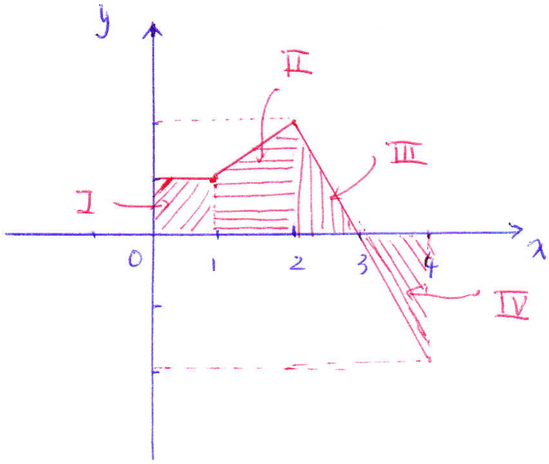
SVE 101, Feuille 1

ex 1.1 :

(2) : $f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ x & 1 < x \leq 2 \\ -2x+6 & 2 < x \leq 4 \end{cases}$, alors $\int_0^4 f(x)\,dx = ?$




$S_f = \text{I} + \text{II} + \text{III} + \text{IV}$

$$\int_0^4 f(x)\,dx = \text{l'aire algébrique de } S_f$$

$$= \text{l'aire de I} + \text{l'aire de II} + \text{l'aire de III} - \text{l'aire de IV}$$

$$= 1 + \tfrac{1}{2}(1+2)\cdot 1 + \tfrac{1}{2}\cdot 2\cdot 1 - \tfrac{1}{2}\cdot 2\cdot 1$$

$$= 1 + \frac{3}{2} + 1 - 1 = \frac{5}{2}$$

1.3 (1) $\int e^{3x+1}\,dx = \frac{1}{3}\int e^{3x+1}\,d(3x+1) = \frac{1}{3}\int d(e^{3x+1})$

$$= \frac{1}{3}e^{3x+1} + C \qquad x \in \mathbb{R}$$

(2) $\int \frac{1}{x-2}\,dx = \int \frac{1}{x-2}\,d(x-2) = \int d(\ln|x-2|)$

$$= \ln|x-2| + C \qquad x \in \mathbb{R}\setminus\{2\}$$

$$= ]-\infty, 2[ \cup ]2, +\infty[$$

(3) : $\int \frac{x^3+1}{x}\, dx = \int \frac{x^3}{x}\, dx + \int \frac{1}{x}\, dx$

$= \int x^2\, dx + \int \frac{1}{x}\, dx = \int d(\frac{1}{3}x^3) + \int d(\ln|x|)$

$= \frac{1}{3}x^3 + \ln|x| + C \qquad x \in \mathbb{R} \setminus \{0\}$

1.5 : (1) pour que $\tan(x) = \frac{\sin x}{\cos x}$ soit bien defini,

il faut et il suffit que $\cos x \neq 0$

$\Leftrightarrow \quad x \neq \frac{\pi}{2} + k\pi, \quad \forall k \in \mathbb{Z}$

Sous la condition précédente,

$(\tan(x))' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \qquad (x \neq \frac{\pi}{2} + k\pi)$

(2) d'après (1), $x \longmapsto \tan(x)$ est une ~~fon~~ primitive pour la fonction $x \longmapsto \frac{1}{\cos^2 x}$ $(x \neq \frac{\pi}{2} + k\pi)$

~~donc~~ en particulier, $x \longmapsto \tan(x)$ est une primitive pour la fonction $x \longmapsto \frac{1}{\cos^2 x}$ sur l'intervalle $[0, \frac{\pi}{4}]$

donc : $\int_0^{\pi/4} \frac{1}{\cos^2 t}\, dt$

$= \left[\frac{\sin t}{\cos t}\right]_0^{\frac{\pi}{4}} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} - \frac{\sin 0}{\cos 0} = 1.$

(3) $\int_0^{\pi/4} \frac{t}{\cos^2 t}\, dt$

$= \int_0^{\pi/4} t \left( \frac{\sin t}{\cos t} \right)' dt$

$= \left[ t \cdot \frac{\sin t}{\cos t} \right]_0^{\frac{\pi}{4}} - \int_0^{\pi/4} \frac{\sin t}{\cos t} \cdot (t)'\, dt.$

$= \frac{\pi}{4} \cdot \frac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} - 0 - \int_0^{\pi/4} \frac{\sin t}{\cos t}\, dt$

$= \frac{\pi}{4} - \int_0^{\pi/4} \frac{\sin t}{\cos t}\, dt$

$= \frac{\pi}{4} - \int_0^{\pi/4} \frac{(-\cos t)'}{\cos t}\, dt$

$= \frac{\pi}{4} + \int_0^{\pi/4} \frac{(\cos t)'}{\cos t}\, dt$

$= \frac{\pi}{4} + \int_{\cos 0}^{\cos \frac{\pi}{4}} \frac{1}{x}\, dx$

$= \frac{\pi}{4} + \int_1^{\frac{\sqrt{2}}{2}} \frac{1}{x}\, dx = \frac{\pi}{4} + \left[ \ln |x| \right]_1^{\frac{\sqrt{2}}{2}}$

$= \frac{\pi}{4} + \ln \left| \frac{\sqrt{2}}{2} \right| = \frac{\pi}{4} + \ln(\sqrt{2}) - \ln 2$

$= \frac{\pi}{4} - \frac{\ln 2}{2}$

**rappels** : quelques propriétés de $\sin x$, $\cos x$.

$\sin 0 = \sin \pi = 0$, $\sin \frac{\pi}{2} = 1$. $\sin \frac{3}{2}\pi = -1$

$\cos 0 = \cos(2\pi) = 1$, $\cos \frac{\pi}{2} = \cos \frac{3}{2}\pi = 0$

$\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos \frac{\pi}{3} = \sin \frac{\pi}{6} = \frac{1}{2}$, $\cos \frac{\pi}{6} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\sin(x + 2\pi) = \sin x \quad \forall x \in \mathbb{R}$

$\cos(x + 2\pi) = \cos x \quad \forall x \in \mathbb{R}.$

$\cos^2 x + \sin^2 x = 1.$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin(x+y) = \cos x \sin y + \sin x \cos y$

1.7

(2) $$\int_0^1 x e^x \, dx = \int_0^1 x (e^x)' \, dx$$

$$= [x e^x]_0^1 - \int_0^1 e^x \cdot (x)' \, dx$$

$$= 1 \cdot e - 0 - \int_0^1 e^x \, dx$$

$$= e - [e^x]_0^1 = e - (e - 1) = 1.$$

$$\int_0^1 x^2 e^x \, dx$$

$$= \int_0^1 x^2 (e^x)' \, dx = \cancel{\int} \; [x^2 e^x]_0^1 - \int_0^1 e^x \cdot (x^2)' \, dx$$

$$= e - 2 \int_0^1 x \cdot e^x \, dx$$

$$= e - 2 \cdot 1 = e - 2$$

(3). pour $n \geq 1$

$$\int_0^1 x^n e^x \, dx = \int_0^1 x^n (e^x)' \, dx$$

$$= [x^n e^x]_0^1 - \int_0^1 e^x \cdot (x^n)' \, dx$$

$$= e - n \int_0^1 x^{n-1} e^x \, dx$$

1.8. posons $A = \int_0^{\pi/2} e^x \sin 2x \, dx$

donc $A = \int_0^{\pi/2} e^x \sin 2x \, dx = \int_0^{\pi/2} \sin 2x \, (e^x)' \cdot dx$

$= [\sin 2x \cdot e^x]_0^{\pi/2} - \int_0^{\pi/2} e^x \, (\sin 2x)' \, dx$

$= \sin \pi \cdot e^{\frac{\pi}{2}} - \sin 0 \cdot e^0 - 2\int_0^{\pi/2} e^x \cos 2x \, dx$

$= 0 - 0 - 2\int_0^{\pi/2} e^x \cos 2x \, dx$

$= -2\int_0^{\pi/2} e^x \cos 2x \, dx$

$= -2 \int_0^{\pi/2} (\cos 2x) \cdot (e^x)' \, dx$

$= -\left\{ 2 \, [\cos 2x \cdot e^x]_0^{\pi/2} - 2 \int_0^{\pi/2} e^x \cdot (\cos 2x)' \, dx \right\}$

$= -2 \, [\cos 2x \cdot e^x]_0^{\pi/2} + 2\int_0^{\pi/2} e^x (-2\sin 2x) \, dx$

$= -2 \left( \cos \pi \cdot e^{\pi/2} - \cos 0 \cdot e^0 \right) - 4\int_0^{\pi/2} e^x (\sin 2x) \, dx$

$= 2 \cdot (1 + e^{\pi/2}) - 4\int_0^{\pi/2} e^x \sin 2x \, dx$

$= 2(e^{\frac{\pi}{2}} + 1) - 4A$ (rappelons que $A = \int_0^{\pi/2} e^x \sin$

on obtient ainsi une equation pour le réel $A$

$$A = 2(e^{\frac{\pi}{2}} + 1) - 4A$$

$$\Rightarrow A = \frac{2}{5}(e^{\pi/2} + 1)$$

1.9

(2) $$\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$$

$$= \int_0^1 \frac{\frac{1}{2}\cdot(x^2+1)'}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int_0^1 \frac{(x^2+1)'}{\sqrt{x^2+1}}\,dx$$

$$= \frac{1}{2}\int_{0^2+1}^{1^2+1} \frac{1}{\sqrt{t}}\,dt = \frac{1}{2}\int_1^2 \frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{2}\left[\,2\sqrt{t}\,\right]_1^2 = \sqrt{2}-1$$

$$\int_1^2 \frac{\ln(x)}{x}\,dx$$

$$= \int_1^2 \ln(x)\,(\ln(x))'\,dx = \int_0^{\ln 2} t\,dt = \frac{1}{2}(\ln 2)^2$$

(N.B. sur l'intervalle $[1, 2]$, $\ln(x)$ est une primitive de la fonction $x \longmapsto \frac{1}{x}$)

(3). $$\int_1^2 \frac{\ln x}{x}\,dx$$

$$= \int_1^2 \underbrace{\ln x}_{v(x)\,:=\,\ln(x)}\cdot\underbrace{(\ln x)'}_{u(x)\,=\,\ln(x)}\,dx = \left[\ln x\cdot\ln x\right]_1^2 - \int_1^2 \underbrace{(\ln x)'}_{v'(x)}\,\underbrace{\ln x}_{u(x)}\,dx$$

$$= \ln^2 2 - \int_1^2 \frac{\ln x}{x}\,dx$$

donc $$2\int_1^2 \frac{\ln x}{x}\,dx = \ln^2 2 \Rightarrow \int_1^2 \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 2$$

1.10.

(1). d'après les relations générales des fonctions trigonométriques

on sait que $\cancel{\cos^2 x = 2\cos 2x +}$

$$\cos 2x = 2\cos^2 x - 1$$

donc $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

donc $$\int_{\pi/6}^{\pi/3} \cos^2 x \, dx = \int_{\pi/6}^{\pi/3} \frac{1}{2}(\cos 2x + 1)\, dx$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} (\cos 2x + 1)\, dx$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} \cos 2x \, dx + \frac{1}{2}\int_{\pi/6}^{\pi/3} 1 \cdot dx$$

$$= \frac{1}{2}\cdot\left[\frac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/3} + \frac{1}{2}[x]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{4}\left(\sin\frac{2}{3}\pi - \sin\frac{1}{3}\pi\right) + \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{1}{4}\cdot\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + \frac{1}{2}\cdot\frac{\pi}{6} = \frac{\pi}{12}.$$

(2). $\cos^4 x \sin^3 x$

$$= \sin x \cdot \cos^4 x \cdot \sin^2 x$$

$$= \sin x \cdot (\cos^4 x)\cdot(1 - \cos^2 x)$$

$$= \sin x \cdot (\cos^4 x - \cos^6 x)$$

Posons $P(T) = T^4 - T^6$, c'est un polynôme de degré 6

qui satisfait à la propriété suivante:

$$\cos^4 x \sin^3 x = \sin x \cdot P(\cos x)$$

$$\Rightarrow \int_{\pi/6}^{\pi/3} \cos^4 x \cdot \sin^3 x \, dx$$

$$= \int_{\pi/6}^{\pi/3} (\cos^4 x - \cos^6 x)\cdot \sin x \, dx$$

$$= -\int_{\pi/6}^{\pi/3} (\cos^4 x - \cos^6 x)\cdot(\cos x)' \, dx$$

$$= -\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} (t^4 - t^6)\, dt = -\left[\frac{1}{5}t^5 - \frac{1}{7}t^7\right]_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}}$$

1.13 :

(1) pour que la fonction $x \longmapsto \frac{1}{x[1-(\ln x)^2]}$ soit bien définie,

il faut et il suffit

$$\begin{cases} x \neq 0 \\ 1-\ln^2 x \neq 0 \\ x > 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x > 0 \\ \ln^2 x \neq 1 \end{cases} \Longleftrightarrow \begin{cases} x > 0 \\ \ln(x) \neq 1 \ \& \ \ln(x) \neq -1 \end{cases}$$

$$\Longleftrightarrow \quad x > 0, \text{ et } x \neq e, \ x \neq \frac{1}{e}$$

(2) $\int_2^3 \frac{x}{x^2-2}\,dx$

6 deux méthodes.

1° : changement de variable

$$\int_2^3 \frac{x}{x^2-2}\,dx = \frac{1}{2}\int_2^3 \frac{(x^2-2)'}{x^2-2}\,dx$$

$$= \frac{1}{2}\int_2^7 \frac{1}{t}\,dt = \frac{1}{2}(\ln 7 - \ln 2)$$

2° : $\frac{x}{x^2-2} = \frac{x}{(x-\sqrt{2})(x+\sqrt{2})}$

$$= \frac{1}{2} \cdot \frac{x-\sqrt{2}+x+\sqrt{2}}{(x-\sqrt{2})(x+\sqrt{2})}$$

$$= \frac{1}{2}\,\frac{1}{x+\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{x-\sqrt{2}}$$

donc. $\int_2^3 \frac{x}{x^2-2}\,dx = \frac{1}{2}\int_2^3 \frac{1}{x+\sqrt{2}}\,dx + \frac{1}{2}\int_2^3 \frac{1}{x-\sqrt{2}}\,dx$

$$= \frac{1}{2}\left[\ln(x+\sqrt{2})\right]_2^3 + \frac{1}{2}\left[\ln(x-\sqrt{2})\right]_2^3$$

$$= \frac{1}{2}\left(\ln(3+\sqrt{2}) - \ln(2+\sqrt{2}) + \ln(3-\sqrt{2}) - \ln(2-\sqrt{2})\right)$$

$$= \frac{1}{2}(\ln 7 - \ln 2)$$

1.14 :



(1) puisque $x^2-2x-3 = (x-3)(x+1)$

$$\Rightarrow \frac{x}{x^2-2x-3} = \frac{x}{(x-3)(x+1)} = \frac{\frac{1}{4}\cdot[(x-3)+3(x+1)]}{(x-3)(x+1)}$$

$$= \frac{1}{4}\frac{1}{x+1} + \frac{3}{4}\frac{1}{x-3}$$

donc :
$$\int \frac{x}{x^2-2x-3}dx = \frac{1}{4}\int \frac{1}{x+1}dx + \frac{3}{4}\int \frac{1}{x-3}dx$$

$$= \frac{1}{4}\ln|x+1| + \frac{3}{4}\ln|x-3|$$

$$= \frac{1}{4}\ln|(x+1)\cdot(x-3)^3| \qquad (x \neq 3 \ \& \ x \neq -1)$$

(2)
$$\int \frac{x^2}{x^2-2x-3}dx = \int \frac{x^2-2x-3+2x+3}{x^2-2x-3}dx$$

$$= \int \frac{x^2-2x-3}{x^2-2x-3}dx + 2\cdot\int \frac{x}{x^2-2x-3}dx + 3\int \frac{1}{x^2-2x-3}dx$$

donc, il suffit de calculer respectivement les trois intégrales ci-dessus.

$$\int \frac{x^2-2x-3}{x^2-2x-3}dx = \int 1\cdot dx = x + C_1 \qquad C_1 \text{ constant}.$$

$$\int \frac{x}{x^2-2x-3}dx = \frac{1}{4}\ln|(x+1)(x-3)^3| + C_2$$

$$\int \frac{1}{x^2-2x-3}dx = \int \frac{1}{(x-3)(x+1)}dx = \int \frac{\frac{1}{4}\cdot[(x+1)-(x-3)]}{(x-3)(x+1)}dx$$

$$= \frac{1}{4}\int \frac{1}{x-3}dx - \frac{1}{4}\int \frac{1}{x+1}dx$$

$$= \frac{1}{4}\ln\left|\frac{x-3}{x+1}\right| + C_3$$

donc
$$\int \frac{x^2}{x^2-2x-3}dx = \int \frac{x^2-2x-3}{x^2-2x-3}dx + 2\int \frac{x}{x^2-2x-3}dx + 3\int \frac{1}{x^2-2x-3}dx$$

$$= x + C_1 + \frac{2}{4}\ln|(x+1)(x-3)^3| + 2C_2 + \frac{3}{4}\ln\left|\frac{x-3}{x+1}\right| + 3C_3$$

$$= x + \frac{1}{4}\ln\left|\frac{(x-3)^9}{x+1}\right| + C_1 + 2C_2 + 3C_3$$

$$= x + \frac{1}{4}\ln\left|\frac{(x-3)^9}{x+1}\right| + C \qquad (\text{avec } C = C_1 + 2C_3 + 3C_3 \text{ est })$$