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Small gaps in coefficients of L-functions and $\mathfrak{B}$-free numbers in short intervals

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Abstract. We discuss questions related to the non-existence of gaps in the series defining modular forms and other arithmetic functions of various types, and improve results of Serre, Balog & Ono and Alkan using new results about exponential sums and the distribution of $\mathfrak{B}$-free numbers.

§ 1. Introduction

The motivation of this paper is a result of Serre ([43, Th. 15]) and the questions he subsequently raises. Let f be a primitive holomorphic cusp form (i.e. a newform in the Atkin-Lehner terminology) of weight k, with conductor N and nebentypus χ. Write

$$f(z) = \sum_{n\geq 1} \lambda_f(n) e(nz) \tag{1.1}$$

its Fourier expansion at infinity, where $e(z) = \exp(2\pi i z)$, so that $\lambda_f(n)$ is also the Hecke eigenvalue of f for the Hecke operator T_n. Serre's result is that

$$|\{p \leq x \mid \lambda_f(p) = 0\}| \ll x(\log x)^{-1-\delta}, \tag{1.2}$$

for $x \geq 2$ and any $\delta < \frac{1}{2}$, the implied constant depending on f and δ, from which he deduces that the series (1.1), or equivalently the L-function

$$L(f, s) = \sum_{n\geq 1} \lambda_f(n) n^{-s} \tag{1.3}$$

is not lacunary, i.e. the set of indices n where $\lambda_f(n) \neq 0$, has a positive density. Serre asked ([43, p. 183]) for more precise statements, in particular for bounding non-trivially the function $i_f(n)$ defined by

$$i_f(n) = \max\{k \geq 1 \mid \lambda_f(n+j) = 0 \text{ for } 0 < j \leq k\}, \tag{1.4}$$

where non-trivial means an estimate of type $i_f(n) \ll n^\theta$ for some $\theta < 1$ and all $n \geq 1$. A stronger form of the problem is to find y as small as possible (as a function of x, say $y = x^\theta$ with $\theta < 1$) such that

$$|\{n \mid x < n \leq x + y \text{ and } \lambda_f(n) \neq 0\}| \gg y \tag{1.5}$$

(where the implied constant can depend on f). Non-lacunarity means $y = x$ is permitted, and one wishes to improve this. Note $i_f(n) \ll y$ so this generalizes the first question.

The history of this problem is somewhat confused. First, Serre could have solved it quite simply in (at least) two ways available at the time. The first is to argue that by multiplicativity

$\lambda_f(n) \neq 0$ if n is squarefree and not divisible by primes p for which $\lambda_f(p) = 0$. The latter have density zero by (1.1), so estimating $i_f(n)$ becomes a special case of a problem in multiplicative number theory, that of counting so-called $\mathfrak{B}$-free numbers in small intervals, where for a set $\mathfrak{B} = \{b_i\}$ of integers with $(b_i, b_j) = 1$ if $i \neq j$ and

$$\sum_i \frac{1}{b_i} < +\infty,$$

one says that $n \geq 1$ is $\mathfrak{B}$-free if it is not divisible by any element in $\mathfrak{B}$. Erdös [11] already showed in 1966 that with no further condition there exists a constant $\theta < 1$ (absolute) such that the interval $(x, x + x^\theta]$ contains a $\mathfrak{B}$-free number for x large enough, thereby solving Serre's first question in the affirmative. A quantitative result proving the analogue of (1.5) for general $\mathfrak{B}$-free numbers was also obtained Szemerédi [44] as early as 1973.

This was apparently first noticed by Balog and Ono [2]. By this time results about $\mathfrak{B}$-free numbers had been refined a number of times, and they deduced from a result of Wu [45] that $i_f(n) \ll n^{17/41+\varepsilon}$ for $n \geq 1$ and any $\varepsilon > 0$, the implied constant depending on f and ε. Using this idea and other results (such as a version of the Chebotarev density theorem in small intervals and the Shimura correspondence), they also get weaker results for modular forms of weight 1 or half-integral weight. The latter is noteworthy in this respect since the Fourier coefficients of half-integral weight forms are highly non-multiplicative (see [7] for a strong quantitative expression of this fact). Alkan [1] has developed and improved the results of [2], tailoring some arguments to the specific instance of $\mathfrak{B}$-free numbers involved for the problem at hand.

A second method of estimating $i_f(n)$ available to Serre was a direct appeal to the properties of the Rankin-Selberg L-function $L(f \otimes \bar{f}, s)$. Specifically this proves [36, 42] (for f any cusp form of integral weight k and level N)

$$\sum_{n \leq x} |\lambda_f(n)|^2 n^{1-k} = c_f x + O(x^{3/5})$$

for some $c_f > 0$, and $x \geq 1$, the implied constant depending only on f. Trivially this implies $i_f(n) \ll n^{3/5}$, and incidentally this fact is implicit in [27] (which Serre quotes as one source for his problems!)

It turns out however that there are still a number of things which seem to have been overlooked. For instance we will show that it is not necessary to sieve by squarefree numbers, and we will explain the applications of the Rankin-Selberg L-functions (in particular to non-congruence subgroups, another of the questions in [43]). We also look at lacunarity in some other Dirichlet series coming from arithmetic or analysis, including one which is really neither fish nor fowl (see Proposition 4). On the other hand (this is our main new contribution), we will improve quite significantly the $\mathfrak{B}$-free number results that can be used. Some of our tools are new estimates for exponential sums and bilinear forms which are of independent interest in analytic number theory.

We of course welcome any further corrections to the picture thus produced about this problem.

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Notation. For any $k \geq 1$, $N \geq 1$ and any character χ modulo N, we denote $S_k(N,\chi)$ the vector space of cusp forms of weight k for the group $\Gamma_0(N)$, with nebentypus χ. If χ is the trivial character modulo N, we simply write $S_k(N)$. We also denote by $S_k^*(N,\chi)$, or $S_k^*(N)$, the set of primitive forms in $S_k(N,\chi)$ or $S_k(N)$, i.e. those forms which are eigenfunctions of all Hecke operators T_n and are normalized by $\lambda_f(1) = 1$, where $\lambda_f(n)$ is the n-th Fourier coefficient, which is then equal to the n-th Hecke eigenvalue. See e.g. [23] for basic analytic facts about modular forms.

For s a complex number, we denote σ its real part and t its imaginary part. Also, we use $f(x) = O(g(x))$ and $f(x) \ll g(x)$ for x in some set X as synonyms, meaning $|f(x)| \leq Cg(x)$ for all $x \in X$, $C \geq 0$ being called the implied constant.

§ 2. Algebraic aspects

We start by noticing that the restriction to squarefree numbers present in [2] and [1] is in fact unnecessary, because the set of primes for which $\lambda_f(p^\nu) = 0$ for *any* ν still satisfies an estimate similar to (1.2). This is partly implicit in [43, p. 178–179].

Lemma 2.1. *Let $f \in S_k^*(N,\chi)$ be a primitive holomorphic cusp form. There exists an integer ν_f depending only on f such that for any prime $p \nmid N$, either $\lambda_f(p^\nu) \neq 0$ for all $\nu \geq 0$, or there exists $\nu \leq \nu_f$ such that $\lambda_f(p^\nu) = 0$.*

Proof. Let $p \nmid N$. By multiplicativity we have the power series expansion

$$\sum_{\nu \geq 0} \lambda_f(p^\nu) X^\nu = \frac{1}{1 - \lambda_f(p)X + \chi(p)p^{k-1}X^2}. \tag{2.1}$$

Let α_p and β_p be the complex numbers such that

$$1 - \lambda_f(p)X + \chi(p)p^{k-1}X^2 = (1 - \alpha_p X)(1 - \beta_p X). \tag{2.2}$$

Thus

$$\alpha_p + \beta_p = \lambda_f(p) \qquad \text{and} \qquad \alpha_p\beta_p = \chi(p)p^{k-1} \neq 0. \tag{2.3}$$

Expanding (2.1) using (2.2) by geometric series gives the well-known expressions

$$\lambda_f(p^\nu) = \frac{\alpha_p^{\nu+1} - \beta_p^{\nu+1}}{\alpha_p - \beta_p} \tag{2.4}$$

if $\alpha_p \neq \beta_p$, and the simpler

$$\lambda_f(p^\nu) = (\nu+1)\alpha_p^\nu = (\nu+1)\tau p^{\nu(k-1)/2} \neq 0 \tag{2.5}$$

if $\alpha_p = \beta_p$, where $\tau^2 = \chi(p)$. So we can assume $\alpha_p \neq \beta_p$. In this case we get by (2.4)

$$\lambda_f(p^\nu) = 0 \quad \text{if and only if} \quad (\alpha_p/\beta_p)^{\nu+1} = 1$$

so that there exists $\nu \geq 0$ for which $\lambda_f(p^\nu) = 0$ if and only if α_p/β_p is a root of unity, and if this ratio is a primitive root of unity of order $d \geq 1$, then $\lambda_f(p^{d-1}) = 0$.

Now we input some more algebraic properties of the Fourier coefficients. The field

$$K_f = \mathbb{Q}(\lambda_f(n), \chi(n))$$

generated by all Fourier coefficients and values of χ is known to be a number field. By (2.2), the "roots" α_p and β_p lie in a quadratic extension of K_f. This extension (say K_p) depends on p, but it has degree $[K_p : \mathbb{Q}] \leq 2[K_f : \mathbb{Q}]$ for all p.

Now we combine both remarks and the fact that a number field $L/\mathbb{Q}$ can only contain a primitive d-th root of unity if $\varphi(d) \leq [L : \mathbb{Q}]$. It follows that if $p \nmid N$ and $\lambda_f(p^\nu) = 0$ for some $\nu \geq 0$, α_p/β_p is a primitive root of unity of some order d such that $\varphi(d) \leq 2[K_f : \mathbb{Q}]$, and then $\lambda_f(p^{d-1}) = 0$. Since $\varphi(d) \gg d/\log\log(3d)$, this proves the lemma. □

It is clear that ν_f is effectively computable. Here are some simple cases.

Lemma 2.2. *Let k be even and $f \in S_k^*(N, \chi)$. There exists $M \geq 1$ such that for any $p \nmid M$, either $\lambda_f(p) = 0$ or $\lambda_f(p^\nu) \neq 0$ for any $\nu \geq 1$. If χ is trivial and f has integer coefficients, one can take $M = N$.*

Proof. If $p \mid N$, the condition $\lambda_f(p^\nu) = 0$ is equivalent to $\lambda_f(p) = 0$ by total multiplicativity, so we can assume that $p \nmid N$. Let p be such a prime with $\lambda_f(p^\nu) = 0$ for some $\nu \geq 2$, but $\lambda_f(p) \neq 0$. Using the same notation as the proof of Lemma 2.1, we have $\alpha_p = \xi\beta_p$ for some root of unity ξ of order $d + 1$, and $\xi \neq -1$. We derive from the second relation of (2.3) that $\alpha_p^2 = \xi\chi(p)p^{k-1}$, hence $\alpha_p = \pm\tau p^{(k-1)/2}$, where $\tau^2 = \xi\chi(p)$. By the second relation of (2.3), we get

$$\lambda_f(p) = (1 + \bar{\xi})\alpha_p = \pm\tau(1 + \bar{\xi})p^{(k-1)/2} \neq 0.$$

In particular, since k is even, $\mathbb{Q}(\tau(1 + \bar{\xi})\sqrt{p}) \subset K_f$. As K_f is a number field, this can happen only for finitely many p, and one can take as M the product of those primes and those $p \mid N$ with $\lambda_f(p) = 0$.

Furthermore, if χ is trivial and f has integer coefficients, then for $p \nmid N$, $\alpha_p/\beta_p = \xi$ is a root of unity $\neq 1$ in a quadratic extension of $\mathbb{Q}$ (see (2.2)), hence $\xi \in \{-1, \pm j, \pm j^2, \pm i\}$ (with $\nu \in \{1, 2, 3, 5\}$). All those except $\xi = -1$ contradict the fact that f has integer coefficients by simple considerations such as the following, for $\xi = j$ say: we have $\alpha_p^2 = jp^{k-1}$, $\alpha_p = \pm j^2 p^{(k-1)/2}$ and

$$\lambda_f(p) = (1 + \bar{j})\alpha_p = \pm(1 + \bar{j})j^2 p^{(k-1)/2} = \pm(j^2 + j)p^{(k-1)/2} \notin \mathbb{Z}$$

(compare [43, p. 178–179]). □

We now prove the analogue of (1.2) for primes p such that $\lambda_f(p^\nu) = 0$ for some ν.

Lemma 2.3. *Let $f \in S_k^*(N, \chi)$ be a primitive cusp form not of CM type, in particular with $k \geq 2$. For $\nu \geq 1$, let*

$$\mathfrak{P}_{f,\nu} = \{p \nmid N \mid \lambda_f(p^\nu) = 0\}. \tag{2.6}$$

For any $\nu \geq 1$ we have

$$\left|\mathfrak{P}_{f,\nu} \cap [1, x]\right| \ll \frac{x}{(\log x)^{1+\delta}} \tag{2.7}$$

for $x \geq 2$ and any $\delta < \frac{1}{2}$, the implied constant depending on f and δ. Let $\mathfrak{P}_f^*$ be the union of $\mathfrak{P}_{f,\nu}$. We have

$$\left|\mathfrak{P}_f^* \cap [1, x]\right| \ll \frac{x}{(\log x)^{1+\delta}} \tag{2.8}$$

for $x \geq 2$ and any $\delta < \frac{1}{2}$, the implied constant depending on f and δ.

Proof. All the tools needed to prove (2.7), if not the exact statements, can be gathered from [43], in particular Section 7.2. By Lemma 2.1, we need only prove (2.7), so let $\nu \geq 1$ be fixed.

Fix a prime number ℓ totally split in the field $K_f = \mathbb{Q}(\lambda_f(n), \chi(n))$ already considered. Thus $K_f \subset \mathbb{Q}_\ell$. There exists an ℓ-adic Galois representation

$$\rho_{f,\ell} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(2, \mathbb{Q}_\ell)$$

constructed by Deligne, such that for $p \nmid N\ell$ we have

$$\mathrm{Tr}\rho_{f,\ell}(\sigma_p) = \lambda_f(p) \qquad \text{and} \qquad \det \rho_{f,\ell}(\sigma_p) = \chi(p)p^{k-1},$$

where σ_p is a Frobenius at p. Let G_ℓ be the image of $\rho_{f,\ell}$. As explained by Serre [43, Prop. 17], it is an open subgroup of $GL(2, \mathbb{Q}_\ell)$, hence an ℓ-adic group of dimension 4.

By symmetry, there exists a polynomial $P_\nu \in \mathbb{Z}[X, Y]$ such that the identity

$$\frac{X^{\nu+1} - Y^{\nu+1}}{X - Y} = P_\nu(X + Y, XY)$$

holds. Consider the set $C \subset G_\ell$ defined by

$$C = \{s \in G_\ell \mid P_\nu(\mathrm{Tr}(s), \det(s)) = 0\}.$$

Note the following facts about C: it is a closed ℓ-adic subvariety of G_ℓ, stable by conjugation, and of dimension ≤ 3. Moreover, C is stable by multiplication by $H_\ell = \{\text{homotheties in } G_\ell\}$, and therefore $C = \pi^{-1}(C')$ for a certain subvariety $C' \subset G_\ell/H_\ell$, where $\pi : G_\ell \to G_\ell/H_\ell$ is the projection. The set C' is an ℓ-adic variety of dimension ≤ 2 and all its elements are regular ([43, Section 5.2]), since they have distinct eigenvalues α, $\xi\alpha$ for some root of unity $\xi \neq 1$ of order $\nu + 1$.

Now remark that if $p \in \mathfrak{P}_{f,\nu}$ and $p \nmid N\ell$, we have $\pi(\sigma_p) \in C'$ (going back to the proof Lemma 2.1 if necessary). Hence our result (2.7) follows from Theorem 12 of [43], as in the proof of the case $h = 0$ of Theorem 15 of loc. cit., p. 177. □

For ease of reference we recall the lemma which allows the extension of the results for $i_f(n)$ to general cusp forms from that of newforms.

Lemma 2.4. *Let $f \in S_k(N, \chi)$ be a cusp form not in the space spanned by CM forms. There exist:*

(i) *an integer $s \geq 1$ and algebraic numbers β_j and positive rational numbers γ_j for $1 \leq j \leq s$;*

(ii) *a divisor $\delta \mid N$ such that χ is induced by χ_1 modulo N/δ and a divisor $\delta_1 \mid \delta$;*

(iii) *a primitive form $g \in S_k^*(N/\delta, \chi_1)$, not of CM type;*

such that

$$\lambda_g(n) = \sum_{1 \leq j \leq s} \beta_j \lambda_f(\gamma_j \delta_1 n)$$

for $n \geq 1$. By convention, we put $\lambda_f(x) = 0$ if $x \in \mathbb{Q}$ is not a positive integer.

This is just a formal restatement of the computations in [2], p. 362, or follows from [43, §7.6].

We now discuss briefly the possibility of extending the results above to higher rank situations. From the proof of Lemma 2.3, it is natural to start from an ℓ-adic representation

$$\rho\,:\,\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\rightarrow GL(V)$$

where $V\simeq\mathbb{Q}_\ell^r$ for some $r\geq 1$. We assume it is "sufficiently geometric", namely that it is unramified outside a finite set of primes S, and that the L-function of ρ, defined as usual by the Euler product

$$L(\rho,s)=\prod_p\det(1-\rho(\sigma_p)p^{-s}\mid V^{I_p})^{-1}=\sum_{n\geq 1}\lambda_\rho(n)n^{-s}, \tag{2.9}$$

(where σ_p is a Frobenius element at p and I_p the inertia group at p) has coefficients in a number field $K_\rho\subset\mathbb{Q}_\ell$. Note that we view this here as a formal Dirichlet series. If the image of ρ is fairly big, one can use the methods of Serre to get

$$|\{p\leq x\mid p\notin S\text{ and }\lambda_\rho(p)=\mathrm{Tr}\,\rho(\sigma_p)=0\}|\ll x(\log x)^{-1-\delta}$$

for some $\delta>0$, see Proposition 1 below. On the other hand, it is not clear if the analogue of Lemma 2.1 holds, and this seems a hard question in general. The analogue of (2.4) does not provide an equation easily solvable to characterize the values of ν for which $\lambda_\rho(p^\nu)=0$. The best that seems doable is to notice that, for fixed (unramified) p, $u_\nu=\lambda_\rho(p^\nu)$ is given by a linear recurrence relation of degree r with "companion polynomial" given by

$$\det(X-\rho(\sigma_p))=X^r-\lambda_\rho(p)X^{r-1}+\cdots+(-1)^r\det\rho(\sigma_p)=\prod_{1\leq i\leq r}(X-\alpha_{p,i})$$

so that

$$u_\nu=\lambda_\rho(p^\nu)=\sum_{1\leq i\leq r}\gamma_{p,i}\alpha_{p,i}^\nu$$

for some $\gamma_{p,i}$. The Skolem-Mahler-Lech theorem (see e.g. [6, p. 88]) says that for any linear recurrence sequence (u_ν), either $u_\nu=0$ for only finitely many values of ν, or there exists an arithmetic progression $a+tv$ with $v\neq 0$ such that $u_{a+tv}=0$ for all $t\geq 0$. In the latter case, spelling this out yields a Vandermonde type linear system for powers of the $\alpha_{p,i}^v$, hence it implies that $\alpha_{p,i}^v=\alpha_{p,j}^v$ for some $i\neq j$. Coming back to ρ, this case implies that an extension of degree $\leq r[K_\rho:\mathbb{Q}]$ contains a v-th root of unity. As in Lemma 2.1, this bounds v, and an analogue of Lemma 2.3 is possible, given ξ a root of unity, to get

$$|\{p\leq x\mid p\notin S\text{ and there are two roots }\alpha_{i,p},\,\alpha_{j,p}=\xi\alpha_{i,p}\text{ with }i\neq j\}|\ll x(\log x)^{-1-\delta}$$

for some $\delta>0$.

However, in the first case where $u_\nu=0$ has only finitely many solutions, despite the remarkable fact that there exists a uniform bound for the number of solutions depending only on r (see [12]), this is insufficient because only the number of solutions, not the value of ν, is bounded, so that an integer ν_0 (independent of p) for which the smallest solution is $\nu\leq\nu_0$ is not known to exist. The question amounts to asking for a bound for the height of the solutions to the relevant linear equations in multiplicative groups [12, p. 820], and is thus in full generality of the same type as asking for effective versions of Roth's theorem, or of Schmidt's Subspace Theorem. (Note that by replacing u_ν by $p^{-(k-1)/2}u_\nu$ one gets a linear recurrence relation with companion polynomial having height absolutely bounded, by the Ramanujan-Petersson conjecture proved by Deligne).

The theory of $\mathfrak{B}$-free numbers does however still apply. Thus we get:

Proposition 1. *Let ρ be an ℓ-adic representation of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *on* $GL(r,\mathbb{Q}_\ell)$*. Assume that ρ is unramified for p outside a finite set S and that its L-function has coefficients in a number field K_ρ. Let $G = \mathrm{Im}\,\rho$, $C = G \cap \{s \in GL(r,\mathbb{Q}_\ell) \mid \mathrm{Tr}\, s = 0\}$. Assume that, as ℓ-adic varieties, we have $\dim C < \dim G$. Then for any $\varepsilon > 0$, $x \geq x_0(\rho,\varepsilon)$ and $y \geq x^{7/17+\varepsilon}$ we have*

$$|\{n \mid x < n \leq x + y \text{ and } \lambda_\rho(n) \neq 0\}| \gg y.$$

In particular $i_\rho(n) \ll n^{7/17+\varepsilon}$.

Proof. One can argue as for modular forms using $\mathfrak{B}$-free numbers (see Proposition 6) with

$$\mathfrak{B} = \{p \mid p \in S \text{ or } \lambda_\rho(p) = 0\} \cup \{p^2 \mid \lambda_\rho(p) \neq 0\},$$

after applying Theorem 10 of [43] to G and C, with $E = \bar{\mathbb{Q}}^{\ker\rho}$, to derive

$$|\{p \leq x \mid p \notin S \text{ or } \lambda_\rho(p) = 0\}| \ll x(\log x)^{-1-\delta}$$

for some $\delta > 0$ depending on the dimensions of G and C (for instance, any $\delta < 1 - \dim C/\dim G$). Strictly speaking, to apply this theorem as stated we must also treat separately the case where G is finite. One can then see ρ as a linear representation of the finite group $G = \mathrm{Gal}(E/\mathbb{Q})$ into $GL(n,\bar{\mathbb{Q}})$, or into $GL(n,\mathbb{C})$. In that case the condition $\dim C < \dim G$ means that the character of ρ does not vanish. By a well-known fact about linear representations of finite groups (see e.g. [17, Ex. 2.39]), this means that the representation ρ is a one-dimensional character of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, which by the Kronecker-Weber theorem corresponds to (i.e. has the same L-function as) a Dirichlet character χ, of conductor N say (divisible only by primes in S). Then $\lambda_\rho(n) \neq 0$ if and only if $(n, N) = 1$. □

This is also implicit in [43, §6.4, 6.5].

§ 3. Maass forms, cofinite groups and the Rankin-Selberg method

In this section, we describe what results follow from the Rankin-Selberg method. Although, for fixed $f \in S_k^*(N,\chi)$, they are weaker than those obtained by means of $\mathfrak{B}$-free numbers, this method has the advantage of yielding quite easily estimates uniform in terms of f, i.e. with explicit dependency on k and N. Those are by no means obvious from the ℓ-adic point of view leading to (1.2). Moreover, the Rankin-Selberg method applies, at least as far as bounding $i_f(n)$, to non-congruence subgroups, as shown by Good [19], Sarnak [41] and Petridis [34]. This answers the last question in [43, p. 183].

Proposition 2. *Let $\Gamma \subset SL(2,\mathbb{R})$ be a discrete subgroup such that the quotient $\Gamma\backslash\mathbb{H}$ has finite hyperbolic volume and Γ contains the integral translation matrices acting by $z \mapsto z + n$. Let f be either a holomorphic cusp form of weight $k \geq 2$ or a Maass cusp form with eigenvalue $\lambda \neq 1/4$. Define $i_f(n)$ by (1.4) where $\lambda_f(n)$ are the Fourier coefficients in the expansion of f at the cusp ∞ of Γ. Then for some $\theta < 1$ we have*

$$i_f(n) \ll n^\theta$$

for $n \geq 1$ where the implied constant depends on f. Specifically, one can take $\theta = 2/3$ if f is holomorphic and any $\theta > 4/5$ if f is non-holomorphic.

Proof. The non-holomorphic case follows from [34] as the holomorphic case follows from [19], so we describe only the latter. Good shows that

$$\sum_{n\leq x}|\lambda_f(n)|^2=\sum_{2/3<s_j\leq 1}\frac{(4\pi x)^{s_j+k-1}}{\Gamma(k+s_j)}\langle r_j,y^k|f|^2\rangle+O(x^{k-1+2/3}) \tag{3.1}$$

for $x\geq 1$, where $1=s_0>s_1\geq\cdots\geq s_r$ are the finitely many poles of the Eisenstein series $E(z,s)$ for Γ in the interval $[1/2,1]$ (those with $s_j>2/3$ go to the error term), $r_j(z)$ is the residue of $E(z,s)$ at s_j and $\langle\cdot,\cdot\rangle$ is the inner product on $L^2(\Gamma\backslash\mathbb{H})$. The pole at $s_0=1$ with residue V^{-1} contributes

$$\frac{(4\pi x)^k\|f\|^2}{k!V}$$

where $\|f\|$ is the Petersson norm of f and V the volume of $\Gamma\backslash\mathbb{H}$. Comparing (3.1) at $x=n$ and $x=n+Cn^{2/3}$, where C is some large constant, shows that $i_f(n)\leq Cn^{2/3}$. □

Remark 1. As for half-integral weight forms, it is not expected that the coefficients of a cusp form for a non-arithmetic group satisfy any multiplicativity properties. In fact, it would be quite interesting to express this in a quantitative manner as done by Duke and Iwaniec [7] for half-integral forms using bilinear forms in the Fourier coefficients.

In the case of congruence subgroup the methods using $\mathfrak{B}$-free numbers yield stronger results such as (1.5) for y quite small. Those however are not uniform in terms of f (i.e. in terms of N and k for holomorphic forms). The Rankin-Selberg method can quite easily yield some uniform estimates. Here are sample statements; note that we have not tried to get the best possible results.

Proposition 3. (1) *Let $N\geq 1$ and f a primitive Maass form of conductor N with eigenvalue $\lambda\neq 1/4$ and trivial nebentypus. Let $\Lambda=\lambda+1$ and let $\lambda_f(n)$ be the Fourier coefficients of f. For any $\varepsilon>0$, there exists $c>0$ depending only on ε such that when*

$$y>x^{37/40}\Lambda^{45/32}N^{19/8}(x\Lambda N)^{\varepsilon}, \tag{3.2}$$

we have

$$|\{n\mid x<n\leq x+y \text{ and } |\lambda_f(n)|^2\geq c(\Lambda N)^{-\varepsilon}\}|\gg y(\log x)^{-1}(\Lambda N)^{-\varepsilon},$$

the implied constants depending only on ε.

(2) *Let $N\geq 1$ and f a primitive holomorphic form of conductor N, weight k with nebentypus χ, not of CM type. There exists an absolute constant $c>0$ such that for any $\varepsilon>0$ and*

$$y>x^{4/5}(kN)^{1/2}(xkN)^{\varepsilon}$$

we have

$$|\{n\mid x<n\leq x+y \text{ and } |\lambda_f(n)|^2n^{(1-k)}\geq c(\log kN)^{-1}\}|\gg y(\log x)^{-14}(\log kN)^{-3}$$

the implied constant depending only on ε.

Proof. We prove (1) and only give some indications for the easier (2) at the end. It turns out to be simpler to reduce to squarefree numbers (so in fact we could impose this condition on n). The result will follow by Cauchy's inequality from the two asymptotic formulas

$$\sideset{}{^\flat}\sum_{n\leq x}|\lambda_f(n)|^2=c_fx+O((\Lambda N)^{25/64}x^{121/146}(x\Lambda N)^{\varepsilon}) \tag{3.3}$$

and

$$\sum_{n\le x}^{\flat}|\lambda_f(n)|^4=d_f x\log x+e_f x+O(\Lambda^{45/32}N^{19/8}x^{37/40}(x\Lambda N)^{\varepsilon}) \tag{3.4}$$

where $\sum^{\flat}$ restricts n to squarefree integers coprime with N. Both hold for any $\varepsilon>0$, with the implied constant depending only on ε, and c_f, d_f, e_f are real numbers with c_f, $d_f>0$ and

$$c_f\gg(\Lambda N)^{-\varepsilon},\quad d_f,\ e_f\ll(\Lambda N)^{\varepsilon} \tag{3.5}$$

for any $\varepsilon>0$, the implied constant depending only on ε.

Indeed, let $\varepsilon>0$ and $\eta>0$ be any positive numbers, and put the integers n with $x<n\le x+y$ in two sets L and S if, respectively, $|\lambda_f(n)|^2>\eta$ or $|\lambda_f(n)|^2\le\eta$. If y satisfies (3.2) we have by (3.3) and (3.5)

$$\sum_{x<n\le x+y}^{\flat}|\lambda_f(n)|^2\gg c_f y\ge Cy(\Lambda N)^{-\varepsilon/4}$$

where C depends only on ε, whereas by positivity and Cauchy's inequality

$$\sum_{x<n\le x+y}^{\flat}|\lambda_f(n)|^2\le\eta|S|+\sum_{x\in L}^{\flat}|\lambda_f(n)|^2\le\eta y+|L|^{1/2}\Big(\sum_{x<n\le x+y}^{\flat}|\lambda_f(n)|^4\Big)^{1/2}.$$

If $\eta<\frac{C}{2}(\Lambda N)^{-\varepsilon/4}$ we derive by (3.4) and (3.5)

$$|L|\gg y^2(\Lambda N)^{-\varepsilon}y^{-1}(\log x)^{-1},$$

as desired.

We give the proof of (3.4) and the upper bounds on d_f, e_f, since (3.3) is easier. The lower bound for c_f is deeper, and follows immediately from the bound $L(F,1)\gg(\Lambda N)^{-\varepsilon}$ of Hoffstein and Lockhart [22] for the adjoint square F of f (which is also its symmetric square since the nebentypus is trivial).

Since f is primitive and has trivial nebentypus, hence real coefficients, we have

$$|\lambda_f(p)|^4=\lambda_f(p)^4=(1+\lambda_f(p^2))^2=1+2\lambda_f(p^2)+\lambda_f(p^2)^2$$

for $p\nmid N$ and thus we find that

$$\begin{aligned}L(s):=\sum_{n\ge1}^{\flat}|\lambda_f(n)|^4n^{-s}&=\prod_{p\nmid N}(1+|\lambda_f(p)|^4p^{-s})\\&=\zeta^{\flat}(s)L^{\flat}(\mathrm{Sym}^2 f,s)^2L^{\flat}(\mathrm{Sym}^2 f\otimes\mathrm{Sym}^2 f,s)H(s)\end{aligned}$$

where

$$\begin{aligned}\zeta^{\flat}(s)&=\prod_{p\nmid N}(1+p^{-s}),\\L^{\flat}(\mathrm{Sym}^2 f,s)&=\prod_{p\nmid N}(1+\lambda_f(p^2)p^{-s}),\\L^{\flat}(\mathrm{Sym}^2 f\otimes\mathrm{Sym}^2 f,s)&=\prod_{p\nmid N}(1+\lambda_f(p^2)^2p^{-s})\end{aligned}$$

and $H(s)$ is an Euler product which converges absolutely for $\sigma > \frac{23}{32}$ by the estimate $|\lambda_f(p)| \leq 2p^{7/64}$ of Kim and Sarnak [26] (any estimate $|\lambda_f(p)| \leq 2p^{\theta}$ with $\theta < 1/4$ would do, at the cost of worsening the exponent), and is moreover uniformly bounded (in terms of f) on any line $\sigma = \sigma_0$ with $\sigma_0 > \frac{23}{32}$. To see this, define H as the obvious ratio for σ large enough, and check on the Euler factors individually that

$$H(s) \ll \zeta(2\sigma - 14/32)^B$$

for some absolute constant $B > 0$ (for a similar argument, see e.g. [9, Prop. 2]).

Each of the three L-functions is obtained by removing non-squarefree coefficients (and those not coprime with N) from an L-function which has analytic continuation and a functional equation of the standard type: the first one is the zeta function, the second one is the adjoint square $F = \mathrm{Sym}^2 f$ of Shimura and Gelbart-Jacquet [18], and the third is the Rankin-Selberg square $F \otimes F$ of the latter (which exists as a special case of convolution of cusp forms on $GL(3)$). The same bound and reasoning already used shows that

$$\zeta^{\flat}(s)L^{\flat}(F,s)^2L^{\flat}(F\otimes F,s) = \zeta(s)L(F,s)^2L(F\otimes F,s)H_1(s)$$

where $H_1(s)$ has the same properties as $H(s)$ above.

In particular we see that $L(s)$ has a pole of order 2 at $s = 1$ (by [31] since $\lambda \neq 1/4$ so that F is a cusp form on $GL(3)$). We can now proceed along classical lines: let $U > 1$ (to be chosen later) and let ψ be a C^{∞} function on $[0, +\infty($ such that $0 \leq \psi \leq 1$ and

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 - U^{-1}, \\ 0 & \text{if } x \geq 1 + U^{-1}. \end{cases}$$

The Mellin transform $\hat{\psi}(s)$ is holomorphic for $\sigma > 0$, it satisfies $\hat{\psi}(s) = s^{-1} + O(|\sigma|U^{-1})$ and by integration by parts

$$\hat{\psi}(s) \ll U^A(1+|t|)^{-A-1} \tag{3.6}$$

for $\sigma \geq 1/2$ and for any $A > 0$, the implied constant depending on A and ψ only.

For suitable choices (say ψ_+ and ψ_-) of ψ we get

$$\sideset{}{^\flat}\sum_{n\geq 1} |\lambda_f(n)|^4\psi_-(n/x) \leq \sideset{}{^\flat}\sum_{n\leq x} |\lambda_f(n)|^4 \leq \sideset{}{^\flat}\sum_{n\geq 1} |\lambda_f(n)|^4\psi_+(n/x).$$

Thus it is enough to prove (3.4) for a sum weighted by $\psi(n/x)$. We have

$$\begin{aligned}\sideset{}{^\flat}\sum_{n\geq 1} |\lambda_f(n)|^4\psi(n/x) &= \frac{1}{2\pi i}\int_{(3)} L(s)x^s\hat{\psi}(s)ds \\ &= \frac{1}{2\pi i}\int_{(3)} \zeta(s)L(F,s)^2L(F\otimes F,s)H(s)H_1(s)\hat{\psi}(s)x^s ds.\end{aligned}$$

For any fixed $\alpha > \frac{23}{32}$ we can move the line of integration (the three L-functions are polynomially bounded in vertical strips and $\hat{\psi}$ decays rapidly) to $\sigma = \alpha$. We pass the double pole at $s = 1$ with residue of the form

$$d_f x\log x + e_f x + O(x(\log x)(\Lambda N)^{\varepsilon}U^{-1})$$

with $d_f = L(F,1)H(1)H_1(1)\mathrm{res}_{s=1}L(F\otimes F,s) > 0$, d_f and e_f being estimated by [32] to get

$$d_f \ll (\Lambda N)^{\varepsilon}, \quad e_f \ll (\Lambda N)^{\varepsilon}$$

for any $\varepsilon > 0$, the implied constant depending only on ε.

Now the integral on $\sigma = \alpha < 1$ is estimated using $H(s)H_1(s) \ll 1$, the uniform convexity bound for automorphic L-functions (see e.g. [24, §5.12]) yielding

$$L(s) \ll (1+|t|)^{8(1-\alpha)+\varepsilon}\Lambda^{5(1-\alpha)+\varepsilon}N^{8(1-\alpha)+\varepsilon}$$

for the product of the three L-functions, the implied constant depends only on α and ε. Then (3.6) with $A = 8(1-\alpha)+1/2$ (to get an absolutely convergent integral) yields

$$\sum_{n\geq 1}^{\flat} |\lambda_f(n)|^4\psi(n/x) = d_f x\log x + e_f x + O(x(\log x)(\Lambda N)^{\varepsilon}U^{-1}) + O(x^{\alpha}\Lambda^{5(1-\alpha)+\varepsilon}N^{8(1-\alpha)+\varepsilon}U^{8(1-\alpha)+1/2+\varepsilon}).$$

Without trying to optimize, we take U so that $xU^{-1} = x^{\alpha}U^{8(1-\alpha)+1/2}$, which gives

$$\sum_{n\geq 1}^{\flat} |\lambda_f(n)|^4\psi(n/x) = d_f x\log x + e_f x + O(\Lambda^{5(1-\alpha)+\varepsilon}N^{8(1-\alpha)+\varepsilon}x^{\beta+\varepsilon})$$

with

$$\beta = 1 - \frac{2(1-\alpha)}{16(1-\alpha)+3}.$$

Taking $\alpha = \frac{23}{32}+\varepsilon$, we get (3.4), up to renaming ε.

For holomorphic forms, we proceed in the a slightly different manner. First since we have a nebentypus we use the adjoint square instead of the symmetric square in proving the analogue of (3.3), namely

$$\sum_{n\leq x}^{\flat} |\lambda_f(n)|^2 = c_f x^k + O(x^{k-1/5+\varepsilon}(kN)^{1/2+\varepsilon}) \tag{3.7}$$

with $c_f \gg (\log kN)^{-1}$ (by Goldfeld, Hoffstein and Lieman, see the Appendix to [22]). Secondly we can avoid proving the analogue of (3.4), for which we require only an upper bound, by means of the Ramanujan-Petersson bound (proved by Deligne)

$$|\lambda_f(n)| \leq d(n)n^{(k-1)/2},$$

where $d(n)$ is the divisor function. In fact it is more efficient then to argue with the third power moment, and use Hölder's Inequality with $(p,q) = (3, 2/3)$ for the final estimates:

$$\sum_{x<n\leq x+y}^{\flat} |\lambda_f(n)|^2 \leq \eta|S| + \sum_{x\in L}^{\flat} |\lambda_f(n)|^2 \leq \eta y + |L|^{1/3}\Big(\sum_{x<n\leq x+y}^{\flat} |\lambda_f(n)|^3\Big)^{2/3}. \tag{3.8}$$

We have

$$\sum_{x<n\leq x+y}^{\flat} |\lambda_f(n)|^3 \leq \sum_{x<n\leq x+y} d(n)^3 n^{3(k-1)/2},$$

and estimating this is classical. Here are the main steps for completeness. The generating Dirichlet series for $d(n)^3$ is

$$L_1(s) := \sum_{n\geq 1} d(n)^3 n^{-s} = \zeta(s)^4 \prod_p (1 + 4p^{-s} + p^{-2s}) = \zeta(s)^8 H_2(s)$$

where H_2 is absolutely convergent, hence holomorphic, for $\sigma > \frac{1}{2}$. Say it has coefficients $\alpha(n)$, and $\zeta(s)^8$ has coefficients $d_8(n)$. By [21, Th. 2] we have

$$\sum_{n\leq x} d_8(n) = xP(\log x) + O(x^{5/8+\varepsilon})$$

where P is some polynomial of degree 7. Hence

$$\begin{aligned}\sum_{n\leq x} d(n)^3 &= \sum_{b\leq x} \alpha(b) \sum_{a\leq x/b} d_8(a)\\ &= xP_1(\log x) + O(x^{5/8+\varepsilon})\end{aligned}$$

for some polynomial P_1 of degree 7 since $H_2(5/8)$ is absolutely convergent. By partial summation we get

$$\sum_{n\leq x} d(n)^3 n^{3(k-1)/2} = x^{3(k-1)/2+1} P_2(\log x) + O(x^{3(k-1)/2+5/8+\varepsilon}),$$

hence the result follows using (3.7), (3.8) since $5/8 < 4/5$. □

Remark 2. We see that this method provides n where a lower bound for $\lambda_f(n)$ holds, and this also seems very hard to get by purely algebraic techniques. In applications to analytic number theory, this can be of crucial importance; see for instance [8], [9]. In these papers the question is somewhat different: one needs to find very small n, compared to some large parameter x (say $n \ll x^\varepsilon$), such that $\lambda_f(n)$ is not too small, and this is solved by using the trick of Iwaniec that for any prime $p \nmid N$, we have $\lambda_f(p)^2 - \lambda_f(p^2) = 1$, so one of $\lambda_f(p)$, $\lambda_f(p^2)$ is at least $1/\sqrt{2}$ in absolute value, and p^2 remains small enough for the application in mind.

There is a strong contrast between the proof of Proposition 3, which depends on quite deep analytic properties of L-functions, and the algebraic approach of the previous section, where not even convergence mattered! It is clear that one can extend Proposition 2 to any cuspidal automorphic form on $GL(n)/\mathbb{Q}$ using its Rankin-Selberg convolution (compare [9]), but Proposition 3 requires either that f satisfies the Ramanujan-Petersson conjecture, or that the adjoint square be automorphic (in which case there is also is a bound of the type $|\alpha_p| \leq p^\theta$ with $\theta < \frac{1}{4}$ for the local parameters of f at unramified primes). This is not known for $n \geq 3$.

It is natural to ask if the property in Lemma 2.1 holds for primitive Maass forms. If the eigenvalue is $\lambda = 1/4$, conjecturally the Fourier coefficients still generate a number field, and in this case the proof goes through without change. If $\lambda \neq 1/4$, the field $K_f = \mathbb{Q}(\lambda_f(n), \chi(n))$ is not expected to be a number field. However we still see that if Lemma 2.1 is false for f, then $\mathbb{Q}(\alpha_p, \beta_p, \chi(p)) \cap \mathbb{Q}^{ab}$ is an infinite extension of $\mathbb{Q}$, where $\mathbb{Q}^{ab}$ is the cyclotomic field generated by all roots of unity. This does not sound very likely, as the field generated by the local roots α_p, β_p could be expected to be mostly transcendental, but it is certainly beyond proof or disproof today! (The corresponding fact is true however, for the field $K_t = \mathbb{Q}(2^{it}, 3^{it}, \ldots, p^{it}, \ldots)$ generated by the local roots of the Eisenstein series $E(z, \frac{1}{2} + it)$, for $SL(2, \mathbb{Z})$ say, for all $t \in \mathbb{R}$ except maybe those in a countable set; it doesn't seem easy to decide if the latter is really empty, but this would follow from Schanuel's Conjecture, as observed by B. Poonen).

One is tempted to confront this with the famous "optimistic" question of Katz ([25, p.15]): is

$$L(s) = \prod_p (1 - S(1,1;p)p^{-s} + p^{1-2s})^{-1} = \sum_{n\geq 1} \lambda_S(n) n^{-s}$$

the L-function of a (primitive) Maass form (of weight 2), even up to finitely many factors, where $S(1,1;p)$ denotes the usual Kloosterman sums? Note that $S(1,1;p)$ generates the maximal real subfield of the field of p-th roots of unity, so in this case the field generated by $\lambda_S(p)$ is an infinite algebraic extension of $\mathbb{Q}$. However we can prove the analogue of Lehmer's conjecture for this Dirichlet series! (Of course, the answer to Katz's question is widely expected to be "No", see [5] for some strong evidence).

Proposition 4. *For any $n \geq 1$, we have $\lambda_S(n) \neq 0$.*

Proof. We give two proofs (suggested by Katz and simpler than our original argument). We need to show that $\lambda_S(p^\nu) \neq 0$ for p prime and $\nu \geq 0$. For the first argument, consider the Euler factor at p as a rational function of $X = p^{-s}$ with coefficients in the cyclotomic field $\mathbb{Q}(e(1/p))$. It is congruent (modulo the ideal generated by p) to

$$\frac{1}{1 - S(1,1;p)X} = \sum_\nu S(1,1;p)^\nu X^\nu.$$

Thus the result follows from the well-known fact that $S(1,1;p)$ is non-zero modulo p, in fact we have

$$S(1,1;p) \equiv -1 \pmod{1 - e(1/p)},$$

and the prime ideal $1 - e(1/p)$ divides p.

For the other argument, notice that since the form of the Euler product is the same as for a holomorphic form of weight 2, we must show that α_p/β_p is not a root of unity, where α_p and β_p satisfy

$$\alpha_p + \beta_p = S(1,1;p) \quad \text{and} \quad \alpha_p\beta_p = p.$$

Hence the product $\alpha_b\beta_p$ is divisible by $1 - e(1/p)$, whereas by the congruence above, the sum is invertible modulo $1 - e(1/p)$. This means one of α_p, β_p must also be invertible while the other is not, which implies that the ratio α_p/β_p is not a p-unit, hence not a root of unity. □

It is probably possible to derive a fancy proof of this proposition (more amenable to generalizations, if desired) using ideas as in [14], Lemma 4.9, applied to some Kloosterman/Gauss sum sheaves on $\mathbb{G}_m/\mathbb{F}_p$ with traces of Frobenius at $\alpha \in \mathbb{G}_m(\mathbb{F}_p)$ given by both sides of (3.9). Note also that if $\nu \geq 1$ and p is odd we do have (see e.g. [23, Lemma 4.1])

$$S(1,1;p^{2\nu}) = p^\nu\Big(e\Big(\frac{2}{p^{2\nu}}\Big) + e\Big(\frac{-2}{p^{2\nu}}\Big)\Big),$$

so Proposition 4 is special to Kloosterman sums with prime modulus.

§ 4. Applications of 𝔅-free numbers

We now come to the technical heart of this paper where we consider the original question of proving (1.5) for a cusp form $f \in S_k(N, \chi)$, not in the space spanned by CM forms. Recall that Balog and Ono [2] proved (1.5) for $y = x^{17/41+\varepsilon}$, $\varepsilon > 0$ being arbitrary. It is interesting to look for smaller exponents, in particular since it is natural to expect that $y = x^{\varepsilon}$ should be sufficient. (By a result of Plaksin [35] on $\mathfrak{B}$-free numbers, this is true for almost all n). For one very natural f, namely the Ramanujan Δ function with coefficients $\tau(n)$, a famous conjecture of Lehmer [29] says that $\tau(n) \neq 0$ for any $n \geq 1$.

Since this problem seems very difficult, approaching it by means of conditional statements based on solid conjectures is also desirable. Very recently Alkan [1] gave two such results: he showed that the exponent 17/41 can be reduced to 69/169 and 1/3 ([1], Theorems 3 and 4) under the generalised Riemann hypothesis (GRH) for Dedekind zeta-function and the Lang-Trotter conjecture [28], respectively.

We will prove a number of results improving the previously known statements, both conditional and unconditional. The following is a general bound, where we recall that $\mathfrak{P}_{f,1}$ is defined in (2.6):

Theorem 1. *Suppose that $k \geq 2$ and $f \in S_k^*(N, \chi)$ is a primitive form not of CM type such that*

$$\big|\mathfrak{P}_{f,1} \cap [1, x]\big| \ll_f x^{\rho} \frac{(\log\log x)^{\Psi_\rho}}{(\log x)^{\Theta_\rho}} \qquad (x \geq 2), \tag{4.1}$$

where $\rho \in [0, 1]$ and Θ_ρ, Ψ_ρ are real constants such that $\Theta_1 > 1$. Define

$$\theta(\rho) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq \rho \leq \frac{1}{3}, \\ \frac{10\rho}{19\rho+7} & \text{if } \frac{1}{3} < \rho \leq \frac{9}{17}, \\ \frac{3\rho}{4\rho+3} & \text{if } \frac{9}{17} < \rho \leq \frac{15}{28}, \\ \frac{5}{16} & \text{if } \frac{15}{28} < \rho \leq \frac{5}{8}, \\ \frac{22\rho}{24\rho+29} & \text{if } \frac{5}{8} < \rho \leq \frac{9}{10}, \\ \frac{7\rho}{9\rho+8} & \text{if } \frac{9}{10} < \rho \leq 1, \end{cases}$$

For every $\varepsilon > 0$, $x \geq x_0(f, \varepsilon)$ and $y \geq x^{\theta(\rho)+\varepsilon}$, we have

$$|\{n \mid x < n \leq x + y \text{ and } \lambda_f(n) \neq 0\}| \gg_{f,\varepsilon} y. \tag{4.2}$$

In particular for any $\varepsilon > 0$ and all $n \geq 1$, we have

$$i_f(n) \ll_{f,\varepsilon} n^{\theta(\rho)+\varepsilon}. \tag{4.3}$$

Theorem 1 follows immediately by multiplicativity from Corollary 10 below which gives a more effective treatment for $\mathfrak{B}$-free numbers in short intervals, applied with

$$\mathfrak{P} = \{p \mid p \mid N \text{ or } \lambda_f(p) = 0\}.$$

The new ideas and new ingredients will be explained in § 5.

According to (1.2), the hypothesis (4.1) holds with $(\rho, \Theta_\rho, \Psi_\rho) = (1, 1 + \delta, 0)$ for any $\delta < \frac{1}{2}$. Thus, applying this result and Lemma 2.4, we immediately obtain an improvement of the result of Balog and Ono.

Corollary 1. *Suppose that $k \geq 2$ and $f \in S_k(N,\chi)$ is not in the space spanned by CM forms. Then for any $\varepsilon > 0$, $x \geq x_0(f,\varepsilon)$ and $y \geq x^{7/17+\varepsilon}$, we have*

$$|\{n \mid x < n \leq x+y \text{ and } \lambda_f(n) \neq 0\}| \gg_{f,\varepsilon} y.$$

In particular

$$i_f(n) \ll_{f,\varepsilon} n^{7/17+\varepsilon}.$$

In proving this we do not exploit Lemma 2.1 (so we could claim that we obtain the correct proportion of squarefree numbers if f is primitive). It can be used to simplify the proof, as we'll see, but it does not influence the strength of the exponent. This is mainly due to the fact that we have $\rho = 1$, and when ρ is close to 1 we do not succeed in getting better results by not imposing the numbers to be squarefree.

However, if one can get ρ quite small, e.g. smaller than the current best results about squarefree numbers in short intervals (see [13]), it is clear that using Lemma 2.1 will yield an improvement. So consider the set of prime numbers

$$\mathfrak{P}_f^* := \{p \mid p|N\} \cup \bigcup_{\nu=1}^{\infty} \mathfrak{P}_{f,\nu},$$

where as before

$$\mathfrak{P}_{f,\nu} = \{p \mid p \nmid N \text{ and } \lambda_f(p^\nu) = 0\}.$$

Clearly $\lambda_f(n) \neq 0$ (for $(n,N)=1$) if and only if n is $\mathfrak{P}_f^*$-free. We then have the following result:

Theorem 2. *Assume that $k \geq 2$, $f \in S_k^*(N,\chi)$ is not a CM form, and that*

$$\left|\mathfrak{P}_f^* \cap [1,x]\right| \ll_f x^\rho \frac{(\log\log x)^{\Psi_\rho}}{(\log x)^{\Theta_\rho}} \qquad (x \geq 2), \tag{4.4}$$

where $\rho \in [0,1]$ and Θ_ρ, Ψ_ρ are real constants such that $\Theta_1 > 1$. Then the inequalities (4.2) and (4.3) hold with $\theta(\rho) = \rho/(1+\rho)$ for $0 \leq \rho \leq 1$.

This theorem gives a better exponent than Theorem 1 when $\rho \leq \frac{1}{3}$ under a slightly stronger hypothesis than (4.1). However recall from Lemma 2.2 that the hypotheses (4.1) and (4.4) are in fact equivalent when k is even. It is of course particularly interesting that this new exponent tends towards 0 when $\rho \to 0$. As for Theorem 1, this result follows directly by multiplicativity from the corresponding result for $\mathfrak{B}$-free numbers, Proposition 9 below, where this time $\mathfrak{P} = \mathfrak{P}_f^*$.

Another consequence of Lemma 2.3 and Corollary 10 is an extension to all symmetric powers:

Corollary 2. *Let $k \geq 2$ and $f \in S_k^*(N,\chi)$ which is not a CM form. Let $m \geq 1$ and define the unramified m-th symmetric power L-function of f by*

$$L_{nr}(\mathrm{Sym}^m f, s) = \prod_{p \nmid N} \prod_{0 \leq j \leq m} (1 - \alpha_p^j \beta_p^{m-j} p^{-s})^{-1} = \sum_{n \geq 1} \lambda_f^{(m)}(n) n^{-s}.$$

Then for any $\varepsilon > 0$, $x \geq x_0(f,\varepsilon)$ and $y \geq x^{7/17+\varepsilon}$, we have

$$|\{n \mid x < n \leq x+y \text{ and } \lambda_f^{(m)}(n) \neq 0\}| \gg_{f,m,\varepsilon} y,$$

and in particular $i_{\mathrm{Sym}^m f}(n) \ll_{f,\varepsilon,m} n^{7/17+\varepsilon}$ *for* $n \geq 1$.

Proof. For $p \nmid N$ prime, we have $\lambda_f^{(m)}(p) = \lambda_f(p^m)$. Hence by Lemma 2.3 we derive

$$|\{p \leq x \mid p \nmid N \text{ and } \lambda_f^{(m)}(p) = 0\}| \ll x(\log x)^{-1-\delta}$$

for any $\delta < \frac{1}{2}$. By multiplicativity and Corollary 10 below, the result follows. □

Note we do not need the automorphy of $\mathrm{Sym}^m f$ (which is known only for $m \leq 4$).

The hypothesis (4.1) is known only with $\rho = 1$, with the one exception of primitive forms $f \in S_2^*(N)$ with integral coefficients. Those are associated to elliptic curves over $\mathbb{Q}$, and Elkies [10] has proved that (4.1) (or (4.4)) holds with $\rho = 3/4$, $\Theta = \Psi = 0$. Theorem 1 is still better for this value of ρ than Theorem 2 and we get:

Corollary 3. *Let* $E/\mathbb{Q}$ *be an elliptic curve without complex multiplication and let* f *be the associated primitive form. Then for every* $\varepsilon > 0$, $x \geq x_0(E, \varepsilon)$ *and* $y \geq x^{33/94+\varepsilon}$, *we have*

$$|\{n \mid x < n \leq x + y \text{ and } \lambda_f(n) \neq 0\}| \gg_{E,\varepsilon} y.$$

In particular for any $\varepsilon > 0$ *and all* $n \geq 1$, *we have*

$$i_f(n) \ll_{E,\varepsilon} n^{33/94+\varepsilon}.$$

This improves Theorem 2 of [1], which requires 69/169 in place of 33/94.

Some well-known conjectures imply that (4.1) holds for smaller values of ρ. For example, Serre ([43, $(182)_{\mathrm{R}}$]) showed that the GRH for Dedekind zeta-functions implies (4.1) with $(\rho, \Theta_\rho, \Psi_\rho) = (\frac{3}{4}, 0, 0)$. Lang and Trotter [28] formulated a conjecture for the size of the set $\mathfrak{P}_{f,1}$, in the case where f is associated to an elliptic curve over $\mathbb{Q}$. This, if true, implies for these forms an estimate (4.1) with $(\rho, \Theta_\rho, \Psi_\rho) = (\frac{1}{2}, 1, 0)$. Generalizations of the Lang-Trotter conjecture (see e.g. Murty's version [33], especially Conjecture 3.4) imply that if $k \geq 2$ and $f \in S_k^*(N, \chi)$ is not of CM type, then we have (4.1) with

$$(\rho, \Theta_\rho, \Psi_\rho) = \begin{cases} (\frac{1}{2}, 1, 0) & \text{if } k = 2 \text{ and } [F_f : \mathbb{Q}] = 2, \\ (0, 0, 1) & \text{if } k = 2 \text{ and } [F_f : \mathbb{Q}] = 3 \\ & \text{or } k = 3 \text{ and } [F_f : \mathbb{Q}] = 2, \\ (0, 0, 0) & \text{otherwise,} \end{cases} \tag{4.5}$$

where F_f is the stable trace field (see § 2 and § 3 of [33]).

Applying Theorem 1, we get the following conditional result, which improves Theorem 1 of [1].

Corollary 4. *Suppose that* $k \geq 2$ *and* $f \in S_k(N, \chi)$ *is not in the space spanned by CM forms.*

(i) *Under the GRH for Dedekind zeta-function, the exponent* 7/17 *of Corollary 1 can be further improved to* 33/94.

(ii) *Under the generalized Lang-Trotter conjecture, the exponent* 7/17 *can be further improved to* 10/33 *if* $k = [F_f : \mathbb{Q}] = 2$, *and to* 1/4 *otherwise.*

We can apply Theorem 2 instead if f satisfies the assumptions of Lemma 2.2, but it is just as simple to extend the Lang-Trotter type conjectures to deal with the sets $\mathfrak{P}_{f,\nu}$ for any $\nu \geq 1$. The heuristics which lead to these conjectures, based on Deligne's estimate $|\alpha_p| = |\beta_p| = p^{(k-1)/2}$, suggest the following:

Conjecture 1. *Let $\nu \geq 1$ be any integer. If $k \geq 2$ and $f \in S_k^*(N,\chi)$ is not of CM type, then*

$$\left|\mathfrak{P}_{f,\nu} \cap [1,x]\right| \ll_f x^{\rho} \frac{(\log\log x)^{\Psi_\rho}}{(\log x)^{\Theta_\rho}} \qquad (x \geq 2)$$

with $(\rho,\Theta_\rho,\Psi_\rho) = (\frac{1}{2},1,0)$ if $k=2$, $(0,0,1)$ if $k=3$ and $(0,0,0)$ if $k \geq 4$.

We only state upper bounds, but one could propose a more precise statement, which involves looking at the possibility of f having "extra twists" and eliminating the all but finitely many ν for which $\mathfrak{P}_{f,\nu}$ is empty. About this conjecture, recall that even under GRH, one can not get a better general result towards the Lang-Trotter conjecture than

$$\left|\mathfrak{P}_{f,1} \cap [1,x]\right| \ll_f x^{3/4}$$

for f of weight $k \geq 2$. The exponent is the same for all weights, so this gets worse (compared to what we expect) as k grows. In particular, this conjecture for $k \geq 3$ seems hopeless for the time being. Lemma 2.1 implies:

Corollary 5. *Let $k \geq 2$ and $f \in S_k^*(N,\chi)$ not of CM type. Assuming Conjecture 1 for f, the inequality (4.4) holds with $(\rho,\Theta_\rho,\Psi_\rho)$ given by $(\rho,\Theta_\rho,\Psi_\rho) = (\frac{1}{2},1,0)$ if $k=2$, $(0,0,1)$ if $k=3$ and $(0,0,0)$ if $k \geq 4$.*

As applications (or cautionary tale...), here are some very impressive-looking results.

Corollary 6. *Suppose that $k \geq 3$ and $f \in S_k(N,\chi)$ is not in the space spanned by CM type. If Conjecture 1 holds for all primitive forms, then the exponent 7/17 of Corollary 1 can be improved to 0. If $k \geq 4$ and f is primitive, then there exists $M \geq 1$ such that $(n,M)=1$ implies $\lambda_f(n) \neq 0$.*

Specializing to the Ramanujan τ-function, which is integer valued, Lemma 2.2 allows us to deduce the following result (implicit in [43]):

Corollary 7. *Assume Conjecture 1, or equivalently the generalized Lang-Trotter conjecture, for $f = \Delta \in S_{12}^*(1)$. There exists $P \geq 1$ such that $\tau(n) = 0$ if and only if (n,P^∞) is a square, i.e. if and only if $v_p(n)$ is even for $p \mid P$. In particular $i_\Delta(n) \leq P$ for $n \geq 1$, and for all $x \geq 2$ and $y \geq 1$, we have*

$$|\{n \mid x < n \leq x+y \text{ and } \tau(n) \neq 0\}| = \prod_{p \mid P}\Big(1+\frac{1}{p}\Big)^{-1} y + O((\log(x+y))^{\omega(P)}) \geq \frac{\varphi(P)}{P} y + O(1)$$

where the implied constant is absolute and $\omega(P)$ is the number of prime divisors of P.

Proof. The first statement is the rephrasing of Lemma 2.2 and Conjecture 1 in this case. Notice that $\tau(n) \neq 0$ if $(n,P)=1$ so $i_\Delta(n) \leq P$ follows (an interval of length P contains elements prime to P) as does the last inequality by trivial counting. For the asymptotic, write

$$|\{n \leq x \mid \tau(n) \neq 0\}| = \sum_{\substack{d \mid P^\infty \\ d \leq x}} \lambda(d) \sum_{n \leq x/d} 1$$

where $\lambda(n)$ is the Liouville function, i.e. $\lambda(p^k) = (-1)^k$. Since

$$\sum_{d \mid P^\infty} \frac{\lambda(d)}{d} = \prod_{p \mid P} (1+p^{-1})^{-1},$$

we get the result after elementary estimates. □

This is of course trivial and of little practical significance towards the Lehmer conjecture.

§ 5. Multiple exponential sums and bilinear forms

This section is devoted to the study of multiple exponential sums and bilinear forms, which will be used in the proofs of our results on $\mathfrak{B}$-free numbers in the next sections, but are also of independent interest. We begin by investigating a double exponential sum of type II:

$$S(M,N) := \sum_{m\sim M}\sum_{n\sim N} \varphi_m\psi_n e\bigg(X\frac{m^\alpha n^\beta}{M^\alpha N^\beta}\bigg),$$

where $e(t) := \exp\{2\pi i t\}$, $X > 0$, $M \geq 1$, $N \geq 1$, $|\varphi_m| \leq 1$, $|\psi_n| \leq 1$, $\alpha, \beta \in \mathbb{R}$ and $m \sim M$ means $M \leq m < 2M$. Such a sum occurs in many arithmetic problems and is studied by many authors (for example, [15] and [40]). We shall estimate this sum by the method of Fouvry & Iwaniec [15] together with the refinement of Robert & Sargos [39]. When $X < N^2$, we need to use an idea in [40].

The following result is an improvement of Theorem 4 in [15] and Theorem 10 in [40].

Proposition 5. *If $\alpha, \beta \in \mathbb{R}\smallsetminus\{0,1\}$, then for any $\varepsilon > 0$ we have*

$$S(M,N) \ll \big\{(XM^6N^6)^{1/8} + M^{1/2}N + MN^{3/4} + X^{-1/2}MN\big\}(MN)^\varepsilon.$$

Proof. We shall distinguish two cases.

A. *The case of $X \geq N^2$*

By applying twice the Cauchy-Schwarz' inequality, it follows that

$$|S(M,N)|^4 \leq (MN)^2 \sum_{n_1\sim N}\sum_{n_2\sim N}\sum_{m_1\sim M}\sum_{m_2\sim M} e\bigg(X\frac{(m_1^\alpha - m_2^\alpha)(n_1^\beta - n_2^\beta)}{M^\alpha N^\beta}\bigg).$$

The double large sieve inequality ([15], Proposition 1) with the choice of

$$\mathcal{X} = \big\{(m_1^\alpha - m_2^\alpha)/M^\alpha\big\}_{m_1,m_2\sim M} \quad \text{and} \quad \mathcal{Y} = \big\{(n_1^\beta - n_2^\beta)/N^\beta\big\}_{n_1,n_2\sim N}$$

leads to the following estimate

$$|S(M,N)|^8 \ll X(MN)^4\mathcal{N}(M,1/X)\mathcal{N}(N,1/X), \tag{5.1}$$

where $\mathcal{N}(M,\Delta)$ is the number of quadruplets $(m_1,m_2,m_3,m_4) \in \{M+1,\dots,2M\}^4$ satisfying

$$\big|m_1^\alpha + m_2^\alpha - m_3^\alpha - m_4^\alpha\big| \leq \Delta M^\alpha.$$

According to Theorem 2 of [39], we have

$$\mathcal{N}(M,1/X) \ll \big(M^2 + X^{-1}M^4\big)M^\varepsilon.$$

Inserting this into (5.1) and simplifying the estimate obtained by using the hypothesis $X \geq N^2$, we find that

$$S(M,N) \ll \big\{(XM^6N^6)^{1/8} + MN^{3/4}\big\}(MN)^\varepsilon.$$

B. *The case of $X \leq N^2$*

By Lemma 2.1 of [40], we deduce that, for any $Q \in [1, M^{1-\varepsilon}]$,

$$|S(M,N)|^2 \ll (MN)^2Q^{-1} + MNQ^{-1}(\log M) \max_{1\le Q_1\le Q} |S(Q_1)|, \tag{5.2}$$

where

$$S(Q_1) := \sum_{q\sim Q_1}\sum_{m\sim M} \varphi_{m,q} \sum_{n\sim N} e\bigg(X'\frac{t(m,q)n^\beta}{TN^\beta}\bigg) \tag{5.3}$$

and

$$t(m,q) := (m+q)^\alpha - m^\alpha, \qquad T := M^{\alpha-1}Q_1, \qquad X' := XM^{-1}Q_1.$$

If $N' := X'/N \ge \frac{1}{2}$, applying Lemma 2.2 of [40] to the sum over n yields

$$\sum_{n\sim N} e\bigg(X'\frac{t(m,q)n^\beta}{TN^\beta}\bigg) \ll X'^{-1/2}N \sum_{n'\in I(m,q)} w_{n'} e\bigg(\tilde\beta X'\frac{u(m,q){n'}^{\beta_1}}{UN'^{\beta_1}}\bigg) + R_1 + R_2 + \log N, \tag{5.4}$$

where

$$I(m,q) := \big[c_1Xt(m,q)M^{-\alpha}N^{-1},\, c_2Xt(m,q)M^{-\alpha}N^{-1}\big],$$
$$R_j := \min\big\{X'^{-1/2}N,\, 1/\|c_j'XM^{-\alpha}N^{-1}t(m,q)\|\big\},$$

$u(m,q) := t(m,q)^{1/(1-\beta)}$, $U := T^{1/(1-\beta)}$, $\beta_1 := \beta/(\beta-1)$, $\tilde\beta := |1-\beta||\beta|^{-\beta_1}$, $|w_{n'}| \le 1$, and $c_j = c_j(\beta)$, $c_j' = c_j'(\beta)$ are some suitable constants. Inserting into (5.3), using Lemma 2.5 of [40] to eliminate multiplicative restrictions and using Lemma 2.3 of [40] with $n = m$ to estimate the related error terms, we find

$$S(Q_1) \ll X'^{-1/2}N \int_{-\infty}^{+\infty} \Xi(r)S(Q_1,r)\,\mathrm{d}r + \big\{(XM^{-1}Q_1^3)^{1/2} + MQ_1\big\}(MN)^\varepsilon,$$

where $\Xi(r) := \max\{M, (\pi r)^{-1}, (\pi r)^{-2}\}$, $\psi_{n'}(r) := w_{n'}e(rn')$ and

$$S(Q_1,r) := \sum_{q\sim Q_1}\sum_{m\sim M}\bigg|\sum_{n'\sim N'} \psi_{n'}(r)\, e\bigg(\tilde\beta X'\frac{u(m,q){n'}^{\beta_1}}{UN'^{\beta_1}}\bigg)\bigg|.$$

If $X'/N \le \frac{1}{2}$, the Kusmin-Landau inequality (see e.g. [20], Theorem 2.1) implies

$$S(Q_1) \ll X'^{-1}MNQ_1.$$

Thus we always have

$$\begin{aligned} S(Q_1) \ll{}& X'^{-1/2}N \int_{-\infty}^{+\infty} \Xi(r)S(Q_1,r)\,\mathrm{d}r \\ &+ \big\{(XM^{-1}Q_1^3)^{1/2} + MQ_1 + X'^{-1}MNQ_1\big\}(MN)^\varepsilon, \end{aligned} \tag{5.5}$$

Now by applying Cauchy-Schwarz' inequality, it follows that

$$|S(Q_1,r)|^2 \le MQ_1 \sum_{q\sim Q_1}\sum_{m\sim M}\sum_{n_1'\sim N'}\sum_{n_2'\sim N'} \psi_{n_1'}(r)\overline{\psi_{n_2'}(r)}e\bigg(\tilde\beta X'\frac{u(m,q)\big({n_1'}^{\beta_1} - {n_2'}^{\beta_1}\big)}{UN'^{\beta_1}}\bigg).$$

The double large sieve inequality with the choice of

$$\mathcal{X} = \{u(m,q)/U\}_{m\sim M, q\sim Q_1} \quad \text{and} \quad \mathcal{Y} = \{({n'_1}^{\beta_1} - {n'_2}^{\beta_1})/{N'}^{\beta_1}\}_{n_1,n_2\sim N'}$$

allows us to deduce

$$|S(Q_1,r)|^4 \ll (MQ_1)^2 X' \mathcal{N}^*(u,U;1/X')\mathcal{N}(N';1/X') \tag{5.6}$$

uniformly for $r\in\mathbb{R}$, where $\mathcal{N}^*(u,U;\Delta)$ is the number of quadruplets $(m_1+q_1, m_2+q_2, m_1, m_2)$ such that $m_1, m_2\sim M$, $q_1,q_2\sim Q_1$ and

$$\big|u(m_1,q_1)-u(m_2,q_2)\big| \le \Delta U,$$

and $\mathcal{N}(N;\Delta)$ is the number of quadruplets $(n_1,n_2,n_3,n_4)\in\{N+1,\dots,2N\}^4$ satisfying

$$\big|n_1^\beta + n_2^\beta - n_3^\beta - n_4^\beta\big| \le \Delta N^\beta.$$

Since

$$\big|u(m_1,q_1)-u(m_2,q_2)\big| \asymp \big|t(m_1,q_1)-t(m_2,q_2)\big| T^{\beta/(1-\beta)},$$

we have, for some suitable constant $C>0$,

$$\mathcal{N}^*(u,U;\Delta) = \mathcal{N}^*(t,T;C\Delta).$$

Noticing that

$$\big|t(m_1,q_1)-t(m_2,q_2)\big| = \big|(m_1+q_1)^\alpha - m_1^\alpha - (m_2+q_2)^\alpha - m_2^\alpha\big|$$

and $(m_1+q_1, m_1, m_2+q_2, m_2)\in\{M+1,\dots,3M\}^4$, clearly we have

$$\mathcal{N}^*(t,T;C\Delta) \ll \mathcal{N}(M;C\Delta).$$

Thus Theorem 2 of [39] implies that

$$\begin{aligned}\mathcal{N}^*(u,U;1/X') &\ll \big(M^2 + X'^{-1}M^4\big)M^\varepsilon,\\ \mathcal{N}(N',1/X') &\ll \big(N'^2 + X'^{-1}N'^4\big)N'^\varepsilon.\end{aligned}$$

Inserting these into (5.6), we obtain uniformly for $r\in\mathbb{R}$,

$$|S(Q_1,r)|^4 \ll \{X'M^4(N'Q_1)^2 + (MN')^4Q_1^2 + M^6(N'Q_1)^2 + X'^{-1}M^6N'^4Q_1^2\}(MN)^\varepsilon.$$

Combining this with (5.5), we find that

$$\begin{aligned}S(Q_1) \ll \{&(XM^3N^2Q_1^3)^{1/4} + (XMQ_1^2)^{1/2} + (M^3NQ_1)^{1/2}\\ &+ (XM^5Q_1^3)^{1/4} + (XM^{-1}Q_1^3)^{1/2} + MQ_1 + X'^{-1}MNQ_1\}(MN)^\varepsilon.\end{aligned}$$

Since $Q\le M^{1-\varepsilon}$, the fifth and sixth terms on the right-hand side are superfluous. Inserting the simplified estimate into (5.2) and taking $Q=M^{1-\varepsilon}$, we find

$$\begin{aligned}|S(M,N)|^2 \ll \{&MN^2 + (XM^6N^6)^{1/4} + (XM^3N^2)^{1/2}\\ &+ (M^4N^3)^{1/2} + (XM^8N^4)^{1/4} + X^{-1}(MN)^2\}(MN)^\varepsilon.\end{aligned} \tag{5.7}$$

Similarly by interchanging the role of M and N, we also have

$$|S(M,N)|^2 \ll \big\{M^2N + (XM^6N^6)^{1/4} + (XM^2N^3)^{1/2} + (M^3N^4)^{1/2} + (XM^4N^8)^{1/4} + X^{-1}(MN)^2\big\}(MN)^{\varepsilon}. \tag{5.8}$$

Now the required estimate follows from (5.7) if $X \le N^2$ and $M \le N$, and from (5.8) when $X \le N^2$ and $M > N$. This completes the proof. □

Next as an application of Proposition 5, we consider a particular triple exponential sum of type I:

$$S_I(H,M,N) := \sum_{h\sim H}\sum_{m\in I}\sum_{n\sim N} \xi_h\psi_n e\bigg(X\frac{h^{\beta}m^{-\beta}n^{\alpha}}{H^{\beta}M^{-\beta}N^{\alpha}}\bigg),$$

where $X > 0$, $H \ge 1$, $M \ge 1$, $N \ge 1$, $|\xi_h| \le 1$, $|\psi_n| \le 1$ and I is a subinterval of $[M, 2M]$.

Corollary 8. *Let $\alpha, \beta \in \mathbb{R}$ satisfy $\beta \ne -1, 0$ and $\alpha/(1+\beta) \ne 0, 1$. For any $\varepsilon > 0$, we have*

$$S_I(H,M,N) \ll \big\{(X^3H^6M^2N^6)^{1/8} + (XH^2N)^{1/2} + HN + (XH^3M)^{1/4}N + X^{-1}HMN\big\}(HMN)^{\varepsilon}, \tag{5.9}$$

$$S_I(H,M,N) \ll \big\{(X^{\kappa+\lambda}H^{1+\kappa+\lambda}M^{1+\kappa-\lambda}N^{2+\kappa})^{1/(2+2\kappa)} + (XH^2N)^{1/2} + (HM)^{1/2}N + HN + X^{-1}HMN\big\}(HMN)^{\varepsilon}, \tag{5.10}$$

where (κ, λ) is an exponent pair.

Proof. If $X/M \le \frac{1}{2}$, the Kusmin-Landau inequality implies

$$S_I(H,M,N) \ll X^{-1}HMN.$$

When $X/M > \frac{1}{2}$, applying Lemma 2.2 of [40] to the sum over m and using Lemma 2.3 of [40] with $n = n$ to estimate the related error terms, we find

$$S_I(H,M,N) \ll X^{-1/2}MS' + (HN + X^{1/2}H)\log M,$$

where

$$S' := \sum_{n\sim N}\sum_{h\sim H}\sum_{m'\in I'(h,n)} \widetilde{\psi}_n\widetilde{\xi}_h\varphi_{m'} e\bigg(\widetilde{\alpha}X\frac{h^{\beta'}m'^{\beta'}n^{\alpha'}}{H^{\beta'}M'^{\beta'}N^{\alpha'}}\bigg),$$

where $I'(h,n)$ is a subinterval of $[M', 2M']$ with $M' := X/M$, $\beta' := \beta/(1+\beta)$, $\alpha' := \alpha/(1+\beta)$, $\widetilde{\alpha} := |1+\beta||\beta|^{\beta'}$, $|\widetilde{\xi}_h| \le 1$, $|\varphi_{m'}| \le 1$ and $|\widetilde{\psi}_n| \le 1$. Noticing that the exponents of h and m' are equal, we can express this new triple sum as a double exponential sum over (h', n) with $h' = hm' \in hI'(h,n)$. We use Lemma 2.5 of [40] to relax the condition $h' = hm' \in hI'(h,n)$ to $h' \sim H' := HM' = XH/M$. Finally applying Proposition 5 with $(M,N) = (N,H')$ yields the desired estimate (5.9). The last inequality follows from (3.11) of [30] with the choice of $(H,M,N) = (H',1,N)$. This completes the proof. □

Finally we study bilinear form of type I:

$$\sum_{m\sim M}\sum_{n\sim N} \psi_n\, r_{mn}(x,y), \tag{5.11}$$

where $|\psi_n| \le 1$ and

$$r_d(x,y) := \sum_{\substack{x<n\le x+y \\ d\,|\,n}} 1 - \frac{y}{d}. \tag{5.12}$$

In the sequel, ε denotes an arbitrarily small positive number and ε' a constant multiple of ε, which may be different in each occurrence.

Corollary 9. *Let $y := x^\theta$ and $|\psi_n| \le 1$. Then for any $\varepsilon > 0$ we have*

$$\sum_{m\sim M}\sum_{n\sim N}\psi_n\, r_{mn}(x,y) \ll_\varepsilon yx^{-\varepsilon} \tag{5.13}$$

provided one of the following two conditions holds

$$\begin{cases} \frac{1}{3} < \theta \le \frac{5}{11},\\ N \le y^{9/4}x^{-3/4-\varepsilon'},\\ MN \le x^{1-\varepsilon'}, \end{cases} \tag{5.14}$$

or

$$\begin{cases} (\kappa+\lambda)/(1+2\kappa+2\lambda) < \theta \le (\kappa+\lambda)/(2\kappa+\lambda),\\ N \le y^{(1+2\kappa+2\lambda)/(1+\lambda)}x^{-(\kappa+\lambda)/(1+\lambda)-\varepsilon'},\\ MN \le x^{1-\varepsilon'}. \end{cases} \tag{5.15}$$

Proof. Without loss of generality, we can suppose that $MN \ge yx^{-\varepsilon}$. By applying (5.9) of Corollary 8, we see that

$$\sum_{h\sim H}\sum_{m\sim M}\sum_{n\sim N}\psi_n\, e\Big(\frac{xh}{mn}\Big) \ll MNx^{-2\varepsilon},$$

provided

$$\tfrac{1}{3} < \theta \le \tfrac{5}{11}, \qquad H \le MNy^{-1}x^{3\varepsilon}, \qquad N \le y^{9/4}x^{-3/4-\varepsilon'}, \qquad MN \le x^{1-\varepsilon'}.$$

Combining this with Lemma 9 of [46] with the choice of $\varphi_m \equiv 1$, we deduce (5.13) provided (5.14) holds. The other one can be proved by using (5.10) of Corollary 8. □

A particular case of (5.11) – linear forms (with $N = 1$) – will be needed in the proof of Corollary 10.

Lemma 5.1. *Let $y := x^\theta$. Then for any $\varepsilon > 0$ we have*

$$\sum_{m\sim M} r_m(x,y) \ll_\varepsilon yx^{-\varepsilon} \tag{5.16}$$

provided one of the following two conditions holds

$$\tfrac{1}{4} < \theta \le \tfrac{9}{29} \qquad \text{and} \qquad M \le y^{19/7}x^{-3/7-\varepsilon'}; \tag{5.17}$$

$$\tfrac{9}{29} < \theta \le \tfrac{1}{2} \qquad \text{and} \qquad M \le y^{4/3}x^{-\varepsilon'}. \tag{5.18}$$

Proof. Without loss of generality, we can suppose that $M \ge yx^{-\varepsilon}$. Theorem 1 of [38] allows us to write

$$\begin{aligned}\sum_{h\sim H}\bigg|\sum_{m\sim M} e\Big(\frac{xh}{m}\Big)\bigg| &\ll \big\{(x^3H^{19}M^6)^{1/18} + (xH^6M)^{1/5} + HM^{3/4} + (x^{-1}H^2M^4)^{1/3}\big\}M^\varepsilon\\ &\ll Mx^{-2\varepsilon},\end{aligned}$$

provided one of the following two conditions holds

$$\tfrac{1}{4} < \theta \le \tfrac{9}{29}, \qquad H \le My^{-1}x^{3\varepsilon}, \qquad M \le y^{19/7}x^{-3/7-\varepsilon'}$$

or

$$\tfrac{9}{29} < \theta \le \tfrac{1}{2}, \qquad H \le My^{-1}x^{3\varepsilon}, \qquad M \le y^{4/3}x^{-\varepsilon'}.$$

This implies (5.16) if (5.17) or (5.18) holds. □

§ 6. $\mathfrak{B}$-free numbers in short intervals

In this section we explain our new results about $\mathfrak{B}$-free numbers. The notion of $\mathfrak{B}$-free numbers, introduced by Erdős [11], is a generalisation of square-free integers. More precisely, let

$$\mathfrak{B} = \{b_k \mid 1 < b_1 < b_2 < \cdots\}$$

be an infinite sequence of integers such that

$$\sum_{k=1}^{\infty} \frac{1}{b_k} < \infty \qquad \text{and} \qquad (b_j, b_k) = 1 \quad (j \neq k). \tag{6.1}$$

The $\mathfrak{B}$-free numbers are the integers that are divisible by no element of $\mathfrak{B}$. We already mentioned that the existence of $\mathfrak{B}$-free numbers in short intervals was proved by Erdős [11], who showed that there is a constant $\theta \in (0,1)$ such that the short interval $(x, x+x^{\theta}]$ with x sufficiently large contains $\mathfrak{B}$-free numbers. Szemerédi [44] showed that $\theta = \frac{1}{2} + \varepsilon$ is admissible. This result was further improved to

$\theta = \frac{9}{20} + \varepsilon$	by Bantle & Grupp [3],
$\theta = \frac{5}{12} + \varepsilon$	by Wu [45],
$\theta = \frac{17}{41} + \varepsilon$	by Wu [46],
$\theta = \frac{33}{80} + \varepsilon$	by Wu [47] and by Zhai [48] (independently),
$\theta = \frac{40}{97} + \varepsilon$	by Sargos & Wu [40].

Inserting our new result on bilinear form ((5.14) of Corollary 9) into the argument of [46], we immediately obtain a slightly better exponent.

Proposition 6. *For any $\varepsilon > 0$, $x \ge x_0(\mathfrak{B}, \varepsilon)$ and $y \ge x^{7/17+\varepsilon}$, we have*

$$\sum_{\substack{x < n \le x+y \\ b \nmid n\ (\forall b \in \mathfrak{B})}} 1 \gg_{\mathfrak{B},\varepsilon} y.$$

Next we shall consider special sets $\mathfrak{B}$, of the type which occurs in the applications to modular forms (Theorems 1 and 2). Let $\mathbb{P}$ be the set of all prime numbers and $\mathfrak{P}$ be a subset of $\mathbb{P}$ for which there is a constant $\rho \in [0,1]$ such that

$$\big|\mathfrak{P} \cap [1, x]\big| \ll \frac{x^{\rho}}{(\log x)^{\Theta_\rho}} \qquad (x \ge 2), \tag{6.2}$$

where Θ_ρ is a real constant such that $\Theta_1 > 1$. Define

$$\mathfrak{B}_{\mathfrak{P}} := \mathfrak{P} \cup \big\{p^2 \mid p \in \mathbb{P} \smallsetminus \mathfrak{P}\big\} = \{b_k \mid b_1 < b_2 < \cdots\}.$$

Clearly the hypothesis (6.2) guarantees that $\mathfrak{B}_{\mathfrak{P}}$ satisfies the condition (6.1). One can hope to obtain a smaller exponent for this special set of integers $\mathfrak{B}_{\mathfrak{P}}$ than in the general case. In this direction, Alkan ([1], Theorems 2.2 and 2.3) proved, by exploiting the structure of the first component $\mathfrak{P}$ of $\mathfrak{B}_{\mathfrak{P}}$, the following result: If $y \geq x^{\theta}$ with

$$\theta = \theta(\rho) = \begin{cases} \frac{1}{3} + \varepsilon & \text{if } \rho = \frac{1}{2}, \\ \max\left\{\frac{7}{19}, \frac{23\rho}{35\rho+16}\right\} + \varepsilon & \text{if } \frac{1}{2} < \rho \leq 1, \end{cases} \tag{6.3}$$

then

$$\sum_{\substack{x<n\leq x+y \\ b\nmid n\,(\forall b\in\mathfrak{B}_{\mathfrak{P}})}} 1 \gg_{\mathfrak{P},\varepsilon} y. \tag{6.4}$$

His proof is based on the method of Bantle & Grupp [3], using the weight of the form

$$w(n) := \sum_{\substack{p_1\in\mathcal{P}_1 \\ p_1p_2|n}} \sum_{p_2\in\mathcal{P}_2} 1,$$

where

$$\mathcal{P}_i := \left\{p \in \mathbb{P} \mid x^{\delta_i} < p_i \leq x^{\delta_i+\varepsilon}\right\}.$$

This leads to estimate a bilinear form of type II:

$$\sum_{m\sim M} \sum_{n\sim N} \varphi_m \psi_n \, r_{mn}(x, y), \tag{6.5}$$

where $|\varphi_m| \leq 1$, $|\psi_n| \leq 1$ and $r_d(x, y)$ is defined in (5.12). Thus (6.3) is a consequence of the following result of Fouvry & Iwaniec [15] with the choice of $x^{\delta_1+\varepsilon} = M$ and $x^{\delta_2+\varepsilon} = N$: If $y = x^{\theta}$, then for any $\varepsilon > 0$ we have

$$\sum_{m\sim M} \sum_{n\sim N} \varphi_m \psi_n \, r_{mn}(x, y) \ll yx^{-2\varepsilon} \tag{6.6}$$

provided

$$\tfrac{7}{19} < \theta \leq \tfrac{11}{23}, \qquad M \leq yx^{-\varepsilon'}, \qquad N \leq y^{19/16}x^{-7/16-\varepsilon'}. \tag{6.7}$$

It is worth indicating that the condition $M \leq yx^{-\varepsilon'}$ forces $\delta_1 < \theta$, which obstructs to exploit fully the second component $\left\{p^2 \mid p \in \mathbb{P}\smallsetminus\mathfrak{P}\right\}$ of $\mathfrak{B}_{\mathfrak{P}}$.

In [45] and [46], the third author proposed an improved weighting device, i.e. replacing $\mathcal{P}_1$ by a set of quasi-prime numbers $\mathcal{M}$ (cf. (7.4) below). Thanks to the fundamental lemma of sieve ([4], Lemma 4), we are brought back to estimate the bilinear form of type I defined in (5.11). Our result (Corollary 9) on bilinear forms of type I has two advantages in comparison of (6.7). Firstly N has a larger range. Secondly there is no condition on M as $M \leq yx^{-\varepsilon'}$. The technique of using weights is more effective if the range of weights can go beyond the natural limit y. In the general case of $\mathfrak{B}$-free numbers, this is a crucial obstruction. However the special structure of the second component $\left\{p^2 \mid p \in \mathbb{P}\smallsetminus\mathfrak{P}\right\}$ of $\mathfrak{B}_{\mathfrak{P}}$ allows us to surmount this difficulty with the result of Filaseta & Trifonov ([13], (4)). These two observations and our new estimate for exponential sums enable us to improve considerably (6.3) of Alkan.

Proposition 7. *Let $0 < \rho \le 1$ and (κ, λ) be an exponent pair. For any $\varepsilon > 0$, $x \ge x_0(\mathfrak{P}, \varepsilon)$ and $y \ge x^{\theta(\rho)}$ with*

$$\theta(\rho) = \max\left\{\frac{1}{3}, \frac{7\rho}{9\rho+8}\right\} + \varepsilon, \tag{6.8}$$

or

$$\theta(\rho) = \max\left\{\frac{\kappa+\lambda}{1+2\kappa+2\lambda}, \frac{(1+\kappa+2\lambda)\rho}{(1+2\kappa+2\lambda)\rho+2+2\lambda}\right\} + \varepsilon, \tag{6.9}$$

we have

$$\sum_{\substack{x<n\le x+y \\ b\nmid n\ (\forall b\in\mathfrak{B}_{\mathfrak{P}})}} 1 \gg_{\mathfrak{P},\varepsilon} y.$$

When $\rho \le 3(\kappa+\lambda)/(3+2\kappa+2\lambda)$ where (κ, λ) is an exponent pair, we can obtain a better exponent than that in Proposition 7.

Proposition 8. *For any $\varepsilon > 0$, $x \ge x_0(\mathfrak{P}, \varepsilon)$ and $y \ge x^{\theta(\rho)}$ with*

$$\theta(\rho) = \begin{cases} \max\left\{\dfrac{1}{4}, \dfrac{10\rho}{19\rho+7}\right\} + \varepsilon & \text{if } 0 \le \rho < \frac{9}{17}, \\ \dfrac{3\rho}{4\rho+3} + \varepsilon & \text{if } \frac{9}{17} \le \rho \le 1, \end{cases} \tag{6.10}$$

we have

$$\sum_{\substack{x<n\le x+y \\ b\nmid n\ (\forall b\in\mathfrak{B}_{\mathfrak{P}})}} 1 \gg_{\mathfrak{P},\varepsilon} y.$$

By combining Propositions 7 and 8, we immediately obtain the following result.

Corollary 10. *For any $\varepsilon > 0$, $x \ge x_0(\mathfrak{P}, \varepsilon)$ and $y \ge x^{\theta(\rho)+\varepsilon}$ with*

$$\theta(\rho) = \begin{cases} \frac{1}{4} & \text{if } 0 \le \rho \le \frac{1}{3}, \\ \frac{10\rho}{19\rho+7} & \text{if } \frac{1}{3} < \rho \le \frac{9}{17}, \\ \frac{3\rho}{4\rho+3} & \text{if } \frac{9}{17} < \rho \le \frac{15}{28}, \\ \frac{5}{16} & \text{if } \frac{15}{28} < \rho \le \frac{5}{8}, \\ \frac{22\rho}{24\rho+29} & \text{if } \frac{5}{8} < \rho \le \frac{9}{10}, \\ \frac{7\rho}{9\rho+8} & \text{if } \frac{9}{10} < \rho \le 1, \end{cases}$$

we have

$$\sum_{\substack{x<n\le x+y \\ b\nmid n\ (\forall b\in\mathfrak{B}_{\mathfrak{P}})}} 1 \gg_{\mathfrak{P},\varepsilon} y.$$

Proof. The intervals $(0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{9}{17}]$ and $[\frac{9}{17}, \frac{15}{28}]$ come from Proposition 8.

The intervals $[\frac{15}{28}, \frac{5}{8}]$ and $[\frac{5}{8}, \frac{9}{10}]$ come from (6.9) of Proposition 7 with $(\kappa, \lambda) = (\frac{4}{18}, \frac{11}{18})$.

The interval $[\frac{9}{10}, 1]$ come from (6.8) of Proposition 7. □

Remark 3. (i) Propositions 7 and 8 improve Alkan's exponent (6.3). It is worth remarking that we have no restriction $\rho \ge \frac{1}{2}$ as in [1].

(ii) The parameter ρ can be considered as a measure of difficulty in the problem of $\mathfrak{B}_{\mathfrak{P}}$-free numbers. Clearly the case $\rho = 1$ is the most difficult and $\rho = 0$ is the simplest. In fact when $\mathfrak{P}$ is empty (so $\rho = 0$) the $\mathfrak{B}_{\emptyset}$-free numbers are the square-free integers. In this case, Filaseta & Trifonov [13] proved that $\theta = \frac{1}{5} + \varepsilon$ is admissible. However our method only gives $\theta = \frac{1}{4} + \varepsilon$. It seems interesting to generalise the method of Filaseta & Trifonov to the case of $\mathfrak{B}_{\mathfrak{P}}$-free numbers (at least for small values of ρ).

(iii) The function $\theta(\rho)$ is continuous, increasing, and $\theta(\frac{9}{17}) = \frac{9}{29}$, $\theta(\frac{15}{28}) = \frac{5}{16}$, $\theta(\frac{9}{10}) = \frac{9}{23}$, $\theta(1) = \frac{7}{17}$.

If we relax the multiplicative constraint by removing the square-free assumption, we can prove a better result for $\rho \le \frac{1}{3}$.

Proposition 9. *Suppose that* $0 \le \rho \le 1$. *For any* $\varepsilon > 0$, $x \ge x_0(\mathfrak{P}, \varepsilon)$ *and* $y \ge x^{\rho/(1+\rho)+\varepsilon}$, *we have*

$$\sum_{\substack{x<n\le x+y \\ b\nmid n\ (\forall b\in\mathfrak{P})}} 1 \gg_{\mathfrak{P},\varepsilon} y.$$

§ 7. Proof of Proposition 7

We begin by describing our weight function. Let θ, δ_1 and δ_2 be some parameters such that

$$\tfrac{1}{4} + \varepsilon \le \theta < \tfrac{1}{2}, \quad \varepsilon < \delta_2 + 2\varepsilon < \delta_1 + \varepsilon < \theta/\rho, \quad \delta_1 + \delta_2 < 1, \quad \delta_1 + \delta_2 + \theta/\rho > 1. \tag{7.1}$$

Introduce two sets

$$\mathcal{M} := \big\{m \in \mathbb{N} \mid x^{\delta_1} < m \le x^{\delta_1+\varepsilon},\, p \mid m \Rightarrow p \ge x^{\eta}\big\}, \tag{7.2}$$

$$\mathcal{P} := \big\{p \in \mathbb{P} \mid x^{\delta_2} < p \le x^{\delta_2+\varepsilon}\big\}, \tag{7.3}$$

where $\eta = \eta(\mathfrak{P}, \varepsilon) > 0$ is a (small) parameter chosen later.

Our weight function is defined by

$$c(n) := \sum_{\substack{m\in\mathcal{M} \\ mp\mid n}} \sum_{p\in\mathcal{P}} 1. \tag{7.4}$$

Put

$$A := \sum_{\substack{x<n\le x+y \\ b\nmid n\ (\forall b\in\mathfrak{B}_{\mathfrak{P}})}} c(n). \tag{7.5}$$

From (7.1), (7.2) and (7.3), it is easy to see that

$$c(n) \le 2^{1/\eta}/\varepsilon \qquad (n \le 2x), \tag{7.6}$$

which implies

$$\sum_{\substack{x<n\le x+y \\ b\nmid n\ (\forall b\in\mathfrak{B}_{\mathfrak{P}})}} 1 \ge \varepsilon 2^{-1/\eta} A. \tag{7.7}$$

In order to prove Proposition 7, it is sufficient to show that

$$A \gg_{\mathfrak{P},\varepsilon} y. \tag{7.8}$$

For this, we let $\ell := \ell(\mathfrak{P}, \varepsilon) \in \mathbb{N}$ be a positive integer such that

$$\sum_{k=\ell+1}^{\infty} \frac{1}{b_k} < \frac{B_{\mathfrak{P}}\varepsilon^3}{\eta 2^{1/\eta+2}}, \tag{7.9}$$

where

$$B_{\mathfrak{P}} := \prod_{p\in\mathfrak{P}} \left(1 - \frac{1}{p}\right) \prod_{p\in\mathbb{P}\smallsetminus\mathfrak{P}} \left(1 - \frac{1}{p^2}\right)$$

is the natural density of the sequence of $\mathfrak{B}_{\mathfrak{P}}$-free numbers.

Clearly we can write

$$A \geq A_1 - A_2 - A_3 \tag{7.10}$$

where

$$\begin{aligned} A_1 &:= \sum_{\substack{x<n\leq x+y \\ b_k\nmid n\ (\forall\, k\leq\ell)}} c(n), \\ A_2 &:= \sum_{\substack{b_\ell<b\leq y \\ b\in\mathfrak{B}_{\mathfrak{P}}}} \sum_{\substack{x<n\leq x+y \\ b\mid n}} c(n), \\ A_3 &:= \sum_{\substack{y<b\leq x \\ b\in\mathfrak{B}_{\mathfrak{P}}}} \sum_{\substack{x<n\leq x+y \\ b\mid n}} c(n). \end{aligned}$$

We shall see that A_2 and A_3 are negligible and A_1 gives the desired principal term. The required estimates for A_2 and A_3 will be offered by the next two lemmas.

Lemma 7.1. *We have*

$$A_2 \leq \frac{B_{\mathfrak{P}}\varepsilon^2}{2\eta} y.$$

Proof. By (7.6), it follows that

$$\begin{aligned} A_2 &\leq \frac{2^{1/\eta}}{\varepsilon} \sum_{\substack{b_\ell<b\leq y \\ b\in\mathfrak{B}_{\mathfrak{P}}}} \sum_{\substack{x<n\leq x+y \\ b\mid n}} 1 \\ &\leq \frac{2^{1/\eta}}{\varepsilon} \sum_{\substack{b_\ell<b\leq y \\ b\in\mathfrak{B}_{\mathfrak{P}}}} \frac{2y}{b}, \end{aligned}$$

which implies the required inequality in view of (7.9). □

Lemma 7.2. *There is a constant $C(\mathfrak{P}, \varepsilon)$ such that*

$$A_3 \leq \frac{C(\mathfrak{P}, \varepsilon)2^{1/\eta}}{(\log x)^{1/2}} y.$$

Proof. According to the definition of $\mathfrak{B}_{\mathfrak{P}}$, we can write

$$\begin{aligned} A_3 = & \sum_{\substack{y<p\le x^{\theta/\rho}(\log x)^{(\Theta_\rho-1/2)/\rho} \\ p\in\mathfrak{P}}} \sum_{\substack{x<n\le x+y \\ p\mid n}} c(n) \\ & + \sum_{\substack{x^{\theta/\rho}(\log x)^{(\Theta_\rho-1/2)/\rho}<p\le x \\ p\in\mathfrak{P}}} \sum_{\substack{x<n\le x+y \\ p\mid n}} c(n) \\ & + \sum_{\substack{y<q^2\le y^2\log x \\ q\in\mathbb{P}\smallsetminus\mathfrak{P}}} \sum_{\substack{x<n\le x+y \\ q^2\mid n}} c(n) \\ & + \sum_{\substack{y^2\log x<q^2\le x \\ q\in\mathbb{P}\smallsetminus\mathfrak{P}}} \sum_{\substack{x<n\le x+y \\ q^2\mid n}} c(n) \\ =: & A_{3,1}+A_{3,2}+A_{3,3}+A_{3,4}. \end{aligned} \tag{7.11}$$

For $p>y$, there is at most an integer $n\in(x,x+y]$ such that $p\mid n$. Thus (7.6) and (6.2) imply that

$$\begin{aligned} A_{3,1} &\le \frac{2^{1/\eta}}{\varepsilon} \sum_{\substack{p\le x^{\theta/\rho}(\log x)^{(\Theta_\rho-1/2)/\rho} \\ p\in\mathfrak{P}}} 1 \\ &\ll \frac{2^{1/\eta}}{\varepsilon} \frac{\big(x^{\theta/\rho}(\log x)^{(\Theta_\rho-1/2)/\rho}\big)^\rho}{(\log x)^{\Theta_\rho}} \\ &\ll \frac{2^{1/\eta}}{\varepsilon(\log x)^{1/2}}y. \end{aligned}$$

The definition of $c(n)$ allows us to write

$$A_{3,2} = \sum_{\substack{x^{\theta/\rho}(\log x)^{(\Theta_\rho-1/2)/\rho}<p\le x \\ p\in\mathfrak{P}}} \sum_{m\in\mathcal{M}} \sum_{p'\in\mathcal{P}} \sum_{\substack{x<n\le x+y \\ p\mid n,\, mp'\mid n}} 1.$$

The hypothesis $\delta_2+2\varepsilon<\delta_1+\varepsilon<\theta/\rho$ and $p\in\mathfrak{P}$ imply $(p,mp')=1$. Thus $pmp'\mid n$. Since

$$pmp' > x^{\theta/\rho+\delta_1+\delta_2}(\log x)^{(\Theta_\rho-1/2)/\rho} \ge 2x,$$

the sum over n must be empty. Therefore $A_{3,2}=0$.

We have

$$A_{3,3} \le \frac{2^{1/\eta}}{\varepsilon} \sum_{\substack{q\le y(\log x)^{1/2} \\ q\in\mathfrak{P}}} 1 \ll \frac{2^{1/\eta}}{\varepsilon(\log x)^{1/2}}y.$$

The term $A_{3,4}$ will be treated by the method of Filaseta & Trifonov [13]. Defining

$$S(t_1,t_2) := \{d\in(t_1,t_2] \mid \text{there is an integer } k \text{ such that } kd^2\in(x,x+y]\},$$

we can deduce, in view of (7.6), that

$$\begin{aligned} A_{3,4} &\le \varepsilon^{-1}2^{1/\eta} \sum_{\substack{y^2\log x<q^2\le x \\ q\in\mathbb{P}\smallsetminus\mathfrak{P}}} \sum_{\substack{x<n\le x+y \\ q^2\mid n}} 1 \\ &\le \varepsilon^{-1}2^{1/\eta}\big|S\big(y(\log x)^{1/2},x^{1/2}\big)\big|. \end{aligned}$$

We split $\big(y(\log x)^{1/2}, x^{1/2}\big]$ into dyadic intervals $(x^\phi, 2x^\phi]$ and write

$$A_{3,4} \le \varepsilon^{-1} 2^{1/\eta} (\log x) \max_{\theta \le \phi \le 1/2} \big|S(x^\phi, 2x^\phi)\big|.$$

According to ([13], (4)), we have

$$\big|S(x^\phi, 2x^\phi)\big| \ll x^{(1-\phi)/3}$$

for $y(\log x)^{1/2} \le x^\phi \le 2x^{1/2}$, and thus infer with the hypothesis $\theta > \frac14 + \varepsilon$ that

$$A_{3,4} \ll \varepsilon^{-1} 2^{1/\eta} x^{-\varepsilon'} y.$$

Now inserting the estimates for $A_{3,j}$ into (7.11), we obtain the required inequality. □

Next we shall treat the principal term A_1. It is convenient to introduce some notation. For each $\sigma = \{k_1, \dots, k_i\} \subset \{1, \dots, \ell\}$, we write $|\sigma| = i$ and $d_\sigma = b_{k_1} b_{k_2} \cdots b_{k_i}$ with the convention $|\emptyset| = 0$ and $d_\emptyset = 1$, where $\emptyset$ denotes the empty set.

Lemma 7.3. *For $x \ge x_0(\mathfrak{P}, \varepsilon)$, we have*

$$A_1 \ge \frac{B_{\mathfrak{P}} \varepsilon^2}{\eta} y + R,$$

where

$$R := \sum_{\sigma \subset \{1,\dots,\ell\}} (-1)^{|\sigma|} \sum_{m \in \mathcal{M}} \sum_{p \in \mathcal{P}} r_{d_\sigma m p}(x, y). \tag{7.12}$$

Proof. Since $(b_j, b_k) = 1$ $(j \ne k)$, we can write

$$\begin{aligned} A_1 &= \sum_{\sigma \subset \{1,\dots,\ell\}} (-1)^{|\sigma|} \sum_{\substack{x < n \le x+y \\ d_\sigma | n}} c(n) \\ &= \sum_{\sigma \subset \{1,\dots,\ell\}} (-1)^{|\sigma|} \sum_{m \in \mathcal{M}} \sum_{p \in \mathcal{P}} \sum_{\substack{x < n \le x+y \\ d_\sigma | n,\ mp | n}} 1. \end{aligned}$$

Clearly for any $\sigma \subset \{1, \dots, \ell\}$, any $m \in \mathcal{M}$ and any $p \in \mathcal{P}$ with $x \ge x_0(\mathfrak{P}, \varepsilon)$, we have $(d_\sigma, mp) = 1$ in view of (7.1)–(7.3). Hence it follows that

$$\begin{aligned} A_1 &= \sum_{\sigma \subset \{1,\dots,\ell\}} (-1)^{|\sigma|} \sum_{m \in \mathcal{M}} \sum_{p \in \mathcal{P}} \sum_{\substack{x < n \le x+y \\ d_\sigma m p | n}} 1 \\ &= y \sum_{\sigma \subset \{1,\dots,\ell\}} \frac{(-1)^{|\sigma|}}{d_\sigma} \sum_{m \in \mathcal{M}} \frac{1}{m} \sum_{p \in \mathcal{P}} \frac{1}{p} + R, \end{aligned} \tag{7.13}$$

where R is defined in (7.12).

It is easy to see that

$$\sum_{\sigma \subset \{1,\dots,\ell\}} \frac{(-1)^{|\sigma|}}{d_\sigma} = \prod_{k=1}^{\ell} \left(1 - \frac{1}{b_k}\right) \ge B_{\mathfrak{P}} \tag{7.14}$$

and

$$\sum_{p\in\mathcal{P}} \frac{1}{p} = \log\left(\frac{\delta_2+\varepsilon}{\delta_2}\right) + O\left(\frac{1}{\log x}\right) \geq \varepsilon \tag{7.15}$$

for $x \geq x_0(\mathfrak{P}, \varepsilon)$.

In order to estimate the sum over m, we need the following result of Friedlander ([16], Lemma 2): *Let $w(t)$ be Buchstab's function*

$$w(t) = 1/t \quad (1 \leq t \leq 2), \qquad \big(tw(t)\big)' = w(t-1) \quad (t \geq 2).$$

Assume $x > 1$ and $z = x^{1/t}$ with $t \geq 1$. Then we have uniformly for $t \geq 2$

$$\sum_{n\leq x,\, p\mid n\Rightarrow p\geq z} 1 = w(t)\frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right).$$

From this, an integration by part deduces that

$$\begin{aligned}\sum_{m\in\mathcal{M}} \frac{1}{m} &= \int_{x^{\delta_1}}^{x^{\delta_1+\varepsilon}} \frac{1}{t}\,\mathrm{d}\bigg(\sum_{n\leq t,\, p\mid n\Rightarrow p\geq x^\eta} 1\bigg) \\ &= \frac{1}{\eta\log x}\left\{w\left(\frac{\delta_1+\varepsilon}{\eta}\right) - w\left(\frac{\delta_1}{\eta}\right)\right\} + O\left(\frac{1}{\eta^2\log x}\right) + \int_{\delta_1/\eta}^{(\delta_1+\varepsilon)/\eta} w(u)\,\mathrm{d}u.\end{aligned}$$

In view of the well known relation

$$w(t) \to e^{-\gamma} \qquad (t \to \infty),$$

where γ is Euler's constant, we immediately see

$$\frac{\varepsilon}{2\eta} \leq \sum_{m\in\mathcal{M}} \frac{1}{m} \leq \frac{\varepsilon}{\eta} \tag{7.16}$$

for $x \geq x_0(\mathfrak{P}, \varepsilon)$.

Now the expected inequality follows from (7.13)–(7.16). □

The next lemma gives the desired estimate for the error term R defined in (7.12).

Lemma 7.4. *Let s be a real number such that*

$$s \geq 3 \qquad \textit{and} \qquad s\eta < \tfrac{1}{2}\varepsilon < \tfrac{1}{4}. \tag{7.17}$$

If

$$\begin{cases} \frac{1}{3} < \theta \leq \frac{5}{11}, \\ \delta_2 \leq (9\theta-3)/4 - \varepsilon', \\ \delta_1+\delta_2 \leq 1-\varepsilon', \end{cases} \tag{7.18}$$

or

$$\begin{cases} (\kappa+\lambda)/(1+2\kappa+2\lambda) < \theta \leq (\kappa+\lambda)/(2\kappa+\lambda), \\ \delta_2 \leq [(1+2\kappa+2\lambda)\theta-\kappa-\lambda]/(1+\lambda) - \varepsilon', \\ \delta_1+\delta_2 \leq 1-\varepsilon', \end{cases} \tag{7.19}$$

then we have

$$|R| \le C_1(\varepsilon) 2^{\ell(\mathfrak{P},\varepsilon)} \big(\eta^{-1} s^{-s} + x^{-\varepsilon/4}\big) y,$$

where $C_1(\varepsilon)$ is a positive constant depending on ε only.

Proof. For each $\sigma \subset \{1, \dots, \ell\}$, we define

$$R(\sigma) := \sum_{m\in\mathcal{M}} \sum_{p\in\mathcal{P}} r_{d_\sigma mp}(x,y).$$

We shall transform $R(\sigma)$ into a bilinear form of type I by using the fundamental lemma of sieve ([4], Lemma 4): *Let $z = x^\eta$ and $Q = z^s$ with $s \ge 3$. There are two sequences $\{\lambda_q^\pm\}_{q\le Q}$ such that*

$$|\lambda_q^\pm| \le 1, \qquad \lambda_q^\pm = 0 \quad (q > Q), \tag{7.20}$$

$$\begin{cases} (\lambda^- * \mathbf{1})(n) = (\lambda^+ * \mathbf{1})(n) = 1 & \text{if } p \mid n \Rightarrow p \ge z, \\ (\lambda^- * \mathbf{1})(n) \le 0 \le (\lambda^+ * \mathbf{1})(n) & \text{otherwise,} \end{cases} \tag{7.21}$$

$$\sum_{q\le Q} \frac{\lambda_q^\pm}{q} = \{1 + O(s^{-s})\} \prod_{p<z} \Big(1 - \frac{1}{p}\Big). \tag{7.22}$$

With the help of (7.21), we can write

$$\begin{aligned} R(\sigma) &= \sum_{m\in\mathcal{M}} \sum_{p\in\mathcal{P}} \sum_{\substack{x<n\le x+y \\ d_\sigma mp \mid n}} 1 - \sum_{m\in\mathcal{M}} \sum_{p\in\mathcal{P}} \frac{y}{d_\sigma mp} \\ &\le \sum_{x^{\delta_1}<m\le x^{\delta_1+\varepsilon}} (\lambda^+ * \mathbf{1})(m) \sum_{p\in\mathcal{P}} \sum_{\substack{x<n\le x+y \\ d_\sigma mp \mid n}} 1 \\ &\qquad - \sum_{x^{\delta_1}<m\le x^{\delta_1+\varepsilon}} (\lambda^- * \mathbf{1})(m) \sum_{p\in\mathcal{P}} \frac{y}{d_\sigma mp} \\ &= \sum_{x^{\delta_1}<m\le x^{\delta_1+\varepsilon}} (\lambda^+ * \mathbf{1})(m) \sum_{p\in\mathcal{P}} r_{d_\sigma mp}(x,y) \\ &\qquad + \sum_{x^{\delta_1}<m\le x^{\delta_1+\varepsilon}} [(\lambda^+ - \lambda^-) * \mathbf{1}](m) \sum_{p\in\mathcal{P}} \frac{y}{d_\sigma mp} \\ &=: R_1(\sigma) + R_2(\sigma). \end{aligned} \tag{7.23}$$

Clearly

$$\begin{aligned} |R_2(\sigma)| &\le \frac{y}{d_\sigma} \sum_{q\le Q} \bigg|\frac{\lambda_q^+ - \lambda_q^-}{q}\bigg| \sum_{x^{\delta_1}/q<m\le x^{\delta_1+\varepsilon}/q} \frac{1}{m} \sum_{p\in\mathcal{P}} \frac{1}{p} \\ &\le \frac{y}{d_\sigma} \sum_{q\le Q} \bigg|\frac{\lambda_q^+ - \lambda_q^-}{q}\bigg| \cdot \big\{\varepsilon \log x + O(qx^{-\delta_1})\big\} \cdot 2\varepsilon \end{aligned}$$

On the other hand, (7.22) implies that

$$\sum_{q\le Q} \bigg|\frac{\lambda_q^+ - \lambda_q^-}{q}\bigg| \ll \frac{s^{-s}}{\log z} \ll \frac{s^{-s}}{\eta \log x}.$$

Inserting it into the preceding estimate, we find that

$$\big|R_2(\sigma)\big| \ll \big(\eta^{-1} s^{-s} + Qx^{-\delta_1}\big) y. \tag{7.24}$$

It remains to estimate $R_1(\sigma)$. Let ψ_n be the characteristic function of the set $\mathcal{P}$. Since

$$r_{d_\sigma qmn}(x,y) = r_{mn}\Big(\frac{x}{d_\sigma q}, \frac{y}{d_\sigma q}\Big),$$

we can write

$$R_1(\sigma) = \sum_{q\le Q} \lambda_q^+ \sum_{x^{\delta_1}/q<m\le x^{\delta_1+\varepsilon}/q} \ \sum_{x^{\delta_2}<n\le x^{\delta_2+\varepsilon}} \psi_n r_{mn}\Big(\frac{x}{d_\sigma q}, \frac{y}{d_\sigma q}\Big).$$

We split $(x^{\delta_1}/q,\, x^{\delta_1+\varepsilon}/q]$ and $(x^{\delta_2},\, x^{\delta_2+\varepsilon}]$ into dyadic intervals $(M,2M]$ and $(N,2N]$, respectively. In view of (7.22), we have for $x\ge x_0(\mathfrak{P},\varepsilon)$

$$1\le q\le Q = x^{s\eta} < x^{\varepsilon/2} \qquad\text{and}\qquad 1\le d_\sigma < x^{\varepsilon/2}.$$

The hypothesis (7.18) and (7.19) imply that

$$N < \Big(\frac{x}{d_\sigma q}\Big)^{(9\theta-3)/4-\varepsilon'} \qquad\text{and}\qquad MN\le \Big(\frac{x}{d_\sigma q}\Big)^{1-\varepsilon'}$$

and

$$N < \Big(\frac{x}{d_\sigma q}\Big)^{[(1+2\kappa+2\lambda)\theta-\kappa-\lambda]/(1+\lambda)-\varepsilon'} \qquad\text{and}\qquad MN\le \Big(\frac{x}{d_\sigma q}\Big)^{1-\varepsilon'},$$

respectively. Thus Corollary 9 allows us to deduce that

$$\begin{aligned}\sum_{x^{\delta_1}/q<m\le x^{\delta_1+\varepsilon}/q} \ \sum_{x^{\delta_2}<n\le x^{\delta_2+\varepsilon}} \psi_n r_{mn}\Big(\frac{x}{d_\sigma q}, \frac{y}{d_\sigma q}\Big) &\ll_\varepsilon \frac{y}{d_\sigma q}\Big(\frac{x}{d_\sigma q}\Big)^{-\varepsilon}(\log x)^2\\ &\ll_\varepsilon x^{-\varepsilon/2}y.\end{aligned}$$

This estimate and (7.20) imply that

$$R_1(\sigma) \ll_\varepsilon Qx^{-\varepsilon/2}y \ll_\varepsilon x^{-\varepsilon/4}y. \tag{7.25}$$

Combining (7.24) and (7.25), there is a positive constant $C_1(\varepsilon)>0$ depending on ε such that

$$R(\sigma) \le C_1(\varepsilon)\big(\eta^{-1}s^{-s} + x^{-\varepsilon/4}\big)y.$$

Similarly we can prove that

$$R(\sigma) \ge -C_1(\varepsilon)\big(\eta^{-1}s^{-s} + x^{-\varepsilon/2}\big)y.$$

Thus

$$|R| \le \sum_{\sigma\subset\{1,\dots,\ell\}} |R(\sigma)| \le C_1(\varepsilon)2^{\ell(\mathfrak{P},\varepsilon)}\big(\eta^{-1}s^{-s} + x^{-\varepsilon/2}\big)y.$$

This completes the proof of Lemma 7.4. □

Now we are ready to complete the proof of Proposition 7.

Without loss of generality, we can assume that

$$C_1(\varepsilon)2^{\ell(\mathfrak{P},\varepsilon)} > 8/B_{\mathfrak{P}}\varepsilon^2 > 16.$$

Take

$$\eta^{-1} = \min\big\{\tfrac{1}{5}\varepsilon^{-2},\, C_1(\varepsilon)2^{\ell(\mathfrak{P},\varepsilon)}\big\}, \qquad s = \eta^{-1/2},$$
$$\theta = \max\big\{\tfrac{1}{3},\, \tfrac{7\rho}{9\rho+8}\big\} + \varepsilon' \quad \text{or} \quad \theta = \max\big\{\tfrac{\kappa+\lambda}{1+2\kappa+2\lambda},\, \tfrac{(1+\kappa+2\lambda)\rho}{(1+2\kappa+2\lambda)\rho+2+2\lambda}\big\} + \varepsilon',$$
$$\delta_1 = \theta/\rho - \varepsilon', \qquad \delta_2 = 1 - 2\theta/\rho + \varepsilon'.$$

It is easy to verify that these choices satisfy the conditions (7.1), (7.17), (7.18) or (7.19). Thus Lemmas 7.1–7.5 imply that

$$\begin{aligned} A &\ge \bigg(\frac{B_{\mathfrak{P}}\varepsilon^2}{2\eta} - C_1(\varepsilon)2^{\ell(\mathfrak{P},\varepsilon)}\big(\eta^{-1}s^{-s} + x^{-\varepsilon/4}\big) - \frac{C(\mathfrak{P},\varepsilon)2^{1/\eta}}{(\log x)^{1/2}}\bigg)y \\ &\gg_{\mathfrak{P},\varepsilon} y \end{aligned}$$

for $x \ge x_0(\mathfrak{P},\varepsilon)$. This completes the proof of (7.8) and hence Proposition 7. □

§ 8. Proof of Proposition 8

The proof is very similar to that of Proposition 7 so we shall mention only the important points. As before let θ and δ be two parameters such that

(8.1) $$\tfrac{1}{4} + \varepsilon \le \theta < \tfrac{1}{2}, \qquad \theta < \delta + 2\varepsilon < \min\{\theta/\rho,\, 1\}, \qquad \delta + \theta/\rho > 1.$$

Let $\eta = \eta(\mathfrak{P},\varepsilon) > 0$ be a (small) parameter determined later. Introduce the set

$$\mathcal{M}' := \big\{m \in \mathbb{N} \;\mid\; x^{\delta} < m \le x^{\delta+\varepsilon},\, p \mid m \Rightarrow p \ge x^{\eta}\big\}.$$

Our weight function is defined to be

$$c'(n) := \sum_{\substack{m\in\mathcal{M}' \\ m\mid n}} 1$$

and the corresponding weighted sum is

$$A' := \sum_{\substack{x<n\le x+y \\ b\nmid n\ (\forall b\in\mathfrak{B}_{\mathfrak{P}})}} c'(n).$$

It is easy to see that

(8.2) $$c'(n) \le 2^{1/\eta} \qquad (n \le 2x)$$

and

(8.3) $$\sum_{\substack{x<n\le x+y \\ b\nmid n\ (\forall b\in\mathfrak{B}_{\mathfrak{P}})}} 1 \ge 2^{-1/\eta}A'.$$

Let $\ell := \ell(\mathfrak{P},\varepsilon) \in \mathbb{N}$ be a positive integer such that

(8.4) $$\sum_{k=\ell+1}^{\infty} \frac{1}{b_k} < \frac{B_{\mathfrak{P}}\varepsilon^2}{\eta 2^{1/\eta+2}}.$$

We can write

$$A' \geq A'_1 - A'_2 - A'_3 \tag{8.5}$$

where

$$A'_1 := \sum_{\substack{x<n\leq x+y \\ b_k \nmid n \, (\forall\, k\leq \ell)}} c'(n),$$
$$A'_2 := \sum_{\substack{b_\ell<b\leq y \\ b\in \mathfrak{B}_{\mathfrak{P}}}} \sum_{\substack{x<n\leq x+y \\ b\mid n}} c'(n),$$
$$A'_3 := \sum_{\substack{y<b\leq x \\ b\in \mathfrak{B}_{\mathfrak{P}}}} \sum_{\substack{x<n\leq x+y \\ b\mid n}} c'(n).$$

Similar to Lemmas 7.1, 7.2 and 7.3, we have, for $x \geq x_0(\mathfrak{P}, \varepsilon)$,

$$A'_2 \leq \frac{B_{\mathfrak{P}}\varepsilon^2}{2\eta} y, \tag{8.6}$$

$$A'_3 \leq \frac{C2^{1/\eta}}{(\log x)^{1/2}} y, \tag{8.7}$$

$$A'_1 \geq \frac{B_{\mathfrak{P}}\varepsilon^2}{\eta} y + R, \tag{8.8}$$

where

$$R' := \sum_{\sigma\subset\{1,\dots,\ell\}} (-1)^{|\sigma|} \sum_{m\in\mathcal{M}'} r_{d_\sigma m}(x,y).$$

Similar to Lemma 7.4, we can prove, by using (5.17) and (5.18) of Lemma 5.1 instead of Corollary 9, that there is a positive constant $C'_1(\varepsilon)$ depending on ε only such that

$$|R'| \leq C'_1(\varepsilon) 2^{\ell(\mathfrak{P},\varepsilon)} \big(\eta^{-1} s^{-s} + x^{-\varepsilon/4}\big) y \tag{8.9}$$

provided

$$s \geq 3, \qquad s\eta < \tfrac{1}{2}\varepsilon < \tfrac{1}{4} \tag{8.10}$$

and

$$\begin{cases} \frac{1}{4} < \theta < \frac{9}{29}, \\ \delta \leq (19\theta - 3)/7 - \varepsilon', \end{cases} \qquad \text{or} \qquad \begin{cases} \frac{9}{29} < \theta < \frac{1}{2}, \\ \delta \leq 4\theta/3 - \varepsilon'. \end{cases} \tag{8.11}$$

Now take

$$\eta^{-1} = \min\big\{\tfrac{1}{5}\varepsilon^{-2},\, C_1(\varepsilon)2^{\ell(\mathfrak{P},\varepsilon)}\big\}, \qquad s = \eta^{-1/2}$$

and

$$\begin{cases} \theta = \max\big\{\frac{1}{4},\, \frac{10\rho}{19\rho+7}\big\} + \varepsilon', \\ \delta = \frac{19\theta-3}{7} - \varepsilon', \end{cases} \qquad \text{or} \qquad \begin{cases} \theta = \frac{3\rho}{4\rho+3} + \varepsilon', \\ \delta = \frac{4\theta}{3} - \varepsilon'. \end{cases}$$

It is straightforward to verify that these choices satisfy the conditions (8.1), (8.10) and (8.11). Thus the relations (8.5)–(8.9) imply

$$A' \gg_{\mathfrak{P},\varepsilon} y$$

for $x \geq x_0(\mathfrak{P}, \varepsilon)$. This completes the proof of Proposition 4. □

§ 9. Proof of Proposition 9

The proof of Proposition 9 (which can in fact be properly described as a sieve problem in the usual sense) is much simpler than that of Proposition 8. So we shall mention only the important points. Let $\theta = \rho/(1+\rho) + 2\varepsilon$ and

$$\mathcal{P}'' := \{p \in \mathbb{P} \mid x^{\theta-2\varepsilon} < p \leq x^{\theta-\varepsilon}\}.$$

Define the weight function

$$c''(n) := \sum_{\substack{p \in \mathcal{P}'' \\ p \mid n}} 1$$

and consider the corresponding weighted sum

$$A'' := \sum_{\substack{x < n \leq x+y \\ b \nmid n \, (\forall b \in \mathfrak{P})}} c'(n).$$

Similarly we can write

$$A'' \geq A''_1 - A''_2 - A''_3,$$

where A''_j is defined as A'_j (replacing $\mathfrak{B}_{\mathfrak{P}}$ by $\mathfrak{P}$). Now A''_3 is easier to treat (without the corresponding parts $A_{3,3}$ and $A_{3,4}$, see (7.11)). In view of $\theta - 2\varepsilon + \theta/\rho > 1$, we can prove the same estimates for A''_2 and A''_3. The error term R'', which comes from A''_1, can be controlled trivially as follows:

$$|R''| \leq \sum_{\sigma \subset \{1,\dots,\ell\}} \sum_{p \in \mathcal{P}''} |r_{d_\sigma p}(x,y)| \ll_{\mathfrak{P}} yx^{-\varepsilon}.$$

This completes the proof of Proposition 9. □

References

[1] E. Alkan, Nonvanishing of Fourier coefficients of modular forms, *Proc. Amer. Math. Soc.* **131** (2002), 1673–1680.

[2] A. Balog & K. Ono, The Chebotarev density theorem in short intervals and some questions of Serre, *J. Number theory* **91** (2001), 356–371.

[3] G. Bantle & F. Grupp, On a problem of Erdős and Szemerédi, *J. Number Theory* **22** (1986), 280–288.

[4] E. Bombieri, J.B. Friedlander & H. Iwaniec, Primes in arithmetic progressions to large moduli, *Acta Math.* **156** (1986), 203–251.

[5] A. Booker, A test for identifying Fourier coefficients of automorphic forms and application to Kloosterman sums, *Experiment. Math.* **9** (2000), 571–581.

[6] J.W.S Cassels, Local fields, Cambridge Univ. Press 1986.

[7] W. Duke & H. Iwaniec, Bilinear forms in the Fourier coefficients of half-integral weight cusp forms and sums over primes, *Math. Ann.* **286** (1990), 783–802.

[8] W. Duke, J. Friedlander & H. Iwaniec, Bounds for automorphic L-functions, II, *Invent. math.* **115** (1994), 219–239.

[9] W. Duke & E. Kowalski, A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations, *Invent. math.* **139** (2000), 1–39.

[10] N. ELKIES, Distribution of supersingular primes, *Astérisque* (1992), 127–132.

[11] P. ERDŐS, On the difference of consecutive terms of sequences, defined by divisibility properties, *Acta Arith.* **12** (1966), 175–182.

[12] J-H EVERTSE, H. SCHLICKEWEI & W. SCHMIDT, Linear equations in variables which lie in a multiplicative group, *Annals of Math.* **155** (2002), 807–836.

[13] M. FILASETA & O. TRIFONOV, On gaps between squarefree numbers II, *J. London Math. Soc.* (2) **45** (1992), 215–221.

[14] B. FISHER Distinctness of Kloosterman sums, in *p-adic methods in number theory and algebraic geometry*, Contemporary Math. 133, A.M.S 1992, 81–102.

[15] E. FOUVRY & H. IWANIEC, Exponential sums with monomials, *J. Number Theory* **33** (1989), 311–333.

[16] J.B. FRIEDLANDER, Integers free from large and small primes, *Proc. London Math. Soc.* (3) **33** (1976), 565–576.

[17] W. FULTON & J. HARRIS, Representation Theory, a first course, Grad. Texts in Math. 129, Springer Verlag, 1991.

[18] S. GELBART & H. JACQUET, A relation between automorphic representations of $GL(2)$ and $GL(3)$, *Ann. Sci. École Norm. Sup. (4)* **11** (1978), 471–542.

[19] A. GOOD, Cusp forms and eigenfunctions of the laplacian, *Math. Ann.* **255** (1981), 523–548.

[20] S.W. GRAHAM & G. KOLESNIK, *Van der Corput's Method of Exponential sums*, Cambridge University Press, Cambridge, 1991.

[21] D.R. HEATH-BROWN, *Mean values of the zeta-function and divisor problems*, in *Recent progress in analytic number theory*, Vol. 1 (Durham, 1979), pp. 115–119, Academic Press, London-New York, 1981.

[22] J. HOFFSTEIN & P. LOCKHART, Coefficients of Maass forms and the Siegel zero, *Annals of Math.* **140** (1994), 161–181.

[23] H. IWANIEC, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, Rhode Island, 1997.

[24] H. IWANIEC & E. KOWALSKI, Analytic Number Theory, Colloquium Publications 53, American Mathematical Society, Providence, Rhode Island, 2004.

[25] N. KATZ, Sommes exponentielles, *Astérisque* **79**, SMF 1980.

[26] H. KIM & P. SARNAK, Refined estimates towards the Ramanujan and Selberg conjectures, *J. American Math. Soc.* **16** (2003), 175–181.

[27] M.I. KNOPP & J. LEHNER, Gaps in the Fourier series of automorphic forms, Analytic Number Theory (Philadelphia 1980), *Lecture Notes in Math.* **899**, Springer Verlag 1981, 360–381.

[28] S. LANG & H. TROTTER, Frobenius distribution in $GL(2)$ extensions, Lecture Notes in Mathematics **504**, Springer-Verlag, Berlin-New York, 1976. iii+274 pp.

[29] D.H. LEHMER, Some functions of Ramanujan, *Math. Student* **27** (1959), 105–116.

[30] H.-Q. LIU & J. WU, Numbers with a large prime factor, *Acta Arith.* **89** (1999), 163–187.

[31] C. MOEGLIN & J-L. WALDSPURGER, Pôles des fonctions L de paires pour $GL(N)$, appendix to Le spectre résiduel de $GL(N)$, *Ann. Sci. École Norm. Sup. (4)* **22** (1989), 605–674.

[32] G. MOLTENI, Upper and lower bounds at $s = 1$ for certain Dirichlet series with Euler product, *Duke Math. J.* **111** (2002), 133–158.

[33] V.K. Murty, Frobenius distributions and Galois representations, in: *Automorphic forms, automorphic representations and arithmetic* (Fort Worth, TX, 1996), 193–211, Proc. Sympos. Pure Math. **66**, Part 1, Amer. Math. Soc., Providence, RI, 1999.

[34] Y. Petridis, On squares of eigenfunctions for the hyperbolic plane and a new bound on certain L-series, *IMRN* **1995**, n. 3, 111–127.

[35] V.A. Plaksin, Distribution de $\mathcal{B}$-free numbers, *Matematicheskie Zametki* **47** (1990), 69–77.

[36] R.A Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions II. The order of the Fourier coefficients of integer modular forms, *Proc. Cambridge Phil. Soc.* **35** (1939), 357–372.

[37] K.A. Ribet, Galois representations attached to eigenforms with Nebentypus, in *Modular functions of one variable V* (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Lecture Notes in Math. **601** (1977), 17–51, Springer & Berlin.

[38] O. Robert & P. Sargos, A third derivative test for mean values of exponential sums with application to lattice point problems, *Acta Arith.* **106** (2003), 27–39.

[39] O. Robert & P. Sargos, Three dimensional exponential sums with monomials, *J. Reine angew Math.*, to appear.

[40] P. Sargos & J. Wu, Multiple exponential sums with monomials and their applications in number theory, *Acta Math. Hungar.* **87 (4)** (2000), 333–354.

[41] P. Sarnak, Integral of products of eigenfunctions, *IMRN* **1994**, n. 6, 251–260.

[42] A. Selberg, Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist, *Arch. Math. Naturvid.* **43** (1940), 47–50.

[43] J.-P. Serre, Quelques applications du théorème de densité de Chebotarev, *Inst. Hautes Études Sci. Publ. Math.* **54** (1981), 323–401.

[44] E. Szemerédi, On the difference of consecutive terms of sequences, defined by divisibility properties II, *Acta Arith.* **23** (1973), 359–361.

[45] J. Wu, *Sur trois questions classiques de crible : nombres premiers jumeaux, nombres P_2 et nombres $\mathcal{B}$–libres*, Thèse de Doctorat, Université Paris-Sud, 1990.

[46] J. Wu, Nombres $\mathcal{B}$–libres dans les petits intervalles, *Acta Arith.* **65** (1993), 97–116.

[47] J. Wu, Distance entre nombres $\mathcal{B}$-libres consécutifs, *Prépublications de l'Institut Élie Cartan* N° **32** (1994), Université Henri Poincaré (Nancy 1).

[48] W.G. Zhai , Number of $\mathcal{B}$-free numbers in short intervals, *Chinese Sci. Bull.* **45** (2000), no. 3, 208–212.

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