Modelization of a Split in a Ferromagnetic Body by an Equivalent Boundary Condition: Part I.

THE CLASSICAL CASE: NO SURFACE ENERGIES PRESENT.

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Abstract. We study the influence of a thin split over the dynamic evolution of a ferromagnetic body. A naive numerical simulation would require a huge number of cells to model the split. To avoid this problem, we introduce an equivalent boundary condition which is obtained by a Taylor expansion in the thickness of the split. We prove the existence of solution to this new problem and then rigorously establish the convergence of the expansion.

Introduction and notations

Ferromagnetism has been studied since antiquity. Nowadays, ferromagnetic materials are widely used in the industry[1]. Optimizing their form is important, as it strongly influences the magnetic behavior. Among the possible configurations, thin-layers[2] and multi-layers are the focus of recent research.

Ferromagnetic bodies may suffer from imperfections such as thin inclusions made of materials with different magnetic properties. Such imperfections strongly alter their behavior. It is important to estimate this alteration to either optimize the form or to compute the maximum tolerance for the fabrication of an object. Such an evaluation can be done via a numerical simulation. However, the thin imperfections are difficult to model as their thickness is typically an order of magnitude less than that of the size of the mesh. Using irregular meshes is unsuitable because it hampers the performance of the evaluation of the demagnetization field via Fast Fourier transform and multidimensional Toeplitz matrices, see S. Labbé and P. Leca [12] or S. Labbé [11]. Decreasing the step-size to match the thickness of the imperfections would prohibitively increase the computation requirements. The aim of this article is to provide an efficient mean of computing the influence of the thickness of the split. To compute the evolution of the magnetization, we will expand the magnetization up to the first order in the thickness of the split. Then, we will derive from this expansion an equivalent boundary condition.

In this article, we consider a simple geometry : two cylindrical ferromagnetic bodies, Figure 1, separated by a thin non-magnetic plane spacer with a small but nonzero thickness. As we may later want to extend the theory to more general geometries, we should avoid, to the maximum possible extent methods depending too strongly on this geometry. Especially, we try to avoid scaling methods. The study of the influence of interactions able to cross the split, such as super-exchange or surface anisotropy[3] to part II [21] of this article.

[1]Among applications of ferromagnetism, we find hard disk, radar protection.

[2]See [3], [14] for some results.

[3]Interested readers may consult [13], [14], and [10], for an introduction to this phenomenon, and [22] for the proof of existence of solutions.

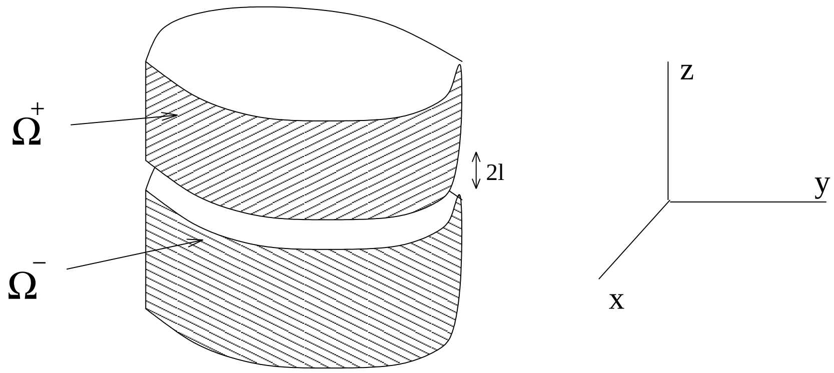


Figure 1: The considered domains

The following notations will be used.

ε the half thickness of the split.

B a bounded convex open set of $\mathbb{R}^2$, with a smooth boundary.

L^+, L^- two nonzero positive numbers.

$\Omega_\varepsilon^+ = B \times (\varepsilon, L^+)$ and $\Omega_\varepsilon^- = B \times (-L^-, -\varepsilon)$ for all $\varepsilon < \min(L^-, L^+)/2$ are domains filled with ferromagnetic material.

$\Omega^+ = B \times (0, L^+)$ and $\Omega^- = B \times (-L^-, 0)$.

$\Omega = \Omega^+ \cup \Omega^-$ and $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$ for all $\varepsilon < \min(L^-, L^+)/2$.

$\mathcal{I}_\varepsilon = (-L^-, -\varepsilon) \cup (\varepsilon, L^+)$.

$Q_T^\varepsilon = \Omega_\varepsilon \times (0, T)$ for all $\varepsilon < \min(L^-, L^+)/2$ and $Q_T = \Omega \times (0, T)$.

$\Gamma = B \times \{\pm 0\}$.

$\mathfrak{I}_\varepsilon = B \times (-\varepsilon, \varepsilon)$.

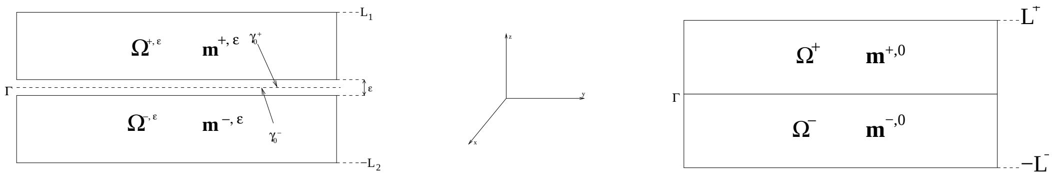


Figure 2: Initial domain and approximate domain

In this simple case, we want to make the simulation over Ω instead of Ω_ε using an approximate model.

First we introduce in section 1 the micro-magnetic model of Brown [4]. Then, we introduce our original model and its formal expansion up to the first order in section 2. We obtain formally the equations satisfied by the terms of order 0 and 1 and in particular the equivalent boundary condition. Then, we prove, using Galerkin's method, the existence and uniqueness of strong solutions to these equations in section 2.3. The convergence is established in section 3. Eventually, we supply some numerical simulations in section 4.

1 The mathematical model

1.1 Some qualitative properties of ferromagnetic materials

The micro-magnetic theory[4] models magnetism at the mesoscopic scale where physical values are mean values over small domains occupied by several millions of atoms. The magnetic state of a ferromagnetic body is given by its magnetization denoted by $\boldsymbol{M}$, a vector field null outside the body. The magnetic excitation, denoted by $\boldsymbol{H}$, characterize the magnetic state of a point in the space. Typically, $\boldsymbol{H}$ is the sum of an applied exterior $\boldsymbol{H}_{\text{app}}$ and a field created by the magnetization $\boldsymbol{H} = \mathcal{H}(\boldsymbol{M})$. The most common form of contributions to $\mathcal{H}(\boldsymbol{M})$ are given in section 1.2. Another relation between $\boldsymbol{H}$ and $\boldsymbol{M}$ models the influence of $\boldsymbol{H}$ on $\boldsymbol{M}$. For some materials, this relation is linear $\boldsymbol{M} = \chi\boldsymbol{H}$, which happens in paramagnetic, $\chi > 0$, and diamagnetic, $\chi < 0$, materials. In contrast, ferromagnetic materials can have a nonzero magnetization even in absence of any exterior excitation. They also have an hysteresis cycle. Moreover, the resulting magnetization and thus the magnetic radiation of ferromagnetic materials is some order of magnitude larger than those of para- or dia-magnetic materials. This behavior can only be explained by quantum mechanics. $\boldsymbol{M}$ being a local mean of the spins of electrons. Recall forces ensure that the spin of neighboring electrons are parallel enough for the magnetization to exist at the mesoscopic scale. Inside the ferromagnetic body, the modulus of the magnetization has a constant local norm M_s inside and is null outside. The evolution of $\boldsymbol{M}$ is modeled by the Landau-Lifshitz equation [16] and equation (1.5a). Hereafter, we denote by $\boldsymbol{m}$ the dimensionless variable[5] $\boldsymbol{m} = \boldsymbol{M}/M_s$ with local norm 1, and the dimensionless excitation $\boldsymbol{h} = \boldsymbol{H}/(\mu_0 * M_s)$, where μ_0 is the magnetic permeability.

1.2 Energies and associated operators

To study ferromagnetic materials, Brown [4] introduced the internal energies of ferromagnetism. These energies allow a complete study of the static problem. The equilibrium states are local minima of the energy among the magnetic states verifying the non-convex constraint (1.6a).

To each interaction, we associate both an energy and a symmetric operator. We use the following notations

- E_p, the energy associated to a contribution p.
- $\mathcal{H}_p$ the operator associated to E_p by

$$\mathrm{E}_p(0) = 0, \qquad \mathrm{DE}_p(\boldsymbol{m})\cdot\boldsymbol{v} = -\int \mathcal{H}_p(\boldsymbol{m})\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}. \tag{1.1}$$

- $\boldsymbol{h}_p = \mathcal{H}_p(\boldsymbol{m})$ the magnetic excitation associated to interaction p.

We model only the exchange, anisotropy and demagnetization field interactions. Their expressions are

$$\begin{aligned}
\mathcal{H}_e(\boldsymbol{m}) &= A\triangle\boldsymbol{m} && \text{exchange},\\
\mathcal{H}_a(\boldsymbol{m}) &= -\mathbf{K}\boldsymbol{m} && \text{anisotropy},\\
\mathcal{H}_d(\boldsymbol{m}) &= \boldsymbol{h}_d \text{ verifying } \begin{cases} \operatorname{div}(\boldsymbol{m}+\boldsymbol{h}_d) = 0,\\ \mathbf{rot}(\boldsymbol{h}_d) = 0,\end{cases} && \text{demagnetization field},
\end{aligned} \tag{1.2}$$

[4] For an introduction to micromagnetism, see Brown [4].

[5] See [11] for an explanation on the relation between physical and dimensionless variables.

where $\mathbf{K}$ is a smooth application from $\overline{\mathcal{O}}$ to the set of definite positive symmetric matrices satisfying a uniform coercivity property. A common form of anisotropy is the uniaxis one, $\mathbf{K}\boldsymbol{m} = K(\boldsymbol{m} - (\boldsymbol{m}\cdot\boldsymbol{u})\boldsymbol{u})$ where $\boldsymbol{u}$ is the privileged direction of magnetization and K isthe anisotropy constant. The expressions of the energies are

$$\begin{aligned} \mathrm{E}_e(\boldsymbol{m}) &= \tfrac{A}{2}\int_{\mathcal{O}}|\nabla\boldsymbol{m}|^2\mathrm{d}\boldsymbol{x} && \text{exchange},\\ \mathrm{E}_a(\boldsymbol{m}) &= \tfrac{1}{2}\int_{\mathcal{O}}(\mathbf{K}\boldsymbol{m})\cdot\boldsymbol{m}\mathrm{d}\boldsymbol{x} && \text{anisotropy},\\ \mathrm{E}_d(\boldsymbol{m}) &= \tfrac{1}{2}\int_{\mathbb{R}^3}|\mathcal{H}_d(\boldsymbol{m})|^2\mathrm{d}\boldsymbol{x}, \\ &= -\tfrac{1}{2}\int_{\mathcal{O}}\boldsymbol{m}\cdot\mathcal{H}_d(\boldsymbol{m})\mathrm{d}\boldsymbol{x} && \text{demagnetization field}, \end{aligned} \tag{1.3}$$

where $\mathcal{O}$ represents a generic open bounded subset of $\mathbb{R}^3$. We keep this notation in this article. We also define the total excitation and energy operator

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_e + \mathcal{H}_d, \qquad \mathrm{E} = \mathrm{E}_a + \mathrm{E}_e + \mathrm{E}_d. \tag{1.4}$$

1.3 The dynamic model: the Landau-Lifshitz equation

The Landau-Lifshitz equation models the evolution of the magnetization [16].

$$\frac{\partial\boldsymbol{m}}{\partial t} = -\boldsymbol{m}\times\mathcal{H}(\boldsymbol{m}) - \alpha(\boldsymbol{m}\times(\boldsymbol{m}\times\mathcal{H}(\boldsymbol{m}))) \text{ in } \mathcal{O}\times(0,T), \tag{1.5a}$$

with Neumann homogenous boundary condition

$$\frac{\partial\boldsymbol{m}}{\partial\boldsymbol{\nu}} = 0 \text{ on } \partial\mathcal{O}\times(0,T), \tag{1.5b}$$

and initial condition

$$\boldsymbol{m}(\cdot,0) = \boldsymbol{m}_0 \text{ in } \mathcal{O}, \tag{1.5c}$$

and the constraints

$$|\boldsymbol{m}| = 1 \text{ in } \mathcal{O}\times(0,T), \qquad \boldsymbol{m} = 0 \text{ in } (\mathbb{R}^3\setminus\mathcal{O})\times(0,T).$$

Multiplying scalarly equation (1.5b) by $\boldsymbol{m}$ or $\mathcal{H}(\boldsymbol{m})$ yields

$$\frac{\mathrm{d}|\boldsymbol{m}|^2}{\mathrm{d}t} = 0, \tag{1.6a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{E}(\boldsymbol{m})\right) = -\alpha\|\boldsymbol{m}\times\mathcal{H}(\boldsymbol{m})\|^2_{\mathbb{L}^2(\mathcal{O})}. \tag{1.6b}$$

Formally, the local norm remains constant and the energy decreases over time, which is in accordance with the qualitative model and the physical observations.

2 The limit problem

In this article, $\mathbb{L}^p(\mathcal{O}) = (\mathrm{L}^p(\mathcal{O}))^3$. We also denote by $\mathrm{W}^{s,p}(\mathcal{O})$ the Sobolev spaces as defined in Adams [1].

Definition 2.1. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. Let $s \geq 0$ and $1 \leq p \leq +\infty$. Then, if s is an integer, $s = m$, $\mathrm{W}^{m,p}(\mathcal{O})$ is the space whose derivatives of order up to m are in $\mathrm{L}^p(\mathcal{O})$ and

$$\|u\|_{\mathrm{W}^{m,p}(\mathcal{O})} = \sum_{|\alpha|\leq m} \|\mathrm{D}^\alpha u\|_{\mathrm{L}^p(\mathcal{O})}.$$

Otherwise, if $s = m+\sigma$ with m an integer and $0 < \sigma < 1$, $\mathrm{W}^{s,p}(\mathcal{O})$ is the subset of $\mathrm{W}^{m,p}(\mathcal{O})$ for which the following quantity is finite

$$\|u\|_{\mathrm{W}^{s,p}(\mathcal{O})} = \|u\|_{\mathrm{W}^{m,p}(\mathcal{O})} + \left(\sum_{|\alpha|=m} \int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|\mathrm{D}^\alpha u(\boldsymbol{x}) - \mathrm{D}^\alpha u(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{n+\sigma p}} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{p}}$$

We define $\mathbb{W}^{s,p}(\mathcal{O}) = (\mathrm{W}^{s,p})^3(\mathcal{O})$. We denote by $\mathrm{H}^s(\mathcal{O})$ the Sobolev space $\mathrm{W}^{s,2}(\mathcal{O})$ and define $\mathbb{H}^s(\mathcal{O})$ as $(\mathrm{H}^s(\mathcal{O}))^3$. For $s > \frac{3}{2}$, we define

$$\widetilde{\mathrm{H}}^s(\mathcal{O}) = \left\{u \in \mathrm{H}^s(\mathcal{O}), \frac{\partial u}{\partial \boldsymbol{\nu}} = 0\right\} \tag{2.1}$$

and $\widetilde{\mathbb{H}}(\mathcal{O})$ as $(\widetilde{\mathrm{H}}(\mathcal{O}))^3$. We also define anisotropic Sobolev spaces

$$\mathrm{H}^{r,s}(B \times (0,T)) = \mathrm{H}^s(0,T;\mathrm{L}^2(B)) \cap \mathrm{L}^2(0,T;\mathrm{H}^r(B)), \tag{2.2}$$

$$\mathbb{H}^{r,s}(B \times (0,T)) = (\mathrm{H}^{r,s}(B \times (0,T)))^3 = \mathrm{H}^s(0,T;\mathbb{L}^2(B)) \cap \mathrm{L}^2(0,T;\mathbb{H}^r(B)). \tag{2.3}$$

By $\widetilde{1}_{\mathcal{O}}$ we denote the characteristic function of the set $\mathcal{O}$.

2.1 The physical problem

We compare the solutions to Landau-Lifshitz system (1.5) with different ferromagnetic domains, Ω or Ω_ε. We search the solutions in the Sobolev spaces $\mathbb{H}^s(\mathcal{O})$. First, for $\varepsilon > 0$ we consider a sequence of initial conditions $\boldsymbol{m}_0^{\varepsilon,(0)}$ belonging to $\widetilde{\mathbb{H}}^2(\Omega_\varepsilon)$, $|\boldsymbol{m}_0^{\varepsilon,(0)}| = 1$. We suppose that there exists $\boldsymbol{m}_0^{(0)}$ in $\widetilde{\mathbb{H}}^2(\Omega)$, and $\boldsymbol{m}_0^{(1)}$ in $\mathbb{H}^1(\Omega)$ such that

$$\|\boldsymbol{m}_0^{(0)} - \boldsymbol{m}_0^{\varepsilon,(0)}\|_{\mathbb{H}^2(\Omega_\varepsilon)} = O(1), \quad \|\boldsymbol{m}_0^{(0)} - \boldsymbol{m}_0^{\varepsilon,(0)}\|_{\mathbb{H}^1(\Omega_\varepsilon)} = O(\varepsilon), \tag{2.4a}$$

$$\frac{\boldsymbol{m}_0^{(0),\varepsilon} - \boldsymbol{m}_0^{(0)}}{\varepsilon} \to \boldsymbol{u}^1_{|\Omega_{\varepsilon_0}} = \boldsymbol{m}_0^{(1)} \text{ weakly in } \mathbb{H}^1(\Omega_{\varepsilon_0}) \text{ for all } \varepsilon_0 > 0. \tag{2.4b}$$

For all $\varepsilon > 0$, we define $\boldsymbol{m}^{\varepsilon,(0)}$ as the solution to the Landau-Lifshitz equation over Ω_ε with initial condition $\boldsymbol{m}_0^{\varepsilon,(0)}$. We also define $\boldsymbol{m}^{(0)}$ as the solution to the Landau-Lifshitz equation over Ω with initial condition $\boldsymbol{m}_0^{(0)}$. These solutions exist by Theorem 3.4 in [22].

Remark 2.2. The construction of such operators must be done for each geometry on a case by case basis. For our simple geometry, a scaling construction works. For example, we may use

$$(x,y,z,t) \mapsto \begin{cases} \zeta(z)\boldsymbol{m}_0^{(0)}(x,y,\varepsilon + \frac{L^+-\varepsilon}{L^+}z,t) & \text{on } \Omega_\varepsilon^+, \\ \zeta(z)\boldsymbol{m}_0^{(0)}(x,y,-\varepsilon + \frac{L^--\varepsilon}{L^-}z,t) & \text{on } \Omega_\varepsilon^-, \end{cases}$$

where ζ is a smooth real function with a compact support included in $(-L^-, L^+)$ and value 1 in $(-L^-/2, L^+/2)$.

2.2 Expansion up to the first order

We develop $\boldsymbol{m}^{\varepsilon,(0)}$ up to the first order in ε. Formally $\boldsymbol{m}^{\varepsilon,(0)} = \chi_{B\times\mathcal{I}_\varepsilon}\boldsymbol{m}^{(0)} + \varepsilon\boldsymbol{m}^{(1)}$. Thus, if we develop (1.5a) and isolate terms of order 0 and 1 in ε, we obtain

$$\frac{\partial \boldsymbol{m}^{(0)}}{\partial t} = -\boldsymbol{m}^{(0)} \times \mathcal{H}(\boldsymbol{m}^{(0)}) - \alpha\boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}(\boldsymbol{m}^{(0)})), \tag{2.5a}$$

$$|\boldsymbol{m}^{(0)}| = 1 \text{ in } \Omega \times (0,T), \tag{2.5b}$$

$$\frac{\partial \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}} = 0 \text{ on } \partial\Omega \times (0,T), \tag{2.5c}$$

$$\boldsymbol{m}^{(0)}(\cdot,0) = \boldsymbol{m}_0^{(0)}. \tag{2.5d}$$

And

$$\begin{aligned}\frac{\partial \boldsymbol{m}^{(1)}}{\partial t} = &-\boldsymbol{m}^{(1)} \times \mathcal{H}(\boldsymbol{m}^{(0)}) - \boldsymbol{m}^{(0)} \times \mathcal{H}(\boldsymbol{m}^{(1)}) - \alpha\boldsymbol{m}^{(1)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}(\boldsymbol{m}^{(0)})) \\ &- \alpha\boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(1)} \times \mathcal{H}(\boldsymbol{m}^{(0)})) - \alpha\boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}(\boldsymbol{m}^{(1)}))\end{aligned} \tag{2.6a}$$

$$\boldsymbol{m}^{(0)} \cdot \boldsymbol{m}^{(1)} = 0, \tag{2.6b}$$

$$\frac{\partial \boldsymbol{m}^{(1)}}{\partial \boldsymbol{\nu}} = \begin{cases} \frac{\partial^2 \boldsymbol{m}^0}{\partial \boldsymbol{\nu}^2} & \text{on } \Gamma^+ \times (0,T), \\ \frac{\partial^2 \boldsymbol{m}^0}{\partial \boldsymbol{\nu}^2} & \text{on } \Gamma^- \times (0,T), \\ 0 & \text{on } (\partial\Omega \setminus \Gamma) \times (0,T), \end{cases} \tag{2.6c}$$

$$\boldsymbol{m}^{(1)}(\cdot,0) = \boldsymbol{m}_0^{(1)}, \tag{2.6d}$$

where $\boldsymbol{\nu}$ is the normal exterior on the boundary. In equation (2.6a), we denote by $\mathrm{d}\sigma(\Gamma))$ the surface measure of Γ, $\gamma^0\boldsymbol{v}$ is in $\mathbb{H}^{\frac{1}{2}}(\Gamma)$ and $\gamma^0\boldsymbol{v}\mathrm{d}\sigma(\Gamma)$ is in $\mathbb{H}^{-\frac{1}{2}}(\Gamma)$. Formally, $\mathcal{H}_d(\gamma^0\boldsymbol{v}\mathrm{d}\sigma(\Gamma))$ is the limit of $\frac{1}{\varepsilon}\mathcal{H}_d(\widetilde{1}_{B\times(-\varepsilon,\varepsilon)}\boldsymbol{m}^{(0)})$ as ε tends to 0. The limit will be justified by Lemma 2.11. Equality (2.6c) is formally derived from

$$0 \approx \frac{\partial(\boldsymbol{m}^{(0)} + \varepsilon\boldsymbol{m}^{(1)})}{\partial z}(,\cdot,\cdot,\varepsilon,\cdot) \approx \varepsilon\left(\frac{\partial^2\boldsymbol{m}^{(0)}}{\partial z^2} + \frac{\partial \boldsymbol{m}^{(1)}}{\partial z}(\cdot,\cdot,0,\cdot)\right),$$

where z is the third variable of space.

2.3 Existence and uniqueness theorems

2.3.1 Inequalities on cylindrical domains and miscellaneous results

We recall some inequalities needed to prove the theorems. In some cases, we prove that Ω and Ω_ε are sufficiently smooth for such inequalities to hold. It is sufficient to verify these inequalities in each connected part, thus in domains of the kind $B \times (0,L)$. In this part, we denote by $\mathcal{O}$ the set $B \times (0,L)$ where B is an open convex bounded set of $\mathbb{R}^2$ with a smooth boundary.

Lemma 2.3 (Elliptic Regularity). *Let* $v \in \mathrm{H}^1_\triangle(\mathcal{O}) = \left\{v \in \mathrm{L}^2(\mathcal{O}) \mid \triangle v \in \mathrm{L}^2(\mathcal{O}), \frac{\partial v}{\partial \boldsymbol{\nu}} = 0\right\}$, *then* v *belongs to* $\mathrm{H}^2(\mathcal{O})$, *and there exists a constant* C *not depending on* $\mathcal{O}$ *such that*

$$\|v\|_{\mathrm{H}^2(\mathcal{O})} \leq C(\|v\|_{\mathrm{L}^2(\mathcal{O})} + \|\triangle v\|_{\mathrm{L}^2(\mathcal{O})}),$$

for all v *in* $\mathrm{H}^2(\mathcal{O})$.

PROOF : $\mathcal{O}$ is convex. Elliptic regularity holds for all convex domain with a constant independent of the open set. See [9]. □

Lemma 2.4 (Sobolev injections)**.** *Sobolev embeddings hold for domain $\mathcal{O}$. The constants can be chosen independent of L as long as $L > L_0 > 0$.*

PROOF : Those domains verify the cone property, i.e. there exists a fixed cone C such that, for all point $\boldsymbol{x}$ in $\mathcal{O}$, there exists a rotation $\mathbf{R}$ with $\boldsymbol{x} + \mathbf{R}C \subset \mathcal{O}$. Therefore Sobolev embeddings hold. See Adams [1], Theorem 5.4. □

Lemma 2.5 (Gagliardo-Nirenberg)**.** *There exists $C > 0$ such that*

$$\|\nabla v\|_{\mathbb{L}^4(\mathcal{O})} \leq C\|v\|^{\frac{1}{2}}_{\mathrm{L}^\infty(\mathcal{O})}\|v\|^{\frac{1}{2}}_{\mathrm{H}^2(\mathcal{O})},$$

for all v in $\mathrm{H}^2(\mathcal{O})$. Moreover, the constant remains bounded as long as $L > L_0 > 0$.

PROOF : Those domains verify the cone property and therefore Gagliardo-Nirenberg inequality. See Maz'ja [20] page 69-70. □

Lemma 2.6. *Let*

$$X(\mathcal{O}) = \{\boldsymbol{u} \in \mathbb{L}^2(\mathcal{O}), \mathrm{div}\boldsymbol{u} \in \mathrm{L}^2(\mathcal{O}), \mathbf{rot}\boldsymbol{u} \in \mathbb{L}^2(\mathcal{O}), \boldsymbol{u}\cdot\boldsymbol{\nu} = 0\}.$$

Then, $X(\mathcal{O}) = \{\boldsymbol{u} \in \mathbb{H}^1(\mathcal{O}), \boldsymbol{u}\cdot\boldsymbol{\nu} = 0\}$. There exists $C > 0$ such that for all $\boldsymbol{u}$ in $X(\mathcal{O})$,

$$\|\boldsymbol{u}\|_{\mathbb{H}^1(\mathcal{O})} \leq C\left(\|\boldsymbol{u}\|_{\mathbb{L}^2(\mathcal{O})} + \|\mathrm{div}\boldsymbol{u}\|_{\mathrm{L}^2(\mathcal{O})} + \|\mathbf{rot}\boldsymbol{u}\|_{\mathbb{L}^2(\mathcal{O})}\right).$$

Moreover, C can be chosen bounded as long as $L > L_0 > 0$.

PROOF : This result is well known for bounded sets with smooth boundaries, see [6]. We generalize the result to cylindrical open sets with a smooth lateral boundary. For $\boldsymbol{u} = [u_x, u_y, u_z]$ in $X(\mathcal{O})$, we define

$$\overline{\boldsymbol{u}}(x,y,z) = \begin{cases} [u_x, u_y, -u_z]\,(x,y,L-z) & \text{on } B\times(L,2L),\\ [u_x,u_y,u_z]\,(x,y,z) & \text{on } B\times(0,L),\\ [u_x,u_y,-u_z]\,(x,y,-z) & \text{on } B\times(-L,0).\end{cases}$$

Thus, $\overline{\boldsymbol{u}}$ belongs to $X(B\times(-L,2L))$. Let ζ be a smooth real function on $\mathbb{R}$ such that

$$\zeta = \begin{cases} 1 & \text{on } \left(-\frac{1}{4}L, \frac{5}{4}L\right),\\ 0 & \text{on } \complement\left(-\frac{1}{2}L, \frac{3}{2}L\right).\end{cases}$$

There exists a bounded open set $\mathcal{O}_2$ with a smooth boundary such that $B\times\left(-\frac{1}{2}L, \frac{3}{2}L\right) \subset \mathcal{O}_2 \subset B\times(-L,2L)$. Thus, $\boldsymbol{x} \mapsto \zeta(z)\overline{\boldsymbol{u}}(\boldsymbol{x})$ belongs to $X(\mathcal{O}_2)$. We apply the already known result on $\mathcal{O}_2$. $\zeta\overline{\boldsymbol{u}}$ belongs to $\mathbb{H}^1(\mathcal{O}_2)$. The restriction $\boldsymbol{u}$ belongs to $\mathbb{H}^1(\mathcal{O})$. The constants depend only on the L^∞ norms of ζ, ζ'. With a good choice of ζ, the constant can be chosen bounded as long as $L \geq L_0 > 0$. □

We generalize the previous lemma. Let $\mathrm{H}^{m-\frac{1}{2}}_{\mathrm{morc}}(\partial\mathcal{O})$ be the set of functions belonging to $\mathrm{L}^2(\partial\mathcal{O})$ whose restrictions on $\partial B\times(0,L)$, $B\times\{0\}$ and $B\times\{L\}$ belong to $\mathrm{H}^{m-\frac{1}{2}}$.

Lemma 2.7. *The following trace application*

$$\gamma^1 : \mathrm{H}^2(\mathcal{O}) \to \mathrm{H}^{\frac{1}{2}}\Big(B\times\{0\}\Big)\times \mathrm{H}^{\frac{1}{2}}\Big(\partial B\times(0,L)\Big)\times \mathrm{H}^{\frac{1}{2}}\Big(B\times\{L\}\Big) = \mathrm{H}^{\frac{1}{2}}(\partial\mathcal{O}),$$
$$u\mapsto \left(\frac{\partial u}{\partial z}, \frac{\partial u}{\partial \boldsymbol{\nu}}, -\frac{\partial u}{\partial z}\right),$$

is onto and has a continuous right inverse.

PROOF : By local map and partition of the unity, we reduce the problem to the half plane and its trace map

$$\gamma^1 : \mathrm{H}^2(\mathbb{R}_x\times\mathbb{R}_y^+\times\mathbb{R}_z^+) \to \mathrm{H}^{\frac{1}{2}}(\mathbb{R}_x\times\mathbb{R}_y^+)\times \mathrm{H}^{\frac{1}{2}}(\mathbb{R}_x\times\mathbb{R}_z^+)$$
$$\boldsymbol{u}\mapsto \left(\frac{\partial u}{\partial y}(\cdot,0,\cdot), \frac{\partial u}{\partial z}(\cdot,\cdot,0)\right),$$

We define

$$\gamma^0 : \mathrm{H}^2(\mathbb{R}_x\times\mathbb{R}_y^+\times\mathbb{R}_z^+) \to \mathrm{H}^{\frac{3}{2}}(\mathbb{R}_x\times\mathbb{R}_y^+)\times \mathrm{H}^{\frac{3}{2}}(\mathbb{R}_x\times\mathbb{R}_z^+)$$
$$\boldsymbol{u}\mapsto (u(\cdot,0,\cdot), u(\cdot,\cdot,0))$$

By Lions-Magenes [18], f_0, g_0, f_1, g_1 is in the image of $\mathrm{H}^2(\Omega)$ by γ^0,γ^1 if and only if $f_0(\cdot,0) = g_0(\cdot,0)$ and $\int|g_1(\cdot,\sigma)-\partial_y f_0(\cdot,\sigma)|\frac{\mathrm{d}\sigma}{\sigma}$ and $\int|f_1(\cdot,\sigma)-\partial_z g_0(\cdot,\sigma)|\frac{\mathrm{d}\sigma}{\sigma}$ are finite. There are no direct compatibility relations between f_1 and g_1, thus we only need to construct f_0 and g_0. Given f_1, we define g_0 as $\widehat{g_0}(\xi,y) = \zeta(y\sqrt{1+|\xi|^2})\int_0^y \widehat{f_1}(\xi,z)\mathrm{d}z$, where ζ is a smooth real function satisfying, $0\le\zeta\le 1$, with $\mathrm{Supp}(\zeta)\subset[0,2]$ and $\zeta = 1$ in $[0,1]$. We construct f_0 by the same formula and f_0, g_0, f_1, g_1 has satisfy all compatibility relations and the map γ^1 is thus onto. As every closed set of a Hilbert spaces has a topological supplementary, there is a right inverse □

Obviously, the result extends to the vectorial case and $\gamma^1(\mathbb{H}^2(\mathcal{O})) = \mathbb{H}^{\frac{1}{2}}_{\mathrm{morc}}(\partial\mathcal{O})$. The following theorem was proved by C. Foias and R. Temam when $\mathcal{O}$ is a bounded open set with a smooth boundary.

Theorem 2.8. *Let* $m\ge 1$. *Let*

$$X^m(\mathcal{O}) = \{\boldsymbol{u}\in\mathbb{L}^2(\mathcal{O}), \mathrm{div}\boldsymbol{u}\in\mathrm{H}^{m-1}(\mathcal{O}), \mathbf{rot}\boldsymbol{u}\in\mathbb{H}^{m-1}(\mathcal{O}), \boldsymbol{u}\cdot\boldsymbol{\nu}\in\mathrm{H}^{m-\frac{1}{2}}_{\mathrm{morc}}(\partial\mathcal{O})\}.$$

Then, $X^m(\mathcal{O}) = \mathbb{H}^m(\mathcal{O})$. *And there exists a constant* $C>0$ *such that*

$$\|\boldsymbol{u}\|_{\mathbb{H}^m(\mathcal{O})}\le C\left(\|\boldsymbol{u}\|_{\mathbb{L}^2(\mathcal{O})} + \|\mathrm{div}\boldsymbol{u}\|_{\mathrm{H}^{m-1}(\mathcal{O})} + \|\mathbf{rot}\boldsymbol{u}\|_{\mathbb{H}^{m-1}(\mathcal{O})} + \|\boldsymbol{u}\cdot\boldsymbol{\nu}\|_{\mathrm{H}^{m-\frac{1}{2}}(\partial\mathcal{O})}\right).$$

The constant C *can be chosen independently of* L *as long as* $L\ge L_0>0$.

PROOF : The proof is adapted from Foias-Temam [7], with no fundamental changes in this case. We proceed by recursion over m.

1. If $m=1$. Let G be the closed subset of $\mathbb{L}^2(\mathcal{O})$ of all gradients of real functions in $\mathrm{H}^1(\mathcal{O})$. The orthogonal of G is HD $= \{\boldsymbol{f}\in\mathbb{L}^2(\mathcal{O}), \mathrm{div}(\boldsymbol{f}) = 0, \boldsymbol{f}\cdot\boldsymbol{\nu} = 0\}$. Let $\boldsymbol{u}$ be in $X^1(\mathcal{O})$, let ∇p the orthogonal projection of $\boldsymbol{u}$ onto G. Then, p is defined by

$$\triangle p = \mathrm{div}(\boldsymbol{u}) \text{ in } \mathcal{O},$$
$$\frac{\partial p}{\partial\boldsymbol{\nu}} = \boldsymbol{u}\cdot\boldsymbol{\nu} \text{ in } \partial\mathcal{O}.$$

   Thus, by elliptic regularity, p belongs to $\mathrm{H}^2(\mathcal{O})$ since $\boldsymbol{u}\cdot\boldsymbol{\nu}$ belongs to $\gamma^1(\mathrm{H}^2(\mathcal{O}))$. Moreover, $\boldsymbol{u}-\nabla p$ verifies the hypothesis of Lemma 2.6. Thus $\boldsymbol{u}-\nabla p$ belongs to $\mathbb{H}^1(\mathcal{O})$ as well as $\boldsymbol{u}$.

2. If $m > 1$. Suppose the theorem holds for $m-1$. By the recursion hypothesis, $\boldsymbol{u}$ belongs to $\mathbb{H}^{m-1}(\mathcal{O})$. Let D^{m-1} be a differential operator with an order $m-1$, then $\boldsymbol{v} = \mathrm{D}^{m-1}\boldsymbol{u}$ belongs to $\mathbb{L}^2(\mathcal{O})$, $\mathrm{div}(\boldsymbol{v})$ belongs to $\mathrm{L}^2(\mathcal{O})$ and $\mathbf{rot}(\boldsymbol{v})$ belongs to $\mathbb{L}^2(\mathcal{O})$. Furthermore,

$$\boldsymbol{v}\cdot\boldsymbol{\nu} = \mathrm{D}^{m-1}(\boldsymbol{u}\cdot\boldsymbol{\nu}) - \sum_{i=1}^{m-1} \mathrm{D}^i\boldsymbol{\nu}\mathrm{D}^{m-1-i}\boldsymbol{u} \in \mathrm{H}^{\frac{1}{2}}_{\mathrm{morc}}(\partial\mathcal{O}).$$

Thus, $\boldsymbol{v}$ belongs to $\mathbb{H}^1(\mathcal{O})$ and $\boldsymbol{u}$ belongs to $\mathbb{H}^m(\mathcal{O})$.

□

Corollary 2.9. *Let $m \geq 1$. The set $\left\{v \in \mathrm{H}^1(\mathcal{O}), \triangle v \in \mathrm{H}^{m-1}(\mathcal{O}), \frac{\partial v}{\partial \boldsymbol{\nu}} = 0\right\}$ is equal to $\widetilde{\mathrm{H}}^{m+1}(\mathcal{O})$. Moreover, there exists a constant C depending only on $\mathcal{O}$ such that*

$$\|v\|_{\mathrm{H}^{m+1}(\mathcal{O})} \leq C\left(\|v\|_{\mathrm{L}^2} + \|\nabla v\|_{\mathbb{L}^2} + \|\triangle v\|_{\mathrm{H}^{m-1}}\right).$$

Moreover, C remains bounded as long as $L > L_0 > 0$.

PROOF : Apply Theorem 2.8 with $\boldsymbol{u} = \nabla v$. □

As a corollary, the eigenfunctions of the Laplace operator with Neumann boundary condition belong to $\mathbb{H}^m(\mathcal{O})$ for all $m \geq 0$ and thus are in $\mathcal{C}^\infty(\overline{\mathcal{O}})$.

The last lemmas establish regularity results for the demagnetization field operator.

Lemma 2.10. *For all $1 < p < +\infty$, $\mathcal{H}_d$ is a continuous operator from $\mathbb{L}^p(\mathcal{O})$ to $\mathbb{L}^p(\mathcal{O})$, and from $\mathbb{W}^{1,p}(\mathcal{O})$ to $\mathbb{W}^{1,p}(\mathcal{O})$.*

PROOF : See M.J. Friedman [8], O. Ladyzhenskaya [15], or G. Carbou and P. Fabrie [5]. □

Lemma 2.11. *If $\boldsymbol{v}$ belongs to $\mathbb{H}^1(B\times(0,L))$, then the $\mathbb{L}^2(B\times(+\varepsilon,L))$ norm of $\mathcal{H}_d(\widetilde{1}_{B\times(0,+\varepsilon)}\boldsymbol{m})$ is dominated by ε near zero. As a direct consequence, $\mathcal{H}_d(\gamma^0\boldsymbol{v}\mathrm{d}\sigma(\Gamma))$ is in $\mathbb{L}^2(B\times(0,L))$.*

PROOF : We denote by h_ε the homothecy that sends

$$\begin{aligned}\mathbb{H}^1(B\times(0;L)) &\to \mathbb{H}^1(B\times(\varepsilon,L))\\ \boldsymbol{v} &\mapsto \begin{cases}\boldsymbol{v}(x,y,+\frac{L}{L-\varepsilon}(z-\varepsilon)) & \text{in } \Omega_+,\\ 0 & \text{in } \mathbb{R}^3\setminus\Omega_+.\end{cases}\end{aligned}$$

Then, if we denote by $\widetilde{1}^*_I$ the characteristic function of $B\times I$,

$$\begin{aligned}\|\mathcal{H}_d(\widetilde{1}^*_{(0,+\varepsilon)}\boldsymbol{v})\|_{\mathbb{L}^2(\Omega_\varepsilon)} &\leq \|\mathcal{H}_d(\widetilde{1}^*_{(0,L)}\boldsymbol{v}) - \mathcal{H}_d(\widetilde{1}^*_{(\varepsilon,L)}\boldsymbol{v})\|_{\mathbb{L}^2(\Omega_\varepsilon)}\\ &\leq \underbrace{\|\mathcal{H}_d(\widetilde{1}^*_{(0,L)}\boldsymbol{v}) - h_\varepsilon(\mathcal{H}_d(\widetilde{1}^*_{(0,L)}\boldsymbol{v}))\|_{\mathbb{L}^2(\Omega_\varepsilon)}}_{I}\\ &\quad+\underbrace{\|h_\varepsilon(\mathcal{H}_d(\widetilde{1}^*_{(0,L)}\boldsymbol{v})) - \mathcal{H}_d(h_\varepsilon(\widetilde{1}^*_{(0,L)}\boldsymbol{v}))\|_{\mathbb{L}^2(\Omega_\varepsilon)}}_{II}\\ &\quad+\underbrace{\|\mathcal{H}_d(h_\varepsilon(\widetilde{1}^*_{(0,L)}\boldsymbol{v}) - \widetilde{1}^*_{(\varepsilon,L)}\boldsymbol{v})\|_{\mathbb{L}^2(\Omega_\varepsilon)}}_{III}.\end{aligned} \tag{2.7}$$

Terms I and III are $O(\varepsilon)$ because $\mathbb{H}^1$ is stable by the action $\mathcal{H}_d$. Estimating II when ε is near 0, is the same as estimating

$$\boldsymbol{w} = \mathcal{H}_d(\widetilde{1}^*_{(0,L)}\boldsymbol{v}) - h_\varepsilon^{-1}\mathcal{H}_d(h_\varepsilon(\widetilde{1}^*_{(0,L)}\boldsymbol{v}))\|_{\mathbb{L}^2(\Omega_\varepsilon)}.$$

The equation satisfied by $\boldsymbol{w}$ is

$$\begin{aligned}
\frac{\partial \boldsymbol{w}_x}{\partial x} + \frac{\partial \boldsymbol{w}_y}{\partial y} + \frac{L}{L-\varepsilon}\frac{\partial \boldsymbol{w}_z}{\partial z} &= \frac{\varepsilon}{L-\varepsilon}\left(\frac{\partial v v_z + \boldsymbol{h}_z}{\partial z}\right),\\
\frac{\partial \boldsymbol{w}_z}{\partial y} - \frac{L}{L-\varepsilon}\frac{\partial \boldsymbol{w}_y}{\partial z} &= -\frac{\varepsilon}{L-\varepsilon}\frac{\partial \boldsymbol{h}_y}{\partial z},\\
\frac{L}{L-\varepsilon}\frac{\partial \boldsymbol{w}_x}{\partial z} - \frac{\partial \boldsymbol{w}_z}{\partial x} &= \frac{\varepsilon}{L-\varepsilon}\frac{\partial \boldsymbol{h}_x}{\partial z},\\
\frac{\partial \boldsymbol{w}_y}{\partial x} - \frac{\partial \boldsymbol{w}_x}{\partial y} &= 0,
\end{aligned}$$

where $\boldsymbol{h} = \mathcal{H}_d(\boldsymbol{v})$. A simple Fourier analysis proves that the $\mathbb{L}^2$ norm of this term is dominated by ε near 0. □

2.3.2 *Existence and uniqueness of* $\boldsymbol{m}^{(0)}$

$\boldsymbol{m}^{(0)}$ satisfies equation (2.5) over Ω. G. Carbou et P. Fabrie proved the existence of strong solutions to the Landau-Lifshitz equation and their uniqueness in [5]. In this section, $\mathcal{O}$ is a bounded open set.

Theorem 2.12 (Regular solutions for Landau-Lifshitz equation)**.** *Suppose $\mathcal{O}$ bounded and "regular" enough. If the initial condition $\boldsymbol{m}_0$ belongs to $\widetilde{\mathbb{H}}^2(\mathcal{O})$ and satisfies*

$$|\boldsymbol{m}_0| = 1 \text{ in } \mathcal{O}, \qquad\qquad \boldsymbol{m}_0 = 0 \text{ in } \mathbb{R}^3 \setminus \mathcal{O},$$

then, there exists $T^* > 0$ and a unique solution $\boldsymbol{m}$ to system (1.6a), (1.5a) (1.5b), *and* (1.5c), *belonging for all $T < T^*$ to* $\mathrm{L}^2(0,T;\mathbb{H}^3(\mathcal{O})) \cap \mathcal{C}(0,T;\mathbb{H}^2(\mathcal{O}))$.

PROOF : See the proof of Carbou-Fabrie [5]. □

We improve slightly Theorem 2.12.

Remark 2.13. The solution $\boldsymbol{m}$ to system (1.5). belongs to $\mathrm{L}^2(0,T;\mathbb{H}^3(\mathcal{O}))\cap\mathrm{H}^{\frac{3}{2}}(0,T;\mathbb{L}^2(\mathcal{O}))\cap\mathcal{C}^1(0,T;\mathbb{L}^2(\mathcal{O}))$ for all $T < T^*$.

PROOF : In [5], it is proved that $\boldsymbol{m}$ belongs to $\mathrm{H}^1(0,T;\mathbb{H}^1(\mathcal{O}))$ and to $\mathrm{L}^2(0,T;\mathbb{H}^3(\mathcal{O}))$. By interpolation, $\boldsymbol{m}$ belongs to $\mathcal{C}([0,T^*);\mathbb{H}^2(\mathcal{O}))$ and to $\mathrm{H}^{\frac{1}{2}}([0,T^*);\mathbb{H}^2(\mathcal{O}))$, see Lions-Magenes [19]. Thus,

$$\begin{aligned}
\boldsymbol{m} &\in \mathcal{C}([0,T^*);\mathbb{L}^\infty(\mathcal{O})), & \triangle\boldsymbol{m} &\in \mathcal{C}([0,T^*);\mathbb{L}^2(\mathcal{O})),\\
\mathcal{H}_d(\boldsymbol{m}) &\in \mathcal{C}([0,T^*);\mathbb{L}^2(\mathcal{O})), & \mathcal{H}_a(\boldsymbol{m}) &\in \mathcal{C}([0,T^*);\mathbb{L}^2(\mathcal{O})).
\end{aligned}$$

But $\frac{\partial \boldsymbol{m}}{\partial t} = -\boldsymbol{m}\times\mathcal{H}(\boldsymbol{m}) - \alpha\boldsymbol{m}\times(\boldsymbol{m}\times\mathcal{H}(\boldsymbol{m}))$, thus $\frac{\partial \boldsymbol{m}}{\partial t}$ belongs to $\mathcal{C}([0,T^*);\mathbb{L}^2(\mathcal{O}))$. Hence $\boldsymbol{m}$ belongs to $\mathcal{C}^1([0,T^*);\mathbb{L}^2(\mathcal{O}))$. We apply corollary 2.15 and $\frac{\partial \boldsymbol{m}}{\partial t}$ belongs to $\mathrm{H}^{\frac{1}{2}}(0,T;\mathbb{L}^2(\mathcal{O}))$. □

Lemma 2.14. *The bilinear form* $(\boldsymbol{m}, f) \mapsto \boldsymbol{m} \times f$ *from* $(\mathrm{H}^{\frac{1}{2}} \cap \mathrm{L}^{\infty})(0,T;\mathbb{L}^{\infty}(\mathcal{O})) \times (\mathrm{H}^{\frac{1}{2}} \cap \mathrm{L}^{\infty})(0,T;\mathbb{L}^2(\mathcal{O}))$ *to* $(\mathrm{H}^{\frac{1}{2}} \cap \mathrm{L}^{\infty})(0,T;\mathbb{L}^2(\mathcal{O}))$ *is continuous.*

PROOF : We recall that the $\mathrm{H}^{\frac{1}{2}}(0,T;X)$ norm is equivalent to the norm

$$\|v\| = \int_{\tau=0}^{T}\int_{\sigma=0}^{T} \frac{\|v(\tau)-v(\sigma)\|_X^2}{|\tau-\sigma|^2}\mathrm{d}\sigma\mathrm{d}\tau.$$

See Lions-Magenes [19]. Then, we have

$$\|\boldsymbol{m}\times f\|_{\mathbb{L}^{\infty}(0,T;\mathbb{L}^2(\mathcal{O}))} \leq \|\boldsymbol{m}\|_{\mathbb{L}^{\infty}(0,T;\mathbb{L}^{\infty}(\mathcal{O}))}\|f\|_{\mathbb{L}^{\infty}(0,T;\mathbb{L}^2(\mathcal{O}))}.$$

$$\begin{aligned}\|\boldsymbol{m}\times f\|_{\mathrm{H}^{\frac{1}{2}}(0,T;\mathbb{L}^2(\mathcal{O}))} &= \int_{\tau=0}^{T}\int_{\sigma=0}^{T} \frac{\|\boldsymbol{m}(\tau)\times f(\tau)-\boldsymbol{m}(\sigma)\times f(\sigma)\|_{\mathbb{L}^2(\mathcal{O})}^2}{|\tau-\sigma|^2}\mathrm{d}\sigma\mathrm{d}\tau \\ &\leq \int_{\tau=0}^{T}\int_{\sigma=0}^{T} \frac{\|\boldsymbol{m}(\tau)\|_{\mathbb{L}^{\infty}(\mathcal{O})}^2\|f(\tau)-f(\sigma)\|_{\mathbb{L}^2(\mathcal{O})}^2}{|\tau-\sigma|^2}\mathrm{d}\sigma\mathrm{d}\tau \\ &\quad + \int_{\sigma=0}^{T} \frac{\|f(\sigma)\|_{\mathbb{L}^2(\mathcal{O})}^2\|\boldsymbol{m}(\tau)-\boldsymbol{m}(\sigma)\|_{\mathbb{L}^{\infty}(\mathcal{O})}^2}{|\tau-\sigma|^2}\mathrm{d}\sigma\mathrm{d}\tau \\ &\leq C\|\boldsymbol{m}\|_{(\mathrm{L}^{\infty}\cap\mathrm{H}^{\frac{1}{2}})(0,T;\mathbb{L}^{\infty}(\mathcal{O}))}\|f\|_{(\mathrm{L}^{\infty}\cap\mathrm{H}^{\frac{1}{2}})(0,T;\mathbb{L}^2(\mathcal{O}))}.\end{aligned}$$

□

Furthermore, $\mathcal{H}$ is linear continuous from $(\mathrm{L}^{\infty}\cap\mathrm{H}^{\frac{1}{2}})(0,T;\mathbb{H}^2(\mathcal{O}))$ to $(\mathrm{L}^{\infty}\cap\mathrm{H}^{\frac{1}{2}})(0,T;\mathbb{L}^2(\mathcal{O}))$. Thus, as a corollary, we have

Corollary 2.15. *The application* $\boldsymbol{m} \mapsto -\boldsymbol{m}\times\mathcal{H}(\boldsymbol{m})-\alpha\boldsymbol{m}\times(\boldsymbol{m}\times\mathcal{H}(\boldsymbol{m}))$ *from* $\mathrm{H}^1(0,T;\mathbb{H}^1(\mathcal{O}))\cap \mathrm{L}^2(0,T;\mathbb{H}^3(\mathcal{O}))$ *to* $(\mathrm{H}^{\frac{1}{2}}\cap\mathrm{L}^{\infty})(0,T;\mathbb{L}^2(\mathcal{O}))$ *is continuous.*

Remark 2.16. The inverse of the time of existence $(T^*)^{-1}$ and the $\mathbb{H}^{3,\frac{3}{2}}$ norm of the solution $\boldsymbol{m}^{(0)}$ remain bounded by a function of the $\mathbb{H}^2$ norm of $\boldsymbol{m}_0^{(0)}$. This function can be chosen independently of L as long as $L > L_0 > 0$.

PROOF : T^* and the estimates on the size of the solutions depend on the size of the initial condition and on the constants of the inequalities of section 2.3.1. Since those constants remain bounded for $L > L_0 > 0$, both T^*, and the size of the solutions remain bounded as long as $L > L_0 > 0$. □

Remark 2.17. The application that sends the initial condition to the solution to the Landau-Lifshitz equation is continuous from $\widetilde{\mathbb{H}}^2(\mathcal{O})$ to $\mathbb{H}^{3,\frac{3}{2}}(\mathcal{O}\times(0,T))$. Moreover, the application $\boldsymbol{m}_0 \to \frac{\partial\boldsymbol{m}}{\partial t}$ is continuous from $\widetilde{\mathbb{H}}^2(\mathcal{O})$ to $\mathcal{C}(0,T;\mathbb{L}^2(\mathcal{O}))$.

PROOF : Theorem 2.12 established in [5] asserts the continuity into $\mathcal{C}(0,T;\mathbb{H}^2(\mathcal{O}))$. However, the provided proof also proves the continuity into $\mathrm{L}^2(0,T;\mathbb{H}^3(\mathcal{O}))$. By interpolation, there is continuity into $\mathrm{H}^1(0,T;\mathbb{H}^1(\mathcal{O}))$. Using corollary 2.15, we obtain the continuity of application $\boldsymbol{m}_0 \mapsto \frac{\partial\boldsymbol{m}}{\partial t}$ into $\mathcal{C}(0,T;\mathbb{L}^2(\mathcal{O}))\cap\mathrm{H}^{\frac{1}{2}}(0,T;\mathbb{L}^2(\mathcal{O}))$. □

The following result is necessary for the proof of the existence of solutions of the linearized Landau-Lifshitz equation.

Remark 2.18. For all $T < T^*$, $\|\nabla\boldsymbol{m}\|_{\mathrm{L}^4(0,T;\mathbb{L}^{\infty}(\mathcal{O}))} < +\infty$.

PROOF : According to Maz'ya [20] page 274, $\|\boldsymbol{u}\|_{\mathbb{L}^{\infty}} \leq C\|\boldsymbol{u}\|_{\mathbb{H}^1(\mathcal{O})}^{\frac{1}{2}}\|\boldsymbol{u}\|_{\mathbb{H}^2(\mathcal{O})}^{\frac{1}{2}}$. Since $\boldsymbol{m}^{(0)}$ belongs to $\mathrm{L}^2(0,T;\mathbb{H}^3(\mathcal{O}))$ and to $\mathrm{L}^{\infty}(0,T;\mathbb{H}^2(\mathcal{O}))$, $\nabla\boldsymbol{m}^{(0)}$ belongs to $\mathrm{L}^4(0,T;\mathbb{L}^{\infty}(\mathcal{O}))$.
□

Remark 2.19. Theorem 2.12, and Remarks 2.16, 2.13, 2.17 and 2.18 hold for bounded open sets with a smooth boundary and for cylindrical open sets of kind $B \times (0, L)$. These results also hold for disjoint finite unions of such open sets. In particular, the theorem holds for Ω and for Ω_ε.

PROOF : The demonstration of Carbou-Fabrie is valid if the Sobolev spaces over $\mathcal{O}$ satisfy some inequalities and the demagnetization field operator $\mathcal{H}_d$ is regular. This was verified in section 2.3.1. □

2.3.3 Existence and uniqueness of $\boldsymbol{m}^{(1)}$

Theorem 2.20. *Let $\boldsymbol{m}^{(0)}$ be a solution to problem* (2.5)*. If*

$$\boldsymbol{m}_0^{(1)} \in \mathbb{H}^1(\Omega), \qquad \boldsymbol{m}_0^{(0)} \cdot \boldsymbol{m}_0^{(1)} = 0,$$

then, System (2.6) *has a unique solution $\boldsymbol{m}^{(1)}$ which for all $T < T^*$ —T^* time of existence of $\boldsymbol{m}^{(0)}$— belongs to* $\mathrm{L}^2(0,T;\mathbb{H}^2(\Omega)) \cap \mathrm{H}^1(0,T;\mathbb{L}^2(\Omega))$. *Moreover, the following application is continuous*

$$\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T)) \times \mathbb{H}^1(\Omega) \to \mathbb{H}^{2,1}(\Omega),$$
$$(\boldsymbol{m}^{(0)}, \boldsymbol{m}_0^{(1)}) \mapsto \boldsymbol{m}^{(1)}.$$

PROOF : System (2.6) is equivalent to

$$\begin{aligned}\frac{\partial \boldsymbol{m}^{(1)}}{\partial t} - \alpha A \triangle \boldsymbol{m}^{(1)} = {} & -\boldsymbol{m}^{(0)} \times A\triangle \boldsymbol{m}^{(1)} - \boldsymbol{m}^{(1)} \times A\triangle \boldsymbol{m}^{(0)} + 2\alpha A(\nabla \boldsymbol{m}^{(0)} \cdot \nabla \boldsymbol{m}^{(1)})\boldsymbol{m}^{(0)} \\ & + \alpha A|\nabla \boldsymbol{m}^{(0)}|^2 \boldsymbol{m}^{(1)} - \boldsymbol{m}^{(1)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(0)}) - \boldsymbol{m}^{(0)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(1)}) \\ & - \boldsymbol{m}^{(0)} \times \mathcal{H}_d(\gamma^0 \boldsymbol{m}^{(0)} \mathrm{d}\sigma(\Gamma)) - \alpha \boldsymbol{m}^{(1)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(0)})) \\ & - \alpha \boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(1)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(0)})) - \alpha \boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(1)})) \\ & - \alpha \boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}_d(\gamma^0 \boldsymbol{m}^{(0)} \mathrm{d}\sigma(\Gamma))),\end{aligned} \tag{2.8a}$$

$$\boldsymbol{m}^{(0)} \cdot \boldsymbol{m}^{(1)} = 0, \tag{2.8b}$$

$$\frac{\partial \boldsymbol{m}^{(1)}}{\partial \boldsymbol{\nu}} = \begin{cases} \frac{\partial^2 \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}^2} & \text{on } \Gamma^{\pm} \times (0,T), \\ 0 & \text{on } (\partial\Omega \setminus \Gamma) \times (0,T), \end{cases} \tag{2.8c}$$

$$\boldsymbol{m}^{(1)}(\cdot, 0) = \boldsymbol{m}_0^{(1)}, \tag{2.8d}$$

where $\mathcal{H}_{d,a} = \mathcal{H}_d + \mathcal{H}_a$. To prove the theorem, our plans comprise n steps

1. Prove the redundancy of (2.8b) in System subeq:LandauLifshitzSympOrdre1 when $\boldsymbol{m}_0^{(0)} \cdot \boldsymbol{m}_0^{(1)}$. This is done in Lemma 2.21.

2. We prove that there exists an expansion $\widetilde{\boldsymbol{f}}$ in $\mathbb{H}^{2,1}(\Omega \times (0,T))$ to the boundary condition 2.8c.

3. We use the expansion $\boldsymbol{f}$ to reduce the well-posedness problem to the case where the boundary condition 2.8c is replaced with the standard zero Neumann boundary condition. This is done by considering the equation satisfied by $\boldsymbol{m}^{(1)} - \boldsymbol{f}$.

4. Finally, we prove the well posedness of this latter system. in Theorem 2.24.

□

Lemma 2.21. *Condition* (2.8b) *may be derived from the other equations of system* (2.8).

PROOF :

$$\frac{\partial(\boldsymbol{m}^{(1)}\cdot\boldsymbol{m}^{(0)})}{\partial t}=\alpha A\left(\triangle(\boldsymbol{m}^{(1)}\cdot\boldsymbol{m}^{(0)})+2|\nabla\boldsymbol{m}^{(0)}|^2(\boldsymbol{m}^{(1)}\cdot\boldsymbol{m}^{(0)})\right). \tag{2.9}$$

But $\boldsymbol{m}_0^{(0)}\cdot\boldsymbol{m}_0^{(1)}=0$ and $\frac{\partial(\boldsymbol{m}^{(0)}\cdot\boldsymbol{m}^{(1)})}{\partial\boldsymbol{\nu}}=0$ on $\partial\Omega$. This is obvious on $\partial\Omega\setminus\Gamma$. On Γ,

$$\begin{aligned}\frac{\partial(\boldsymbol{m}^{(0)}\cdot\boldsymbol{m}^{(1)})}{\partial\boldsymbol{\nu}}=\boldsymbol{m}^{(0)}\cdot\frac{\partial\boldsymbol{m}^{(1)}}{\partial\boldsymbol{\nu}}&=\boldsymbol{m}^{(0)}\cdot\frac{\partial^2\boldsymbol{m}}{\partial\boldsymbol{\nu}^2}\\&=\frac{\partial\left(\boldsymbol{m}^{(0)}\cdot\sum_{i=1}^3\alpha_i(\boldsymbol{x})\frac{\partial\boldsymbol{m}^{(0)}}{\partial x_i}\right)}{\partial\boldsymbol{\nu}}-\left(\frac{\partial\boldsymbol{m}^{(0)}}{\partial\boldsymbol{\nu}}\right)^2=0.\end{aligned}$$

We then multiply (2.9) by $\boldsymbol{m}^{(1)}\cdot\boldsymbol{m}^{(0)}$, and we integrate over $\Omega\times(0,T)$.

$$\begin{aligned}\left[\int_\Omega|\boldsymbol{m}^{(1)}\cdot\boldsymbol{m}^{(0)}|^2\mathrm{d}\boldsymbol{x}\right]_0^T+\alpha A\int_{Q_T}\left|\nabla(\boldsymbol{m}^{(1)}\cdot\boldsymbol{m}^{(0)})\right|^2\mathrm{d}\boldsymbol{x}\\=2\alpha A\int_{Q_T}|\nabla\boldsymbol{m}^{(0)}|^2(\boldsymbol{m}^{(1)}\cdot\boldsymbol{m}^{(0)})^2\mathrm{d}\boldsymbol{x},\\\leq 2\alpha A\int_0^T\|\nabla\boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega)}\int_\Omega|\boldsymbol{m}^{(0)}\cdot\boldsymbol{m}^{(1)}|^2\mathrm{d}\boldsymbol{x}.\end{aligned}$$

By Gronwall's inequality

$$\|\boldsymbol{m}^{(1)}\cdot\boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^2(\Omega)}(T)\leq\|\boldsymbol{m}_0^{(1)}\cdot\boldsymbol{m}_0^{(0)}\|^2_{\mathbb{L}^2(\Omega)}\exp\left(2\alpha A\int_0^T\|\nabla\boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega)}\mathrm{d}\boldsymbol{x}\right)$$

Hence, $\boldsymbol{m}^{(1)}\cdot\boldsymbol{m}^{(0)}=0$ almost everywhere. □
Also, we have an extension result.

Lemma 2.22. *There exists* $\widetilde{\boldsymbol{f}}$ *in* $\mathbb{H}^{2,1}(\Omega\times(0,T))$ *for all* $T<T^*$*, such that*

$$\frac{\partial\widetilde{\boldsymbol{f}}}{\partial\boldsymbol{\nu}}=\begin{cases}\frac{\partial^2\boldsymbol{m}^{(0)}}{\partial\boldsymbol{\nu}^2} & \textit{on } \Gamma^\pm\times(0,T),\\ 0 & \textit{on } (\partial\Omega\setminus\Gamma)\times(0,T),\end{cases} \tag{2.10}$$

PROOF : Define $\widetilde{\boldsymbol{f}}$ as

$$\widetilde{\boldsymbol{f}}(x,y,z,t)=-\zeta(z)\frac{\partial\boldsymbol{m}^{(0)}}{\partial z}(x,y,z,t)\text{ in }\Omega^+, \tag{2.11}$$

$$\widetilde{\boldsymbol{f}}(x,y,z,t)=\zeta(z)\frac{\partial\boldsymbol{m}^{(0)}}{\partial z}(x,y,z,t)\text{ in }\Omega^-, \tag{2.12}$$

with ζ a smooth real function with support included in $(-L^-,L^+)$ and with value 1 in a neighborhood of 0. □
Also, we have the following

Lemma 2.23. *The quantity* $\boldsymbol{m}^{(0)}\times\mathcal{H}_d(\gamma^0\boldsymbol{m}^{(0)}\mathrm{d}\sigma(\Gamma))+\alpha\boldsymbol{m}^{(0)}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}_d(\gamma^0\boldsymbol{m}^{(0)}\mathrm{d}\sigma(\Gamma)))$ *is in* $\mathbb{L}^2(\Omega\times(0,T))$ *for all* $T<T^*$.

PROOF : $\boldsymbol{m}^{(0)}$ belongs in $\mathbb{L}^\infty(\Omega\times(0,T^*))$. We only need to prove that $\mathcal{H}_d(\gamma^0\boldsymbol{m}^{(0)}\mathrm{d}\sigma(\Gamma))$ is in $\mathbb{L}^2(\Omega\times(0,T))$ for all $T<T^*$. This is a consequence of Lemma 2.11. □

We have to consider the following system and proved its well-posedness under certain conditions.

$$\frac{\partial \boldsymbol{w}}{\partial t}-\alpha A\triangle\boldsymbol{w}=-\boldsymbol{m}^{(0)}\times A\triangle\boldsymbol{w}-\boldsymbol{w}\times A\triangle\boldsymbol{m}^{(0)}+2\alpha A(\nabla\boldsymbol{m}^{(0)}\cdot\nabla\boldsymbol{w})\boldsymbol{m}^{(0)}+\alpha A|\nabla\boldsymbol{m}^{(0)}|^2\boldsymbol{w}$$
$$-\boldsymbol{w}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)})-\boldsymbol{m}^{(0)}\times\mathcal{H}_{d,a}(\boldsymbol{w})-\alpha\boldsymbol{w}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)}))$$
$$-\alpha\boldsymbol{m}^{(0)}\times(\boldsymbol{w}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)}))-\alpha\boldsymbol{m}^{(0)}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}_{d,a}(\boldsymbol{w}))+\boldsymbol{\theta}, \tag{2.13a}$$

$$\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{\nu}}=\beta\text{ on }\partial\Omega\times(0,T), \tag{2.13b}$$

$$\boldsymbol{w}(\cdot,0)=\boldsymbol{w}_0. \tag{2.13c}$$

Theorem 2.24. *Let $\boldsymbol{m}^{(0)}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ for all $T<T^*$. Let β be in $\mathbb{L}^2(\partial\Omega\times(0,T))$ such that there exists $\widetilde{\boldsymbol{f}}$ in $\mathbb{H}^{2,1}(\Omega\times(0,T))$ with $\beta=\frac{\partial f}{\partial\boldsymbol{\nu}}$. Let $\boldsymbol{\theta}$ be in $\mathbb{L}^2(\Omega\times(0,T))$. Then, system* (2.13) *has a unique solution $\boldsymbol{w}$ that belongs to $\mathbb{H}^{2,1}(\Omega\times(0,T))$. Moreover, $\boldsymbol{w}$ depends continuously on the data $\boldsymbol{\theta}$,$\boldsymbol{w}_0$, $\widetilde{\boldsymbol{f}}$ and $\boldsymbol{m}^{(0)}$.*

PROOF : First, we reduce the case to $\beta=0$, we notice that if $\boldsymbol{w}$ exists then $\boldsymbol{w}-\widetilde{\boldsymbol{f}}$ must verify system (2.13) once we have replaced β by 0, $\boldsymbol{w}_0$ by $\boldsymbol{w}_0-\widetilde{\boldsymbol{f}}(\cdot,0)$, and $\boldsymbol{\theta}$ by

$$\boldsymbol{\theta}-\frac{\partial\widetilde{\boldsymbol{f}}}{\partial t}+\alpha A\triangle\widetilde{\boldsymbol{f}}+2\alpha A(\widetilde{\boldsymbol{f}}\cdot\boldsymbol{m}^{(0)})\boldsymbol{m}^{(0)}+\alpha A|\nabla\boldsymbol{m}^{(0)}|^2\widetilde{\boldsymbol{f}}-\boldsymbol{m}^{(0)}\times\mathcal{H}(\widetilde{\boldsymbol{f}})-\widetilde{\boldsymbol{f}}\times\mathcal{H}(\boldsymbol{m}^{(0)})$$
$$-\alpha\widetilde{\boldsymbol{f}}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)}))-\alpha\boldsymbol{m}^{(0)}\times(\widetilde{\boldsymbol{f}}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)}))-\alpha\boldsymbol{m}^{(0)}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}_{d,a}(\widetilde{\boldsymbol{f}})).$$

which belongs to $\mathbb{L}^2(\Omega\times(0,T))$ by Hölder inequality. Conversely, if this new system has a unique solution, which we denote by $\boldsymbol{w}'$, $\boldsymbol{w}'+\widetilde{\boldsymbol{f}}$ is the unique solution of the more general system. We now prove the existence, uniqueness and stability when $\beta=0$, i.e.. when the Neumann boundary condition is homogenous.

Preliminary inequalities For each contribution p, we define

$$\boldsymbol{F}^{p,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{w})=-\boldsymbol{w}\times\mathcal{H}_p(\boldsymbol{m}^{(0)})-\boldsymbol{m}^{(0)}\times\mathcal{H}_p(\boldsymbol{w})-\alpha\boldsymbol{w}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}_p(\boldsymbol{m}^{(0)}))$$
$$-\alpha\boldsymbol{m}^{(0)}\times(\boldsymbol{w}\times\mathcal{H}_p(\boldsymbol{m}^{(0)}))-\alpha\boldsymbol{m}^{(0)}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}_p(\boldsymbol{w})). \tag{2.14}$$

We need the following inequality for the term defined in equation (2.14)

$$\|\boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{w})\|_{\mathbb{L}^2(\Omega)}\leq C'(1+\|\boldsymbol{m}^{(0)}\|_{\mathbb{H}^1(\Omega)})\|\boldsymbol{w}\|_{\mathbb{H}^1(\Omega)}. \tag{2.15}$$

This is a consequence of Hölder inequality and $|\boldsymbol{m}^{(0)}|=1$ over Ω.

Galerkin's method We recall that $\boldsymbol{m}^{(0)}$ belongs to $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$. and is a known strong solution to the Landau-Lifshitz equation.

Let $(w_j)_{j\in\mathbb{N}^*}$ be the eigenfunctions of the Laplace operator with Neumann boundary condition with $(\lambda_j)_{j\in\mathbb{N}^*}$ the corresponding eigenvalues. $(w_j)_{j\in\mathbb{N}^*}$ is an orthonormal basis of $\mathrm{L}^2(\Omega)$ and also an orthogonal basis of $\mathbb{H}^1(\Omega)$. By Lemma 2.3, if $\mathcal{O}$ is an open set with a smooth boundary or a convex domain, $\mathbb{H}^1_\triangle=\mathbb{H}^2$ and the eigenfunctions w_i

are in $\mathcal{C}^\infty(\overline{\mathcal{O}})$. We define V_n as $\mathrm{Vect}(w_1,\dots,w_n)$ for each $n \geq 0$. We define $\mathcal{P}_n$ as the orthogonal projection on V_n in L^2, the projection is also orthogonal in H^1 and $\widetilde{\mathrm{H}}^2$. We look for $\boldsymbol{w}^n$ in $V_n \otimes \mathcal{C}^\infty(0,T;\mathbb{R}^3))$ such that

$$\frac{\partial \boldsymbol{w}^n}{\partial t} - \alpha A \triangle \boldsymbol{w}^n = \mathcal{P}_n\left(-A\boldsymbol{w}^n \times \triangle \boldsymbol{m}^{(0)} - A\boldsymbol{m}^{(0)} \times \triangle \boldsymbol{w}^n + 2\alpha A(\nabla \boldsymbol{m}^{(0)} \cdot \nabla \boldsymbol{w}^n)\boldsymbol{m}^{(0)}\right) + \mathcal{P}_n\left(\alpha A |\nabla \boldsymbol{m}^{(0)}|^2 \boldsymbol{w}^n + \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{w}^n) + \boldsymbol{\theta}\right), \tag{2.16a}$$

$$\boldsymbol{w}^n{}_0 = \mathcal{P}_n(\boldsymbol{w}_0). \tag{2.16b}$$

We decompose $\boldsymbol{w}^n = \sum_{i=1}^n \boldsymbol{\varphi}_i^n(t) w_i$ where $\boldsymbol{\varphi}_i$ are functions from $(0,T)$ to $\mathbb{R}^3$. System (2.16) is equivalent to

$$\boldsymbol{\varphi}_i(0) = \langle \boldsymbol{w}_0, w_i\rangle \quad \forall i \in [\![, n]\!],$$

$$\begin{aligned}\boldsymbol{\varphi}_i'(t) = &-A\lambda_i \boldsymbol{\varphi}_i^n(t) + A\sum_{j=1}^n \lambda_j \boldsymbol{\varphi}_j^n(t) \times \int_\Omega \boldsymbol{m}^{(0)} w_i w_j \mathrm{d}\boldsymbol{x} - A\sum_{j=1}^n \boldsymbol{\varphi}_j^n(t) \times \int_\Omega \triangle \boldsymbol{m}^{(0)} w_i w_j \mathrm{d}\boldsymbol{x}\\ &+ \alpha A \sum_{j=1}^n \left(\boldsymbol{\varphi}_j^n(t) \int_\Omega |\nabla \boldsymbol{m}^{(0)}|^2 w_j w_i \mathrm{d}\boldsymbol{x} + 2\sum_{k=1}^3 \boldsymbol{\varphi}_j^{n,k}(t) \int_\Omega (\nabla \boldsymbol{m}^{(0)}{}_{x_k} \cdot \nabla w_j)\boldsymbol{m}^{(0)} w_i \mathrm{d}\boldsymbol{x}\right)\\ &+ \sum_{j=1}^n \sum_{k=1}^3 \boldsymbol{\varphi}_j^{n,k}(t)\left(\int_\Omega \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{u}_k w_j) w_i \mathrm{d}\boldsymbol{x}\right) + \int_\Omega \boldsymbol{\theta} w_i \mathrm{d}\boldsymbol{x} \quad \text{pour tout } i,\ 1 \leq i \leq n,\end{aligned}$$

where $(\boldsymbol{u}_k)$ is the canonical basis of $\mathbb{R}^3$.

This is an ordinary differential linear equation on $(\boldsymbol{\varphi}_i^n)_i$ whose coefficients depend continuously on the time. The affine term is in $\mathbb{L}^1(0,T)$. Thus, there is local and global existence of $\boldsymbol{w}^n$ over $[0,T^*)$ where T^* is the time of existence of $\boldsymbol{m}^{(0)}$. We now estimate the size of $\boldsymbol{w}^n$. In these estimates, η is a positive number that can be chosen arbitrarily small. C is a generic constant depending only on domain Ω_ε, uniformly bounded as long as $\varepsilon < \varepsilon_0 = \min(L^+, L^-)$.

First estimate We multiply (2.16) by $\boldsymbol{w}^n$, integrate, and obtain using Green's formula

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} + \alpha A \int_\Omega |\nabla \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} &= -A\underbrace{\int_\Omega (\boldsymbol{m}^{(0)} \times \triangle \boldsymbol{w}^n)\cdot \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{I} + \alpha A \underbrace{\int_\Omega |\nabla \boldsymbol{m}^{(0)}|^2 |\boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x}}_{II}\\ &+ 2\alpha A \underbrace{\int_\Omega (\nabla \boldsymbol{m}^{(0)} \cdot \nabla \boldsymbol{w}^n)(\boldsymbol{m}^{(0)} \cdot \boldsymbol{w}^n)\mathrm{d}\boldsymbol{x}}_{III} + \underbrace{\int_\Omega \boldsymbol{\theta}\cdot \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{IV} + \underbrace{\int_\Omega \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{w}^n)\cdot \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{V}\end{aligned} \tag{2.17}$$

Let's estimate each separate term of the sum. First, we estimate $I = \int_\Omega (\boldsymbol{m}^{(0)} \times \triangle \boldsymbol{w}^n)\cdot \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned}|I| &= \left|\sum_{k=1}^3 \int_\Omega \left(\boldsymbol{w}^n \times \frac{\partial \boldsymbol{w}^n}{\partial x_k}\right)\cdot \frac{\partial \boldsymbol{m}^{(0)}}{\partial x_k}\right| \mathrm{d}\boldsymbol{x}\\ &\leq \|\nabla \boldsymbol{m}^{(0)}\|_{\mathbb{L}^\infty(\Omega)} \|\boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \|\nabla \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)}\\ &\leq \frac{1}{4\eta}\|\nabla \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\nabla \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{2.18a}$$

We evaluate $II = \int_\Omega |\nabla \boldsymbol{m}^{(0)}|^2 |\boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x}$.

$$|II| \leq \|\nabla \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \tag{2.18b}$$

Estimating $III = \int_\Omega (\nabla \boldsymbol{m}^{(0)} \cdot \nabla \boldsymbol{w}^n)(\boldsymbol{m}^{(0)} \cdot \boldsymbol{w}^n) \mathrm{d}\boldsymbol{x}$ yields

$$|III| \leq \frac{1}{4\eta} \|\nabla \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\nabla \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \tag{2.18c}$$

Then, we estimate $IV = \int_\Omega \boldsymbol{\theta} \cdot \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$

$$|IV| \leq \frac{1}{2} \|\boldsymbol{\theta}\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2} \|\boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \tag{2.18d}$$

Using inequality (2.15) on $V = \int_\Omega \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{w}^n) \cdot \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$, we obtain

$$\begin{aligned} |V| &\leq \|\boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{w}^n)\|_{\mathbb{L}^2(\Omega)} \|\boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \\ &\leq C\left(1 + \|\boldsymbol{m}^{(0)}\|_{\mathbb{H}^1(\Omega)}\right) \|\boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + C\left(1 + \|\boldsymbol{m}^{(0)}\|_{\mathbb{H}^1(\Omega)}\right) \|\nabla \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \|\boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \\ &\leq C'\left(1 + \frac{1}{\eta}\right)\left(1 + \|\boldsymbol{m}^{(0)}\|^2_{\mathbb{H}^1(\Omega)}\right) \|\boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\nabla \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \end{aligned} \tag{2.18e}$$

Combining inequalities (2.18), we obtain for all $\eta > 0$

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} + \alpha A \int_\Omega |\nabla \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} \leq C\left(1 + \frac{1}{\eta}\right) (\|\nabla \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty} + \|\boldsymbol{m}^{(0)}\|^2_{\mathbb{H}^1}) \|\boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} \\ + \|\boldsymbol{\theta}\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\nabla \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \end{aligned} \tag{2.19}$$

We choose $\eta = \frac{\alpha A}{2}$. By Gronwall's inequality, for all $T < T^*$, there exists $C_T > 0$ such that for all $n \geq 0$

$$\|\boldsymbol{w}^n\|_{\mathbb{L}^\infty(0,T;\mathbb{L}^2(\Omega))} \leq C_T, \qquad \|\nabla \boldsymbol{w}^n\|_{\mathbb{L}^2(0,T\times\Omega)} \leq C_T, \tag{2.20}$$

where T^* is the time of existence of $\boldsymbol{m}^{(0)}$.

Second estimate We multiply (2.16) by $\boldsymbol{\psi}$ by $-\triangle \boldsymbol{w}^n$ and obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} + \alpha A \int_\Omega |\triangle \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} = A \underbrace{\int_\Omega (\boldsymbol{w}^n \times \triangle \boldsymbol{m}^{(0)}) \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{I} \\ -2\alpha A \underbrace{\int_\Omega (\nabla \boldsymbol{m}^{(0)} \cdot \nabla \boldsymbol{w}^n) \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{II} - \alpha A \underbrace{\int_\Omega |\nabla \boldsymbol{m}^{(0)}|^2 \boldsymbol{w}^n \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{III} - \underbrace{\int_\Omega \boldsymbol{\theta} \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{IV} \\ - \underbrace{\int_\Omega \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{w}^n) \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{V} \end{aligned} \tag{2.21}$$

We estimate $I = \int_\Omega (\boldsymbol{w}^n \times \triangle \boldsymbol{m}^{(0)}) \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned}|I| &\leq \|\boldsymbol{w}^n\|_{\mathbb{L}^3(\Omega)} \|\triangle \boldsymbol{m}^{(0)}\|_{\mathbb{L}^6(\Omega)} \|\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)}\\ &\leq \frac{1}{4\eta}\|\triangle \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^6(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{L}^3(\Omega)} + \eta \|\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}\\ &\leq \frac{C}{\eta}\|\triangle \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^6(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{H}^1(\Omega)} + \eta \|\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{2.22a}$$

Then, we estimate $II = \int_\Omega (\nabla \boldsymbol{m}^{(0)} \cdot \nabla \boldsymbol{w}^n) \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned}|II| &\leq \|\nabla \boldsymbol{m}^{(0)}\|_{\mathbb{L}^\infty(\Omega)} \|\nabla \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \|\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)}\\ &\leq \frac{1}{4\eta}\|\nabla \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega)} \|\nabla \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{2.22b}$$

Then, we evaluate $III = \int_\Omega |\nabla \boldsymbol{m}^{(0)}|^2 \boldsymbol{w}^n \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$

$$\begin{aligned}|III| &\leq \|\nabla \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega)} \|\boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \|\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)}\\ &\leq \frac{1}{4\eta}\|\nabla \boldsymbol{m}^{(0)}\|^4_{\mathbb{L}^\infty(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{2.22c}$$

Estimating $IV = \int_\Omega \boldsymbol{\theta} \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$ yields

$$|IV| \leq \frac{1}{4\eta}\|\boldsymbol{\theta}\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \tag{2.22d}$$

Using inequality (2.15) $V = \int_\Omega \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{w}^n) \cdot \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$, we obtain

$$\begin{aligned}|V| &\leq \|\boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}^{(0)}}(\boldsymbol{w}^n)\|_{\mathbb{L}^2(\Omega)} \|\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)}\\ &\leq C(1 + \|\boldsymbol{m}^{(0)}\|_{\mathbb{H}^1(\Omega)}) \|\boldsymbol{w}^n\|_{\mathbb{H}^1(\Omega)} \|\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)}\\ &\leq \frac{C}{\eta}(1 + \|\boldsymbol{m}^{(0)}\|^2_{\mathbb{H}^1(\Omega)}) \|\boldsymbol{w}^n\|^2_{\mathbb{H}^1(\Omega)} + \eta \|\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{2.22e}$$

Combining equations (2.22), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} + \alpha A \int_\Omega |\triangle \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} \leq{}& \frac{C}{\eta}\left(\|\nabla \boldsymbol{m}^{(0)}\|^4_{\mathbb{L}^\infty} + \|\boldsymbol{m}^{(0)}\|^2_{\mathbb{H}^3}\right) \|\boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}\\ &+ \frac{C}{\eta}(\|\nabla \boldsymbol{w}^n\|^2_{\mathbb{L}^\infty} + \|\boldsymbol{m}^{(0)}\|^2_{\mathbb{H}^3}) \|\nabla \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}\\ &+ \eta \|\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{\theta}\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{2.23}$$

Choosing η small enough, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla \boldsymbol{w}^n|^2 + \int_\Omega |\triangle \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} \leq f(t) + g(t)\int_\Omega |\nabla \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} + \eta \|\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)},$$

where f and g belongs to $\mathrm{L}^1(0,T)$. By Gronwall's inequality, there exists $C_T > 0$ such that for all $n \geq 0$

$$\|\nabla \boldsymbol{w}^n\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))} \leq C_T, \qquad \|\triangle \boldsymbol{w}^n\|_{\mathrm{L}^2((0,T)\times\Omega)} \leq C_T.$$

Convergence of the sequence to the solution We have proved that for all $T < T^*$, there exists C_T such that

$$\|\boldsymbol{w}^n\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))} \leq C_T, \qquad \|\nabla \boldsymbol{w}^n\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} \leq C_T,$$
$$\|\triangle \boldsymbol{w}^n\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \leq C_T, \qquad \left\|\frac{\partial \boldsymbol{w}^n}{\partial t}\right\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \leq C_T.$$

The last inequality being a consequence of the other estimates and the contractivity of $\mathcal{P}_n$ in L^2 in equation (2.16a). According to elliptic regularity Lemma 2.3 over Ω, $\|\mathrm{D}^2 \boldsymbol{w}^n\|_{\mathrm{L}^2((0,T)\times\Omega)} \leq C_T$. Thus, there exists a subsequence such that, for all $T < T^*$, $\boldsymbol{w}_k^n$ converges weakly to $\boldsymbol{w}$ in $\mathbb{H}^{2,1}(\Omega \times (0,T))$. Since $\bigcup_{n=1}^\infty V_n$ is dense in $\mathrm{L}^2(\Omega)$ and in $\mathrm{H}^1(\Omega)$, $\boldsymbol{w}$ is a solution to system (2.13).

Uniqueness Let $\boldsymbol{w}$ and $\boldsymbol{w}'$ be solutions to the system (2.13) then $\delta\boldsymbol{w} = \boldsymbol{w}' - \boldsymbol{w}$ is solution to (2.13) with affine term $\boldsymbol{\theta} = 0$ and initial condition $\boldsymbol{w}_0 = 0$. After multiplying this equation by $\delta\boldsymbol{w}$ and integrating over $\Omega \times (0,T)$, we obtain the following estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\delta\boldsymbol{w}|^2 \mathrm{d}\boldsymbol{x} + (\alpha A - \eta)\int_\Omega |\delta\nabla\boldsymbol{w}|^2 \mathrm{d}\boldsymbol{x} \leq C(\eta)\left(\|\boldsymbol{m}^{(0)}\|^2_{\mathbb{H}^1(\Omega)} + \|\nabla\boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega)}\right)\|\delta\boldsymbol{w}\|^2_{\mathbb{L}^2(\Omega)}.$$

The uniqueness follows from choosing $\eta = \alpha A/2$ and using Gronwall's lemma.

Stability We define by S the map that sends $(\boldsymbol{m}^{(0)}, \boldsymbol{w}_0, \boldsymbol{\theta})$ to the solution $\boldsymbol{w}$. $S(\boldsymbol{m}^{(0)}, \cdot, \cdot)$ is linear continuous and its norm only depend on the size of $\boldsymbol{m}^{(0)}$ in $\mathbb{H}^{3,\frac{3}{2}}$. Thus S is continuous in $(\boldsymbol{w}_0, \boldsymbol{\theta})$, uniformly when $\boldsymbol{m}^{(0)}$ remains bounded. Thus, to prove that S is Lipschitz, we only need to prove that S is Lipschitz when only $\boldsymbol{m}^{(0)}$ vary. Given $(\boldsymbol{m}^{(0)}, \boldsymbol{w}_0, \boldsymbol{\theta})$ and $(\boldsymbol{m}^{(0)\prime}, \boldsymbol{w}_0, \boldsymbol{\theta})$, we define $\boldsymbol{w}$ and $\boldsymbol{w}'$ as the solutions to the linearized Landau-Lifshitz equation with data $(\boldsymbol{m}^{(0)}, \boldsymbol{w}_0, \boldsymbol{\theta})$ and $(\boldsymbol{m}^{(0)\prime}, \boldsymbol{w}_0', \boldsymbol{\theta}')$. If the data remain bounded, so does the solution. We define

$$\delta\boldsymbol{w} = \boldsymbol{w}' - \boldsymbol{w}, \qquad \delta\boldsymbol{m}^{(0)} = \boldsymbol{m}^{(0)\prime} - \boldsymbol{m}^{(0)}.$$

Since $\boldsymbol{w}$ and $\boldsymbol{w}'$ both satisfies (2.13a), we make estimates by subtracting both equations. Then, we can prove the stability making the following estimates.

- We multiply this equation by $\delta\boldsymbol{w}$ and integrate over Ω. This gives an estimate on the $\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))$ norm of $\boldsymbol{w}$ and the $\mathbb{L}^2(\Omega \times (0,T))$ norm of $\nabla\boldsymbol{w}$.
- We multiply this equation by $\triangle\delta\boldsymbol{w}$ and integrate over Ω. This gives an estimate on the $\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))$ norm of $\nabla\boldsymbol{w}$ and the $\mathbb{L}^2(\Omega \times (0,T))$ norm of $\triangle\boldsymbol{w}$.

Reusing the equation, we obtain an estimate on the $\mathbb{L}^2(\Omega \times (0,T))$ norm of $\frac{\partial \boldsymbol{w}}{\partial t}$. Those estimates proves the stability and are very similar to the ones necessary to prove the existence.

□

We make the following remark on when β verify the extension criteria in Theorem (2.24).

Remark 2.25. The image of the trace application $\gamma^1 : \widetilde{f} \mapsto \frac{\partial \widetilde{f}}{\partial \boldsymbol{\nu}}$ is the space of functions whose restriction to $B \times \{0\}$ and to $\partial B \times (0, +\infty)$ are respectively in $\mathrm{H}^{\frac{3}{2},\frac{3}{4}}(B \times \{0\})$ and in $\mathrm{H}^{\frac{3}{2},\frac{3}{4}}(\partial B \times (0, +\infty))$.

PROOF : This can be proved by the same kind of arguments as in Lemma (2.7). The complete proof is in the appendix of Part 2 of this article A.6[21]. □

3 Convergence of the expansion

3.1 Convergence of $\boldsymbol{m}^{\varepsilon,(0)}$ to $\boldsymbol{m}^{(0)}$

To prove the convergence, we must first introduce an artificial term of first order to discard some boundary condition. We need the following proposition

Proposition 3.1. *Let*

$$\begin{aligned} &\boldsymbol{m}^{(0)} \in \mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T)), && \boldsymbol{m}^{(0)\prime} \in \mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T)),\\ &\beta^{+} \in \mathbb{H}^{\frac{3}{4},\frac{1}{4}}(B \times \{0\} \times (0,T)), && \beta^{-} \in \mathbb{H}^{\frac{3}{4},\frac{1}{4}}(B \times \{0\} \times (0,T)),\\ &\boldsymbol{m}_0^{(1)} \in \mathbb{H}^{1}(\Omega \times (0,T)), && \boldsymbol{\theta} \in \mathbb{L}^{2}(\Omega \times (0,T)). \end{aligned}$$

Then, the following equation with unknown $\boldsymbol{m}^{(1)}$ on Ω_ε

$$\begin{aligned} \frac{\partial \boldsymbol{m}^{(1)}}{\partial t} - \alpha A \triangle \boldsymbol{m}^{(1)} = {}& -A\boldsymbol{m}^{(1)} \times \triangle \boldsymbol{m}^{(0)} - A\boldsymbol{m}^{(0)\prime} \times \triangle \boldsymbol{m}^{(1)} + \alpha A |\nabla \boldsymbol{m}^{(0)}|^2 \boldsymbol{m}^{(1)}\\ & + \alpha A((\nabla \boldsymbol{m}^{(0)} + \nabla \boldsymbol{m}^{(0)\prime}) \cdot \nabla \boldsymbol{m}^{(1)})\boldsymbol{m}^{(0)\prime} - \boldsymbol{m}^{(1)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(0)})\\ & - \boldsymbol{m}^{(0)\prime} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(1)}) - \alpha \boldsymbol{m}^{(0)\prime} \times (\boldsymbol{m}^{(0)\prime} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(1)}))\\ & - \alpha \boldsymbol{m}^{(0)\prime} \times (\boldsymbol{m}^{(1)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(0)})) - \alpha \boldsymbol{m}^{(1)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(0)})) + \boldsymbol{\theta}, \end{aligned} \tag{3.1}$$

and both initial and boundary conditions

$$\frac{\partial \boldsymbol{m}^{(1)}}{\partial \boldsymbol{\nu}} = \begin{cases} \beta^{+} & \text{on } B \times \{+\varepsilon\} \times (0,T^*),\\ \beta^{-} & \text{on } B \times \{-\varepsilon\} \times (0,T^*),\\ 0 & \text{on } \partial\Omega_\varepsilon \setminus (B \times \{\pm\varepsilon\} \times (0,T^*)), \end{cases} \tag{3.2}$$

$$\boldsymbol{m}^{(1)}(\cdot,\cdot,0) = \boldsymbol{m}_0^{(1)}. \tag{3.3}$$

has a unique solution in $\mathbb{H}^{2,1}(\Omega \times (0,T))$. Moreover, the solution is stable in $\mathbb{H}^{2,1}(\Omega \times (0,T))$ with respect to the data $(\boldsymbol{m}^{(0)}, \boldsymbol{m}^{(0)\prime}, \beta^+, \beta^-, \boldsymbol{m}_0^{(1)}, \boldsymbol{\theta})$. Furthermore,

- *the application that sends $(\boldsymbol{m}^{(0)}, \boldsymbol{m}^{(0)\prime}, \beta^+, \beta^-, \boldsymbol{m}_0^{(1)})$ to the solution $\boldsymbol{m}^{(1)}$ is Lipschitz over bounded sets.*
- *Lipschitz constants depend on the open set Ω_ε, but they remain bounded when ε tends to 0.*

PROOF : This is a straightforward adaptation of the proof of Theorem 2.20. □

Proposition 3.2. *Let $\widetilde{1}_{\mathcal{O}}$ be the characteristic function of $\mathcal{O}$. Let $\boldsymbol{m}^{\varepsilon,(1)}$ be the solution over Ω_ε to system* (3.1) *with data*

$$\begin{gathered} \boldsymbol{\theta} = \frac{1}{\varepsilon}\boldsymbol{m}^{(0)} \times \mathcal{H}_{d,a}(\widetilde{1}_{B\times(-\varepsilon,+\varepsilon)}\boldsymbol{m}^{(0)}) + \frac{1}{\varepsilon}\alpha\boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}_{d,a}(\widetilde{1}_{B\times(-\varepsilon,+\varepsilon)}\boldsymbol{m}^{(0)})),\\ \boldsymbol{m}_0^{\varepsilon,(1)} = 0, \quad \boldsymbol{m}^{(0)\prime} = \boldsymbol{m}^{\varepsilon,(0)}, \quad \boldsymbol{m}^{(0)\prime} = \boldsymbol{m}^{0,(0)},\\ \beta^{+} = -\frac{1}{\varepsilon}\frac{\partial \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}}(\cdot,\cdot,+\varepsilon,\cdot), \quad \beta^{-} = -\frac{1}{\varepsilon}\frac{\partial \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}}(\cdot,\cdot,-\varepsilon,\cdot). \end{gathered}$$

Then, $\boldsymbol{m}^{\varepsilon,(1)}$ exists and is unique in $\mathbb{H}^{2,1}(\Omega_\varepsilon)$ with a $\mathbb{H}^{2,1}(\Omega_\varepsilon)$ norm that remains bounded when ε tends to 0.

PROOF : The initial data and the boundary conditions are in the required spaces. We apply Proposition 3.1. The size of the data remains bounded, the boundary condition remains bounded because

$$\left\|\frac{\partial \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}}(0,\varepsilon,0)\right\|_{\mathbb{H}^{\frac{3}{4},\frac{1}{4}}} \leq \|\tau_{-\varepsilon}\boldsymbol{m}^{(0)} - \boldsymbol{m}^{(0)}\|_{\mathbb{H}^{2,1}(B\times(0,+\infty)\times(0,T))}, \\ \leq \varepsilon\|\boldsymbol{m}^{(0)}\|_{\mathbb{H}^{3,\frac{3}{2}}(B\times(0,\varepsilon)\times(0,T))},$$

after extension of $\boldsymbol{m}^{(0)}$ over $B\times\mathbb{R}^{+,*}\times(0,T)$ and where $\tau_{-\varepsilon}$ is the translation operator over the third coordinate of space defined by $\tau_{-\varepsilon}f = (x,y,z,t)\mapsto f(x,y,z+\varepsilon,t)$. The affine term $\boldsymbol{\theta}$ remains bounded by Lemma 2.11. □

We define $\delta_\varepsilon\boldsymbol{m} = \boldsymbol{m}^{\varepsilon,(0)} - \boldsymbol{m}^{(0)}$, and $\delta_\varepsilon^1\boldsymbol{m} = \boldsymbol{m}^{\varepsilon,(0)} - \boldsymbol{m}^{(0)} - \varepsilon\boldsymbol{m}^{\varepsilon,(1)}$. Then, we estimate $\delta_\varepsilon^1\boldsymbol{m}$. Direct estimates of $\delta_\varepsilon\boldsymbol{m}$ would be more difficult because of the nonhomogenous boundary conditions.

Theorem 3.3. *We have*

$$\|\boldsymbol{m}^{(0)} - \boldsymbol{m}^{\varepsilon,(0)} - \varepsilon\boldsymbol{m}^{\varepsilon,(1)}\|_{\mathbb{H}^{2,1}(\Omega_\varepsilon\times(0,T))} = O(\varepsilon), \\ \|\boldsymbol{m}^{(0)} - \boldsymbol{m}^{\varepsilon,(0)}\|_{\mathbb{H}^{2,1}(\Omega_\varepsilon\times(0,T))} = O(\varepsilon).$$

PROOF : Since $\boldsymbol{m}^{\varepsilon,(1)}$ is bounded in $\mathbb{H}^{2,1}(\Omega\times(0,T)))$ independently of ε, the second inequality is a consequence of the first. We prove the first inequality. $\delta_\varepsilon^1\boldsymbol{m}$ satisfies equation (3.1) with $\boldsymbol{\theta} = 0$. The $\mathbb{H}^1(\Omega)$ norm of the initial condition is dominated by ε, by hypothesis (2.4a).

Estimates on anisotropic and demagnetization field terms we define

$$\begin{aligned}\delta_\varepsilon^1\boldsymbol{F}^{a,d} &= -\delta_\varepsilon^1\boldsymbol{m}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)}) - \boldsymbol{m}^{\varepsilon,(0)}\times\mathcal{H}_{d,a}(\delta_\varepsilon^1\boldsymbol{m}) \\ &- \alpha\boldsymbol{m}^{\varepsilon,(0)}\times(\boldsymbol{m}^{\varepsilon,(0)}\times\mathcal{H}_{d,a}(\delta_\varepsilon^1\boldsymbol{m})) \\ &- \alpha\boldsymbol{m}^{\varepsilon,(0)}\times(\delta_\varepsilon^1\boldsymbol{m}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)})) - \alpha\delta_\varepsilon^1\boldsymbol{m}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)})).\end{aligned}$$

But $\mathcal{H}_{d,a} = \mathcal{H}_a + \mathcal{H}_d$ is continuous from L^q to L^q for all $1\leq q<+\infty$, and in particular for $q=4$ or $q=6$. Moreover, $\boldsymbol{m}^{\varepsilon,(1)}$, $\boldsymbol{m}^{(0)}$ and $\boldsymbol{m}^{\varepsilon,(0)}$ belongs to $\mathrm{L}^\infty(0,T;\mathbb{L}^6(\Omega_\varepsilon))$ and to $\mathrm{L}^\infty(0,T;\mathbb{L}^4(\Omega_\varepsilon))$, for all $T<T^*$. Thus, for all $T<T^*$, there exists $C(T)$ independent of ε such that

$$\|\delta_\varepsilon^1\boldsymbol{F}^{a,d}\|_{\mathbb{L}^2(\Omega_\varepsilon)} \leq C(T)\|\delta_\varepsilon^1\boldsymbol{m}\|_{\mathbb{H}^1(\Omega_\varepsilon)}.$$

First estimate We multiply equation (3.1) by $\delta_\varepsilon^1\boldsymbol{m}$ and integrate over Ω

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\varepsilon}|\delta_\varepsilon^1\boldsymbol{m}|^2\mathrm{d}\boldsymbol{x} + \alpha A\int_{\Omega_\varepsilon}|\nabla\delta_\varepsilon^1\boldsymbol{m}|^2\mathrm{d}\boldsymbol{x} &= -A\underbrace{\sum_i\int_{\Omega_\varepsilon}\left(\frac{\partial\delta_\varepsilon^1\boldsymbol{m}}{\partial x_i}\times\frac{\partial\boldsymbol{m}^{\varepsilon,(0)}}{\partial x_i}\right)\cdot\delta_\varepsilon^1\boldsymbol{m}\mathrm{d}\boldsymbol{x}}_{I} \\ &+ \alpha A\underbrace{\int_{\Omega_\varepsilon}(\nabla\boldsymbol{m}^{(0)}\cdot\nabla(\boldsymbol{m}^{(0)}+\boldsymbol{m}^{\varepsilon,(0)}))(\boldsymbol{m}^{\varepsilon,(0)}\cdot\delta_\varepsilon^1\boldsymbol{m})\mathrm{d}\boldsymbol{x}}_{II} \\ &+ \alpha A\underbrace{\int_{\Omega_\varepsilon}|\nabla\boldsymbol{m}^{(0)}|^2|\delta_\varepsilon^1\boldsymbol{m}|^2\mathrm{d}\boldsymbol{x}}_{III} + \underbrace{\int_{\Omega_\varepsilon}\delta_\varepsilon^1\boldsymbol{F}^{a,d}\cdot\delta_\varepsilon^1\boldsymbol{m}\mathrm{d}\boldsymbol{x}}_{IV}. \quad (3.4)\end{aligned}$$

First, we estimate $I = \sum_i \int_{\Omega_\varepsilon} \left(\frac{\partial \delta^1_\varepsilon \boldsymbol{m}}{\partial x_i} \times \frac{\partial \boldsymbol{m}^{\varepsilon,(0)}}{\partial x_i}\right) \cdot \delta^1_\varepsilon \boldsymbol{m} \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned} |I| &\leq \|\nabla \boldsymbol{m}^{(0)}\|_{\mathbb{L}^\infty(\Omega_\varepsilon)} \|\delta^1_\varepsilon \boldsymbol{m}\|_{\mathbb{L}^2(\Omega_\varepsilon)} \|\nabla \delta^1_\varepsilon \boldsymbol{m}\|_{\mathbb{L}^2(\Omega_\varepsilon)} \\ &\leq \frac{1}{4\eta} \|\nabla \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega_\varepsilon)} \|\delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \eta \|\nabla \delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)}. \end{aligned} \tag{3.5a}$$

We also estimate $II = \int_{\Omega_\varepsilon} (\nabla \boldsymbol{m}^{(0)} \cdot \nabla(\boldsymbol{m}^{(0)} + \boldsymbol{m}^{\varepsilon,(0)}))(\boldsymbol{m}^{\varepsilon,(0)} \cdot \delta^1_\varepsilon \boldsymbol{m}) \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned} |II| \leq \frac{1}{4\eta} &\|\nabla(\boldsymbol{m}^{(0)} + \boldsymbol{m}^{\varepsilon,(0)})\|^2_{\mathbb{L}^\infty(\Omega_\varepsilon)} \|\delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} \\ &+ \eta \|\nabla(\delta^1_\varepsilon \boldsymbol{m})\|^2_{\mathbb{L}^2(\Omega_\varepsilon)}. \end{aligned} \tag{3.5b}$$

Estimating $III = \int_{\Omega_\varepsilon} |\nabla \boldsymbol{m}^{(0)}|^2 |\delta^1_\varepsilon \boldsymbol{m}|^2 \mathrm{d}\boldsymbol{x}$ yields

$$|III| \leq \|\nabla \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^\infty(\Omega_\varepsilon)} \|\delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)}. \tag{3.5c}$$

We estimate $IV = \int_{\Omega_\varepsilon} \delta^1_\varepsilon \boldsymbol{F}^{a,d} \cdot \delta^1_\varepsilon \boldsymbol{m} \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned} |IV| &\leq \|\delta^1_\varepsilon \boldsymbol{F}^{a,d}\|_{\mathbb{L}^2(\Omega_\varepsilon)} \|\delta^1_\varepsilon \boldsymbol{m}\|_{\mathbb{L}^2(\Omega_\varepsilon)} \\ &\leq C\left(1 + \frac{1}{\eta}\right) \|\delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \eta \|\nabla \delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)}. \end{aligned} \tag{3.5d}$$

We combine all inequalities (3.5), we choose η small enough. By Gronwall's lemma, there exists a constant C_T such that

$$\|\delta^1_\varepsilon \boldsymbol{m}\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))} \leq C_T \varepsilon \qquad \|\nabla \delta^1_\varepsilon \boldsymbol{m}\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \leq C_T \varepsilon \tag{3.6}$$

Second estimate We multiply equation (3.1) by $-\triangle \delta^1_\varepsilon \boldsymbol{m}$ and integrate over Ω

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\varepsilon} |\nabla(\delta^1_\varepsilon \boldsymbol{m})|^2 \mathrm{d}\boldsymbol{x} + \alpha A \int_{\Omega_\varepsilon} |\triangle(\delta^1_\varepsilon \boldsymbol{m})|^2 \mathrm{d}\boldsymbol{x} = \\ &\underbrace{A \int_{\Omega_\varepsilon} (\triangle \boldsymbol{m}^{(0)} \times \triangle \delta^1_\varepsilon \boldsymbol{m}) \cdot \delta^1_\varepsilon \boldsymbol{m} \mathrm{d}\boldsymbol{x}}_{I} - \alpha A \underbrace{\int_{\Omega_\varepsilon} (\nabla \delta^1_\varepsilon \boldsymbol{m} \cdot \nabla(\boldsymbol{m}^{\varepsilon,(0)} + \boldsymbol{m}^{(0)})) \boldsymbol{m}^{(0)} \cdot \triangle \delta^1_\varepsilon \boldsymbol{m}) \mathrm{d}\boldsymbol{x}}_{II} \\ &\qquad - \alpha A \underbrace{\int_{\Omega_\varepsilon} |\nabla \boldsymbol{m}^{(0)}|^2 \delta^1_\varepsilon \boldsymbol{m} \cdot \triangle \delta^1_\varepsilon \boldsymbol{m} \mathrm{d}\boldsymbol{x}}_{III} - \underbrace{\int_{\Omega_\varepsilon} \delta^1_\varepsilon \boldsymbol{F}^{a,d} \cdot \triangle \delta^1_\varepsilon \boldsymbol{m} \mathrm{d}\boldsymbol{x}}_{IV}. \end{aligned} \tag{3.7}$$

We estimate $I = \int_{\Omega_\varepsilon} (\triangle \boldsymbol{m}^{(0)} \times \triangle \delta^1_\varepsilon \boldsymbol{m}) \cdot \delta^1_\varepsilon \boldsymbol{m} \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned} |I| &\leq \|\triangle \boldsymbol{m}^{(0)}\|_{\mathbb{L}^3(\Omega_\varepsilon)} \|\triangle(\delta^1_\varepsilon \boldsymbol{m})\|_{\mathbb{L}^2(\Omega_\varepsilon)} \|\delta^1_\varepsilon \boldsymbol{m}\|_{\mathbb{L}^6(\Omega_\varepsilon)} \\ &\leq \frac{1}{4\eta} \|\triangle \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^3(\Omega_\varepsilon)} \|\delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^6(\Omega_\varepsilon)} + \eta \|\triangle(\delta^1_\varepsilon \boldsymbol{m})\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} \\ &\leq \frac{C}{4\eta} \|\triangle \boldsymbol{m}^{(0)}\|^2_{\mathbb{L}^3(\Omega_\varepsilon)} \|\delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{H}^1(\Omega_\varepsilon)} + \eta \|\triangle(\delta^1_\varepsilon \boldsymbol{m})\|^2_{\mathbb{L}^2(\Omega_\varepsilon)}. \end{aligned} \tag{3.8a}$$

Then, we estimate $II = \int_{\Omega_\varepsilon} (\nabla \delta^1_\varepsilon \boldsymbol{m} \cdot \nabla(\boldsymbol{m}^{\varepsilon,(0)} + \boldsymbol{m}^{(0)})) \boldsymbol{m}^{(0)} \cdot \triangle \delta^1_\varepsilon \boldsymbol{m}) \mathrm{d}\boldsymbol{x}$.

$$|II| \leq \frac{1}{4\eta} \|\nabla(\boldsymbol{m}^{\varepsilon,(0)} + \boldsymbol{m}^{(0)})\|^2_{\mathbb{L}^\infty(\Omega_\varepsilon)} \|\nabla \delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \eta \|\triangle \delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)}. \tag{3.8b}$$

Estimating $III = \int_{\Omega_\varepsilon} |\nabla \boldsymbol{m}^{(0)}|^2 \delta^1_\varepsilon \boldsymbol{m} \cdot \triangle \delta^1_\varepsilon \boldsymbol{m} \mathrm{d}\boldsymbol{x}$ yields

$$|III| \leq \frac{1}{4\eta} \|\nabla \boldsymbol{m}^{(0)}\|^4_{\mathbb{L}^\infty(\Omega_\varepsilon)} \|\delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \eta \|\triangle \delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)}. \tag{3.8c}$$

Then, we estimate $IV = \int_{\Omega_\varepsilon} \delta^1_\varepsilon \boldsymbol{F}^{a,d} \cdot \triangle \delta^1_\varepsilon \boldsymbol{m} \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned} |IV| &\leq \|\delta^1_\varepsilon \boldsymbol{F}^{a,d}\|_{\mathbb{L}^2(\Omega_\varepsilon)} \|\triangle \delta^1_\varepsilon \boldsymbol{m}\|_{\mathbb{L}^2(\Omega_\varepsilon)} \\ &\leq \frac{C}{\eta} \|\delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \eta \|\triangle \delta^1_\varepsilon \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega_\varepsilon)}. \end{aligned} \tag{3.8d}$$

We combine all equations (3.8), choose η small enough. By Gronwall's lemma, there exists a constant C_T such that

$$\|\nabla \delta^1_\varepsilon \boldsymbol{m}\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))} \leq \varepsilon C_T \qquad \|\triangle \delta^1_\varepsilon \boldsymbol{m}\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \leq \varepsilon C_T \tag{3.9}$$

Thus by elliptic regularity the $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ of $\mathrm{D}^2 \delta^1_\varepsilon \boldsymbol{m}$ is also bounded by $C'_T \varepsilon$.

Estimate of the time derivative It only remains to estimate $\frac{\partial \delta^1_\varepsilon \boldsymbol{m}}{\partial t}$. The previous estimates and (3.1) imply that $\left\| \frac{\partial \delta^1_\varepsilon \boldsymbol{m}}{\partial t} \right\|_{\mathbb{L}^2(0,T;\mathbb{L}^2(\Omega_\varepsilon))}$ is a $O(\varepsilon)$.

□

3.2 *Convergence of* $\boldsymbol{m}^{\varepsilon,(0)} - \boldsymbol{m}^{(0)} - \varepsilon \boldsymbol{m}^{(1)}$

To compare solutions existing on Ω or Ω_ε, we need a sensible comparison criteria. We introduce an extension operator in order to compare $\boldsymbol{m}^{\varepsilon,(0)}$ and $\boldsymbol{m}^{(0)}$ over Ω. We define $\widetilde{\boldsymbol{m}^\varepsilon} = \mathrm{Prol}_\varepsilon(\boldsymbol{m}^{\varepsilon,(0)})$ by reflection.

$$\widetilde{\boldsymbol{m}^\varepsilon}(x,y,z) = \begin{cases} \boldsymbol{m}^{\varepsilon,(0)} & \text{on } \Omega_\varepsilon, \\ \boldsymbol{m}^{\varepsilon,(0)}(x,y,\varepsilon - z) & \text{on } B \times (0,\varepsilon) \times (0,T), \\ \boldsymbol{m}^{\varepsilon,(0)}(x,y,-\varepsilon - z) & \text{on } B \times (-\varepsilon,0) \times (0,T). \end{cases}$$

Then, $\widetilde{\boldsymbol{m}^\varepsilon}$ belongs to $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$, verifies the Neumann homogenous boundary condition over the lateral boundary and also almost verifies Landau-Lifshitz equation with a small correction for the demagnetization field operator.

Lemma 3.4. *Our extension operator satisfy*

$$\begin{aligned} \|\mathrm{Prol}_\varepsilon(\boldsymbol{m}_{|\Omega_\varepsilon})\|_{\mathbb{H}^2(\Omega)} &\leq C\|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)} + C\varepsilon \|\boldsymbol{m}\|_{\mathbb{H}^3(\Omega)}, \\ \|\mathrm{Prol}_\varepsilon(\boldsymbol{m}_{|\Omega_\varepsilon})\|_{\mathbb{L}^2(\Omega)} &\leq \|\boldsymbol{m}\|_{\mathbb{L}^2(\Omega)} + C\varepsilon \|\boldsymbol{m}\|_{\mathbb{H}^1(\Omega)}. \end{aligned}$$

PROOF : The result follows from

$$\begin{aligned} \|\mathrm{Prol}_\varepsilon(\boldsymbol{m})\|^2_{\mathbb{L}^2} &\leq 2\|\boldsymbol{m}\|^2_{\mathbb{L}^2} + 2\int_\Gamma \int_{z=0}^{\varepsilon} \left| \int_{s=\varepsilon-z}^{\varepsilon+z} \frac{\partial \boldsymbol{m}}{\partial z} \mathrm{d}s \right|^2 \mathrm{d}z \mathrm{d}\sigma(\boldsymbol{x}), \\ &\leq 2\|\boldsymbol{m}\|^2_{\mathbb{L}^2} + 4\varepsilon^2 \int_\Gamma \int_{z=0}^{2\varepsilon} \left| \frac{\partial \boldsymbol{m}}{\partial z} \right|^2 \mathrm{d}z \mathrm{d}\sigma(\boldsymbol{x}). \end{aligned}$$

□

Theorem 3.5. *The quantity* $\frac{\mathrm{Prol}_\varepsilon(\boldsymbol{m}^{\varepsilon,(0)})-\boldsymbol{m}^{(0)}}{\varepsilon}$ *converges weakly to* $\boldsymbol{m}^{(1)}$ *in* $\mathbb{H}^{2,1}(\Omega\times(0,T))$.

PROOF : According to Theorem 3.3 $\|\boldsymbol{m}^{\varepsilon,(0)}-\boldsymbol{m}^{(0)}\|_{\mathbb{H}^{2,1}}$ is a $O(\varepsilon)$ thus according to Lemma 3.4 $\|\widetilde{\boldsymbol{m}^\varepsilon}-\boldsymbol{m}^{(0)}\|_{\mathbb{H}^{2,1}}=O(\varepsilon)$. Thus, there exists a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and $\overline{\boldsymbol{m}^{(1)}}$ in $\mathbb{H}^{2,1}(\Omega\times(0,T))$ such that,

$$\frac{\widetilde{\boldsymbol{m}^{\varepsilon_n}}-\boldsymbol{m}^{(0)}}{\varepsilon_n}\to\overline{\boldsymbol{m}^{(1)}}\text{ weakly in }\mathbb{H}^{2,1}(\Omega\times(0,T)).$$

We now prove that $\overline{\boldsymbol{m}^{(1)}}=\boldsymbol{m}^{(1)}$ and that the whole sequence converges.

We introduce $\delta'_\varepsilon\boldsymbol{m}=\frac{\widetilde{\boldsymbol{m}^\varepsilon}-\boldsymbol{m}^{(0)}}{\varepsilon}$. Then, for all $k\geq 0$

$$\begin{aligned}\frac{\partial\delta'_{\varepsilon_{n_k}}\boldsymbol{m}}{\partial t}-\alpha A\triangle\delta'_{\varepsilon_{n_k}}\boldsymbol{m}&=-\widetilde{\boldsymbol{m}^\varepsilon}\times A\triangle\delta'_{\varepsilon_{n_k}}\boldsymbol{m}-\delta'_{\varepsilon_{n_k}}\boldsymbol{m}\times A\triangle\boldsymbol{m}^{(0)}+\alpha A|\nabla\boldsymbol{m}^{(0)}|^2\delta'_{\varepsilon_{n_k}}\boldsymbol{m}\\&+\alpha A((\nabla\boldsymbol{m}^{(0)}+\nabla\widetilde{\boldsymbol{m}^\varepsilon})\cdot\nabla\delta'_{\varepsilon_{n_k}}\boldsymbol{m})\widetilde{\boldsymbol{m}^\varepsilon}-\delta'_{\varepsilon_{n_k}}\boldsymbol{m}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)})\\&-\widetilde{\boldsymbol{m}^\varepsilon}\times\mathcal{H}_{d,a}(\delta'_{\varepsilon_{n_k}}\boldsymbol{m})-\alpha\delta'_{\varepsilon_{n_k}}\boldsymbol{m}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)}))\\&-\alpha\widetilde{\boldsymbol{m}^\varepsilon}\times(\delta'_{\varepsilon_{n_k}}\boldsymbol{m}\times\mathcal{H}_{d,a}(\boldsymbol{m}^{(0)}))-\alpha\widetilde{\boldsymbol{m}^\varepsilon}\times(\widetilde{\boldsymbol{m}^\varepsilon}\times\mathcal{H}_{d,a}(\delta'_{\varepsilon_{n_k}}\boldsymbol{m}))\\&-\boldsymbol{m}^{(0)}\times\frac{1}{\varepsilon}\mathcal{H}_d(\widetilde{1}^*_{(-\varepsilon,\varepsilon)}\boldsymbol{m}^{(0)})-\alpha\boldsymbol{m}^{(0)}\times(\boldsymbol{m}^{(0)}\times\frac{1}{\varepsilon}\mathcal{H}_d(\widetilde{1}^*_{(-\varepsilon,\varepsilon)}\boldsymbol{m}^{(0)}))\text{in }\Omega_\varepsilon.\quad(3.10)\end{aligned}$$

We take the limit in the sense of distributions, in (3.10). $\overline{\boldsymbol{m}^{(1)}}$ satisfies equation (2.6a). Since $\delta'_{\varepsilon_{n_k}}\boldsymbol{m}\cdot(\boldsymbol{m}^{(0)}+\widetilde{\boldsymbol{m}^\varepsilon})=0$, $\overline{\boldsymbol{m}^{(1)}}$ satisfies constraint (2.6b). Moreover, over $\partial\Omega\setminus\Gamma$, we have $\frac{\partial\delta'_{\varepsilon_{n_k}}\boldsymbol{m}}{\partial\boldsymbol{\nu}}=0$. Thus, $\overline{\boldsymbol{m}^{(1)}}$ satisfies equation (2.6c) on $\partial\Omega\setminus\Gamma$. On Γ,

$$\begin{aligned}&\iint_{\Gamma\times(0,T)}\left|\frac{\partial\boldsymbol{m}^{(0)}}{\partial z}(\boldsymbol{x},\varepsilon,t)+\frac{\partial\widetilde{\boldsymbol{m}^\varepsilon}}{\partial z}(\boldsymbol{x},0^+,t)\right|^2\mathrm{d}\sigma(\boldsymbol{x})\mathrm{d}t\\&\qquad\leq\iiint\left|\int_{z=0}^{\varepsilon}\frac{\partial^2(\widetilde{\boldsymbol{m}^\varepsilon}-\boldsymbol{m}^{(0)})}{\partial z^2}(\boldsymbol{x},z,t)\mathrm{d}z\right|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t\\&\qquad\leq\varepsilon\iiint\int_{z=0}^{\varepsilon}\left|\frac{\partial^2(\widetilde{\boldsymbol{m}^\varepsilon}-\boldsymbol{m}^{(0)})}{\partial z^2}(\boldsymbol{x},z,t)\right|^2\mathrm{d}z\mathrm{d}\boldsymbol{x}\mathrm{d}t\leq\varepsilon^3\left\|\frac{\widetilde{\boldsymbol{m}^\varepsilon}-\boldsymbol{m}^{(0)}}{\varepsilon}\right\|^2_{\mathbb{H}^{0,2,0}}.\end{aligned}$$

Thus,

$$\int_0^T\int_\Gamma\left|\frac{1}{\varepsilon}\frac{\partial\boldsymbol{m}^{(0)}}{\partial z}(\boldsymbol{x},\varepsilon,t)+\frac{1}{\varepsilon}\frac{\partial\widetilde{\boldsymbol{m}^\varepsilon}}{\partial z}(\boldsymbol{x},0^+,t)\right|^2\mathrm{d}\sigma(\boldsymbol{x})\mathrm{d}t=O(\varepsilon).$$

Hence,

$$\frac{\partial\overline{\boldsymbol{m}^{(1)}}}{\partial z}(\cdot,0^+,\cdot)=\lim_{\varepsilon_k\to0}\frac{1}{\varepsilon_k}\frac{\partial\widetilde{\boldsymbol{m}^\varepsilon}}{\partial z}(\cdot,0^+,\cdot),\;=-\lim_{\varepsilon_k\to0}\frac{1}{\varepsilon_k}\frac{\partial\boldsymbol{m}^{(0)}}{\partial z}(\cdot,\varepsilon,\cdot),\;=-\frac{\partial^2\boldsymbol{m}^{(0)}}{\partial z^2}(\cdot,0^+,\cdot).$$

Hence, $\overline{\boldsymbol{m}^{(1)}}$ satisfies (2.6c) on Γ. By equality (2.4b), $\overline{\boldsymbol{m}^{(1)}}$ also satisfies (2.6d). Thus, $\overline{\boldsymbol{m}^{(1)}}$ is the unique solution to system (2.6). Thus, $\overline{\boldsymbol{m}^{(1)}}=\boldsymbol{m}^{(1)}$. The whole sequence converges. □

4 Numerical simulations

We now have an equivalent boundary condition that allow us to simulate the evolution of a ferromagnetic body crossed by a thin split. The finite volume scheme introduced by S. Labbé [11] is adapted to the first order equation.

4.1 Space discretization

The space discretization is done via finite volume. We use a regular cubic mesh. This choice is primordial to compute the demagnetization field excitation via Toeplitz matrices. We must discretize the anisotropy operator, the exchange operator with both kind of Neumann conditions and the demagnetization field operator. All this work can be found in [11]. The exchange operator is discretized as the standard Laplace operator with Neumann homogenous boundary condition. We also discretize the anisotropy operator, denoting by $\mathbf{K}_i$ the mean of $\mathbf{K}$ over cell i.

$$\mathcal{H}_{a,h}\boldsymbol{m}_i = \mathbf{K}_i\boldsymbol{m}_i, \qquad \mathcal{H}^0_{e,h}(\boldsymbol{m})_i = \frac{A}{h^2}\sum_{j\in V(i)}(\boldsymbol{m}_i - \boldsymbol{m}_j), \tag{4.1}$$

where $V(i)$ is the set of all the neighbors of cell i in the mesh. We also define the discretization of the correction of the exchange operator with conditions (2.6c).

$$\mathcal{H}^{1,\mathrm{co}}_{e,h}(\boldsymbol{m})_i = \delta(i)\frac{A}{h^3}\left(\boldsymbol{m}^{(0)}_{N(i)} - \boldsymbol{m}^{(0)}_i\right), \tag{4.2}$$

where $\delta(i)$ is 1 if cell i is adjacent to the interface Γ, and 0 otherwise. In the former case, cell $N(i)$ is the adjacent cell to cell i such that cell i is between cell $N(i)$ and Γ. This discretization require at least two cells of depth in the mesh on each side of the split.

The discretization of the demagnetization field operator is done by defining the operators

$$\begin{aligned} R_h &: \mathbb{R}^n \to \mathbb{L}^2(\Omega), \quad P_h : \mathbb{L}^2(\Omega) \to \mathbb{R}^n. \\ R_h(v) &\mapsto \sum_{i=1}^n v_i\widetilde{1}_{\omega_i}, \qquad P_h(u)_i \mapsto \frac{1}{|\omega_i|}\int_{\omega_i} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{4.3}$$

where n is the number of cells. We would ideally define $\mathcal{H}_{d,h}$ as $P_h\circ\mathcal{H}_d\circ R_h$. In practice, P_h is computed by Gauss integration with some corrections close to singularities[6]. The operator is Toeplitz and can thus be computed by fast Fourier transform.

4.2 Time discretization

The equation is discretized in time by a second order scheme.

$$\begin{aligned} \boldsymbol{m}^{(0)}_{i+1} - \boldsymbol{m}^{(0)}_i &= \Delta t_i\boldsymbol{F}_h(\boldsymbol{m}^{(0)}_i) + \frac{{\Delta t_i}^2}{2}\mathrm{D}_{\boldsymbol{m}^{(0)}_i}\boldsymbol{F}_h\cdot\boldsymbol{F}_h(\boldsymbol{m}^{(0)}_i), \\ \boldsymbol{F}_h(\boldsymbol{m}^{(0)}) &= -\boldsymbol{m}^{(0)}\times\mathcal{H}^0_h(\boldsymbol{m}^{(0)}) - \alpha\boldsymbol{m}^{(0)}\times(\boldsymbol{m}^{(0)}\times\mathcal{H}^0_h(\boldsymbol{m}^{(0)})). \end{aligned}$$

$\mathcal{H}^0_h$ being equal to $\mathcal{H}_{d,h}+\mathcal{H}_{a,h}+\mathcal{H}^0_{e,h}$. The time step size is chosen to maximize energy loss.

For the first order term, we use an analogous scheme.

$$\begin{aligned} \boldsymbol{m}^{(1)}_{i+1} - \boldsymbol{m}^{(1)}_i = {}& \Delta t_i\boldsymbol{F}^1_h(\boldsymbol{m}^{(0)}_i, \boldsymbol{m}^{(1)}_i, \mathcal{H}^{1,\mathrm{co}}_{e,h}(\boldsymbol{m}^{(0)}_i) - \mathcal{H}_{d,h}(\gamma^0\boldsymbol{m}^{(0)}_i\mathrm{d}\sigma(\Gamma))) \\ &+ \frac{{\Delta t_i}^2}{2}\mathrm{D}_{\boldsymbol{m}^{(0)}_i}\boldsymbol{F}^1_h(\boldsymbol{m}^{(0)}_i, \boldsymbol{m}^{(1)}_i, \mathcal{H}^{1,\mathrm{co}}_{e,h}(\boldsymbol{m}^{(0)}_i))\cdot\boldsymbol{F}_h(\boldsymbol{m}^{(0)}_i) \\ &+ \frac{{\Delta t_i}^2}{2}\boldsymbol{F}^1_h(\boldsymbol{m}^{(0)}_i, \boldsymbol{F}^1_h(\boldsymbol{m}^{(0)}_i, \boldsymbol{m}^{(1)}_i, \mathcal{H}^{1,\mathrm{co}}_{e,h}(\boldsymbol{m}^{(0)}_i)), \\ &\qquad \mathcal{H}^{1,\mathrm{co}}_{e,h}(\boldsymbol{F}_h(\boldsymbol{m}^{(0)}_i)) - \mathcal{H}_{d,h}(\gamma^0\boldsymbol{F}_h(\boldsymbol{m}^{(0)}_i)\mathrm{d}\sigma(\Gamma)))), \end{aligned}$$

[6]The interested reader should refer to [11] or [12] for details.

where

$$\begin{aligned}\boldsymbol{F}_h^1(\boldsymbol{m}_i^{(0)},\boldsymbol{m}_i^{(0)},\boldsymbol{h}^{\rm co}) = -\boldsymbol{m}_{h,i}^{(0)}\times(\mathcal{H}_h(\boldsymbol{m}_{h,i}^{(1)})+\boldsymbol{h}^{\rm co}) - \boldsymbol{m}_{h,i}^{(1)}\times\mathcal{H}_h(\boldsymbol{m}_{h,i}^{(0)})\\ -\alpha\boldsymbol{m}_{h,i}^{(0)}\times(\boldsymbol{m}_{h,i}^{(0)}\times(\mathcal{H}_h(\boldsymbol{m}_{h,i}^{(1)})+\boldsymbol{h}^{\rm co}))\\ -\alpha\boldsymbol{m}_{h,i}^{(0)}\times(\boldsymbol{m}_{h,i}^{(1)}\times\mathcal{H}_h(\boldsymbol{m}_{h,i}^{(0)})) - \alpha\boldsymbol{m}_{h,i}^{(1)}\times(\boldsymbol{m}_{h,i}^{(0)}\times\mathcal{H}_h(\boldsymbol{m}_{h,i}^{(0)})).\end{aligned}$$

We use the same step size for the first order as for the zero order.

4.3 Simulations: results

In our simulations, the aim is to compute final equilibrium states for various initial conditions. Thus, we are only interested in the final states of the dynamic numerical simulation. We stop the simulation when the derivative of the discrete energy cross a threshold. By h, we denote the space step-size.

4.3.1 Physical parameters

We consider a thin plate with a mesh $256\times128\times1$, hence 32768 grid points, with a step size of 2.3nm. Their magnetic parameters are

$$M_s = 1.4 * 10^6, \qquad A = 10^{-11}/\mu_0, \qquad \mathbf{K} = \mathbf{0}.$$

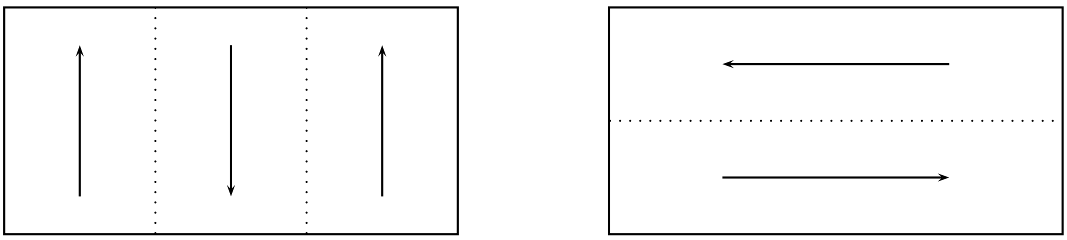
For initial conditions and the position of the split, we choose among those represented in Figure 3, and Figure 4. We prefer to represent $h\boldsymbol{m}^{(1)}$ instead of $\boldsymbol{m}^{(1)}$ in the numerical results. We make the following simulations

Simulation 0a First initial condition without split.

Simulation 0b Second initial condition without split.

Simulation 1 First initial condition and longitudinal split.

Simulation 2 Second initial condition and transversal split.



First initial condition *Second initial condition*

Figure 3: Possible initial conditions for $\boldsymbol{m}^{(0)}$

Simulations $0a$ et $0b$, Figure 5, serve as a basis of comparison. In each simulation, we iterated 7000 times. The analysis of the energy graph imply that we are reasonably close from equilibrium after a thousand iterations. On a PC with a single processor and 784 Mo of RAM, the Fortran program needed 6 hours to compute the equilibrium states for each configuration.

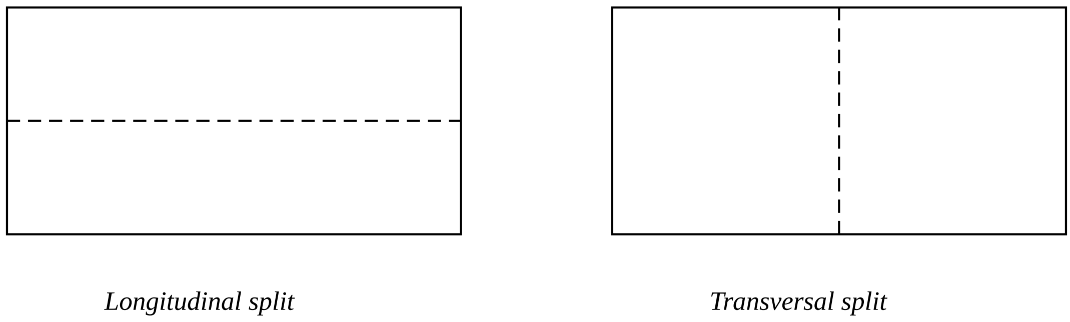


Figure 4: Possible positions for the spacer

4.3.2 Analysis of the results

We first compare the results between geometries with and without splits. First, in the presence of a split, final energies are lower. Magnetization reversal is easier across the split. Also, the magnetization at the equilibrium states lies parallel to the thin plate.

We analyze the equilibrium state of the magnetization obtained during the first simulation, Figure 6. Like the initial state, the final state is symmetric with respect to the split. We notice a reversal of the magnetization when crossing the split. There are two vortices in the whole ferromagnetic body. Walls are thick except when crossing the split. We also offer a zoom on the center left, Figure 7, and we notice that the vortex is stretched along the split. The correction term is almost entirely concentrated on the split. Four incomplete vortices are present, two on each side of the split at one fourth of the length of the split and the same pair at the three fourth on the split.

We analyze the equilibrium state of the magnetization obtained during the second simulation, Figure 8. The initial term of order 0 is antisymmetric with respect to the transversal axis. This antisymmetry remains in the equilibrium state. We can see one global vortex in the whole plate and the split is clearly visible. We also present a zoom on the central vortex, Figure 9. We notice that the vortex is stretched in the direction of the split. This is due to the absence of exchange between both faces of the split. Let's analyze the correction term. It is almost entirely concentrated on the split. Two incomplete vortices appear on the split at respectively one third and two third of the length of the split. The size of the correction is small.

Conclusion

We have established an equivalent boundary condition for the equations of micromagnetism. We can now compute the behavior of a ferromagnetic material without having to mesh the split. Our next work will include in this model physical interactions such as super-exchange and surface-anisotropy as presented in [14]. This will be the second part of this article. We could also consider the same configuration in the non quasi-stationary case with the full Maxwell's equations and some other generalizations such as non plane splits and a split filled with a magnetic material.

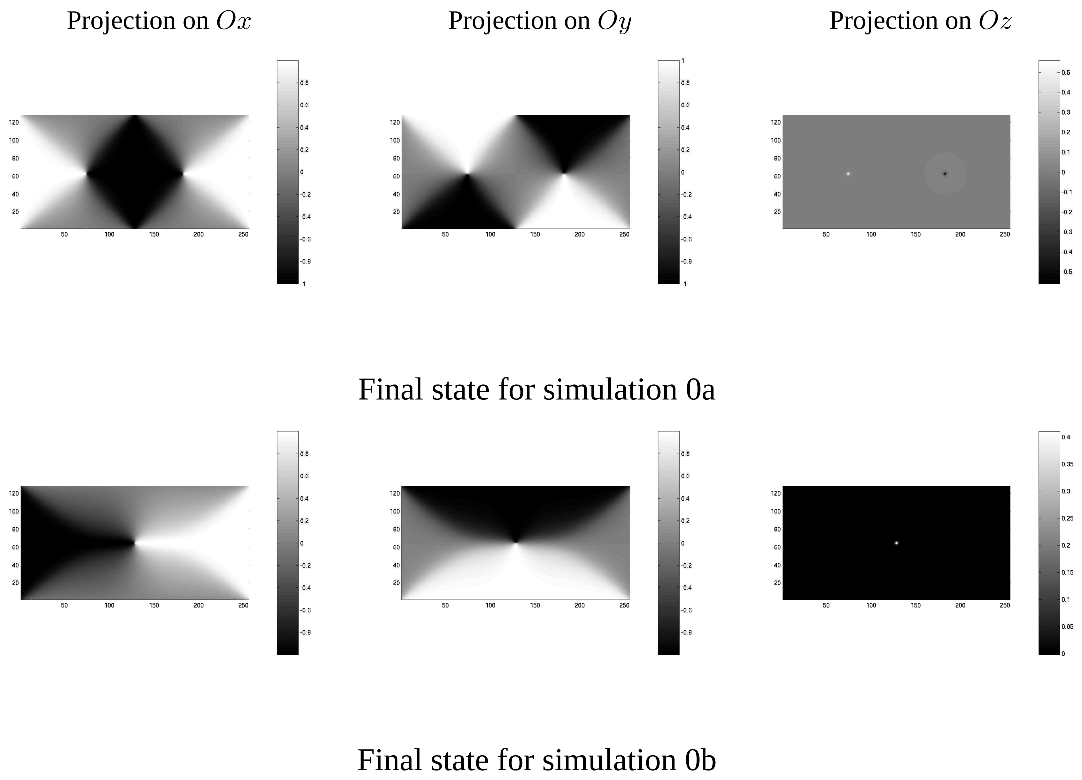


Figure 5: Simulations with no split

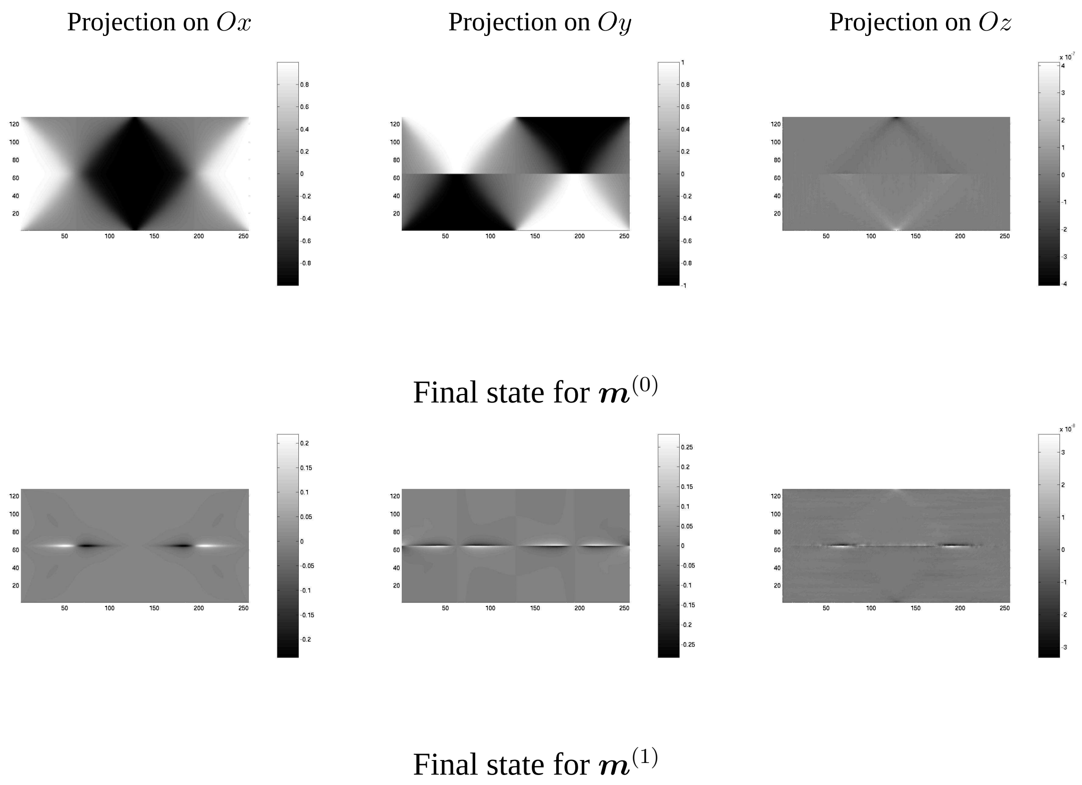


Figure 6: First Simulation

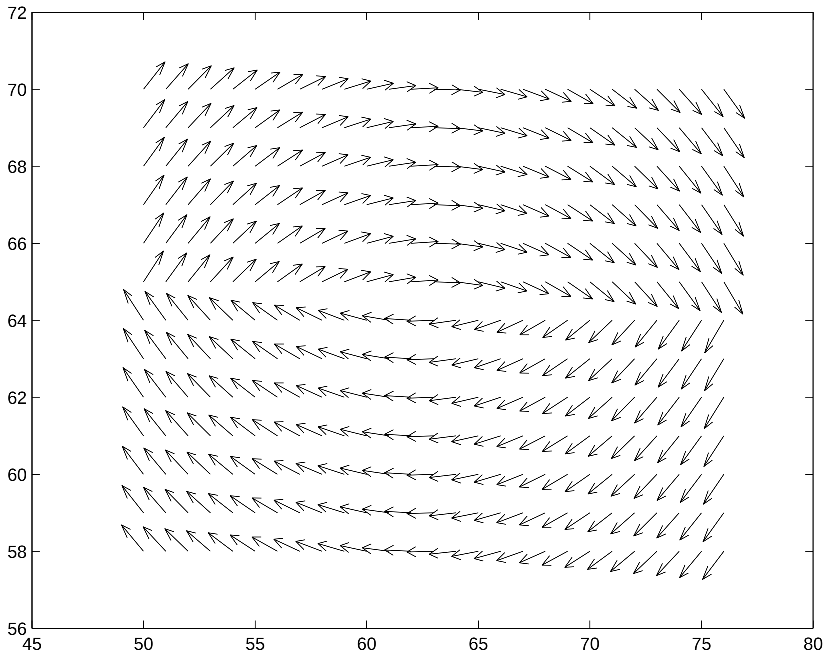


Figure 7: Zoom on the left center vortex for $\boldsymbol{m}^{(0)}$ for the first simulation

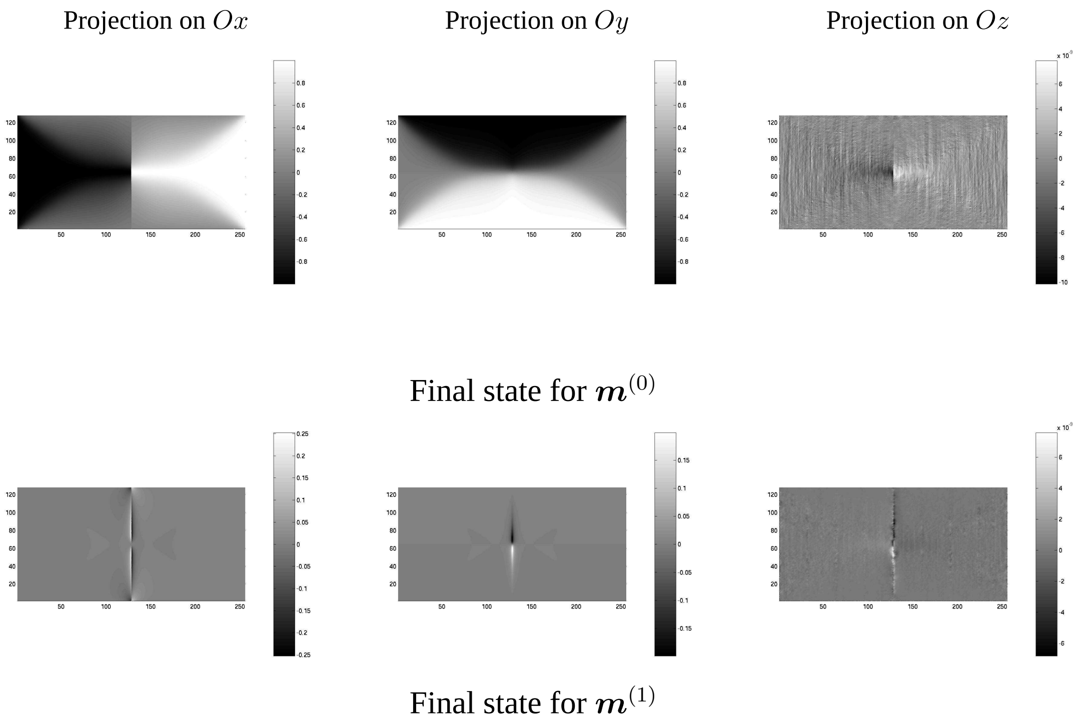


Figure 8: Second Simulation

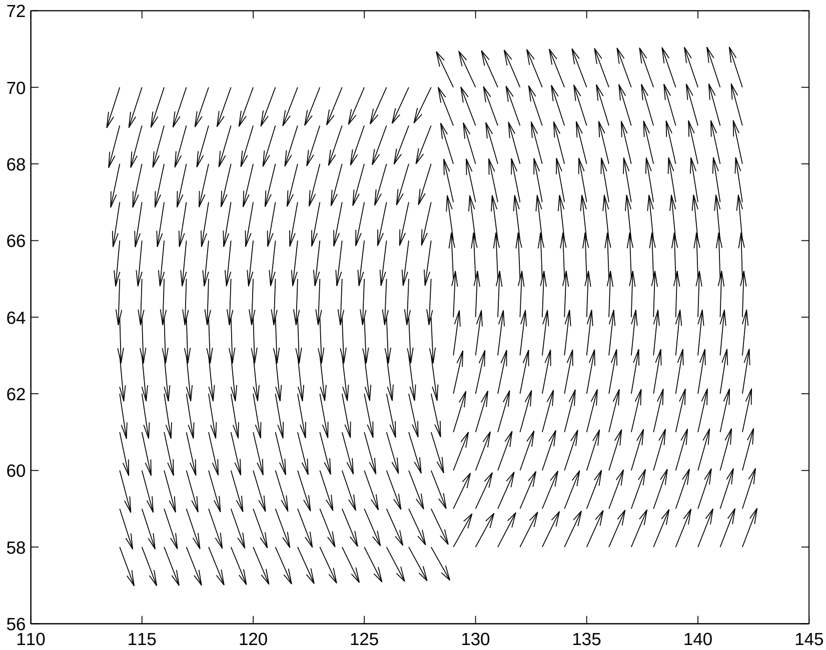


Figure 9: Zoom on the center vortex for $\boldsymbol{m}^{(0)}$ for the second simulation

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