Arithmetic Quantum Unique Ergodicity for Symplectic Linear Maps of the Multidimensional Torus

DUBI KELMER

25.04.06

Overview

- QM on Arithmetic surfaces

- Model:Quantum cat map

- AQUE in dimension 2

- AQUE for higher dimensions

QM on Arithmetic surfaces

- **Classical phase space:** $M = \Gamma \setminus \mathbb{H}$
  Γ- discrete co-compact (arithmetic) subgroup of PSL$(2, \mathbb{R})$. (more precisely it's unit cotangent bundle S^*M)

- **Classical dynamics:** Geodesic flows
  Ergodic and displays chaotic features.

Quantum system

- **Hilbert space:** $L^2(M)$

- **Quantum propagator:**
  $\triangle$-The Laplace-Beltrami operator.

- For ψ_i eigenfunction of $\triangle$ corresponding quantum measure: $d\mu_i = |\psi_i|^2 d\text{vol}$.

Theorem((QE) Schnirelman, Zeldich, Colin de Verdiere). *As* $\lambda_i \to \infty$ *for almost all* i *the measure* $d\mu_i \to d$*vol*

Conjecture((QUE) (Rudnick, Sarnak 93)). *As* $\lambda_i \to \infty$ *the measure* $d\mu_i \to d$*vol*

Arithmetic symmetries:

Hecke operators T_n, $n \in \mathbb{N}$, commute with $\triangle$.

Consider joint eigenfunctions.

Theorem ((AQUE) (Lindenstrauss (03))). *For* ψ_i *joint eigenfunctions of* $\triangle$ *and all Hecke operators. As* $\lambda_i \to \infty$ *the measure* $d\mu_i \to d$*vol*

Model:Quantum Cat Map

- **Classical phase space:** Torus

  $$\mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^{2d}.$$

  Coordinates: $x = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{T}^{2d}$.

- **Test functions:** $C^{\infty}(\mathbb{T}^{2d})$.

  $$\text{Basis: } \left\{ e_n | n = (n_1, n_2) \in \mathbb{Z}^{2d} \right\}$$

  $$e_n(x) = \exp[2\pi i (n_1 \cdot p + n_2 \cdot q)].$$

- **Classical dynamics:** $A \in \mathrm{Sp}(2d, \mathbb{Z})$,

  $$x = \begin{pmatrix} p \\ q \end{pmatrix} \mapsto Ax \quad (\text{mod } 1).$$

- **Chaotic dynamics:** If A has no eigenvalues that are roots of unity then the classical dynamics is ergodic and mixing.

Quantum system (Hannay, Berry 80):

- Planck's constant $h = 1/N$.

- Hilbert space of states:

$$\mathcal{H}_N = L^2[(\mathbb{Z}/N\mathbb{Z})^d] \cong \mathbb{C}^{N^d}.$$

- Quantization procedure

$$f \mapsto \mathsf{Op}_N(f)$$

$$A \mapsto U_N(A)$$

Satisfying

$$\|\mathsf{Op}_N(f)\mathsf{Op}_N(g) - \mathsf{Op}_N(fg)\| = O(\frac{1}{N})$$

$$U_N(A)^{-1} Op_N(f) U_N(A) = Op_N(f \circ A)$$

Quantum Operators:

- For basis: $e_n(x) = \exp(2\pi i(n_1 \cdot p + n_2 \cdot q))$

  $$\mathrm{Op}(e_n)\psi(Q) = e\Big(\frac{n_1 \cdot n_2}{2N}\Big) e\Big(\frac{n_2 \cdot Q}{N}\Big)\psi(Q+n_1)$$

- For $f = f(q)$ function of position only:

  $$\mathrm{Op}(f)\psi(Q) = f\Big(\frac{Q}{N}\Big)\psi(Q)$$

- Commutation relation:

  $$\mathrm{Op}(e_n)\mathrm{Op}(e_m) = e\Big(\frac{\omega(n,m)}{N}\Big)\mathrm{Op}(e_m)\mathrm{Op}(e_n)$$

  $$\omega(n,m) = n_1 \cdot m_2 - n_2 \cdot m_1$$

- Representation of the Heisenberg group.

Quantum propagator:

- Quantization depends only on A (mod N)

- The map $A \mapsto U_N(A)$ is a representation of $Sp(2d, \mathbb{Z}/N\mathbb{Z})$. (for N prime this is the Weil representation).

- Examples:

$$U_N \begin{pmatrix} B^t & 0 \\ 0 & B^{-1} \end{pmatrix} \psi(Q) = \sigma(B)\psi(BQ)$$

$$U_N \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \psi(Q) = e\Big(\frac{Q^t BQ}{2N}\Big)\psi(Q)$$

$$U_N \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \psi(Q) = \frac{1}{N^{d/2}} \sum_y e\Big(\frac{y \cdot Q}{N}\Big)\psi(y)$$

Quantum Ergodicity

- Take $\psi \in \mathcal{H}_N$ eigenfunction of $U_N(A)$.

- Quantum distribution

$$f \mapsto \langle \mathrm{Op}(f)\psi, \psi \rangle$$

In the limit $N \to \infty$ converges to an invariant measure.

Theorem((QE) Bouzouina, De Bievre). *For "almost all" eigenfunctions*

$$\langle Op_N(f)\psi_j, \psi_j \rangle \to \int_{T^{2d}} f d\mathit{vol}$$

- Are there other possible limiting measures?

Arithmetic Symmetries

- For $A \in \mathrm{Sp}(2d, \mathbb{Z})$ take $C_A(N)$– centralizer of $A \in Sp(2d, \mathbb{Z}/N\mathbb{Z})$

- Commutative family of (Hecke) operators:

$$U_N(B),\ B \in C_A(N)$$

- In dimension 2 ($d = 1$), Quantum cat map is AQUE **(Kurlberg, Rudnick (2000))**

- Quantum cat map is not QUE **(Faure, Nonnenmacher, De Bièvre (2003)).**

Higher dimensions

- For dimension $d > 1$, Cat Map is not necessarily AQUE

- Example:

  $A = \begin{pmatrix} B^t & 0 \\ 0 & B^{-1} \end{pmatrix}$ for any $B \in \mathrm{GL}(d, \mathbb{Z})$

  $\psi(Q) = \sqrt{N}\delta_0(Q)$ is a Hecke eigenfunction.

  $$U_N(A)\psi(Q) = \sigma(B)\psi(BQ) = \sigma(B)\psi(Q)$$

  For $f = f(q)$ of position only:

  $$Op(f)\psi(Q) = f\Big(\frac{Q}{N}\Big)\psi(Q) = f(0)\psi(Q).$$

- Matrix element:

  $$\langle \mathrm{Op}(f)\psi, \psi\rangle = f(0) \neq \int f d\mathrm{vol}$$

Theorem (K. (05)).

1. *A sufficient condition for AQUE, is the absence of rational isotropic invariant subspaces.*
2. *For any invariant rational isotropic subspace, there are Hecke eigenfunctions evenly distributed on dual invariant (co-isotropic) manifolds.*

<u>Previous example:</u>

Invariant manifold

$$\left\{ \begin{pmatrix} p \\ q \end{pmatrix} \Big| q = 0 \right\} \subset \mathbb{T}^{2d}$$

Invariant co-isotropic manifolds.

- For $V \subset \mathbb{Q}^{2d}$-invariant rational isotropic, let $U \subset \mathbb{Q}^{2d}$ invariant s.t. $V \oplus U = \mathbb{Q}^{2d}$.

- $U_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}(U) \subset \mathbb{R}^{2d}$ invariant real co-isotropic subspace, and $\Lambda = U \cap \mathbb{Z}^{2d}$ invariant lattice.

- $X = U_{\mathbb{R}}/\Lambda \subset \mathbb{T}^{2d}$ invariant co-isotropic submanifold.

- There is a sequence of Hecke eigenfunctions $\psi = \psi^{(N)} \in \mathcal{H}_N$ such that

$$\langle \mathrm{Op}_N(f)\psi, \psi\rangle \to \int_X f_{|X} d\mathrm{vol}_X$$

Outline of Proof

- It is sufficient to check for basis e_n

$$\langle \mathsf{Op}_N(e_n)\psi, \psi\rangle \to \begin{cases} 1 & n \in V \cap \mathbb{Z}^{2d} \\ 0 & \text{otherwise} \end{cases}$$

- Consider the family of operators

$$\mathcal{A} = \left\{ \mathsf{Op}_N(e_n) | n \in V \cap \mathbb{Z}^{2d} \right\}$$

- V isotropic $\Rightarrow \mathcal{A}$ is commutative.

- Decomposition into joint eigenspaces

$$\mathcal{H}_N = \bigoplus \mathcal{H}_\lambda.$$

  Each of dimension $N^{d-\dim V}$.

- Hecke operators act on eigenspaces:

$$U_N(B) : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda \circ B}$$

- The trivial eigenspace $\mathcal{H}_1$, is preserved by Hecke operators.

- $\mathcal{H}_1$ has a basis of Hecke eigenfunctions.

- For $\psi \in \mathcal{H}_1$ Hecke eigenfunction:

  For $\vec{n} \in V \cap \mathbb{Z}^{2d}$

  $$\langle \mathrm{Op}(e_n)\psi, \psi\rangle = 1$$

  For $\vec{n} \in V^* \cap \mathbb{Z}^{2d}$ in the isotropic dual subspace

  $$\langle \mathrm{Op}(e_n)\psi, \psi\rangle = 0$$

  Proposition. *For* $\vec{n} \in \mathbb{Z}^{2d}$ **not** *contained in any isotropic invariant subspace and any sequence of Hecke eigenfunctions:*

  $$\langle Op(e_n)\psi, \psi\rangle \to 0$$

Matrix elements as exponential sums

- For $\nu \in \mathbb{F}_q$ and χ a character of $\mathbb{F}_q^*$ define:

$$E_q(\nu,\chi) = \frac{1}{q-1}\sum_{x\neq 1} e_q(\nu\frac{x+1}{x-1})\chi(x)$$

- Matrix elements of elementary operators in Hecke basis are products of such sums

$$\langle \mathrm{Op}_N(e_n)\psi,\psi\rangle = \prod E_{q_i}(\nu_i,\chi_i)$$

- Each term in the product corresponds to an invariant symplectic irreducible subspace for A (mod p).

- The parameter $\nu_i = 0$ if and only if the projection of n (mod p) to the corresponding subspace lies in an isotropic subspace.

Equidistribution Conjecture

Conjecture (Kurlberg, Rudnick). *For* $\nu \neq 0$*, the normalized exponential sums* $\sqrt{q}E_q(\nu, \chi)$ *all lie in* $[-2, 2]$*, and when* χ *varies, become equidistributed with respect to Sato-Tate measure.*

- This implies a conjecture for the limiting distribution of matrix elements.

- Denote by P_k the set of primes for which $\mathbb{F}_p^{2d}$ decomposes into k invariant irreducible symplectic subspaces.

Conjecture. *As* $N \to \infty$ *through primes from* $\mathbf{P}_k$*, the limiting distribution of normalized matrix elements* $N^{d/2}\langle \mathit{Op}_N(e_n)\psi_i, \psi_i\rangle$*, is that of a product of* k *independent random variables, each obeying the semi-circle law.*