Z}_K$, the ideal $(\alpha, \beta)$ is not principal. Otherwise, we have $(\alpha, \beta) = \nu \mathbb{Z}_K$ with $\nu \in \mathbb{Z}_K$. By hypothesis $h_K(\xi) \geq 1$ so that for every $X, Y \in \mathbb{Z}_K$ with $X\xi \notin \mathbb{Z}_K$ we have
$$|N_{K/\mathbb{Q}}(X\alpha - Y\beta)| \geq \left|N_{K/\mathbb{Q}}(\beta)\right|.$$ 

Now $\nu$ can be written $\nu = X\alpha - Y\beta$ with $X, Y \in \mathbb{Z}_K$ and $X\xi \notin \mathbb{Z}_K$. Otherwise $\nu \in \beta \mathbb{Z}_K$ so that $\beta | \nu$. But this implies that $\nu$ and $\beta$ are associates and we have $(\alpha, \beta) = \beta \mathbb{Z}_K$ which implies $\beta | \alpha$ and $\xi \in \mathbb{Z}_K$, which is impossible. We deduce from this that $|N_{K/\mathbb{Q}}(\nu)| \geq \left|N_{K/\mathbb{Q}}(\beta)\right|$. Since $N_{K/\mathbb{Q}}(\nu) | N_{K/\mathbb{Q}}(\beta)$ we have $|N_{K/\mathbb{Q}}(\nu)| = \left|N_{K/\mathbb{Q}}(\beta)\right|$, and since $\nu | \beta, \nu$ and $\beta$ are associates, which is impossible by the previous argument.

Case 3. Theorem 3.5 indicates that $K$ is not principal. The second assertion is a consequence of Proposition 3.1. Indeed, as previously, if $h_K(\xi) \geq 1$ and $\xi = \alpha/\beta$ then $(\alpha, \beta)$ is not principal and this case is not an obstruction for $K$ to be G.E. Finally, the only possibilities for contradicting G.E. come from the $\xi = \alpha/\beta \in S$ such that $h_K(\xi) < 1$ and $(\alpha, \beta)$ is principal.

Corollary 3.7. Suppose that $K$ is not norm-Euclidean and that, with the above notation, $S$ modulo $\mathbb{Z}_K$ is composed of a single orbit under the (multiplicative) action of $\mathbb{Z}_K^*$ modulo $\mathbb{Z}_K$, i.e. that if $\xi, \mu \in S$ there exists an $\varepsilon \in \mathbb{Z}_K^*$ such that $\mu = \varepsilon \xi + \alpha$. Then either $K$ is principal and not G.E. or $K$ is not principal but is G.E.

Proof. If $K$ is principal, we are in case 1. Otherwise, since all the elements of $S$, which are in the same orbit, have the same image by $h_K$ (Proposition 3.4), we cannot be in case 3 of Theorem 3.6. Finally, we are in case 2 and $K$ is G.E.

Remark 2. To simplify notation and vocabulary, we shall often, by abuse of language, speak of $\xi \in K$ to mean $\xi \in \mathbb{Z}_K$ mod $\mathbb{Z}_K$. For instance we shall speak of orbits in $S$ under the action of $\mathbb{Z}_K^*$; in this context $S$ and these orbits should be understood modulo $\mathbb{Z}_K$.

Corollary 3.8. The totally real number fields of degree 3 and discriminants 1957, 2777, 3981 (see Table 1) are G.E. The totally real number fields of degree 4 and discriminants 46400 and 51200 (see Table 1) are G.E.

Proof. In fact, in all these cases, our algorithm establish that we are under the previous hypotheses. For discriminant 1957, we have $M(K) = 2$ and one orbit with one element in $S$. For discriminant 2777, we have $M(K) = 5/3$ and one orbit with 2 elements in $S$. For discriminant 3981, we have $M(K) = 3/2$ and one orbit with one element in $S$. For discriminant 46400, we have $M(K) = 5/4$ and one orbit with 3 elements in $S$. For discriminant 51200, we have $M(K) = 7/2$ and one orbit with one element in $S$.

Now, if there are several orbits in $S$, and we want to use Theorem 3.6, we have to see whether, for one element $\xi$ by orbit, and for every orbit, we have $h_K(\xi) \geq 1$, in which case necessarily $K$ is G.E. The problem is now: how can we compute $h_K(\xi)$? Our algorithm gives us every such $\xi$ by its coordinates in a $\mathbb{Z}$-basis of $\mathbb{Z}_K$. These coordinates are of the form $(a_1/d, a_2/d, \ldots, a_n/d)$ where $a_i \in \mathbb{Z}$ for every $i$ and $d \in \mathbb{Z}_{>0}$. Furthermore we can compute $m_K(\mu)$ for every $\mu \in K$. Hence, it is
easy to see that, to compute \( h_K(\xi) \), it is sufficient to compute \( m_K(\Upsilon \xi) \) for every \( \Upsilon \) with coordinates in \( \{0, 1, \ldots, d - 1\} \) for our basis, such that \( \Upsilon \xi \notin \mathbb{Z}_K \). This is easy to check. By definition, the value of \( h_K(\xi) \) will be the minimum of these \( m_K(\Upsilon \xi) \).

Of course if for every \( \xi \) and every such \( \Upsilon \) we have \( \Upsilon \xi \in S \mod \mathbb{Z}_K \), then \( K \) is G.E. Using this last approach we have established the following result.

**Theorem 3.9.** The following totally real number fields of degree \( n \) are G.E. but not norm-Euclidean:

- when \( n = 3 \), the fields in Table 1 with discriminants 2597, 4212, 4312, 5684;
- when \( n = 4 \), the fields in Table 1 with discriminants 21025, 32625.

**Proof.** We just give a typical example. For \( n = 3 \) and discriminant 2597, we have two orbits in \( S \), the first one \( O_1 \) with two elements \( (\pm(e_1 + 2e_2 + 2e_3)/3 \mod \mathbb{Z}_K \) where \( (e_i) \) is the \( \mathbb{Z} \)-basis of \( \mathbb{Z}_K \) returned by PARI [1]) and the second one \( O_2 \) with one element \( (e_1 + 2e_2 + e_3)/2 \mod \mathbb{Z}_K \). Then we can easily check that \( \mathbb{Z}_K \cdot O_1 = O_1 \cup \{0\} \) and that \( \mathbb{Z}_K \cdot O_2 = O_2 \cup \{0\} \). The same thing happens in other cases with sometimes more complicated equalities but always with \( \mathbb{Z}_K \cdot O \subseteq S \cup \{0\} \).

**Remark 3.** If we want to treat all the non-principal number fields of degree 3 and discriminant \( > 6000 \), it remains to study the two number fields with discriminant 3969. In these cases, our previous method does not work, because we have some \( \xi = \alpha/\beta \in S \) such that \( h_K(\xi) < 1 \). The first one, \( K_1 \), is defined by \( x^3 - 21x - 28 \). For this field, \( S \) is composed of five orbits \( O_i \), \( 1 \leq i \leq 5 \). For four of them, say for \( 1 \leq i \leq 4 \), we have \( \mathbb{Z}_K \cdot O_i \subseteq S \cup \{0\} \); but for the last one, \( O_5 \), this is not true. Take an element \( \alpha/\beta \) of \( O_5 \); here we can take \( \alpha = 3e_1 + 2e_2 + 2e_3 \) and \( \beta = 6 \) where \( (e_1, e_2, e_3) \) is the \( \mathbb{Z} \)-basis returned by PARI [1]. We can then prove directly as in Section 3.2 that the ideal \( (\alpha, \beta) \) is not principal. We conclude that \( K_1 \) is G.E.

For the second field, \( K_2 \), defined by \( x^3 - 21x - 35 \) the situation is different. Here \( S \) is composed of seven orbits \( O_i \), \( 1 \leq i \leq 7 \) and four of them, say \( O_i \), \( 1 \leq i \leq 4 \), are such that \( \mathbb{Z}_K \cdot O_i \subseteq S \cup \{0\} \). Now if we look at the three others, we find that two of them contain an \( \alpha/\beta \) for which \( (\alpha, \beta) \) is principal. For completeness these \( (\alpha, \beta) \) are \( (7e_1 + 12e_2 + 4e_3, 21) \) and \( (7e_1 + 5e_2 + 11e_3, 21) \) with the usual notation. Consequently \( K_2 \) is not G.E. All the computations, which are long and complicated – in particular for \( K_2 \) – have been done by hand and checked using PARI [1]. We do not give them here for lack of space; anyway they are not especially enlightening.

Finally, we put all these results together to give us Theorem 1.4.

### 4. The 2-stage Euclidean number fields

Let us begin with an example. Let \( K \) be the totally real cubic number field with discriminant 3988, defined by \( x^3 - 16x - 4 \). Using our algorithm we see that the upper part of the Euclidean spectrum of \( K \) has five elements:

\[
\text{sp}(K) \cap [1, \infty) = \{19/8, 11/8, 5/4, 19/16, 133/128\}.
\]

The set \( S \) is composed of five orbits, respectively the orbits of \( ae_1 + be_2 + ce_3 \) with \( (a, b, c) = (0, 1/2, 1/2), (1/2, 1/2, 0), (1/2, 1/2, 1/2), (0, 3/4, 1/2) \) and \( (0, 3/8, 1/2) \), where \( (e_1, e_2, e_3) \) is the \( \mathbb{Z} \)-basis of \( \mathbb{Z}_K \) returned by PARI [1]. These orbits have respectively 1, 1, 1, 2 and 4 elements. For one element \( \xi \) by orbit, we try to find
$q_1, q_2 \in \mathbb{Z}_K$ such that

$$
|N_{K/Q}(\xi - q_1 - \frac{1}{q_2})| < \frac{1}{|N_{K/Q}(q_2)|},
$$

by testing “small” $q_1 \in \mathbb{Z}_K$ and “small” $q_2 \in \mathbb{Z}_K \setminus \{0\}$. In each case this is possible, so that for every $\xi \in S$, (6) is true. Finally this implies that $K$ is 2-stage norm-Euclidean. Using exactly the same approach we have established the results of Theorem 1.5.

**Remark 4.** Obviously these fields, which are principal and not norm-Euclidean, are not G.E.

5. **Acknowledgements**

I owe a particular debt of gratitude to the anonymous referee and to Pr. Karim Belabas for the many suggestions that they made, and which enabled me to improve the presentation of the paper.

**References**