Three hours with toric varieties

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1 Introduction

Our motivation to study toric varieties is P. Scholze's proof of Deligne's weight-monodromy conjecture in the case of complete intersection subvarieties of projective smooth toric varieties over a local field ([4], Theorem 9.6).

The main reference for this text is Fulton [2]. Cox-Little-Schenck [1] treats toric varieties in great details. Oda [3] is also useful. All these books consider toric varieties only $\mathbb{C}$, so we had to check that all proofs here are correct over any field.

2 Rational convex polyhedral cones

Notation

(1) N is a free $\mathbb{Z}$-module of rank d;

(2) $\mathbb{R}_+$ is the set of non-negative real numbers;

(3) $M = \mathrm{Hom}(N, \mathbb{Z})$ is the linear dual of N;

(4) $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, and $e_1, \ldots, e_d$ is a basis of N.

2.1 Basic definitions

Definition 2.1 A *convex polyhedral cone* in $N_\mathbb{R}$ is a subset of the form

$$\sigma = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_s$$

where $v_1, \ldots, v_s$ are some vectors in $N_\mathbb{R}$. If there are generators $v_i \in N$, we say σ is a *rational convex polyhedral cone*. We say σ is *strongly convex* if σ doesn't contain a line $\mathbb{R}v$.

The set $\sigma + (-\sigma) := \{v + (-v') \mid v, v' \in \sigma\}$ is a vector subspace of $N_\mathbb{R}$, its dimension is called the *dimension* $\dim \sigma$ *of* σ.

Example 2.2 (1) $\sigma = \{0\}$;

(2) $\sigma = \sum_{1 \le i \le d'} \mathbb{R}_+ e_i$ for some $d' \le d$;

(3) $d = 2$, $\sigma = \mathbb{R}_+(2e_1 - 3e_2) + \mathbb{R}_+ e_2$.

They are all strongly convex.

(4) $\sigma = \mathbb{R}_+ e_1 + \cdots + \mathbb{R}_+ e_{d'} + \mathbb{R} e_{d'+1} + \cdots + \mathbb{R} e_d$. It is not strongly convex if $d' < d$.

2.2 Dual

Recall $M = N^*$ is the dual of N. So $M_\mathbb{R}$ is the dual of $N_\mathbb{R}$. Put

$$\sigma^\vee := \{u \in M_\mathbb{R} \mid u(v) \ge 0, \ \forall v \in \sigma\}.$$

Note that if $\sigma = \sum_i \mathbb{R}_+ v_i$, then $\sigma^\vee = \cap_i (\mathbb{R}_+ v_i)^\vee$ and $\sigma^\vee$ is the intersection of the half-spaces in $(\mathbb{R}_+ v_i)^\vee$ in $M_\mathbb{R}$.

Example 2.3 Denote by $e_1^*, \ldots, e_d^*$ the dual basis of $e_1, \ldots, e_d$.

(1) $\{0\}^\vee = M_\mathbb{R}$.

(2) We have

$$\Big(\sum_{1 \le i \le d'} \mathbb{R}_+ e_i\Big)^\vee = \sum_{1 \le i \le d'} \mathbb{R}_+ e_i^* + \sum_{d'+1 \le j \le d} \mathbb{R} e_j^*.$$

It is not strongly convex if $d' < d$.

(3) $d = 2$, $(\mathbb{R}_+(2e_1 - 3e_2) + \mathbb{R}_+e_2)^* = \mathbb{R}_+e_1^* + \mathbb{R}_+(3e_1^* + 2e_2^*)$.

Let $u \in M_\mathbb{R}$. Dente by $u^\perp = \{v \in N_\mathbb{R} \mid u(v) = 0\}$.

Definition 2.4 Let σ be a rational convex polyheadral cone. A *face* τ *of* σ is a subset of σ of the form

$$\tau = \sigma \cap u^\perp$$

for some $u \in \sigma^\vee$.

Remark 2.5 Let τ be a face of σ. Then there exists $u \in M$ such that $\tau = \sigma \cap u^\perp$ ([2], §1.2, Prop. 2). We will always chose $u \in M$ when dealing with faces of σ.

Proposition 2.6. *Let $\sigma \subset N_\mathbb{R}$ be a rational convex polyhedral cone.*

(1) $(\sigma^\vee)^\vee = \sigma$.

(2) *σ has finitely many faces.*

(3) *A face τ of σ is a rational convex polyhedral cone, strongly convex if σ is strongly convex.*

(4) *A face of a face of σ is a face of σ*

(5) *The intersection of two faces of σ is a face of σ.*

Proof. (1) Clearly $\sigma \subseteq (\sigma^\vee)^\vee$. The converse is a classical theorem on convex bodies in $\mathbb{R}^d$: if σ is a convex subset of $\mathbb{R}^d$ and $v_0 \in \mathbb{R}^d \setminus \sigma$, then there exists a half-space containing σ but not v_0.

(2)-(3) Write $\tau = \sigma \cap u^\perp$. We have $\sigma = \sum_{1 \le i \le s} \mathbb{R}_+v_i$. So

$$\tau = \sum_{i \le s, u(v_i)=0} \mathbb{R}_+v_i.$$

This implies (2) and that τ is a rational polyhedral cone. If σ is strongly convex, it doesn't contain real line, so *a fiortiori* τ doesn't contain real line.

(4) Let $\tau = \sigma \cap u^\perp$ and $\tau' = \tau \cap u'^\perp$ with $u' \in \tau^\vee$. There exists $n \ge 0$, such that $u' + nu \in \sigma^\vee$ and $u'(v_i) + nu(v_i) > 0$ if $v_i \notin \tau$. Indeed, if $v_i \in \tau$, then $u'(v_i) \ge 0$ and $u(v_i) = 0$, so $(u' + nu)(v_i) \ge 0$ for all $n \ge 0$. If $v_i \notin \tau$, then $u(v_i) > 0$, so $u'(v_i) + nu(v_i) > 0$ if n is big enough. The linear form $u' + nu \in \sigma^\vee$.

We have clearly $\tau' \subseteq \sigma \cap (u' + nu)^\perp$. Conversely let $v = \sum_i \lambda_i v_i \in \sigma \cap (u' + nu)^\perp$ (so $\lambda_i \in \mathbb{R}_+$), as $(u' + nu)(v_i) \ge 0$ and is > 0 for those $v_i \in\notin \tau$, we find $\lambda_i = 0$ if $v_i \notin \tau$. So $v \in \tau$ and then $v \in u'^\perp$. Thus $v \in \tau'$.

(5) Let $\tau_1 = \sigma \cap u_1^\perp$, $\tau_2 = \sigma \cap u_2^\perp$. Then $\tau_1 \cap \tau_2 = \sigma \cap (u_1 + u_2)^\perp$. □

Proposition 2.7. (Farkas's theorem) *Let σ be a rational convex polyhedral cone in $N_\mathbb{R}$. Then $\sigma^\vee$ is a rational convex polyhedral cone in $M_\mathbb{R}$.*

Proof. First suppose that $\dim \sigma = d$. It is easy to see that any proper face is contained in a face of dimension $d-1$ ([2], 1.2(5)). Let $\sigma \cap u_1^{\perp}, \dots, \sigma \cap u_r^{\perp}$ be the faces of dimension $d-1$. Then

$$\sigma = \cap_{1 \leq j \leq r} \{v \in N_{\mathbb{R}} \mid u_j(v) \geq 0\}$$

([2], 1.2(8)). Let $S = \sum_{j \leq r} \mathbb{R}_+ u_j \subseteq \sigma^{\vee}$. Let $v \in S^{\vee}$. Then $u_j(v) \geq 0$ for all $j \leq r$ and $v \in \cap_j \{u_j \geq 0\} = \sigma$. Therefore $S^{\vee} \subseteq \sigma$ and $\sigma^{\vee} \subseteq (S^{\vee})^{\vee} = S$. This implies that $\sigma^{\vee} = S$ is a rational convex polyhedral cone in $M_{\mathbb{R}}$.

In general, let $W = \sigma + (-\sigma)$. Then $M_{\mathbb{R}}/W^{\perp} = W^*$ (linear dual space) and $\sigma^{\vee}$ (as a cone) is generated by the lifting of a system of generators of $\sigma_W^{\vee}$ defined by the cone $\sigma \subset W_{\mathbb{R}}^*$, and $\pm$ a system of generators of $W^{\perp}$. $\square$

Definition 2.8 Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational convex polyhedeal cone. We define

$$S_{\sigma} := \sigma^{\vee} \cap M.$$

This is a sub-semigroup of M with 0.

Proposition 2.9. (Gordan's lemma) *S_{σ} is finitely generated.*

Proof. By Farkas's theorem, $\sigma^{\vee} = \sum_{1 \leq j \leq r} \mathbb{R}_+ u_j$ with $u_j \in M$. Let $K = \sum_j [0,1] u_j \subset \sigma^{\vee}$. It is compact. As $K \cap M$ is discrete in a compact, it is finite. As $\mathbb{R}_+ = \mathbb{N} + [0,1]$, we find $S_{\sigma} \subseteq \sum_j \mathbb{N} u_j + (K \cap M)$. Hence S_{σ} is finitely generated. $\square$

Example 2.10 (1) If $\sigma = \{0\}$, then $S_{\sigma} = M$.

(2) If $\sigma = \sum_{1 \leq i \leq d'} \mathbb{R}_+ e_i$, then $S_{\sigma} = \sum_{1 \leq i \leq d'} \mathbb{N} e_i^* + \sum_{d'+1 \leq r \leq d} \mathbb{Z} e_i^*$.

(3) Let σ be the cone given in 2.2(3). Then $S_{\sigma} = \mathbb{N} e_1^* + \mathbb{N}(2e_1^* + e_2^*) + \mathbb{N}(3e_1^* + 2e_2^*)$.

Proposition 2.11. *The semigroup $S_{\sigma} \subseteq M$ is saturated (if $n \geq 1$, $u \in M$ satisfy $nu \in S_{\sigma}$, then $u \in S_{\sigma}$), finitely generated. If σ is strongly convex. Then $S_{\sigma} + (-S_{\sigma}) = M$.*

Proof. Only the last property has to be proved. First we have $\sigma^{\vee} + (-\sigma^{\vee}) = M_{\mathbb{R}}$. Indeed, if this is not true, then $\sigma^{\vee} \subseteq \ker(v)$ for some non-zero linear form $v \in M_{\mathbb{R}}^* = N_{\mathbb{R}}$. Therefore $\mathbb{R} v = (\ker(v))^{\vee} \subseteq (\sigma^{\vee})^{\vee} = \sigma$. Contradiction with σ strongly convex. $\square$

3 Affine toric varieties

We define the affine scheme associated to a rational convex polyhedral cone.

We fix a (commutative unitary) ring R. Most of the time R is a field. But in applications we have in mind (Scholze's theorem), we have to deal with R a discrete valuation ring.

3.1 Algebra of a semigroup

Let S be a commutative semigroup. We denote by $R[S]$ the direct sum

$$R[S] = \oplus_{s \in S} R\chi^s$$

where χ^s denotes the basis indexed by s. It has a natural structure of commutative R-algebra by setting

$$\chi^s \cdot \chi^t = \chi^{s+t}.$$

Example 3.1 $R[\mathbb{N}^d] \simeq R[T_1, \dots, T_d]$; $R[\mathbb{Z}^d] \simeq R[T_1^{\pm}, \dots, T_d^{\pm}]$.
If $S = 2\mathbb{N} + 3\mathbb{N} \subset \mathbb{N}$. Then $R[S] = R[T^2, T^3] \subseteq R[T]$.

Lemma 3.2. *Let S_1, S_2 be semigroups contained in M.*

(1) *If $S_1 \subseteq S_2$, the we have canonically $R[S_1] \subseteq R[S_2] \subseteq R[M]$.*

(2) *If S is finitely generated, then $R[S]$ is a finitely generated algebra over R.*

(3) $R[S_1 + S_2] = R[S_1]R[S_2]$*;*

(4) $R[S_1 \cap S_2] = R[S_1] \cap R[S_2]$.

Proof. Immediate from the definition. □

Let $X = \operatorname{Spec} R[S]$. Let us decribe the points of X. Let A be an R-algebra. Consider a homomorphism $\phi : R[S] \to A$. Then we have a map

$$S \to A, \quad s \mapsto \phi(\chi^s).$$

It is a morphism of semigroups (A is considered as a semigroup with its multiplication law).

Proposition 3.3. *The above process induces a canonical bijection*

$$X(A) \to \operatorname{Hom}_{sg}(S, A)$$

from the set of A-valued points of X to the set of morphisms of semigroups from S to $(A, \times)$.

Proof. If $\psi : S \to A$ is a morphism of semigroups, we define an R-linear map $\phi : R[S] \to A$ by $\phi(\chi^s) = \psi(s)$. We check easily that ϕ is a morphism of R-algebras, and $\psi \mapsto \phi$ is the reciprocal map of $X(A) \to \operatorname{Hom}(S, A)$. □

3.2 Affine toric varieties

Let σ be a rational convex polyhedral cone in $N_{\mathbb{R}}$.

Definition 3.4 Let $S_\sigma \subseteq M$ be the semigroup associated to σ (2.8). We define

$$U_\sigma = \operatorname{Spec} R[S_\sigma].$$

If necessary, we add R in the subscript to indicate the scheme is defined over R.

Example 3.5 (1) $U_{\{0\}} = \operatorname{Spec} R[M] \simeq \mathbb{G}^d_{m,R}$. Denote by $T_N := U_{\{0\}}$.

(2) If $\sigma = \sum_{1\le i\le d'} \mathbb{R}_+ e_i$ in $N_\mathbb{R}$, then $U_\sigma \simeq \mathbb{A}^{d'}_R \times_R \mathbb{G}^{d-d'}_{m,R}$.

(3) Let σ be the cone given in 2.2(3). The semigroup S_σ has been computed in 2.10(3). Denote by $T_1 = \chi^{e_1}, T_2 = \chi^{e_2}$. Then

$$R[S_\sigma] = R[T_1, T_1^2 T_2, T_1^3 T_2^2] \subseteq R[T_1^{\pm 1}, T_2^{\pm 1}].$$

Denote by $U = T_1^2 T_2$ and $V = T_1^3 T_2^2$, then $R[S_\sigma] = R[T_1, U, V]$ with the relation $T_1 V - U^2$. So if k is a field, then $U_{\sigma,k}$ is a rational surface isomorphic to $\operatorname{Spec} k[T,U,V]/(TV - U^2)$.

Lemma 3.6. *Let $\tau = \sigma \cap u^\perp$ be a face of σ with $u \in S_\sigma$. Then*

(1) $\tau^\vee = \sigma^\vee + \mathbb{R}_+(-u)$;

(2) $S_\tau = S_\sigma + \mathbb{N}(-u)$.

(3) *The inclusion $S_\sigma \subseteq S_\tau$ induces an open immesion $U_\tau \to U_\sigma$ which identifies U_τ with the principal open subset $D(\chi^u)$ of U_σ.*

(4) *If R is an integral domain, then U_σ is integral. If moreover σ is strongly convex, then $T_N \to U_\sigma$ is birational.*

Proof. (1) We have

$$\tau^\vee = \sigma^\vee + \mathbb{R}u = \sigma^\vee + \mathbb{R}_+ u + \mathbb{R}_+(-u) = \sigma^\vee + \mathbb{R}_+(-u).$$

(2) Obviously $S_\sigma + \mathbb{N}(-u) \subseteq S_\tau$. Let $u' \in S_\tau$. We saw in the proof of 2.6(4) that there exists $n \ge 1$ such that $u' + nu \in \sigma^\vee \cap M$. So $u' \in S_\sigma + \mathbb{N}(-u)$.

(3) follows from

$$R[S_\tau] = R[S_\sigma][\chi^{-u}] = R[S_\sigma][(\chi^u)^{-1}] = R[S_\sigma]_{\chi^u}.$$

(4) As $S_\sigma + (-S_\sigma) = M$ (Proposition 2.11), for any $u \in M$, there exist $u_1, u_2 \in S_\sigma$ such that $u = u_1 - u_2$. So $\chi^u = \chi^{u_1}(\chi^{u_2})^{-1}$. This implies that $\operatorname{Frac}(R[M]) = \operatorname{Frac}(R[S_\sigma])$. □

The next lemma will be used in §4.

Lemma 3.7. ([2], §1.2, Proposition 3) *Let σ, σ' be two rational convex polyhedral cones sharing a common face τ. Then $S_\tau = S_\sigma + S'_\sigma$.*

Remark 3.8 The R-scheme U_σ has a *distinguished section* $x_\sigma \in U_\sigma(R)$ defined by the morphism of semigroups $S_\sigma \to R$, $u \mapsto 1$ if $u|_\sigma = 0$ and $u \mapsto 0$ otherwise.

If $\sigma = \{0\}$, then x_σ correspond to the unit section $(1, \dots, 1) \in \mathbb{G}^d_{m,R}$. If $\sigma = \sum_{1\le i\le d'} \mathbb{R}_+ e_i$, then $x_\sigma = (0, .., 0, 1, \dots, 1)$ (with 0 repeated d' times and 1 repeated $d - d'$ times). In Example 3.5.(3) it corresponds to $T_1 = U = V = 0$.

3.3 Local properties of U_σ

Proposition 3.9. *The scheme U_σ is smooth over R if and only if σ is generated by a subset of a basis of N.*

Proof. Suppose σ is generated by a basis of N. Then U_σ is isomorphic to a product of $\mathbb{A}_R^{d'}$ and a $\mathbb{G}_{m,R}^{d-d'}$ by Example 3.5(2). So it is smooth over R.

Suppose $U_{\sigma,R}$ is smooth over R. Base change to a residue field, we find that $U_{\sigma,k}$ is regular for some field k. Suppose first $\dim\sigma = d$. Consider the distinguished rational point x_σ (Remark 3.8). The maximal ideal $\mathfrak{m}_\sigma$ of $k[S_\sigma]$ corresponding to x_σ is generated by all χ^u for $u \in S_\sigma$ non-zero. By hypothesis, $\mathfrak{m}_\sigma/\mathfrak{m}_\sigma^2$ has dimension $d = \dim U_\sigma$ over k. So there exist $u_1, \dots, u_d \in S_\sigma$ such that $\mathfrak{m}_\sigma = \sum_i k\chi^{u_i} + \mathfrak{m}_\sigma^2$. For any $u \in S_\sigma$ non-zero, $\chi^u = \sum_i \lambda_i \chi^{u_i} + \sum_j f_j g_j$, $f_j, g_j \in \mathfrak{m}_\sigma$. This implies that

$$u \in \{u_1, \dots, u_d\} \cup (T + T)$$

where $T = S_\sigma \setminus \{0\}$ and $u \in (\sum_i \mathbb{N}u_i) \cup (T + \cdots + T)$. As $\sigma^\vee$ is a strongly convex rational polyhedral cone in $M_\mathbb{R}$, the sum of r vectors in T has norm tending to $+\infty$ when r tends to $+\infty$, so $u \in \sum_{1\le i\le d} \mathbb{N}u_i$ andd $S_\sigma = \sum_i \mathbb{N}u_i$. As S_σ generates M, this forces $\{u_1, \dots, u_d\}$ to be free over $\mathbb{R}$, hence free over $\mathbb{Z}$. Up to scaling, we see that $\sigma^\vee$ is generated by a basis of M, therefore $\sigma = (\sigma^\vee)^\vee$ is generated by a basis of N.

In the general case, let L be the sub-module of N generated by $\sigma \cap N$. Then N/L is free. Let Write $N = N' \oplus L$ with $N' \simeq N/L$. Then $S_\sigma = (S'_\sigma) \oplus M'$ where S'_σ is defined by σ viewed as a polyhedral cone in $L_\mathbb{R}$ of dimension equal to $\dim L_\mathbb{R}$. As schemes we than have $U_\sigma = U'_\sigma \times \mathbb{G}_{m,k}^{d'}$ with $d' = d - \dim L_\mathbb{R}$. This implies that U'_σ is regular, hence σ is generated by a basis of L. The latter can be completed into a basis of N. □

Proposition 3.10. *Suppose R is integral and integrally closed. Then U_σ is integral and normal.*

Proof. Write $\sigma = \sum_{1\le i\le s} \mathbb{R}_+ v_i$. So $\sigma^\vee = \cap_i (\mathbb{R}_+ v_i)^\vee$ and $S_\sigma = \cap_i S_{\tau_i}$ where $\tau_i = \mathbb{R}_+ v_i$. We can replace v_i by a suitable vector in $\mathbb{Q}_+ v_i$ so that v_i can be completed into a basis of N over $\mathbb{Z}$. So U_{τ_i} is smooth over R (Proposition 3.9), hence normal. Therefore

$$R[S_\sigma] = \cap_i R[S_{\tau_i}]$$

is integrally closed. □

Remark 3.11 One can show that U_σ over a field satisfies further the following properties:

(1) U_σ is Cohen-Macaulay ([2], page 30).

(2) U_σ is monomial (*i.e.*, closed subscheme of an affine space defined by equations of the type one monomial = other monomial ([1], Proposition 1.1.9). For example the U_σ in 3.5(3) is defined by $T_1 V = U^2$.

(3) U_σ has only rational singularities ([1], Theorem 11.4.2; [2], §3.5, p. 76).

(4) Every projective module of finite type over $k[S_\sigma]$ (k can be replaced by any PID) is free (Gubeladze, *The Anderson conjecture and a maximal class of monoids over which projective modules are free*, (Russian) Mat. Sb. (N.S.) 135(177) (1988), 169–185,). This generalizes Quillen-Suslin's theorem for polynomial rings.

3.4 Torus action

Recall $T_N = U_{\{0\}} \simeq \mathbb{G}^d_{m,R}$. This is a group scheme over R, the co-multiplication law is given by

$$R[M] \to R[M] \otimes_R R[M], \quad \chi^u \mapsto \chi^u \otimes \chi^u.$$

For any σ, the inclusion $R[S_\sigma] \subseteq R[M]$ induces

$$R[S_\sigma] \to R[S_\sigma] \otimes_R R[M], \quad \chi^u \mapsto \chi^u \otimes \chi^u,$$

hence a morphism

$$T_N \times_R U_\sigma \to U_\sigma$$

which satisfies all axioms of an action of T_N on U_σ (we have to check that some diagrams are commutative, but it is enough to see they are commutative with $R[M]$).

On the sections, the action

$$T_N(R) \times U_\sigma(R) \to U_\sigma(R)$$

is described as follows. Let $t \in T_N(R) = \mathrm{Hom}_{sg}(M, R)$ (morphisms of semigroups, see Prop. 3.3), $x \in U_\sigma(R) = \mathrm{Hom}_{sg}(S_\sigma, R)$, then $tx \in U_\sigma(R)$ is $u \mapsto t(u)x(u)$.

Let $S_\sigma = \sum_{i \leq r} \mathbb{N}u_i$. Then each point $x \in U_\sigma$ has coordinates $(x_1, \ldots, x_r)$ when U_σ is embedded in $\mathbb{A}^r_R$ using the χ^{u_i}'s. Write $u_i = \ell_{i1}e_1^* + \cdots + \ell_{id}e_d^*$. Then the above action is

$$((t_1, \ldots, t_d), (x_1, \ldots, x_r)) \mapsto (t_1^{\ell_{11}} \cdots t_d^{\ell_{1d}} x_1, \; \ldots, \; t_1^{\ell_{r1}} \cdots t_d^{\ell_{rd}} x_d).$$

Remark 3.12 The morphism $R[S_\sigma] \to R[S_\sigma] \otimes_R R[S_\sigma]$ defined by $\chi^u \mapsto \chi^u \otimes \chi^u$ induces a morphism $U_\sigma \times_R U_\sigma \to U_\sigma$ which makes U_σ into a "monoidal scheme" (the law is associative, commutative with unit).

4 Toric varieties

A toric variety is obtained by glueing suitable sets of affine toric varieties U_σ. Recall that N is a free $\mathbb{Z}$-module of rank d.

4.1 Construction from fans

Definition 4.1 A *fan* Σ *in* $N_{\mathbb{R}}$ is a finite and non-empty set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ with the following properties:

(i) If $\sigma \in \Sigma$, then all faces of σ belong to Σ;

(ii) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of σ and of σ'.

Example 4.2 Any σ induces a fan Σ which consists in all faces of σ.

Let Σ be a fan. Let $\sigma, \sigma', \sigma'' \in \Sigma$. By Lemma 3.6(3), we have canonical open immersions $p_{\sigma,\sigma'} : U_{\sigma\cap\sigma'} \to U_\sigma$ and $p_{\sigma',\sigma''} : U_{\sigma'\cap\sigma''} \to U_{\sigma'}$. It is easy to check that on $U_{\sigma\cap\sigma'\cap\sigma''}$, the open immersions $p_{\sigma,\sigma'}, p_{\sigma,\sigma''}$ and $p_{\sigma',\sigma''}$ coincide. This allows us to define a unique scheme by glueing the U_σ's.

Definition 4.3 Let Σ be a fan in $N_{\mathbb{R}}$. We denote by $X_{\Sigma,R}$ or simply by X_Σ the R-scheme obtained by glueing as above.

Proposition 4.4. (1) *The R-scheme $X_{\Sigma,R}$ is separated.*

(2) *If $R \to R'$ is ring homomorphism, then $X_{\Sigma,R'} = X_{\Sigma,R} \times_R \operatorname{Spec} R'$.*

(3) *The morphism $X_{\Sigma,R} \to \operatorname{Spec} R$ is faithfully flat with integral and normal geometric fibers.*

Proof. (1) It is enough to show that for any pair of $\sigma, \sigma' \in \Sigma$, the canonical map

$$R[S_\sigma] \otimes_R R[S_{\sigma'}] \to R[S_{\sigma\cap\sigma'}]$$

is surjective. By Lemma 3.7, $S_{\sigma\cap\sigma'} = S_\sigma + S_{\sigma'}$. So $R[S_{\sigma\cap\sigma'}]$ is generated by $R[S_\sigma]$ and $R[S_{\sigma'}]$ and the above map is surjective.

(2) is immediate and (3) follows from (2) and Proposition 3.10. □

Definition 4.5 Le k be a field. A *toric variety over k* is an integral normal separated variety X over k, endowed with the action of a torus $T \simeq \mathbb{G}_{m,k}^d$:

$$\mu : T \times_k X \to X$$

and a rational point $x_0 \in X(k)$ such that $T \simeq T \times \{x_0\} \xrightarrow{\mu} X$ is an open immersion. In other words, T endowed with the natural action of T extends equivariantly to X.

The torus T_N acts on each U_σ (§3.4). By construction, this action is compatible with its action on U_τ if τ is a face of σ. Therefore T_N acts on X_Σ compatibly with its action on the U_σ. As the action of T_N on $U_{\{0\}}$ is free, if x_0 denotes the distinguished section of $U_{\{0\}}$, we see that

$$T_N \simeq T_N \times_R \{x_0\} \to U_{\{0\}}$$

is an isomorphism. By Proposition 4.4, X_Σ is a toric variety when R is a field.

Conversely, one can show that any toric variety is isomorphic to some X_Σ ([1], Corollary 3.1.8[1]).

Example 4.6 (1) Any affine U_σ is a toric variety: just consider the fan consisting in all faces of σ. In particular affine spaces ($\sigma = \sum_{1\le i\le d} \mathbb{R}_+ e_i$) and split tori are toric varieties.

(2) Let $d = 1$ and let Σ be generated by $\mathbb{R}_+e_1$ and $\mathbb{R}_+(-e_1)$. Then X_Σ is obtained by glueing two copies of $\mathbb{A}^1$: $\operatorname{Spec} R[\chi^{e_1}]$ and $\operatorname{Spec} R[(\chi^{e_1})^{-1}]$, so $X_\Sigma = \mathbb{P}^1_R$.

(3) One dimensional toric varieties over a field are $\mathbb{G}_m$, $\mathbb{A}^1$ and $\mathbb{P}^1$ (each U_σ is normal and contains a copy of $\mathbb{G}_m$).

(4) Let $d = 2$, let Σ be generated by $\mathbb{R}_+e_1 + \mathbb{R}_+e_2$, $\mathbb{R}_+(-e_1) + \mathbb{R}_+e_2$, $\mathbb{R}_+e_1 + \mathbb{R}_+(-e_2)$ and $\mathbb{R}_+(-e_1) + \mathbb{R}_+(-e_2)$. Then $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$.

(5) Products of toric varieties are toric varieties.

Example 4.7 (*Projective space*) Let $N = \mathbb{Z}^{d+1}/(1,\dots,1)\mathbb{Z}$. Let $e_0,\dots,e_d$ be the canonical basis of $\mathbb{Z}^{d+1}$. The canonical pairing

$$(\mathbb{Z}^{d+1})^* \times \mathbb{Z}^{d+1} \to \mathbb{Z}, \quad (u,v) \mapsto u(v)$$

induces a perfect pairing

$$M \times N \to \mathbb{Z}$$

where $M = \{x_0e_0^* + \cdots + x_de_d^* \mid \sum_i x_i = 0\}$. Let v_i be the image of e_i in N. For any $i \le d$, $\{v_j\}_{j\ne i}$ is a basis of N and its dual basis is $\{e_j^* - e_i^*\}_{j\ne i} \subset M$.

Let $\sigma_i = \sum_{j\ne i} \mathbb{R}_+ v_i$. Then $S_{\sigma_i} = \sum_{j\ne i} \mathbb{N}(e_j^* - e_i^*)$ and

$$R[S_{\sigma_i}] = R[T_j/T_i]_{j\ne i} \subset R[(\mathbb{Z}^{d+1})^*], \quad T_j = \chi^{e_j^*}.$$

Now any face of σ_i is generated by a subset of $\{v_j\}_{j\ne i}$ (see the proof of Proposition 2.6(2)). So the set Σ of the faces of the various σ_i is a fan. The scheme X_Σ is obtained by just glueing the U_{σ_i}, $0 \le i \le d$. The above presentation of $R[S_{\sigma_i}]$ shows that $X_\Sigma = \mathbb{P}^d$.

In this example, the torus T_N is $U_{\{0\}} = \operatorname{Spec} R[\chi^{e_i^*-e_j^*}]_{i,j}$. In terms of rational points over a field k, the action is

$$((t_0,\dots,t_d),[x_0,\dots,x_d]) \mapsto [t_0x_0,\dots,t_dx_d].$$

[1]The book [1] only treats varieties over $\mathbb{C}$. I have not checked whether the proof works over any field. Anyway this result is not need in the sequel.

4.2 Proper morphisms and proper toric varieties

Let $\phi : N \to N'$ be a linear map of free $\mathbb{Z}$-modules of finite ranks. Let $\phi^* : M' \to M$ be the dual of ϕ. The map ϕ extends to $\phi_{\mathbb{R}} : N_{\mathbb{R}} \to N'_{\mathbb{R}}$. Similarly for ϕ^*.

Let Σ, Σ' be respectively fans in $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$. Suppose:

$$\text{for any } \ \sigma \in \Sigma, \quad \phi_{\mathbb{R}}(\sigma) \ \text{ is contained in some } \ \sigma' \in \Sigma'.$$

Then $\phi^*(S_{\sigma'}) \subseteq S_\sigma$ and we get a morphism of schemes $U_\sigma \to U_{\sigma'}$. This morphism is clearly independent on the choice of $\sigma' \supseteq \phi_{\mathbb{R}}(\sigma)$, and we have a morphism $U_\sigma \to X_{\Sigma'}$ and finally a morphism $f : X_\Sigma \to X_{\Sigma'}$. We have a canonical morphism $T_N \to T_{N'}$ and f is compatible with the action of T_N on X_Σ and that of $T_{N'}$ on $X_{\Sigma'}$

When $N' = 0$, we have $X_{\Sigma'} = \operatorname{Spec} R$ and f is just the structure morphism.

Definition 4.8 For any fan Σ in $N_{\mathbb{R}}$, the *support of* Σ is $|\Sigma| := \cup_{\sigma\in\Sigma}\sigma \subseteq N_{\mathbb{R}}$.

Proposition 4.9. *Let Σ, Σ' be as above (this implies that $|\Sigma| \subseteq \phi_{\mathbb{R}}^{-1}(|\Sigma'|)$). Then the induced morphism*

$$f : X_\Sigma \to X_{\Sigma'}$$

is proper if and only if $\phi_{\mathbb{R}}^{-1}(|\Sigma'|) = |\Sigma|$. In particular, X_Σ is proper over R if and only if $|\Sigma| = N_{\mathbb{R}}$.

Proof. First suppose f is proper. Then it is proper when base changed to a residue field k of R. So we suppose $R = k$. Let $\sigma' \in \Sigma'$. Then $f^{-1}(U_{\sigma'}) \to U_{\sigma'}$ is proper. Let $v \in \phi_{\mathbb{R}}^{-1}(\sigma')$. We have to show $v \in |\Sigma|$. Consider the evaluation map

$$e_v : k[M] \to k[\mathbb{Z}], \quad \chi^u \mapsto u(v).$$

The composition of $k[S_{\sigma'}] \subseteq k[M'] \to k[M]$ with e_v takes values in $k[\mathbb{N}]$ because $\phi_{\mathbb{R}}(v) \in \sigma'$. Therefore we have a commutative diagram

$$\begin{array}{ccc} k[\mathbb{Z}] & \xleftarrow{e_v} & k[M] \\ \uparrow & & \uparrow \\ k[\mathbb{N}] & \longleftarrow & k[S_{\sigma'}] \end{array}$$

As $f(T_N) \subseteq T_{N'} \subseteq U_{\sigma'}$, we have $T_N \subseteq f^{-1}(U_{\sigma'})$ and a commutative diagram

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\psi_v} & f^{-1}(U_{\sigma'}) \\ \downarrow & & \downarrow f \\ \mathbb{A}^1 & \longrightarrow & U_{\sigma'} \end{array}$$

which, by the valuative criterion of properness, can be completed with a morphism $\bar{\psi}_v : \mathbb{A}^1 \to f^{-1}(U_{\sigma'})$. Let $U_\sigma \subseteq X_\Sigma$ containing $\bar{\psi}_v(0)$. Then $\bar{\psi}_v : \mathbb{A}^1_k \to$

U_σ. So $e_v : k[M] \to k[\mathbb{Z}]$ restricts to $k[S_\sigma] \to k[\mathbb{N}]$. This means that $e_v(u) \geq 0$ for all $u \in S_\sigma$, therefore $v \in (\sigma^\vee)^\vee = \sigma \subseteq |\Sigma|$. So $\phi_{\mathbb{R}}^{-1}(|\Sigma'|) = |\Sigma|$.

Now let us prove the converse. We can suppose $R = \mathbb{Z}$ because f over R is obtained by base change from f over $\mathbb{Z}$. So we can suppose R integral. We will again use the valuative criterion of properness. Let $\mathcal{O}_K$ be a discrete valuation ring with field of fractions K and consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\rho} & X_\Sigma \\ \downarrow & & \downarrow f \\ \operatorname{Spec} \mathcal{O}_K & \xrightarrow{g} & X_{\Sigma'} \end{array} \tag{1}$$

We have to show that it can be completed with a morphism $\bar{\rho} : \operatorname{Spec} \mathcal{O}_K \to X_\Sigma$. By EGA, II.7.3.10, we can restrict to those ρ with image equal to the generic point of X_Σ. Let $U_{\sigma'} \supseteq g(\operatorname{Spec} \mathcal{O}_K)$. So we have a commutative diagram

$$\begin{array}{ccc} K & \xleftarrow{\rho^\#} & R[M] \\ \uparrow & & \uparrow \\ \mathcal{O}_K & \longleftarrow & R[S_{\sigma'}] \end{array}$$

where $R[S_{\sigma'}] \to R[M]$ is the restriction of $R[M'] \to R[M]$. The map

$$v : M \to \mathbb{Z}, \quad u \mapsto \nu_K(\rho^\#(\chi^u)),$$

where $\nu_K : K \to \mathbb{Z} \cup \{+\infty\}$ is the valuation of K, is a linear form on M, so $v \in N$. Moreover, the above commutative diagram implies that for all $u' \in \sigma'$, we have $v(\phi_{\mathbb{R}}^*(u')) \geq 0$, so $v \in \phi_{\mathbb{R}}^{-1}(\sigma') \subseteq |\Sigma|$. Let $\sigma \in \Sigma$ be a face containing v. Then for all $u \in S_\sigma$, we have $\nu_K(\rho^\#(\chi^u)) = v(u) \geq 0$ hence $\rho^\#(\chi^u) \in \mathcal{O}_K$. So $\rho^\#$ restricts to $R[S_\sigma] \to \mathcal{O}_K$. This means we succeed to complete our original diagram (1) into a diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\rho} & U_\sigma \\ \downarrow & \overset{h}{\nearrow} & \downarrow f \\ \operatorname{Spec} \mathcal{O}_K & \xrightarrow{g} & X_{\Sigma'} \end{array}$$

whose upper triangle is commutative. The lower triangle is commutative because $f \circ h$ and g coincide on the generic point and $X_{\Sigma'}$ is separated (Proposition 4.4). □

Example 4.10 Let $N = N'$. Suppose that any $\sigma \in \Sigma$ is contained in a $\sigma' \in \Sigma'$ and any $\sigma' \in \Sigma'$ is a union of $\sigma \in \Sigma$ (Σ is called a *refinement of* Σ'). Then $f : X_\Sigma \to X_{\Sigma'}$ is proper, T_N-equivariant, and (fiberwise) birational.

Remark 4.11 (*Resolution of singularities*) Using successive proper birational morphisms $X_\Sigma \to X_{\Sigma'}$ induced by refinements of fans, one can solve the singularities of a given toric variety ([2], §2.6).

Remark 4.12 Let X_Σ be a proper toric variety over a field k. A natural question is under which condition X_Σ is projective. There are criteria using either polytopes or the notion of strictly convex functions. An example of proper smooth non-projective toric variety of dimension 3 can be found in [2], §3.4, p. 71.

5 Divisors on toric varieties

The aim of this section is to compute the global sections of the sheaf $\mathcal{O}_{X_\Sigma}(D)$ associated to some special Weil divisor D on X_Σ (Proposition 5.5).

5.1 Structure of the class group of X_Σ

Recall that if X is a normal integral noetherian scheme, a *Weil divisor on* X is a formal linear combination of integral closed subschemes of codimension 1 in X. The group of Weil divisors is denoted by $Z^1(X)$. Modulo linear equivalence, they define the Chow group $A^1(X)$, also denoted by $\mathrm{Cl}(X)$ and called *the class group of* X).

We fix a fan Σ in $N_\mathbb{R}$. We consider $X = X_\Sigma$ the associated toric variety over a field k. There are some remarkable Weil divisors on X.

Example 5.1 Let τ be a *ray* (cone of dimension 1) in Σ. Then $\tau = \mathbb{R}_+ v$ for some $v \in N$ a generator of $\tau \cap N$. This v can be completed into a basis of N. So $U_\tau \simeq \mathbb{A}^1 \times \mathbb{G}_m^{d-1}$ (Example 3.5(2)) and $U_\tau \setminus T_N \simeq \{0\} \times \mathbb{G}_m^{d-1}$ is an integral Weil divisor in U_τ. Let $V(\tau)$ be the Zariski closure of $U_\tau \setminus T_N$ in X, endowed with the structure of a reduced closed subvariety. Note that $V(\tau)$ is geometrically integral over k because $U_\tau \setminus T_N$ is geometrically integral.

Proposition 5.2. *Let $\tau_1, \dots, \tau_\ell$ be the rays in Σ. Write $D_i = V(\tau_i)$ and denote by v_i a generator of $\tau_i \cap N$ over $\mathbb{N}$. We have*

(1) $\cup_{1\le i\le \ell} D_i = X \setminus T_N$.

(2) *For any $u \in M$,* $\mathrm{div}(\chi^u) = \sum_{1\le i\le l} u(v_i) D_i$.

Proof. (1) postponed to Theorem 6.4.

(2) Let τ be a ray in Σ. The generic point of $V(\tau)$ belongs to U_τ and it is enough to compute the order of $\mathrm{div}(\chi_u)$ in U_τ. Let $v_1, \dots, v_d$ be a basis of N such that $\tau = \mathbb{R}_+ v_1$. Write $u = a_1 v_1^* + \cdots + a_d v_d^*$ in the dual basis. We saw in the example above that $V(\tau) \cap U_\tau$ is defined by $\chi^{v_1^*}$. So the order (of zero or pole) of χ^u at the generic point of $V(\tau)$ is $a_1 = u(v_1)$. □

Proposition 5.3. *We have an exact sequence*

$$M \to \oplus_{1\le i\le \ell}\mathbb{Z}D_i \to \mathrm{Cl}(X) \to 0.$$

It is exact at left if $|\Sigma|$ is not contained in a proper subspace of $N_{\mathbb{R}}$.

Proof. The map $M \to Z^1(X)$, $u \mapsto \mathrm{div}(\chi^u)$ is $\mathbb{Z}$-linear and has image in $\oplus_i \mathbb{Z}D_i$ by Proposition 5.2(2). If $D = \sum_i a_i D_i = \mathrm{div}(f)$ for some $f \in k(X)^*$, then $f|_{T_N} \in \mathcal{O}_X(T_N)^* = k[M]^*$ and $f|_{T_N} = \lambda\chi^u$ for some $\lambda \in k^*$ and $u \in M$. So $D = \mathrm{div}(\chi^u)$. This proves the exactness at middle.

By general results, we have an exact sequence

$$\oplus_i \mathbb{Z}D_i \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus \cup_i D_i) \to 0.$$

By Proposition 5.2(1), $X \setminus \cup_i D_i = T_N$. As $\mathrm{Cl}(T_N) = \{1\}$ because $k[M]$ is UFD, we have the exactness at right.

Finally, suppose $\mathrm{div}(\chi^u) = 0$. Then $u(v_i) = 0$. As we see easily that every σ is generated by some one-dimensional cones in Σ, this implies that u vanishes at the vector subspace of $N_{\mathbb{R}}$ generated by Σ. So if Σ is not contained in a proper subspace of $N_{\mathbb{R}}$, then $u = 0$ and $M \to \oplus_i \mathbb{Z}D_i$ is injective. □

5.2 Action of $T_N(k)$

As T_N acts on $X = X_\Sigma$ by some morphism $\mu : T_N \times X \to X$, we have a morphism of groups $T_N(k) \to \mathrm{Aut}_k(X)$, the image of $t \in T_N(k)$ is

$$X \to X, \quad x \mapsto \mu(t, x).$$

So $T_N(k)$ also acts on $Z^1(X)$ and on $k(X)$ the function field of X.

Lemma 5.4. *Let $t \in T_N(k)$.*

(1) *Let $u \in M$. Then*
$$t.\chi^u = t^{-1}(u)\chi^u,$$
where $t^{-1}(u)$ is $t^{-1} \in T_N(k) = \mathrm{Hom}_{sg}(M, k)$ applied to u.

(2) *Let D_i be a divisor on X as in Proposition 5.2. Then $t.D_i = D_i$.*

Proof. (1) Let us use a basis $e_1, \dots, e_d$ of N. Then $t = (t_1, \dots, t_d)$ with $t_i \in k^*$. Write $u = \sum_i a_i e_i^*$ with $a_i \in \mathbb{Z}$. We have

$$t.\chi^u = \prod_i (t.\chi^{e_i^*})^{a_i} = \prod_i (t_i^{-1}\chi^{e_i^*})^{a_i} = (\prod_i t_i^{-a_i})\chi^u,$$

where $t.\chi^{e_i^*} = t_i\chi^{e_i^*}$ because the action of t on the coordinates is multiplication by t_i on the i-th coordinate.

(2) As t acts on U_{τ_i} and on $T_N = U_{\{0\}} \subset U_{\tau_i}$, t fixes $U_{\tau_i} \setminus T_N$, hence t fixes $D_i = \overline{U_{\tau_i} \setminus T_N}$. □

5.3 Global sections of $\mathcal{O}_X(D)$

Keep the notation of Proposition 5.2.

Proposition 5.5. *Let* $D = \sum_{1\leq i\leq \ell} a_i D_i$. *Let*

$$P_D = \{u \in M_{\mathbb{R}} \mid u(v_i) \geq -a_i, i = 1, \ldots, \ell\}.$$

Then

$$H^0(X, \mathcal{O}_X(D)) = \oplus_{u\in P_D\cap M} k\chi^u.$$

Proof. As the D_i are defined independently of k, $H^0(X, \mathcal{O}_X(D))$ commutes with base change. It is thus enough to show the equality for some extension of k. In particular we can suppose k is infinite.

The restriction to T_N induces a canonical map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(T_N, \mathcal{O}_{T_N}(D|_{T_N})) = H^0(T_N, \mathcal{O}_{T_N}) = k[M]$$

of subspace of $k(X)$. So we can view $H^0(X, \mathcal{O}_X(D))$ as a subspace of $k[M] = \oplus_{u\in M} k\chi^u$. If $u \in M$, then $\chi^u \in H^0(X, \mathcal{O}_X(D))$ if and only if $u \in P_D \cap M$ by Proposition 5.2(2). In particular $\oplus_{u\in P_D\cap M} k\chi^u \subseteq H^0(X, \mathcal{O}_X(D))$.

Let $f \in H^0(X, \mathcal{O}_X(D))$. There exists a finite subset F of M such that $f \in E := \oplus_{u\in F} k\chi^u$. The commutative group $T_N(k)$ acts on the finite dimensional vector space E through diagonalizable automorphisms. Moreover, as $T_N(k)$ fixes each D_i (Lemma 5.4), it acts on $H^0(X, \mathcal{O}_X(D))$. So $E \cap H^0(X, \mathcal{O}_X(D))$ is a subspace of E globally invariant by $T_N(k)$. As $T_N(k)$ is commutative with diagonalizable actions, $E\cap H^0(X, \mathcal{O}_X(D))$ is a direct sum of common eigenspaces of $T_N(k)$. Using the description of the action of $T_N(k)$ on χ^u (Lemma 5.4) and the hypothesis that k is infinite, we find $E \cap H^0(X, \mathcal{O}_X(D))$ is a direct sum of $k\chi^u$. But $\chi^u \in H^0(X, \mathcal{O}_X(D))$ implies that $u \in P_D \cap M$ (Proposition 5.2(2)), so

$$f \in E \cap H^0(X, \mathcal{O}_X(D)) \subset \oplus_{u\in P_D\cap M} k\chi^u.$$

□

6 Orbits under T_N

This section is added after the three hours talks. Here we prove Proposition 5.2, see Theorem 6.4. We work over a field k.

We saw in §3.4 that T_N acts on $X = X_\Sigma$. This action will allows us to decompose X as a union of finitely many orbits. Denote by $\mu : T_N \times X \to X$ the morphism defining the action.

Definition 6.1 Let $x \in X(k)$. We define the *orbit of* x *under* T_N or T_N*-orbit of* x the image of the canonical morphism $\tau_x : T_N \to T_N \times \{x\} \xrightarrow{\mu} X$.

Notation Let $\sigma \in \Sigma$ and let x_σ be the distinguished rational point of U_σ (Remark 3.8). We denote by O_σ the orbit of x_σ under T_N. As $T_N \times U_\sigma \to U_\sigma$, we have $O_\sigma \subseteq U_\sigma$.

Example 6.2 (1) If $\sigma = \{0\}$. Then $O_\sigma = T_N$.

(2) If $\dim \sigma = 1$, then we saw that $U_\sigma = \mathbb{A}^1 \times \mathbb{G}_m^{d-1}$, $x_\sigma = (0, 1, \ldots, 1)$ and so $O_\sigma = \{0\} \times \mathbb{G}_m^{d-1} \simeq \mathbb{G}_m^{d-1}$.

(3) If $\dim \sigma = d$, then $x_\sigma : u \mapsto 0$ for all $u \in S_\sigma$ non-zero because $u \notin \sigma^\perp = \{0\}$. For all $t \in T_N(k)$, we have then $t.x_\sigma = x_\sigma$ and $O_\sigma = \{x_\sigma\}$.

In all these cases, O_σ is a subvariety of U_σ of dimension $\dim O_\sigma = d - \dim \sigma$.

Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone. Denote by

$$N(\sigma) = N/(\sigma \cap N + (-\sigma \cap N))$$

the quotient by the submodule of N generated by $\sigma \cap N$. Denote by

$$\sigma^\perp := \{u \in M_\mathbb{R} \mid u(\sigma) = \{0\}\}.$$

The canonical pairing $M \times N \to \mathbb{Z}$ induces a perfect pairing

$$(\sigma^\perp \cap M) \times N(\sigma) \to \mathbb{Z}.$$

Lemma 6.3. *Let $\sigma \in \Sigma$.*

(1) *Consider the linear projection $p_\sigma : k[S_\sigma] \to k[\sigma^\perp \cap M]$, $\chi^u \mapsto \chi^u$ if $u \in \sigma^\perp \cap M$ and 0 otherwise. Then p_σ is a surjective homomorphism of k-algebras.*

(2) *The orbit O_σ is $V(\ker p_\sigma) \simeq T_{N(\sigma)}$. This is a closed subvariety of dimension $d - \dim \sigma$.*

(3) $O_\tau(k) = \{x \in U_\sigma(k) = \mathrm{Hom}_{sg}(\sigma, k) \mid x(u) = 0, \forall u \in S_\sigma \setminus (\sigma^\perp \cap M)\}$.

Proof. (1) Let $u_1, u_2 \in S_\sigma$. Then $u_1 + u_2 \in \sigma^\perp$ if and only if $u_1, u_2 \in \sigma^\perp$. This implies that the map $S_\sigma \to k[\sigma^\perp \cap M]$ defined by $u \mapsto \chi^u$ if $u \in \sigma^\perp \cap M$ and $u \mapsto 0$ otherwise is a morphism of semigroups, so the induced map p_σ is a morphism of k-algebras.

(2) The morphism τ_{x_σ} (see 6.1) corresponds to

$$k[S_\sigma] \to k[M] \otimes k(x_\sigma) \simeq k[M], \quad \chi^u \mapsto \chi^u \otimes \chi^u(x_\sigma) \mapsto x_\sigma(u)\chi^u.$$

This is nothing but p_σ. So τ_{x_σ} factorizes into the surjective morphism $T_N \to T_{N(\sigma)} = U_{\sigma^\perp \cap M}$ and the closed immersion $T_{N(\sigma)} \to U_\sigma$ defined by p_σ.

(3) Let $x \in U_\sigma(k)$. Then $x \in O_\sigma = V(\ker p_\sigma)$ if and only if $\ker p_\sigma \subseteq \ker f_x$ where $f_x : k[S_\sigma] \to k$ is defined by $\chi^u \mapsto x(u)$ (when x is viewed as an element of $\mathrm{Hom}_{sg}(S_\sigma, k)$). This is equivalent to $x(u) = 0$ for all $\chi^u \in \ker p_\sigma$, but the latter condition is nothing but $u \in S_\sigma \setminus (\sigma^\perp \cap M)$. ☐

Theorem 6.4. *Let $X = X_\Sigma$ be a toric variety over a field k. Let $\sigma \in \Sigma$.*

(1) *The orbit $O_\sigma \subseteq U_\sigma$ is a closed subvariety of dimension $d - \dim \sigma$.*

(2) *We have* $U_\sigma = \cup_{\tau \le \sigma} O_\tau$, *where the union runs through the faces* τ *of* σ.

(3) *Let* $\tau_1, \dots, \tau_\ell$ *be the rays (one-dimensional cones) of* Σ. *Let* $D_i = \overline{U_{\tau_i} \setminus T_N}$ *(Zariski closure). Then*

$$X_\Sigma \setminus T_N = \cup_{1 \le i \le \ell} D_i.$$

Proof. (1) This is Lemma 6.3.

(2) As the construction of O_σ is compatible with base changes, we can suppose k is algebraically closed. Let $x \in U_\sigma(k) = \mathrm{Hom}_{sg}(S_\sigma, k)$. Consider

$$x^{-1}(k^*) := \{u \in S_\sigma \mid x(u) \in k^*\}.$$

Let $u_1, u_2 \in S_\sigma$, then $x(u_1 + u_2) = x(u_1)x(u_2)$ and $u_1 + u_2 \in x^{-1}(k^*)$ if and only if $u_1, u_2 \in x^{-1}(k^*)$. Such a sub-semigroup of S_σ is automatically equal to $\tau^\perp \cap S_\sigma$ for some face τ of σ ([2], page 15, Exercise and [1], Proposition 1.2.10). As $x(u) = 0$ for all $u \in S_\sigma \setminus (\tau^\perp \cap M)$, $x \in O_\tau$ by Lemma 6.3.

(3) For any τ_i, $U_{\tau_i} \setminus T_N \subseteq X_\Sigma \setminus T_N$. The latter being closed, we have $D_i \subseteq X_\Sigma \setminus T_N$. On the other hand, for any $\sigma \in \Sigma$, by (2), U_σ is the union of $T_N = O_{\{0\}}$ and locally closed subsets O_τ of dimension $\dim O_\tau = d - \dim \tau \le d - 1$. Therefore the points of codimension 1 in $U_\sigma \setminus T_N$ are in the orbits of rays τ. For a ray τ, we have $O_\tau = U_\tau \setminus T_N$ by (2). Hence $U_\sigma \setminus T_N$ is contained in $\cup_{i \le \ell} D_i$ and $X_\Sigma \setminus T_N \subseteq \cup_{1 \le i \le \ell} D_i$. This proves (3). □

Remark 6.5 It follows from the theorem that the orbit of any rational point in U_σ is of the form O_τ for some face τ of σ (take τ such that $x \in O_\tau$).

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