

DEPENDENCY ON THE GROUP IN AUTOMORPHIC SOBOLEV INEQUALITIES

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1. INTRODUCTION

In [1] and [2], Bernstein and Reznikov have introduced a new way of estimating the coefficients in the spectral expansion of ϕ^2 , where ϕ is a Maass cusp of norm 1 on a quotient $Y = \Gamma \backslash \mathbf{H}$ of the Poincaré upper half-plane with finite volume. The question of obtaining the precise exponential decay of those coefficients had been posed by Selberg, and first solved by Good [5] (for holomorphic forms) and Sarnak [13] (for Maass forms). Bernstein and Reznikov obtain in fact the right polynomial growth conjectured by Sarnak: let (φ_i) be an orthonormal basis of the space of cusp forms on Y , eigenfunctions of the Laplace operator with eigenvalue

$$\mu_i = \frac{1 - \lambda_i^2}{4} \geq 0, \quad 0 < \mu_1 \leq \mu_2 \leq \dots$$

and $c_i = (\phi^2 | \varphi_i)$ the coefficients in question. Then one has

$$(1) \quad \sum_{|\lambda_i| \leq T} |c_i|^2 \exp\left(\frac{\pi}{2} |\lambda_i|\right) \leq C(\Gamma, \mu) (\log T)^3$$

for all $T \geq 2$, $C(\Gamma, \mu) \geq 0$ being a “constant” depending only on Γ and the eigenvalue μ of ϕ . This is essentially best possible.¹

For certain arithmetic applications, particularly because of relations between the c_i and special values of triple-product L -functions, it is important to have some control on $C(\Gamma, \mu)$. Especially in the arithmetic case where Γ is a congruence subgroup, one asks for at least a polynomial bound in terms of the level. We indicate two arguments leading to such bounds (indeed, rather better) in this note, following closely the proof of (1) in [2].

Proposition 1. *We have*

$$C(\Gamma, \mu) \ll_{\mu} 1$$

for all congruence subgroups $\Gamma < SL(2, \mathbf{Z})$, the implied constant depending only on μ .

We deduce from this an extension of (1) which is also arithmetically significant in studying the shifted convolution sums arising in analytic investigations of Rankin-Selberg type L -functions (see e.g. [11], [4], [7], [13]). Here we work (as usual in this situation) with the Hecke groups $\Gamma_0(q)$.

Corollary 2. *Let ϕ be an L^2 -normalized Maass cusp form for $\Gamma_0(q)$ with eigenvalue μ , l_1 and $l_2 \geq 1$ be integers. Let*

$$\Psi(z) = \phi(l_1 z) \overline{\phi(l_2 z)}$$

and $c_i = (\Psi | \varphi_i)$,

where (φ_i) is an orthonormal basis of Maass forms of the space of cusp forms on $\Gamma_0(q l_1 l_2)$ with eigenvalues μ_i as above.

We have

$$\sum_{|\lambda_i| \leq T} |c_i|^2 \exp\left(\frac{\pi}{2} |\lambda_i|\right) \ll_{\mu} [\Gamma_0(q) : \Gamma_0(q l_1 l_2)]^2 (\log T)^3$$

for any $T \geq 2$, the implied constant depending only on μ .

To prove Proposition 1, it will be convenient to summarize the argument leading to (1) as the conjunction of the following two propositions:

¹. A lower bound for (1) is given in [12]; however Bernstein and Reznikov give a different method in [3], removing the $(\log T)^3$ term, and Krötz and Stanton [8] have also extended and slightly improved the method of [2].

Proposition 3. *Let Γ be a subgroup of $SL(2, \mathbf{R})$ with finite covolume and let $Y = \Gamma \backslash \mathbf{H}$, $X = \Gamma \backslash SL(2, \mathbf{R})$. For any Maass cusp form $\phi \in L^2(Y)$ with eigenvalue μ and norm 1, we have*

$$(2) \quad \sum_{|\lambda_i| \leq T} |c_i|^2 \exp\left(\frac{\pi}{2} |\lambda_i|\right) \ll_{\mu} \|e_{\Gamma}\|^2 (\log T)^3$$

for $T \geq 2$, where

$$e_{\Gamma} : H_0^3(X) \rightarrow L^{\infty}(X)$$

is the possibly unbounded embedding of the third Sobolev space of cuspidal functions on X into $L^{\infty}(X)$, and the implied constant depends only on μ .

Precisely, our $H_0^3(X)$ is the space of cuspidal functions on X with all partial derivatives (with respect to standard x, y, θ coordinates [9, VI-4], say) of order ≤ 3 in L^2 , and the Sobolev norm is denoted $\|\cdot\|_3$ and defined to be

$$(3) \quad \|f\|_3^2 = \sum_{|I| \leq 3} \|\partial_I(f)\|_{L^2}^2,$$

where the measure defining the L^2 -norm on X is induced by the Haar measure on $G = SL(2, \mathbf{R})$, normalized to give the standard Poincaré measure $y^{-2} dx dy$ on $\mathbf{H} = G/SO(2, \mathbf{R})$.

This proposition is a rephrasing of [2, Prop. 2.2, 2.3] together with [2, Sect. 3 and Prop. 4.1]. To get precisely our statement, note that we need to relate the Sobolev norm $\|\cdot\|_3$ on X to that denoted S_3 in [2, 3.2], which is defined using the Lie-algebra action of $G = SL(2, \mathbf{R})$ on the space V of smooth vectors of a representation π of G . If V is unitarily embedded in $L^2(X)$, say by $\nu : V \hookrightarrow L^2(X)$, the norms are equivalent, i.e. we have

$$(4) \quad c_V S_3(e) \leq \|\nu(e)\|_3 \leq C_V S_3(e) \text{ for } e \in V,$$

for some constants $c_V, C_V \geq 0$ depending only on V . In particular (notation as in [2, Prop. 4.1])

$$N_{sup}(e) = \|\nu(e)\|_{\infty} \leq \|e_{\Gamma}\| \|\nu(e)\|_3 \leq C_V \|e_{\Gamma}\| S_3(e),$$

i.e. the constant C in (loc. cit.) is $\leq C_V \|e_{\Gamma}\|$. Then the proof of Bernstein-Reznikov gives our Proposition (because $N_{sup} \leq C_V \|e_{\Gamma}\| S_3^G$, see Sect. 3.4, 2.3...)

The inequality (4), on the other hand, holds because of the formulae linking the Lie algebra action and the partial derivatives for functions on G and in induced representations (see e.g. [9, VI-4]). The dependency of this constant on V (i.e. on the eigenvalue of the corresponding Maass form) is of no matter to us because this inequality need only be applied for the representation corresponding to Maass forms of eigenvalue μ (see [2, 2.3]), and we allow a constant depending on μ .

The proof of Proposition 3 uses two beautiful ideas of analytic continuation of representations and invariant semi-norms on representations, which are the crucial points in [2]; note it is important to work with cuspidal forms for the next proposition. When Γ is cocompact, one can replace the third Sobolev space by the second Sobolev space.

Proposition 4. *For Γ as above, we have*

$$\|e_{\Gamma}\| < +\infty,$$

i.e. e_{Γ} is a bounded linear operator.

This is proved in [2, App. B]. Proposition 1 thus follows immediately from the following two facts:

Proposition 5. (1) *The implied constant in Proposition 3 depends only on μ .*

(2) *For any finite index subgroup $\Gamma < SL(2, \mathbf{Z})$, we have*

$$(5) \quad \|e_{\Gamma}\| = \|e_{SL(2, \mathbf{Z})}\|.$$

Of these, (1) is easy, as it follows from reading through the argument of Bernstein-Reznikov, as already partly sketched above. It is mainly due to the fact that the constant in Proposition 3 arises from representation-theoretic computations which only depend on properties of the unitary representation of $SL(2, \mathbf{R})$ corresponding to a Maass form (see [2, Prop. 2.2, (1), (2) and 3.4]).

Thus our much more modest goal is to prove (5), which justifies the title of this note. We will prove a bit more, and give a second argument which applies in more general circumstances, showing that

$$(6) \quad \|e_{\Gamma_q}\| \ll \sqrt{q} \text{ hence } C(\Gamma_q, \mu) \ll_{\mu} q$$

for any $q \geq 3$, where Γ_q is the Hecke triangle group (see e.g [6, 2.3]), taken as a simple concrete example.

2. SOBOLEV INEQUALITIES

Lemma 6. *Let Γ_1 and Γ_2 be commensurable subgroups in $SL(2, \mathbf{R})$ with finite covolume. Then*

$$\|e_{\Gamma_1}\| = \|e_{\Gamma_2}\|.$$

This clearly implies (5), as well as the analogue for the family of congruence subgroups of a cocompact quaternion group (associated to a fixed quaternion algebra).

Proof. It suffices by transitivity to show the result for $\Gamma_1 < \Gamma_2$ of finite index. We denote $e_i = e_{\Gamma_i}$ and let $F_2 \subset G = SL(2, \mathbf{R})$ be an open fundamental domain for the action of Γ_2 . Then the open set

$$F_1 = \bigcup_{\gamma \in \Gamma_1 \setminus \Gamma_2} \gamma F_2$$

is a (possibly disconnected) fundamental domain for the action of Γ_1 . Let $f \in H_0^3(X)$ (where $X = \Gamma_1 \setminus G$), which we assume to be smooth and compactly supported (such functions are dense in $H_0^3(X)$). We let g denote the corresponding function on F_1 , smooth and compactly supported in F_1 . Denoting as above (3) by $\|\cdot\|_3$ the Sobolev norm, we have

$$\begin{aligned} \|f\|_{\infty} &= \|g\|_{\infty} = \max_{\gamma} \|g_{\gamma}\|_{\infty} \\ \|f\|_3^2 &= \|g\|_3^2 = \sum_{\gamma} \|g_{\gamma}\|_3^2 \end{aligned}$$

where we define g_{γ} for $\gamma \in \Gamma_1 \setminus \Gamma_2$ by

$$g_{\gamma}(x) = g(\gamma x) \text{ for } x \in F_2.$$

The second relation follows from the definition (3) of $\|\cdot\|_3$, and the normalization of Haar measure on G , the norm on the right for g_{γ} being the sup or Sobolev norm for Γ_2 . For any γ , g_{γ} is smooth and compactly supported (hence cuspidal) for Γ_2 , so we have

$$\|g_{\gamma}\|_{\infty} \leq \|e_2\| \|g_{\gamma}\|_3$$

by definition. Hence for some γ_0 among the finitely many $\gamma \in \Gamma_1 \setminus \Gamma_2$, we have

$$\|f\|_{\infty} = \|g_{\gamma_0}\|_{\infty} \leq \|e_2\| \|g_{\gamma_0}\|_3 \leq \|e_2\| \left(\sum_{\gamma} \|g_{\gamma}\|_3^2 \right)^{1/2} = \|e_2\| \|f\|_3$$

by positivity, which proves the inequality $\|e_1\| \leq \|e_2\|$. The converse is obtained by considering functions (smooth, compactly supported) on F_1 which vanish on all γF_2 , $\gamma \neq 1$. \square

We now come to the second argument, which follows that of [2, App. B], keeping track of the dependency on the group and requires a simple hyperbolic lattice-point counting lemma which we take from [6].

Proposition 7. *For any integer $q \geq 3$, let Γ_q be the Hecke triangle group generated by*

$$(7) \quad \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \cos(\pi/q) \\ 0 & 1 \end{pmatrix}.$$

We have

$$\|e_{\Gamma_q}\| \ll \sqrt{q} \text{ and } C(\Gamma_q, \mu) \ll_{\mu} q$$

for all $q \geq 3$, the implied constant depending only on μ .

Since Γ_q is a maximal discrete subgroup (see e.g. [14]) and (with few exceptions for $q \leq 6$) they are pairwise non-commensurable [10], this statement is essentially disjoint from Lemma 6.

Let $G = SL(2, \mathbf{R})$, $\mathbf{H} = SL(2, \mathbf{R})/SO(2, \mathbf{R})$ (where $SO(2, \mathbf{R})$ is the stabilizer of $i \in \mathbf{H}$), $Y(q) = \Gamma_q \backslash SL(2, \mathbf{R})$, and $X(q) = \Gamma_q \backslash \mathbf{H}$. We have the commutative square

$$\begin{array}{ccc} G & \xrightarrow{\pi} & \mathbf{H} \\ p \downarrow & & \downarrow \\ Y(q) & \longrightarrow & X(q) \end{array}$$

and because of personal habit, we will work mostly on \mathbf{H} and $X(q)$. The upper-horizontal projection map π is simply $g \mapsto gi$.

A fundamental domain for Γ_q in \mathbf{H} is the interior of

$$F_q = \{z \mid |z| \geq 1 \text{ and } |x| \leq \cos(\pi/q)\}.$$

Let $S_q = \pi^{-1}(\{z \in F_q \mid \text{Im}(z) \geq 2\}) \subset G$ be a ‘‘Siegel domain’’ for Γ_q , and let $C_q = \pi^{-1}(F_q) - S_q$, so that C_q is compact.

Now we have the following lemma adapted from [2, App. B.3], where

$$B = \{g \in G \mid d(gi, i) < 1\}$$

is a fixed neighborhood of the identity in G (d denotes the hyperbolic metric on \mathbf{H}).

Lemma 8. *Let φ be a smooth cuspidal function on $Y(q)$. We have for $x \in F_q$*

$$\begin{aligned} |\varphi(x)| &\ll v(x)^{1/2} \|\varphi\|_3 \text{ if } g \in C_q \\ |\varphi(x)| &\ll o(x)v(x)^{1/2} \|\varphi\|_3 \text{ if } g \in S_q, \end{aligned}$$

where both implied constants are absolute, $o(x)$ is the length of the shortest horocycle through x and $v(x)$ is the maximal cardinality of a fiber $p_x^{-1}(y)$ of the map

$$p_x \begin{cases} B \rightarrow Y(q) \\ g \mapsto p(xg) \end{cases}$$

for $y \in Y(q)$.

Since $SO(2, \mathbf{R}) \subset B$, the function v is $SO(2, \mathbf{R})$ -invariant; so is in fact o . Our goal is now to estimate the L^∞ -norm of v and o^2v on $Y(q)$, or equivalently on $X(q)$. Let $x, y \in Y(q)$, say $y = p(\tilde{y})$. The fiber $p_x^{-1}(y)$ is given by

$$\begin{aligned} p_x^{-1}(y) &= \{g \in B \mid p(xg) = y\} \\ &= \{g \in B \mid xg = \gamma\tilde{y} \text{ for some } \gamma \in \Gamma_q\} \\ &\simeq \{\gamma \in \Gamma_q \mid x^{-1}\gamma\tilde{y} \in B\} \text{ (where } \simeq \text{ indicates the obvious bijection } g \mapsto \gamma) \\ &= \{\gamma \in \Gamma_q \mid d(\gamma\tilde{y}i, xi) < 1\}. \end{aligned}$$

Thus estimating $v(x)$ is essentially a hyperbolic lattice point problem. Those can be notoriously tricky, but we seek only an upper bound. To apply Lemma 2.11 of [6], we write

$$|p_x^{-1}(y)| = \{\gamma \in \sigma^{-1}\Gamma_q\sigma \mid d(\gamma\sigma^{-1}\tilde{y}i, \sigma^{-1}xi) < 1\}$$

hence by loc. cit. we have

$$(8) \quad v(x) \ll \text{Im}(\sigma^{-1}xi) + c^{-1} + (c \text{Im}(\sigma^{-1}xi))^{-1} + 1$$

with an absolute implied constant, where σ is a scaling matrix for the cusp ∞ of Γ_q , namely σ conjugates a generator of the stabilizer of ∞ in Γ_q to the standard parabolic element $z \mapsto z + 1$, and

$$c = \inf \left\{ c \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma^{-1}\Gamma_q\sigma \text{ for some } a, b, c \right\}.$$

Clearly one can take

$$\sigma = \begin{pmatrix} \sqrt{2 \cos(\pi/q)} & 0 \\ 0 & 1/\sqrt{2 \cos(\pi/q)} \end{pmatrix} \text{ acting by } z \mapsto 2z \cos(\pi/q)$$

and since $\sigma^{-1}F_q$ is easily seen to be the Ford polygon for $\sigma^{-1}\Gamma_q\sigma$ (see e.g. [6, p. 44]), we see from the geometric description of c^{-1} as the radius of the largest isometric circle in the Ford polygon (loc. cit., p. 53) that

$$c^{-1} = (2 \cos \pi/q)^{-1}, \text{ in particular } c \geq 1$$

and (8) and the value of σ give

$$(9) \quad v(x) \ll (2 \cos \pi/q)^{-1} \operatorname{Im}(xi) + \operatorname{Im}(xi)^{-1} + 1 \ll \operatorname{Im}(xi) + \operatorname{Im}(xi)^{-1} + 1.$$

We now apply this to deduce the required estimates:

Lemma 9. *We have*

$$v(x) \ll q \text{ for } x \in C_q \text{ and } o(x)^2 v(x) \ll 1 \text{ for } x \in S_q$$

with absolute implied constants.

Proof. First if $x \in C_q$ we get by (9)

$$v(x) \ll \operatorname{Im}(xi) + \operatorname{Im}(xi)^{-1} + 1 \ll (\sin(\pi/q))^{-1} \ll q$$

with an absolute implied constant, since $\sin(\pi/q) \leq \operatorname{Im}(xi) \leq 2$ by definition of C_q .

On the other hand, if $x \in S_q$ is in the Siegel domain, we observe that an horocycle through x is given by

$$O = \{u + i \operatorname{Im}(xi) \mid |u| \leq 2 \cos(\pi/q)\},$$

so that $o(x) \leq \text{length}(O) = 2 \cos(\pi/q) \operatorname{Im}(xi)^{-1} \leq 2 \operatorname{Im}(xi)$ by definition of the Poincaré metric. We get from (9)

$$o(x)^2 v(x) \ll \operatorname{Im}(xi)^{-1} + \operatorname{Im}(xi)^{-3} + \operatorname{Im}(xi)^{-2} \ll 1$$

(with an absolute implied constant) since $\operatorname{Im}(xi) \geq 2$ from the definition of S_q . \square

Putting together Lemma 9 with Lemma 8, we immediately get $\|e_{\Gamma_q}\| \ll \sqrt{q}$ hence Proposition 7.

3. PROOF OF THE COROLLARY

In addition to the notation in the statement of Corollary 2, we let $\psi_1(z) = \phi(l_1 z)$ and $\psi_2(z) = \overline{\phi(l_2 z)}$. It is well-known that ψ_1 is a Maass form on $\Gamma_0(q l_1)$ with the same Laplace-eigenvalue as ϕ , and a fortiori it is a Maass form on $\Gamma_0(q l_1 l_2)$, and so is ψ_2 . We consider the L^2 -normalizations

$$\tilde{\psi}_i = \frac{\psi_i}{\|\psi_i\|}$$

where $\|\cdot\|$ is the L^2 -norm on $\Gamma_0(q l_1 l_2) \backslash \mathbf{H}$. We want to estimate on average

$$c_i = (\Psi \mid \varphi_i) = (\psi_1 \psi_2 \mid \varphi_i) = \|\psi_1\| \|\psi_2\| (\tilde{\psi}_1 \tilde{\psi}_2 \mid \varphi_i).$$

One can either extend (easily) the method of [2] or proceed by simple polarization: let $\tilde{\psi} = \tilde{\psi}_1 \pm \tilde{\psi}_2$, where the sign is chosen so that $\tilde{\psi} \neq 0$. Then $\tilde{\psi}$ is a Maass form on $\Gamma_0(q l_1 l_2)$ with the same eigenvalue as ϕ and

$$\tilde{\psi}^2 = \tilde{\psi}_1^2 + \tilde{\psi}_2^2 + 2\tilde{\psi}_1 \tilde{\psi}_2, \quad (\text{taking the } + \text{ sign for instance})$$

hence

$$\begin{aligned} (\tilde{\psi}_1 \tilde{\psi}_2 \mid \varphi_i) &= \frac{1}{2} \left((\tilde{\psi}^2 \mid \varphi_i) - (\tilde{\psi}_1^2 \mid \varphi_i) - (\tilde{\psi}_2^2 \mid \varphi_i) \right) \\ &= \frac{1}{2} \left(\|\tilde{\psi}\|^2 \left(\left(\frac{\tilde{\psi}}{\|\tilde{\psi}\|} \right)^2, \varphi_i \right) - (\tilde{\psi}_1^2 \mid \varphi_i) - (\tilde{\psi}_2^2 \mid \varphi_i) \right) \end{aligned}$$

so (with obvious notation)

$$\frac{|c_i|^2}{\|\psi_1\|^2 \|\psi_2\|^2} \leq 12 \left\{ \left| c_i \left(\frac{\tilde{\psi}}{\|\tilde{\psi}\|} \right)^2 \right|^2 + |c_i(\tilde{\psi}_1)|^2 + |c_i(\tilde{\psi}_2)|^2 \right\}$$

and by Proposition 1 and Proposition 5 (which deals with the passage from $\Gamma_0(q)$ to $\Gamma_0(q l_1 l_2)$) we obtain

$$\sum_{|\lambda_i| \leq T} |c_i|^2 \exp\left(\frac{\pi}{2} |\lambda_i|\right) \ll \|\psi_1\|^2 \|\psi_2\|^2 (\log T)^3$$

the implied constant depending only on μ . Our Corollary 2 is thus a consequence of the following well-known formula:

Lemma 10. *Let ϕ be a Maass form on $\Gamma_0(q)$, $q \geq 1$, $l \geq 1$ and $d \geq 1$ two integers, $\psi(z) = \phi(lz)$. Then, seeing ψ as a Maass form on $\Gamma_0(qld) \backslash \mathbf{H}$, we have*

$$\|\psi\| = [\Gamma_0(q) : \Gamma_0(qld)]^{1/2} \|\phi\|$$

where on the left we have the L^2 -norm on $\Gamma_0(qld) \backslash \mathbf{H}$ and on the right the norm on $\Gamma_0(q) \backslash \mathbf{H}$.

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