Numerical data concerning zeta functions of curves over finite fields

This is a short explanation of the data currently available at

`www.math.u-bordeaux1.fr/~kowalski/arithmatrics/zeta-functions.html`

1. Exhaustive families of small genus

– This part of the data concerns families of hyperelliptic curves given by the smooth projective models of affine curves with equation of the type

$$y^2 = f(x)(x - t)$$

where f is a fixed squarefree polynomial of degree $2g$ defined over a finite field $\mathbf{F}_q$ with q elements of characteristic $p \neq 2$, and t is the parameter of the family. Such families are families of hyperelliptic of genus g. By a theorem of Yu, they have "large monodromy" modulo all odd primes distinct from p. This means they are good testing ground for Katz-Sarnak questions involving the symplectic symmetry type.

– Using Magma, we computed the numerators of the zeta functions for some such families and $t \in \mathbf{F}_{q^k}$ for some small q and k. The data sets contain only parameters $t \neq 0$ which are not zeros of f, and moreover each file typically lists those t with degree exactly k relative to $\mathbf{F}_q$, and among those it gives only one for each Galois-orbit under the action of $t \mapsto t^q$ (since the curves with conjugate parameters are isomorphic over $\mathbf{F}_{q^k}$, hence have the same zeta function).

– The data files with name such as `zeta-genre-g-deg-k-?.gp` contain the numerators of zeta functions for all parameters t of degree k for curves in some family of genus g. This is a text file readable directly by Pari/Gp, but which may easily be transformed for use by other programs. This file has the following structure: the first line is of the type

```
info="L-functions for Y^2=(X^6+X-1)(X-t) for t in F_{5^8}, one per conjugate";\
```

which gives information on the exact family and base field used. The rest of the file loads the data into the vector `lfs`, which is then accessible for standard operation. Each component of the vector is a polynomial in `x`.

Here is an example of loading a data set and checking the Riemann Hypothesis for one of the zeta functions.

```
[emmanuel:zeta-functions]$ gp -s 500M
         GP/PARI CALCULATOR Version 2.4.1 (development CHANGES-1.1661)
                i686 running linux (ix86 kernel) 32-bit version
          compiled: Jan 27 2007, gcc-4.1.1 20070105 (Red Hat 4.1.1-51)
                 (readline v5.1 enabled, extended help enabled)

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Type ? for help, \q to quit.
Type ?12 for how to get moral (and possibly technical) support.
```

```
parisize = 500000000, primelimit = 500000
? \r zetas-genre-3-deg-8-1.gp
? info
%2 = "L-functions for Y^2=(X^6+X-1)(X-t) for t in F_{5^8}, one per conjugate"
? length(lfs)
%3 = 48750
? pol=lfs[1729]
%4 = 59604644775390625*x^6 + 109405517578125*x^5 + 172618750000*x^4 +
83953584*x^3 + 441904*x^2 + 717*x + 1
? rev=x^6*subst(pol,x,1/x)
%5 = x^6 + 717*x^5 + 441904*x^4 + 83953584*x^3 + 172618750000*x^2 +
109405517578125*x + 59604644775390625
? abs(polroots(rev))
%6 = %5 = [625.000000000000000000000000, 625.000000000000000000000000,
625.000000000000000000000000, 625.000000000000000000000000,
625.000000000000000000000000, 625.000000000000000000000000]~
? sqrt(5^8)
%7 = 625.000000000000000000000000
?
```

– The data files with name such as `angles-genre-g-deg-k-?.gp` contain the *normalized* angles of Frobenius of the eigenvalues for the curves in the corresponding data set `zetas-genre-g-deg-k-?.gp`, i.e., the values of $\frac{2\pi}{2g}\theta_i$, $1 \leq i \leq 2g$, where $\theta_i \in [0, 2\pi[$ are the angles of Frobenius. Each file starts with a line `info=....` as above, describing the family, the degree and the cardinality of the finite field involved. Then the rest loads a vector `angles`, each element of which is itself a vector of length $2g$ containing all the angles, which are moreover sorted in increasing order. (This is the normalization in Katz-Sarnak). Here is an example of loading the file and looking for the smallest normalized eigenvalue in the whole data set:

```
[emmanuel:zeta-functions]$ gp -s 500M
         GP/PARI CALCULATOR Version 2.4.1 (development CHANGES-1.1661)
                i686 running linux (ix86 kernel) 32-bit version
          compiled: Jan 27 2007, gcc-4.1.1 20070105 (Red Hat 4.1.1-51)
                 (readline v5.1 enabled, extended help enabled)

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Type ?12 for how to get moral (and possibly technical) support.

parisize = 500000000, primelimit = 500000
? \r angles-genre-3-deg-8-1.gp
? info
%2 = "L-functions for Y^2=(X^6+X-1)(X-t) for t in F_{5^8}, one per conjugate"
? small=100;for(i=1,length(angles),for(j=1,length(angles[i]),
small=min(small,angles[i][j])));small
```

```
%3 = 0.015139846067668431986752435 87
?
```

– Finally, the data files with name such as `angles-genre-g-deg-k-?.data` contain the same normalized angles of Frobenius as above, except that they are simply listed in six space-separated columns, which means that they can be looked at easily using the statistic package R (see `www.r-project.org`). Here is an example where we get some basic statistic information for each of the six columns, and then plot an histogram of the distribution of the smallest normalized eigenvalue (which is not necessarily the closest to 1, since we do not look if the largest is closer on the circle).

```
[emmanuel:zeta-functions]$ R

R version 2.4.1 (2006-12-18)
Copyright (C) 2006 The R Foundation for Statistical Computing
ISBN 3-900051-07-0

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  Natural language support but running in an English locale

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Type 'contributors()' for more information and
'citation()' on how to cite R or R packages in publications.

Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

> angles=read.table("angles-genre-3-deg-8-1.data")
> summary(angles)
       V1                V2               V3              V4
 Min.   :0.01514   Min.   :0.4841   Min.   :1.268   Min.   :3.307
 1st Qu.:0.51370   1st Qu.:1.4148   1st Qu.:2.362   1st Qu.:3.801
 Median :0.71123   Median :1.6472   Median :2.580   Median :3.999
 Mean   :0.73139   Mean   :1.6451   Mean   :2.560   Mean   :4.019
 3rd Qu.:0.92959   3rd Qu.:1.8731   3rd Qu.:2.778   3rd Qu.:4.218
 Max.   :1.98894   Max.   :2.8401   Max.   :3.273   Max.   :5.311
       V5              V6
 Min.   :3.740   Min.   :4.591
 1st Qu.:4.707   1st Qu.:5.650
 Median :4.933   Median :5.869
 Mean   :4.935   Mean   :5.848
 3rd Qu.:5.165   3rd Qu.:6.066
 Max.   :6.096   Max.   :6.565
> plot(hist(angles$V1,nclass=100))
```
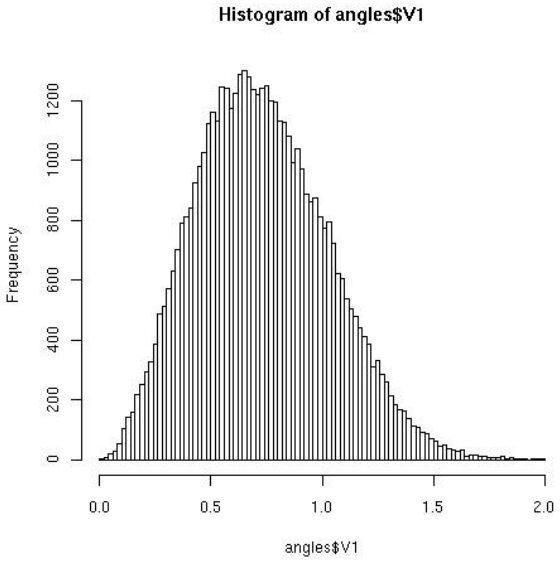

Here is what the resulting picture looks like



2. Random sample with large genus

– This second type of computations involves fixing small base field $\mathbf{F}_q$ ($q = 5$ for instance) and computing a "small" sample of zeta functions of hyperelliptic curves by selecting a random sample of irreducible polynomials $f \in \mathbf{F}_q[X]$ of given ("large") degree N over $\mathbf{F}_q$ and looking at the curve $y^2 = f(x)$.

– Using fairly fast Opteron machines, it takes about one hour for Magma to compute one zeta function for genus 49 over $\mathbf{F}_5$.

– The data files `zetas-random-genus-g-?.gp` contain the numerators of the zeta functions of a given sample of curves of genus g (over $\mathbf{F}_5$ usually). The file is readable with GP; the first line is an information string

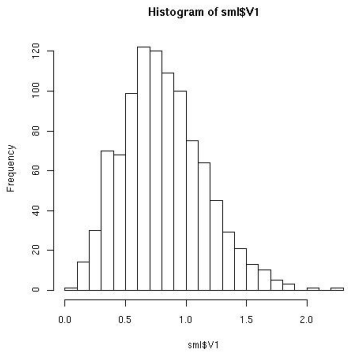
```
info="Random sample of ? zeta functions of genus g over F_q'';\
```

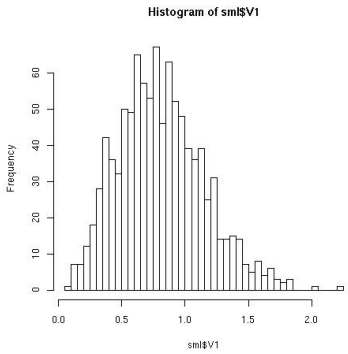
and the remainder is a vector `lfs`, each element of which is a polynomial in x representing one of the zeta functions.

– The data files `arguments-genus-g-?.gp` contain the lists of normalized arguments of Frobenius (as above, but in $]-\pi, \pi]$) of the numerators of the sample of curves of genus g in the corresponding `zetas-random-genus-g-?.gp`. The arguments are given with a precision of 100 digits, and the file is readable with GP; the first line is the same information string as in the file of zeta functions, and the remainder leads to a vector `args`, each element of which is a vector of length $2g$, containing the normalized arguments in increasing order.

– As a sample of analysis, the data files `smallest-genus-g-?.data` contain the normalized angle closest to 0, suitable for reading in R.

– Here are histograms with 30 and 50 bins of the smallest angle in one random sample of 1000 genus 49 hyperelliptic curves:


Histogram of sml$V1
Frequency
0
20
40
60
80
100
120
0.0
0.5
1.0
1.5
2.0
sml$V1


Histogram of sml$V1
Frequency
0
10
20
30
40
50
60
0.0
0.5
1.0
1.5
2.0
sml$V1