

# ABOUT THE MINIMAL VALUE OF SOME POLYNOMIALS

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## Statements

Here are two problems raised by E. Kowalski; solving the second solves trivially the first one. They appeared as Problem 11155 in the May, 2005, issue of the American Math. Monthly.

(1) Let  $P_n$  denote the polynomial

$$P_n(x) = \sum_{j=0}^n \binom{n}{j}^2 x^{2j} (1-x)^{2(n-j)}.$$

Show that the minimal value of  $P_n$  on the closed interval  $[0, 1]$  is attained at  $x = \frac{1}{2}$ .

(2) With  $P_n$  as above and  $Q_n(u) = P_n(u + \frac{1}{2})$ , show that in the Taylor expansion

$$Q_n(u) = \sum_{p=0}^{2n} Q_{n,p} u^p$$

one has  $Q_{n,p} = 0$  if  $p$  is odd and  $Q_{n,p} \geq 0$  if  $p$  is even.

These polynomials arise as the variance of the derivative of the  $(n+1)$ -st Bernstein polynomial of a standard Brownian motion  $B(t)$ ,  $0 \leq x \leq 1$ . See [1, §5] for this and the application of the problem to a new simple proof that for fixed  $x$ , Brownian motion is almost surely not differentiable at  $x$  (with a proof of a statement weaker than (1) which is sufficient for this purpose).

## Solution

Here is a solution to the second problem, due to T. Rivoal.

Start from the identity

$$\sum_{j=0}^n (-1)^j \binom{n}{j}^2 x^j (1-x)^{n-j} = \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{n+j}{n} x^j,$$

(for instance, the right-hand side is the hypergeometric function or polynomial  $F(n+1, -n; 1; x)$  where  $F$  is the Gauss hypergeometric function, and the left-hand side is seen to be  $L_n(1-2x)$ , where  $L_n$  is the Legendre polynomial, hence both are seen to satisfy the same differential equation

$$x(1-x)u'' + (1-2x)u' + n(n+1)u = 0$$

by definition, and one concludes easily). Making the change of variable

$$\frac{x}{x-1} = \frac{u^2}{(1-u)^2}$$

one gets after simplification that

$$P_n(u) = (-1)^n \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{n+j}{n} u^{2j} (2u-1)^{n-j}.$$

Hence

$$\begin{aligned}
Q_n(v) &= P_n\left(v + \frac{1}{2}\right) = (-1)^n \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{n+j}{n} \left(v + \frac{1}{2}\right)^{2j} (2v)^{n-j} \\
&= (-1)^n \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{n+j}{n} \sum_{\ell=0}^{2j} \binom{2j}{\ell} 2^{\ell+(n-j)} v^{(2j-\ell)+(n-j)} \\
&= \sum_{p=0}^n \left[ (-1)^n 2^p \sum_{j=|n-p|}^n \binom{n}{j} \binom{n+j}{n} \binom{2j}{n+j-p} (-4)^{-j} \right] v^p.
\end{aligned}$$

The coefficient  $Q_{n,p}$  of  $v^p$  is then identified with a value of a hypergeometric function

$$\begin{aligned}
Q_{n,p} &= \frac{(-1)^p}{2^{2|n-p|-p}} \binom{n}{|n-p|} \binom{n+|n-p|}{n} \binom{2|n-p|}{n-p+|n-p|} \\
&\quad \times {}_4F_3 \left[ \begin{matrix} 1, \frac{1}{2} + |n-p|, 1+n+|n-p|, |n-p|-n \\ 1+|n-p|, 1+n-p+|n-p|, 1+p-n+|n-p| \end{matrix} \right].
\end{aligned}$$

(with the convention that in the hypergeometric function the value of  $z$  is 1), e.g. using Maple's call

$$\begin{aligned}
&\text{sum(binomial(n,j)*binomial(n+j,n)*} \\
&\quad \text{binomial(2*j,n+j-p)*(-4)^(-j),j=abs(n-p)..infinity)}
\end{aligned}$$

This  ${}_4F_3$  value is in fact a  ${}_3F_2$  because one of the bottom parameters is always equal to 1 and simplifies with the top 1.

We then need to discuss two cases, depending on whether  $n \geq p$  or  $n \leq p$ .

If  $n \geq p$ , we have

$$Q_{n,p} = \frac{(-1)^p}{2^{2n-3p}} \binom{n}{p} \binom{2n-p}{n} {}_3F_2 \left[ \begin{matrix} -p, 1+2n-p, \frac{1}{2}+n-p \\ 1+n-p, 1+2n-2p \end{matrix} \right].$$

If  $n \leq p$ , we have

$$Q_{n,p} = \frac{(-1)^p}{2^{p-2n}} \binom{n}{p-n} \binom{p}{n} {}_3F_2 \left[ \begin{matrix} -2n+p, 1+p, \frac{1}{2}+p-n \\ 1+p-n, 1+2p-2n \end{matrix} \right].$$

In both expressions one can use the summation formula of Watson (see e.g. [2, III.23, p. 245]):

$${}_3F_2 \left[ \begin{matrix} a, b, c \\ 1 + \frac{1}{2}a + \frac{1}{2}b, 2c \end{matrix} \right] = \Gamma \left[ \begin{matrix} \frac{1}{2}, c + \frac{1}{2}, \frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b, \frac{1}{2} - \frac{1}{2}a - \frac{1}{2}b + c \\ \frac{1}{2} + \frac{1}{2}a, \frac{1}{2} + \frac{1}{2}b, \frac{1}{2} - \frac{1}{2}a + c, \frac{1}{2} - \frac{1}{2}b + c \end{matrix} \right],$$

where the right-hand side is interpreted as

$$\frac{\Gamma(1/2)\Gamma(c+1/2)(\cdots)}{\Gamma(\frac{1}{2} + \frac{1}{2}a)\Gamma(\frac{1}{2} + \frac{1}{2}b)(\cdots)}.$$

This formula is valid under certain conditions on the complex parameters  $a, b, c$  which are satisfied here. In particular, if a  $\Gamma(-m)$  occurs in the denominator for some integer  $m \geq 0$ , this means the value of the expression is 0.

If  $n \geq p$ , we use Watson's formula with  $a = -p$ ,  $b = 1 + 2n - p$  and  $c = \frac{1}{2} + n - p$  and if  $n \leq p$ , we use it with  $a = p - 2n$ ,  $b = 1 + p$  et  $c = \frac{1}{2} + p - n$ . In the end, after simplification, we get

$$\begin{aligned}
Q_{n,p} &= \frac{(-1)^p}{2^{2n-3p}} \frac{\Gamma(1+2n-p)\Gamma(\frac{1}{2})^2}{\Gamma(1+p)\Gamma(1+\frac{2n-p}{2})^2 \Gamma(\frac{1}{2}-\frac{p}{2})^2} \quad \text{if } n \geq p \\
Q_{n,p} &= \frac{(-1)^p}{2^{p-2n}} \frac{\Gamma(1+p)\Gamma(\frac{1}{2})^2}{\Gamma(1+2n-p)\Gamma(1+\frac{p}{2})^2 \Gamma(\frac{1}{2}-\frac{2n-p}{2})^2} \quad \text{if } n \leq p.
\end{aligned}$$

If  $n$  is odd,  $Q_{n,p} = 0$  since there is a  $\Gamma(-m)^2$  with  $m$  integer in the denominator, and if  $n$  is even, we see that  $Q_{n,p} > 0$ .

#### REFERENCES

- [1] E. Kowalski, *Bernstein polynomials and Brownian motion*, American Math. Monthly, to appear.
- [2] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.

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