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Stable reduction of finite covers of curves

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Abstract

Let K be the function field of a connected regular scheme S of dimension 1, and let $f : X \to Y$ be a finite cover of projective smooth and geometrically connected curves over K with $g(X) \geqslant 2$. Suppose that f can be extended to a finite cover $\mathcal{X} \to \mathcal{Y}$ of semi-stable models over S (it is known that this is always possible up to finite separable extension of K). Then there exists a unique minimal such cover. This gives a canonical way to extend $X \to Y$ to a finite cover of semi-stable models over S.

Let S be a Dedekind scheme (i.e. a connected Noetherian regular scheme of dimension 1), with field of functions $K := K(S)$. Let $f : X \to Y$ be a finite morphism of smooth geometrically connected projective curves over K. We can ask how to extend, in some canonical way, the morphism f to a morphism of models of X and Y over S. It is proved in [LL99, Proposition 4.4] that if X and Y have stable models $\mathcal{X}^{\mathrm{st}}, \mathcal{Y}^{\mathrm{st}}$ over S, then f extends uniquely to a morphism $\mathcal{X}^{\mathrm{st}} \to \mathcal{Y}^{\mathrm{st}}$. However, we will in general lose the finiteness of f. On the other hand, after a finite separable extension of K, f extends to a finite morphism of semi-stable models $\mathcal{X} \to \mathcal{Y}$ over S (see [Col03]; [LL99, Remark 4.6] and Corollary 3.10 below). Following Coleman [Col03], such a pair $\mathcal{X} \to \mathcal{Y}$ is called a *semi-stable model* of f, and it is called *stable* if, moreover, it is minimal among the semi-stable models of f (cf. Definition 3.1). The stable model of f (if it exists) is unique up to isomorphism.

Theorem 0.1 (Corollary 4.6). *Suppose that either $g(X) \geqslant 2$, or $g(X) = 1$ and X has potentially good reduction. Then there exists a finite separable extension K' of K such that $X_{K'} \to Y_{K'}$ admits a stable model $\mathcal{X}' \to \mathcal{Y}'$ over S', where S' is the integral closure of S in K'. Moreover, for any Dedekind scheme T dominating S', $\mathcal{X}'_T \to \mathcal{Y}'_T$ is the stable model of $X_{K(T)} \to Y_{K(T)}$ over T.*

This gives a canonical way to extend a finite cover of projective smooth curves over K to a finite cover of semi-stable models over (some finite cover of) S. The last part of the theorem states that the stable model of f commutes with flat base change. The stable model of f should be seen as an analogue of the stable model of a curve. In [Mau05], this theorem is used to construct a compactification of Hurwitz moduli spaces of finite covers of curves. We also prove the theorem for smooth marked curves X, Y. Note that, in general, $\mathcal{X}', \mathcal{Y}'$ are not the respective stable models of the curves $X_{K'}, Y_{K'}$. The proof of Theorem 0.1 is based on the following more general result.

Theorem 0.2 (Theorem 4.5). *Let $f : X \to Y$ be a finite morphism of smooth geometrically connected projective curves over K. Let $\mathcal{X}, \mathcal{Y}$ be respective models of X and Y over S. Then there exists a finite separable extension K' of K, and models $\mathcal{X}', \mathcal{Y}'$ of $X_{K'}$ and $Y_{K'}$ over S' (integral closure of S in K') such that:*

(1) *$\mathcal{X}', \mathcal{Y}'$ dominate $\mathcal{X}_{S'}, \mathcal{Y}_{S'}$, respectively, and are semi-stable over S';*

(2) *$X_{K'} \to Y_{K'}$ extends to a finite morphism $\mathcal{X}' \to \mathcal{Y}'$;*

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(3) *the pair $(\mathcal{X}', \mathcal{Y}')$ is minimal: for any pair $(\mathcal{X}'', \mathcal{Y}'')$ satisfying properties (1) and (2), $\mathcal{X}''$ and $\mathcal{Y}''$ dominate $\mathcal{X}'$ and $\mathcal{Y}'$, respectively; the cover $\mathcal{X}' \to \mathcal{Y}'$ is called the* stable hull of $\mathcal{X}_{S'} \dashrightarrow \mathcal{Y}_{S'}$;

(4) *the formation of $\mathcal{X}', \mathcal{Y}'$ commutes with flat base change $T \to S'$: if T is a Dedekind scheme dominating S', then $\mathcal{X}'_T \to \mathcal{Y}'_T$ is the stable hull of $\mathcal{X}_T \dashrightarrow \mathcal{Y}_T$.*

The paper is organized as follows. In § 1, $S = \operatorname{Spec} \mathcal{O}_K$ is local. We discuss some sufficient conditions for a model of $X_{\hat{K}}$ over $\hat{\mathcal{O}}_K$ (completion of $\mathcal{O}_K$) to be defined over $\mathcal{O}_K$. For example, this is true for semi-stable or regular models. This is used to reduce the proof of Proposition 4.4 to the case of a complete local base S. In § 2, we prove the special case of Theorem 0.2 when $X = Y, \mathcal{X} = \mathcal{Y}$ and $f = \mathrm{Id}$. Parts (1) and (2) of Theorem 0.2 are proved in § 3. The existence of a stable model (part (3)) is proved in § 4.

Convention

Through this paper, S is a Noetherian regular connected scheme of dimension 1 (Dedekind scheme), and, unless otherwise specified, X, Y are projective, smooth and geometrically connected curves over $K := K(S)$. When we state that a property (P) holds for some model $\mathcal{X}$ of X *after finite separable extension of* K, this means that there exists a finite separable extension of K'/K, such that (P) is satisfied for $\mathcal{X}_{S'}$, where S' is the normalization of S in K'.

Note that the hypothesis X, Y geometrically connected (instead of connected) is not serious. Actually, X is geometrically connected over the finite separable extension $H^0(X, \mathcal{O}_X)$ of K.

1. Descent from the completion

Let $\mathcal{O}_K$ be a discrete valuation ring, and let X be a geometrically connected smooth projective curve K. We give some sufficient conditions for a model of $X_{\hat{K}}$ over $\hat{\mathcal{O}}_K$ to be defined over $\mathcal{O}_K$.

Lemma 1.1. *Let $\mathcal{O}_K$ be a discrete valuation ring, let A be a flat local $\mathcal{O}_K$-algebra, localization of a finitely generated $\mathcal{O}_K$-algebra. Suppose that* $\operatorname{Spec}(A \otimes \hat{K})$ *is integral, one-dimensional and smooth over $\hat{K}$. Let $\mathcal{W} \to \operatorname{Spec} \hat{A}$ be a projective birational morphism. Then under any of the following conditions, $\mathcal{W}$ and the morphism $\mathcal{W} \to \operatorname{Spec} \hat{A}$ are defined over A:*

(a) $\operatorname{Pic}(\hat{A} \otimes \hat{K})$ *is a torsion group (i.e. every element has finite order);*

(b) *$\hat{A}$ is $\mathbb{Q}$-factorial.*

Proof. The morphism $\mathcal{W} \to \operatorname{Spec}(\hat{A})$ is the blowing-up along a closed subscheme $V(\mathfrak{I})$ of $\operatorname{Spec}(\hat{A})$. Let t be a uniformizing element of $\mathcal{O}_K$. Let us show that $\mathfrak{I}$ can be chosen in such a way that $t^n \in \mathfrak{I}$ for some $n \geqslant 1$. Let us suppose that condition (a) is satisfied. Then there exists some $m \geqslant 1$ such that the restriction of $\mathfrak{I}^m$ to the generic fiber is principal, generated by a $f \in \mathfrak{I}^m \otimes \hat{K}$. Under condition (b), let D be the scheme theoretical closure of $V(\mathfrak{I}) \cap \operatorname{Spec}(\hat{A} \otimes \hat{K})$ in $\operatorname{Spec}(\hat{A})$, considered as a Weil divisor. Then mD is principal for some $m > 0$, generated by a $f \in \mathfrak{I}^m$.

In both cases, there exist $a, b \in \mathbb{Z}$ such that $t^a \mathfrak{I}^m \subseteq f\hat{A} \subseteq t^b \mathfrak{I}^m$. Replacing $\mathfrak{I}$ by the ideal $t^a f^{-1} \mathfrak{I}^m$ (which does not change the blowing-up along $\mathfrak{I}$), we can suppose that the restriction of $\mathfrak{I}$ to the generic fiber is trivial, and $t^n \in \mathfrak{I}$ for some $n \geqslant 0$ ($n = a - b$). Since $\mathcal{O}_K$ is dense in $\hat{\mathcal{O}}_K$, $\mathfrak{I}$ is then generated by an ideal I of A. Let $\mathcal{X} \to \operatorname{Spec} A$ be the blowing-up along $V(I)$, then $\mathcal{W} \to \operatorname{Spec}(\hat{A})$ is obtained from $\mathcal{X} \to \operatorname{Spec} A$ by the base change $\hat{\mathcal{O}}_K / \mathcal{O}_K$. □

Remark 1.2. Let $\mathcal{W}$ be a projective regular model of $X_{\hat{K}}$ over $\hat{\mathcal{O}}_K$ dominating $\mathcal{X}_{\hat{\mathcal{O}}_K}$ for some model $\mathcal{X}$ of X over $\mathcal{O}_K$. Then $\mathcal{W}$ and $\mathcal{W} \to \mathcal{X}_{\hat{\mathcal{O}}_K}$ are defined over $\mathcal{O}_K$. Actually, if $\widetilde{\mathcal{X}} \to \mathcal{X}$ is the minimal desingularization of $\mathcal{X}$, then $\widetilde{\mathcal{X}}_{\hat{\mathcal{O}}_K} \to \mathcal{X}_{\hat{\mathcal{O}}_K}$ is the minimal desingularization of $\mathcal{X}_{\hat{\mathcal{O}}_K}$ (same proof as in [Liu02, Proposition 9.3.28]). Hence, $\mathcal{W} \dashrightarrow \widetilde{\mathcal{X}}_{\hat{\mathcal{O}}_K}$ is a birational morphism. It is a sequence of

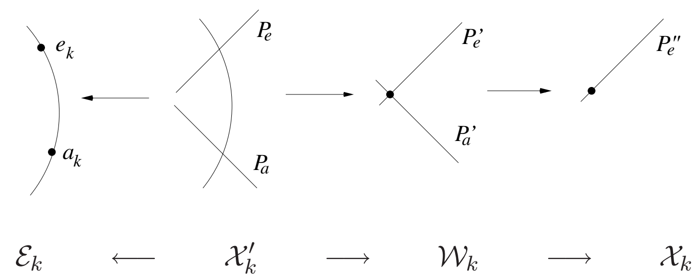

Figure 1. A birational morphism over $\hat{\mathcal{O}}_K$ not defined over $\mathcal{O}_K$ (see Remark 1.4).

blowing-ups of closed points [Liu02, 9.2.2]. Hence, $\mathcal{W}$ is defined over $\mathcal{O}_K$, and so is the morphism $\mathcal{W} \to \mathcal{X}_{\hat{\mathcal{O}}_K}$ by faithfully flat descent of rational maps.

Proposition 1.3. *Let $S = \operatorname{Spec} \mathcal{O}_K$ be local. Let $\mathcal{X}$ be a model of X over S dominating some semi-stable or regular model. Let $\varphi : \mathcal{W} \to \mathcal{X}_{\hat{\mathcal{O}}_K}$ be a projective birational morphism over $\hat{\mathcal{O}}_K$. Then $\mathcal{W}$ and φ are defined over S.*

Proof. It is enough to show that $\mathcal{W}$ is defined over S. We can suppose that $\mathcal{X}$ itself is semi-stable or regular. The morphism φ is an isomorphism outside of a finite set F of closed points of $(\mathcal{X}_{\hat{\mathcal{O}}_K})_s = \mathcal{X}_s$. Let $x \in F$, and let $A_x = \mathcal{O}_{\mathcal{X},x}$. If $\mathcal{X}$ is regular, then $\hat{A}_x$ is regular and thus factorial. If $\mathcal{X}$ is semi-stable, there exists a finite (étale) extension $\mathcal{O}_L/\hat{\mathcal{O}}_K$ such that $\hat{A}_x \otimes_{\hat{\mathcal{O}}_K} \mathcal{O}_L$ is a finite direct sum of rings $\mathcal{O}_L[[u]]$ or $\mathcal{O}_L[[u,v]]/(uv-a)$, $a \in \mathcal{O}_L$. It is well known that Pic of these rings tensored by $\hat{L}$ are trivial (see, for instance, [Hen99, Corollary 2.2], for $\mathcal{O}_L[[u,v]] \otimes \hat{L}/(uv-a)$). Hence, $\operatorname{Pic}(\hat{A}_x)$ is torsion [Liu02, Theorem 7.2.18 and Remark 7.2.19]. By Lemma 1.1, $\mathcal{W} \times_{\mathcal{X}_{\hat{\mathcal{O}}_K}} \operatorname{Spec} \hat{A}_x \to \operatorname{Spec} \hat{A}_x$ is defined over A_x. By glueing the morphisms above $\operatorname{Spec} A_x$, when x varies in F, and the isomorphism above $\mathcal{X} \setminus F$, we find a morphism over $\mathcal{X}$ which is equal to φ when base changed to $\hat{\mathcal{O}}_K$. ☐

Remark 1.4. In general, not every birational projective morphism $\mathcal{W} \to \mathcal{X}_{\hat{\mathcal{O}}_K}$ is defined over $\mathcal{O}_K$. In other words, even if $\mathcal{W} \to \mathcal{X}_{\hat{\mathcal{O}}_K}$ is an isomorphism outside of the special fiber, it is not necessarily the blowing-up along a closed subscheme with support in the special fiber. To construct such a counterexample, we will imitate the example of a non-contractible component given in [BLR90, Lemma 6.7.6]. Let us consider the smooth elliptic curve $\mathcal{E}/\mathcal{O}_K$ and the point $a_k \in \mathcal{E}_k(k)$ as given in [BLR90, p. 171]. The point a_k satisfies the property that no multiple (in the sense of the group law on $\mathcal{E}$) na_k, $n > 0$, can be lifted to a section in $\mathcal{E}(\mathcal{O}_K)$.

Let $\mathcal{X}' \to \mathcal{E}$ be the blowing-up of $\mathcal{E}$ along $\{a_k, e_k\}$, where e is the unit section of $\mathcal{E}$. Let P_a and P_e be the respective inverse images of a_k, e_k in $\mathcal{X}'$. They are projective lines over k. Let $\widetilde{\mathcal{E}}_k$ be the strict transform of $\mathcal{E}_k$ in $\mathcal{X}'$. The unit section of E_K gives rise to a section of $\mathcal{X}'$ which meets $\mathcal{X}'_k$ at an interior point of P_e. Hence, there exists a contraction map $\mathcal{X}' \to \mathcal{X}$ of $P_a \cup \widetilde{\mathcal{E}}_k$ (see [BLR90, Corollary 6.7.3]). Now over $\hat{\mathcal{O}}_K$, we can contract $\widetilde{\mathcal{E}}_k$ in $\mathcal{X}'_{\hat{\mathcal{O}}_K}$ (see [BLR90, 6.7.4]), which gives a model $\mathcal{W}/\hat{\mathcal{O}}_K$ and we have birational projective morphisms $\mathcal{X}'_{\hat{\mathcal{O}}_K} \to \mathcal{W} \to \mathcal{X}_{\hat{\mathcal{O}}_K}$. (See Figure 1.) We want to show that $\mathcal{W}$ is not defined over $\mathcal{O}_K$, or, equivalently, that $\widetilde{\mathcal{E}}_k$ cannot be contracted over $\mathcal{O}_K$. Suppose that $\mathcal{W}$ exists over $\mathcal{O}_K$, then there exists a relative effective Cartier divisor D on $\mathcal{X}'$ such that $\operatorname{Supp} D$ meets P_a and P_e but not $\widetilde{\mathcal{E}}_k$. Taking the Zariski closure of D_K in $\mathcal{E}$ gives a relative effective Cartier divisor D' on $\mathcal{E}$ such that $D'_k = na_k + me_k$ for some integers $n, m > 0$. However, $D' - (n+m)e \in \operatorname{Pic}^0_{\mathcal{E}/\mathcal{O}_K}(\mathcal{O}_K) \simeq \mathcal{E}(\mathcal{O}_K)$ then defines a section $b \in \mathcal{E}(\mathcal{O}_K)$ such that $D' - (n+m)e \sim b - e$. In the special fiber we then have $n(a_k - e_k) \sim b_k - e_k$; hence $na_k = b_k$ in the group $\mathcal{E}_k(k)$, contradicting the assumption on a_k.

LEMMA 1.5. *Suppose that X has semi-stable reduction over S. Then every relatively minimal semi-stable model $\mathcal{W}$ of $X_{\hat{K}}$ over $\hat{\mathcal{O}}_K$ is defined over $\mathcal{O}_K$.*

Proof. If $g(X) \geqslant 2$, then the unique relatively minimal semi-stable model of X is the stable model of X over S. Since the stable model is unique and commutes with flat base change [Liu02, 10.3.36], $\mathcal{W}$ over $\hat{\mathcal{O}}_K$ is defined over $\mathcal{O}_K$. The same is true if $g(X) = 1$ and X has good reduction.

Suppose that $g(X) = 1$ and X has multiplicative reduction. Let $\mathcal{X}$ be a relatively minimal semi-stable model. Let $\widetilde{\mathcal{X}}$ be its minimal desingularization. Let us show that $\widetilde{\mathcal{X}}$ is the minimal regular model of X over S. Let Γ be an irreducible component of $\mathcal{X}_s$ and let $\widetilde{\Gamma}$ be its strict transform in $\widetilde{\mathcal{X}}$. By Lemma 2.13(a), $\deg \omega_{\mathcal{X}/S}|_\Gamma > 0$ if $\mathcal{X}_s$ is reducible. If $\mathcal{X}_s$ is irreducible, then $\deg \omega_{\mathcal{X}/S}|_\Gamma = 2g(X) - 2 = 0$. So in any case $\deg \omega_{\widetilde{\mathcal{X}}/S}|_{\widetilde{\Gamma}} = \deg \omega_{\mathcal{X}/S}|_\Gamma \geqslant 0$. Hence, $\widetilde{\mathcal{X}}$ has no exceptional divisor. Since any strict subset of the set of irreducible components of $\widetilde{\mathcal{X}}_s$ can be contracted [Liu02, Example 9.4.19] into a semi-stable model (Lemma 2.13(a)), $\mathcal{X}_s$ is irreducible. So the relatively minimal semi-stable models of X correspond bijectively to the irreducible components of the minimal regular model of X. The same is true over $\hat{\mathcal{O}}_K$. Since the minimal regular model commutes with the base change $\hat{\mathcal{O}}_K/\mathcal{O}_K$ (see [Liu02, 9.3.28]), we see that the relatively minimal semi-stable models of $X_{\hat{K}}$ are exactly those of X base changed to $\hat{\mathcal{O}}_K$.

Suppose that $g(X) = 0$. Let $\mathcal{X}$ be a relatively minimal semi-stable model of X over S and let $\Gamma_1, \ldots, \Gamma_n$ be the irreducible components of $\mathcal{X}_s$. Then

$$\sum_i \deg \omega_{\mathcal{X}/S}|_{\Gamma_i} = 2g(X) - 2 < 0.$$

So $\omega_{\mathcal{X}/S}$ has negative degree on at least one component Γ_i. By Lemma 2.13(a), $n = 1$, thus $\mathcal{X}_s$ is irreducible, semi-stable and has arithmetic genus 0. So $\mathcal{X}_s$ is smooth. The same reasoning shows that $\mathcal{W}$ is smooth. Let $\mathcal{L}$ be the dual of the dualizing sheaf on $\mathcal{W}$. Let $s_0, s_1 \in H^0(\mathcal{W}, \mathcal{L})$ be a basis with $s_i \in H^0(X, \omega_{X/K}^\vee)$. Then s_0, s_1 define a closed immersion $\mathcal{W} \to \mathbb{P}^2$ over $\hat{\mathcal{O}}_K$. Its image is a conic defined by a polynomial with coefficients in K. Hence, $\mathcal{W}$ is defined over $\mathcal{O}_K$. ☐

The next corollary is weaker than Proposition 1.3, but sufficient for the purpose of Theorem 4.5. It is an immediate consequence of Proposition 1.3 and Lemma 1.5. However, we give a direct proof without using Proposition 1.3.

COROLLARY 1.6. *Let $S = \operatorname{Spec} \mathcal{O}_K$ be local. Let X be a smooth geometrically connected projective curve over K. Then every semi-stable model $\mathcal{W}$ of $X_{\hat{K}}$ over $\hat{\mathcal{O}}_K$ is defined over $\mathcal{O}_K$.*

Proof. The model $\mathcal{W}$ dominates a relatively minimal semi-stable model $\mathcal{Z}$ of $X_{\hat{K}}$ which is then defined over $\mathcal{O}_K$ by Lemma 1.5. Let $\widetilde{\mathcal{W}} \to \mathcal{W}$ be the minimal desingularization of $\mathcal{W}$. Then $\widetilde{\mathcal{W}} \to \mathcal{Z}$ is defined over $\mathcal{O}_K$ (1.2). The irreducible components of $\widetilde{\mathcal{W}}_s$ in the exceptional locus of $\widetilde{\mathcal{W}} \to \mathcal{W}$ are (-2)-curves and can be contracted over $\mathcal{O}_K$ (Lemma 2.13(a)). Hence, $\mathcal{W}$ is defined over $\mathcal{O}_K$. ☐

Remark 1.7. If $X = \mathbb{P}^1_K$, then every normal model $\mathcal{W}$ of $X_{\hat{K}}$ is defined over $\mathcal{O}_K$. Indeed, if $\widetilde{\mathcal{W}}$ is the minimal desingularization of $\mathcal{W}$, then $\widetilde{\mathcal{W}}$ dominates a relatively minimal regular model. The latter is smooth and defined over $\mathcal{O}_K$. Hence, $\widetilde{\mathcal{W}}$ is defined over $\mathcal{O}_K$. Now every strict subset of the set of irreducible components of $\widetilde{\mathcal{W}}_s$ is contractible over $\mathcal{O}_K$ (see [Liu02, Exercise 9.4.5]). So $\mathcal{W}$ is defined over $\mathcal{O}_K$.

2. Stable hull of a model

DEFINITION 2.1. Let S be a connected Noetherian regular scheme of dimension 1 (i.e. a *Dedekind scheme*). Let X be an integral projective variety over K. A *model $\mathcal{X}$ of X over S* is an integral projective scheme over S whose generic fiber is isomorphic to X. Recall that $\mathcal{X}$ is said to be *semi-stable* if its geometric fibers are reduced with only ordinary double points as singularities. A morphism of models is defined in an obvious way.

Definition 2.2. Let X be a connected projective smooth curve over K, and let $\mathcal{X}$ be a model of X over S. A *stable hull* of $\mathcal{X}$ is a semi-stable model $\mathcal{W}$ of X dominating $\mathcal{X}$, and minimal for these properties (i.e. every semi-stable model dominating $\mathcal{X}$ dominates $\mathcal{W}$).

The aim of this section is to prove the following result.

Theorem 2.3. *Let X be a geometrically connected projective smooth curve over K and let $\mathcal{X}$ be a model of X over S.*

(a) *The stable hull of $\mathcal{X}$ is unique (up to isomorphism) when it exists. In general, there exists a finite separable extension K'/K such that $\mathcal{X}_{S'}$ (where S' is the integral closure of S in K') has a stable hull over S'.*
(b) *The stable hull commutes with flat base change: suppose that $\mathcal{X}$ admits a stable hull $\mathcal{W}$ over S and let $S' \to S$ be a flat morphism of Dedekind schemes, then $\mathcal{W}_{S'}$ is the stable hull of $\mathcal{X}_{S'}$ over S'.*

The proof of the theorem is postponed to § 2.17.

Lemma 2.4. *Let G be a finite group acting on X. Let $\mathcal{X}$ be a model of X over S. Then there exists a model $\mathcal{Z}$ of X dominating $\mathcal{X}$, endowed with an action of G, and minimal for these properties.*

Proof. (See [Dej96, 7.6].) Let $\sigma \in G$, then there exists a model $\mathcal{X}^\sigma$ such that $\sigma : X \to X$ extends to an isomorphism $\sigma : \mathcal{X} \to \mathcal{X}^\sigma$. If $\tau \in G$, by composing $(\sigma^{-1}\tau) : \mathcal{X}^{\tau^{-1}\sigma} \to \mathcal{X}$ with $\sigma : \mathcal{X} \to \mathcal{X}^\sigma$, we obtain an isomorphism $\mathcal{X}^{\tau^{-1}\sigma} \to \mathcal{X}^\sigma$ denoted by τ. Let $\mathcal{P}$ be the fiber product $\prod_{S,\sigma\in G} \mathcal{X}^\sigma$ over S. Then we can make G act on $\mathcal{P}$ by

$$\tau : (x_\sigma)_\sigma \mapsto (\tau(x_{\tau^{-1}\sigma}))_\sigma.$$

Moreover, the diagonal morphism $\Delta : X \to \mathcal{P}_K, x \mapsto (x, \dots, x)$ is G-equivariant. Let $\mathcal{Z}$ be the Zariski closure of $\Delta(X)$ in $\mathcal{P}$, endowed with the reduced structure. Then G leaves stable $\mathcal{Z}$. Note that $\mathcal{Z}$ dominates $\mathcal{X}$ because the projection morphism $\mathcal{P} \to \mathcal{X}$ induces a morphism $\mathcal{Z} \to \mathcal{X}$ which is an isomorphism on the generic fibers.

Let us prove that $\mathcal{Z}$ is minimal. Let $\mathcal{W}$ be a model endowed with an action of G and a birational morphism $\mathcal{W} \to \mathcal{X}$. Then we have a morphism $\mathcal{W} \to \mathcal{X}^\sigma$ for all σ and, hence, a morphism $h : \mathcal{W} \to \mathcal{P}$. Since $h(\mathcal{W})$ is irreducible and contains Z, h induces a morphism of models $\mathcal{W} \to \mathcal{Z}$. □

Let us give a corollary of Theorem 2.3.

Corollary 2.5. *Let G be a finite group acting on X. Let $\mathcal{X}$ be a model of X over S. Then after finite separable extension of K, $\mathcal{X}$ is dominated by a semi-stable (respectively, semi-stable and regular) model $\mathcal{W}$ such that the action of G extends to $\mathcal{W}$. Moreover, there exists a minimal such a model $\mathcal{W}$.*

Proof. Let $\mathcal{Z}$ be the model defined in Lemma 2.4. Let $\mathcal{W}$ be the stable hull of $\mathcal{Z}_{S'}$ over some S'/S (Theorem 2.3(a)). By the uniqueness property, the action of G on $\mathcal{Z}_{S'}$ extends to $\mathcal{W}$. It is clear that $\mathcal{W}$ is minimal with respect to the required properties. To have a minimal semi-stable regular model, it is enough to take the minimal desingularization of $\mathcal{W}$. □

Lemma 2.6. *Let S be local with separably closed residue field. Let $\mathcal{X}$ be a model of X with minimal desingularization $\widetilde{\mathcal{X}} \to \mathcal{X}$. Suppose that $\widetilde{\mathcal{X}}$ dominates a regular model $\mathcal{Z}$ and that $\mathcal{X}_s$, $\mathcal{Z}_s$ are geometrically reduced. Then $\widetilde{\mathcal{X}}_s$ is geometrically reduced.*

Proof. Note that if $\mathcal{W}$ is a normal model of X, then $\mathcal{W}_s$ verifies the property (S_1), thus $\mathcal{W}_s$ is geometrically reduced if and only if every irreducible component of $\mathcal{W}_s$ has geometric multiplicity [BLR90, Definition 9.1.3] equal to 1 in $\mathcal{W}_s$. The latter condition depends only on the generic points of $\mathcal{W}_s$. We can decompose $\widetilde{\mathcal{X}} \to \mathcal{Z}$ into a sequence of blowing-ups

$$\widetilde{\mathcal{X}} =: \mathcal{Z}_0 \to \mathcal{Z}_1 \to \cdots \to \mathcal{Z}_n := \mathcal{Z}$$

such that $\mathcal{Z}_i \to \mathcal{Z}_{i+1}$ consists of blowing-down an exceptional divisor Θ_i contained in $\mathcal{Z}_i$. We will show by induction that Θ_i has multiplicity 1 in $(\mathcal{Z}_i)_s$ and $\Theta_i \simeq \mathbb{P}^1_{k(s)}$. Since $(\mathcal{Z}_n)_s$ is geometrically reduced, this will imply that $\widetilde{\mathcal{X}}_s$ is geometrically reduced.

By the minimality of $\widetilde{\mathcal{X}} \to \mathcal{X}$, Θ_0 is not mapped to a closed point in $\mathcal{X}$. Thus, $\widetilde{\mathcal{X}} \to \mathcal{X}$ is an isomorphism in a neighborhood of the generic point of Θ_0. In particular, Θ_0 has geometric multiplicity 1. Since Θ_0 is a projective line by the Castelnuovo criterion, it is isomorphic to $\mathbb{P}^1_{k(s)}$. Suppose that the same holds for Θ_j, $0 \leqslant j \leqslant i-1$. Let $\Theta_{i,j}$ be the strict transform of Θ_i in $\mathcal{Z}_j$, $0 \leqslant j \leqslant i-1$. If $\Theta_{i,j}$ meets Θ_j for some j, then the computation of $\Theta_j^2 = -1$ shows that $\Theta_{i,j}$ has multiplicity 1 and cuts Θ_j at a rational point. Thus, Θ_i has multiplicity 1 and is isomorphic to $\mathbb{P}^1_{k(s)}$. Otherwise, $\mathcal{Z}_0 \to \mathcal{Z}_i$ is an isomorphism in a neighborhood of $\Theta_{i,0}$. In particular, $\Theta_{i,0}$ is an exceptional divisor. Then we can conclude exactly as for Θ_0. □

Proposition 2.7. *Let $\mathcal{X}$ be a normal model of X over S. Then the following properties are equivalent:*

(i) *$\mathcal{X}$ is dominated by a semi-stable model over S;*

(ii) *X admits a semi-stable model over S and $\mathcal{X}_s$ is geometrically reduced for all $s \in S$;*

(iii) *the minimal desingularization $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ is semi-stable over S.*

Proof. (i) $\Longrightarrow$ (ii) If $\mathcal{X}$ is dominated by a semi-stable model $\mathcal{X}'$, then any irreducible component Γ of $\mathcal{X}_s$ is birational to an irreducible component of $\mathcal{X}'_s$. The latter being geometrically reduced, $\mathcal{X}_s$ is geometrically reduced.

(ii) $\Longrightarrow$ (iii) We know that $\widetilde{\mathcal{X}}$ dominates a relatively minimal regular model $\mathcal{Z}$. The semi-stable reduction hypothesis implies that $\mathcal{Z}$ is semi-stable (see [Liu02, 10.3.34(a)] if $g(X) \geqslant 1$; if $g(X) \leqslant 0$, then $\mathcal{Z}$ is smooth). Since the minimal desingularization commutes with étale base change (see the proof of [Liu02, Proposition 9.3.28]), we can suppose that S is local with separably closed residue field. By Lemma 2.6, $\widetilde{\mathcal{X}}_s$ is geometrically reduced. The map $\widetilde{\mathcal{X}} \to \mathcal{Z}$ consists of blowing-up successively closed points. The fact that $\widetilde{\mathcal{X}}_s$ is geometrically reduced implies that we only blow-up rational points in the smooth locus. Since $\mathcal{Z}$ is semi-stable, then so is $\widetilde{\mathcal{X}}$. □

Corollary 2.8. *There exists a finite separable extension K'/K such that $\mathcal{X}_{S'}$, where S' is the normalization of S in K', is dominated by a semi-stable model of $X_{K'}$.*

Proof. We can suppose that X has semi-stable reduction over S. Since X has good reduction over an open dense subset of S, we can suppose that S is local. By the finiteness theorem of Grauert–Remmert [GR66] (see also [BLR95, Theorem 1.3]) applied to the formal completion of $\mathcal{X}$ along its special fiber, there exists a finite Galois extension $L/\hat{K}$ such that the normalization of $\mathcal{X}_{\mathcal{O}_L}$ has geometrically reduced special fiber. See [Epp73, p. 247] for how to descend the result to K (note that [Kuh03] fills a gap in the proof of a main theorem in [Epp73]). We then apply Proposition 2.7. □

Remark 2.9. The corollary is useful in a recent work of Deninger and Werner on vector bundles and representations of the fundamental group of p-adic curves [DW04]. In fact, de Jong [Dej97, Theorem 2.4] already proved it in the situation when S is a Noetherian integral excellent scheme of any dimension, and when X is an integral curve over $K(S)$. The scheme S' is then proper and generically finite over S. The proof here for one-dimensional S is simpler and more effective in some sense.

Remark 2.10. When S is local and complete, the corollary can be reformulated in terms of rigid analytic geometry as follows: let $\mathcal{U}$ be a formal covering of X. Then after finite separable extension of K, $\mathcal{U}$ can be refined to a distinguished formal covering $\mathcal{V}$ with semi-stable reduction. As such, the statement can be easily worked out using Theorem 5.5, and step 2 in the proof of Lemma 7.3 of [BL85]. The non-complete case can then be obtained using Proposition 1.3.

Remark 2.11 (Effective reduced fiber theorem). In the case of Corollary 2.8, we can give an effective method to eliminate the multiplicities of $\mathcal{X}_s$, without using Grauert–Remmert's theorem. Suppose that X has semi-stable reduction. Let Γ be an irreducible component of $\mathcal{X}_s$ of geometric multiplicity $d > 1$. If we choose two closed points P_1, P_2 of X which specialize to two distinct points in the interior of Γ, and if we take $L = K(P_1, P_2)$, then the irreducible components of $(\mathcal{X}_{\mathcal{O}_L})'s$ (where $(\mathcal{X}_{\mathcal{O}_L})'$ denotes the normalization of $\mathcal{X}_{\mathcal{O}_L}$) lying above Γ are geometrically reduced. If K is strictly Henselian, we can bound $[L : K]$ by d^2. Note that L can be chosen to be separable over K. If X has not necessarily semi-stable reduction, then we can bound $[L : K]$ by the max of d^2 and a constant depending only on g.

2.12 The stable hull

Now let us construct a minimal semi-stable model dominating $\mathcal{X}$. Let $\mathcal{Z}$ be a locally complete intersection (e.g. regular or semi-stable) model of X over S. Let $\omega_{\mathcal{Z}/S}$ be the (invertible) dualizing sheaf of $\mathcal{Z}/S$. Recall that a (-2)-*curve* on $\mathcal{Z}$ is an irreducible component Γ of a closed fiber $\mathcal{Z}_s$ such that $\deg \omega_{\mathcal{Z}/S}|_\Gamma = 0$. If $\mathcal{Z}$ is semi-stable and $k(s)$ is algebraically closed, and Γ is not a connected component of $\mathcal{Z}_s$, then this is equivalent to $\Gamma \simeq \mathbb{P}^1_{k(s)}$ and Γ meets the other irreducible components at exactly two points. Recall that the *exceptional locus* of a birational projective morphism $\pi : \mathcal{Z} \to \mathcal{X}$ is by definition the complementary of $\pi^{-1}(U)$, where U is the largest open subscheme of $\mathcal{X}$ such that $\pi^{-1}(U) \to U$ is an isomorphism. When $\mathcal{X}$ is normal, the exceptional locus is equal to the union of the prime divisors of $\mathcal{Z}$ which map to closed points in $\mathcal{X}$. A semi-stable model $\mathcal{Z}$ dominating $\mathcal{X}$ will be called *relatively minimal* if there is no semi-stable model between $\mathcal{X}$ and $\mathcal{Z}$, except $\mathcal{Z}$ itself.

Lemma 2.13. *Let $\mathcal{Z}$ be a semi-stable model of X over S. Let V be an effective vertical divisor on $\mathcal{Z}$ such that for all $s \in S$, no connected component of $\mathcal{Z}_s$ is contained in V.*

(a) *If $\deg \omega_{\mathcal{Z}/S}|_\Gamma \leqslant 0$ for all components Γ of V, then there exists a contraction map $\mathcal{Z} \to \mathcal{W}$ of V and $\mathcal{W}$ is semi-stable.*

(b) *If there exists a contraction map $\mathcal{Z} \to \mathcal{W}$ of V with $\mathcal{W}$ semi-stable, then $\deg \omega_{\mathcal{Z}/S}|_\Gamma \leqslant 0$ for at least one irreducible component Γ of V.*

Proof. (a) This is well known but we were not able to find a proper reference. We can suppose that S is local. Let $\rho : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ be the minimal desingularization. Then the components Θ of the exceptional locus E of ρ are (-2)-curves. Let $\widetilde{V}$ be the strict transform of V in $\widetilde{\mathcal{Z}}$. Let us show that $V' := E + \widetilde{V}$ can be contracted. We have $\omega_{\widetilde{\mathcal{Z}}/S} = \rho^* \omega_{\mathcal{Z}/S}$ because $\mathcal{Z}$ is semi-stable. Hence, $\deg \omega_{\widetilde{\mathcal{Z}}/S}|_{\Gamma'} \leqslant 0$ for all components Γ' of V'. If there exists a Γ' such that $\deg \omega_{\widetilde{\mathcal{Z}}/S}|_{\Gamma'} < 0$, then Γ' is an exceptional divisor. Let $\pi : \widetilde{\mathcal{Z}} \to \mathcal{Z}'$ be the contraction of Γ'. Then $\omega_{\widetilde{\mathcal{Z}}/S} = \pi^* \omega_{\mathcal{Z}'/S}(\Gamma')$. We deduce easily that $\deg \omega_{\mathcal{Z}'/S}|_{\Gamma''} \leqslant 0$ for all Γ'' in $\pi(V')$. So by successively blowing-down exceptional divisors, we can suppose that V' consists only of (-2)-curves. By Artin's criterion of contractibility [Liu02, Corollary 9.4.7], V' can be contracted. Therefore, V can be contracted.

It remains to see that $\mathcal{W}$ is semi-stable. Let $\mathcal{O}_{K'}$ be a discrete valuation ring containing $\mathcal{O}_S$ and let $S' = \operatorname{Spec} \mathcal{O}_{K'}$. It is enough to show that $\mathcal{W}_{S'}$ is semi-stable. The map $\mathcal{Z}_{S'} \to \mathcal{W}_{S'}$ is the contraction of $V_{k(s')}$. We have $\omega_{\mathcal{Z}_{S'}/S'} = \omega_{\mathcal{Z}/S} \otimes \mathcal{O}_{S'}$. If Γ' is a component of $\mathcal{Z}_{s'}$ lying over $\Gamma \subseteq V$, then

$$[k(s') : k(s)] \deg_{k(s')} \omega_{\mathcal{Z}_{S'}/S'}|_{\Gamma'} = [k(\Gamma') : k(\Gamma)] \deg_{k(s)} \omega_{\mathcal{Z}/S}|_\Gamma \leqslant 0.$$

So we can reduce the lemma to the case $k(s)$ algebraically closed. Let $\Gamma \subseteq V$. Then $p_a(\Gamma) = 0$, $\Gamma \simeq \mathbb{P}^1_{k(s)}$, and Γ meets the other components of $\mathcal{Z}_s$ in at most two points. Now it is well known that $\mathcal{W}_s$ is semi-stable (see, for instance, [Liu02, Lemma 10.3.31]).

(b) The previous computations show that $\deg \omega_{\mathcal{Z}/S}|_\Gamma = \deg \omega_{\widetilde{\mathcal{Z}}/S}|_{\widetilde{\Gamma}}$ for any irreducible component Γ of $\mathcal{Z}_s$. Let $\widetilde{\mathcal{W}}$ be the minimal desingularization of $\mathcal{W}$. Suppose that $\widetilde{\mathcal{Z}} \to \widetilde{\mathcal{W}}$ is not an isomorphism. Let Γ be a component of $\mathcal{Z}_s$ whose strict transform in $\widetilde{\mathcal{Z}}$ is an exceptional divisor contracted into a closed point in $\widetilde{\mathcal{W}}$. Then $\Gamma \subseteq V$ and $\deg \omega_{\mathcal{Z}/S}|_\Gamma = \deg \omega_{\widetilde{\mathcal{Z}}/S}|_{\widetilde{\Gamma}} < 0$. If $\widetilde{\mathcal{Z}} = \widetilde{\mathcal{W}}$, then $\deg \omega_{\mathcal{Z}/S}|_\Gamma = \deg \omega_{\widetilde{\mathcal{W}}/S}|_{\widetilde{\Gamma}} = 0$ for all Γ in V because $\widetilde{\Gamma}$ (the strict transform of Γ) is a (-2)-curve in $\widetilde{\mathcal{W}}$. ∎

Proposition 2.14. *Let $\mathcal{X}$ be a model of X over S dominated by a semi-stable model.*

(a) *A semi-stable model $\mathcal{Z}$ dominating $\mathcal{X}$ is relatively minimal if and only if for all irreducible components of the exceptional locus of $\mathcal{Z} \to \mathcal{X}$, we have $\deg \omega_{\mathcal{Z}/S}|_\Gamma > 0$.*

(b) *Let $\widetilde{\mathcal{X}} \to \mathcal{X}$ be the minimal desingularization of $\mathcal{X}$. Let $\widetilde{\mathcal{X}} \to \mathcal{W}$ be the contraction of the (-2)-curves contained in the exceptional locus of $\widetilde{\mathcal{X}} \to \mathcal{X}$. Then $\mathcal{W}$ is the stable hull of $\mathcal{X}$.*

Proof. (a) This is an immediate consequence of Lemma 2.13.

(b) By Lemma 2.13(a), $\mathcal{W}$ is semi-stable and dominates $\mathcal{X}$. Let $\mathcal{Z}$ be a semi-stable model dominating $\mathcal{X}$, and relatively minimal. Let us first show that $\widetilde{\mathcal{X}}$ dominates $\mathcal{Z}$. Let $\widetilde{\mathcal{Z}}$ be the minimal desingularization of $\mathcal{Z}$. Then $\widetilde{\mathcal{Z}}$ dominates $\widetilde{\mathcal{X}}$. Suppose that $\widetilde{\mathcal{Z}} \to \widetilde{\mathcal{X}}$ is not an isomorphism. Let Θ be an exceptional divisor of $\widetilde{\mathcal{Z}}$ mapped to a closed point of $\widetilde{\mathcal{X}}$. Since $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is minimal, Θ maps to an irreducible component Γ of $\mathcal{Z}_s$ which is then contained in the exceptional locus of $\mathcal{Z} \to \mathcal{X}$. We have $\deg \omega_{\mathcal{Z}/S}|_\Gamma = \deg \omega_{\widetilde{\mathcal{Z}}/S}|_\Theta < 0$ – contradiction. Therefore, $\widetilde{\mathcal{Z}} \simeq \widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{X}} \to \mathcal{Z}$ consists in contracting some (-2)-curves in $\mathcal{X}_s$. Hence, $\mathcal{Z}$ dominates $\mathcal{W}$. ∎

Remark 2.15. A *Du Val model* of X over S is a model $\mathcal{X}$ such that $\deg \omega_{\widetilde{\mathcal{X}}/S}|_\Gamma = 0$ where $\widetilde{\mathcal{X}} \to \mathcal{X}$ is the minimal desingularization and where Γ is any irreducible component of the exceptional locus of $\widetilde{\mathcal{X}} \to \mathcal{X}$. The above results (Lemma 2.13 and Proposition 2.14) still hold when 'semi-stable' is replaced by 'Du Val', except the base change property. The point is that Du Val models do not commute with base change.

Remark 2.16. Let $\mathcal{W}$ be semi-stable and dominating $\mathcal{X}$. Then $\mathcal{W}$ is the stable hull of $\mathcal{X}$ if and only if $\mathrm{Aut}_{\mathcal{X}_{\bar{s}}}(\mathcal{W}_{\bar{s}})$ is finite for all $s \in S$.

2.17 Proof of Theorem 2.3

(a) This is contained in Corollary 2.8 and Proposition 2.14.

(b) It is enough to show that $\mathcal{W}_{S'}$ is relatively minimal. Let Γ' be an irreducible component of $\mathcal{W}_{s'}$ contained in the exceptional locus of $\mathcal{W}_{S'} \to \mathcal{X}_{S'}$. The image of Γ' in $\mathcal{W}$ is in the exceptional locus of $\mathcal{W} \to \mathcal{X}$. Similarly to the proof of Lemma 2.13(a), we have $\deg \omega_{\mathcal{W}_{S'}/S'}|_{\Gamma'} > 0$. Therefore, $\mathcal{W}_{S'}$ is relatively minimal by Proposition 2.14(a). ∎

Remark 2.18. Suppose that $g(X) \geqslant 1$ and S is affine. Let $\mathcal{X}$ be the minimal regular model of X over S and let $\mathcal{W}$ be the stable hull of $\mathcal{X}_{S'}$ over some extension S'/S. Let $\mathcal{X}'$ be the stable or minimal regular model of $X_{K'}$ over S'. Then $H^0(\mathcal{W}, \omega_{\mathcal{W}/S'}) = H^0(\mathcal{X}', \omega_{\mathcal{X}'/S'})$. One should be able to recover some arithmetic information on $\mathcal{X}$ from the sheaf $\omega_{\mathcal{W}/S'} \otimes ((\omega_{\mathcal{X}/S})^\vee \otimes \mathcal{O}_{S'})$. Let us consider the ideal

$$\mathcal{O}_{S'}(-\delta) := H^0(\mathcal{W}, \omega_{\mathcal{W}/S'}) \otimes (H^0(\mathcal{X}, \omega_{\mathcal{X}/S})^\vee \otimes \mathcal{O}_{S'}).$$

For example, if X is an elliptic curve over K, then for every closed point $s \in S$, we can show that

$$12 \operatorname{ord}_s(\delta) = \operatorname{ord}_s(\Delta) + a_s \operatorname{ord}_s(j),$$

where Δ is the minimal discriminant divisor of X over S, and $a_s = 0$ if X has potentially good reduction at s, $a_s = 1$ otherwise.

2.19 Marked curves

Recall that a (proper) *marked curve* $Z \to T$ over a scheme T is a proper flat scheme of relative dimension 1 over T endowed with a finite set $M \subset Z(T)$ of sections with pairwise disjoint supports contained in the smooth locus of Z/T (for our purpose, it is not necessary to order these sections). Note that if T is irreducible with generic point ξ, then M is determined by its generic fiber $M \cap Z_\xi$. We say that (Z, M) is *semi-stable* if $Z \to T$ is semi-stable. We say that (Z, M) is *stable* if it is semi-stable and if for any geometric point $\bar{t}$ of T, $Z_{\bar{t}}$ is connected and for any irreducible component Γ of $Z_{\bar{t}}$, Γ meets the other components in at least $1 - (2p_a(\Gamma) - 2) - |M \cap \Gamma|$ points. This amounts to saying that $\omega_{Z/T}(M)$ is ample.

A *morphism* of marked curves $(Z, M) \to (Z', M')$ over T is a T-morphism $f : Z \to Z'$ such that $f(M) \subseteq M'$.

Let (X, M) be a smooth marked curve over $K = K(S)$. A *marked model* of (X, M) over S is a marked curve $(\mathcal{X}, \mathcal{M})$ over S whose generic fiber is isomorphic to (X, M). Since $\mathcal{M}$ is uniquely determined by M and $\mathcal{X}$, we will omit $\mathcal{M}$ in the notation $(\mathcal{X}, \mathcal{M})$ and we will simply say that $\mathcal{X}$ is a marked model of (X, M).

Let $\mathcal{X}$ be a model of X over S. The *stable marked hull* of $\mathcal{X}$ is the minimal semi-stable marked model of (X, M) dominating $\mathcal{X}$. Note that a stable marked hull is not necessarily a stable marked curve.

Corollary 2.20. *Let (X, M) be a smooth marked curve over K and let $\mathcal{X}$ be a (non-marked) model of X. Then after finite separable extension of K, $\mathcal{X}$ admits a stable marked hull. More precisely, if $\mathcal{X}$ has a stable hull over some extension S'/S, then it has a stable marked hull over S'. Moreover, the formation of stable marked hull commutes with flat base change of Dedekind schemes.*

Proof. We can suppose that $\mathcal{X}$ has a stable hull $\mathcal{Z}$ over S. Let $\widetilde{\mathcal{Z}}$ be a desingularization of $\mathcal{Z}$, let $\overline{M}$ be the Zariski closure of M in $\widetilde{\mathcal{Z}}$ and let $\widetilde{\mathcal{Z}}_{s_1}, \dots, \widetilde{\mathcal{Z}}_{s_n}$ be the fibers such that $\overline{M} \to \overline{M} \cap \widetilde{\mathcal{Z}}_{s_i}$ is not injective. Let $\mathcal{Z}' \to \widetilde{\mathcal{Z}}$ be an embedded resolution of $\overline{M} + \sum_i \widetilde{\mathcal{Z}}_{s_i}$ in $\widetilde{\mathcal{Z}}$ so that the Zariski closure $\mathcal{M}'$ of M in $\mathcal{Z}'$ is a disjoint union of sections (contained in the smooth locus because $\mathcal{Z}'$ is regular). Then $\mathcal{Z}'$ is a semi-stable marked model dominating $\mathcal{X}$.

Let $\mathcal{W}$ be any semi-stable marked model of (X, M) over S. Similarly to the non-marked case, we can show that $\mathcal{W}$ is relatively minimal if and only if $\omega_{\mathcal{W}/S}(\mathcal{M})|_\Gamma$, where $\mathcal{M}$ is the Zariski closure of M in $\mathcal{W}$, has positive degree for all Γ in the exceptional locus of $\mathcal{W} \to \mathcal{X}$. This then implies that the stable marked hull is obtained by contracting prime divisors Γ in the exceptional locus of $\mathcal{Z}' \to \mathcal{Z}$ such that $\deg(\omega_{\mathcal{Z}'/S}(\mathcal{M}')|_\Gamma) \leqslant 0$, and that the stable marked hull commutes with flat base change. □

Remark 2.21. If (X, M) is stable (meaning that $2g(X) - 2 + |M| \geqslant 1$) and if X has semi-stable reduction over S, then there exists a semi-stable marked model $\mathcal{X}$ of (X, M) over S and minimal for this property. This model is the stable marked model of (X, M). It is characterized by the property that for all irreducible components Γ of $\mathcal{X}_s$, one has $\deg \omega_{\mathcal{X}/S}(\mathcal{M})|_\Gamma > 0$, where $\mathcal{M}$ is the Zariski closure of M in $\mathcal{X}$. As above, this implies that the stable marked model commutes with base change.

3. Semi-stable models of finite covers

We (re)prove that any finite morphism of projective smooth curves over K extends, after finite separable extension of K, to a finite morphism of semi-stable models.

Definition 3.1. Let $f : X \to Y$ be a finite morphism of smooth connected projective curves over $K(S)$. A *model (or extension) of f over S* consists of a morphism $\mathcal{X} \to \mathcal{Y}$ over S extending f, where $\mathcal{X}$ and $\mathcal{Y}$ are models over S of X and Y, respectively. A model of f is said to be *finite* if it is a finite morphism, and *semi-stable* (see [Col03]) if it is finite and if $\mathcal{X}$, $\mathcal{Y}$ are semi-stable. We say that a model $\mathcal{X} \to \mathcal{Y}$ of f dominates another model $\mathcal{X}' \to \mathcal{Y}'$ if there are birational morphisms $\mathcal{X} \to \mathcal{X}'$, $\mathcal{Y} \to \mathcal{Y}'$ making the following diagram commutative.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{Y}' \end{array}$$

A model of f is *stable* if it is semi-stable and minimal (for the domination relation) among semi-stable models of f. If $\mathcal{X}, \mathcal{Y}$ are respective models of X, Y. Then the semi-stable model $\mathcal{X}' \to \mathcal{Y}'$ of f such that $\mathcal{X}'$ dominates $\mathcal{X}$, $\mathcal{Y}'$ dominates $\mathcal{Y}$, and which is minimal for this property, is called the *stable hull* of (the rational map) $\mathcal{X} \dashrightarrow \mathcal{Y}$. We can obviously make similar definitions for marked curves.

Remark 3.2. For a given $f : X \to Y$, the semi-stable models are not unique: let $\mathcal{X} \to \mathcal{Y}$ be a semi-stable model of f, let $\mathcal{Y}' \to \mathcal{Y}$ be a blowing-up along a closed point, then the stable hull of $\mathcal{X} \dashrightarrow \mathcal{Y}'$ (see Theorem 4.5) is a new semi-stable model.

3.3 Decomposition of inseparable morphisms

Let us first deal with purely inseparable morphisms $X \to Y$. The next two statements are well known at least over perfect base fields.

Lemma 3.4. *Let K be a field of characteristic $p > 0$. Let E/F be a finite extension of function fields of one variable over K, with E separable over K. Then there exists a unique purely inseparable sub-extension L/F of E/F such that E/L is separable. Moreover, $F = KL^{p^r}$ for some $r \geqslant 0$.*

Proof. Let F_s be the separable closure of F in E. By [Liu02, Corollary 3.2.27] (here we use the hypothesis E separable over K), there exists $r \geqslant 0$ such that $F_s = KE^{p^r}$. Let

$$L := \{e \in E \mid e^{p^r} \in F\}.$$

Then L/F is a purely inseparable extension, $F = KL^{p^r}$ and E/L is separable because otherwise $L \subseteq KE^p$ and $F_s \subseteq KE^{p^{r+1}}$. The uniqueness of L is obvious because it is necessarily equal to the radical closure of F in E. □

Proposition 3.5. *Let $f : X \to Y$ be a finite morphism of normal connected curves over a field K of characteristic $p > 0$. Suppose that X is smooth. Then f can be decomposed into a finite separable morphism $X \to Z$ followed by $Z \to Y$ which can be identified to a Frobenius map $Z \to Z^{(p^r)}$ for some $r \geqslant 0$. Moreover, Z is smooth.*

Proof. Let L be the radicial closure of $K(Y)$ in $K(X)$, and let Z be the normalization of Y in L. Then f induces a finite separable morphism $X \to Z$. It is flat because Z is regular of dimension 1. Let $\bar{K}$ be an algebraic closure of K, then $X_{\bar{K}} \to Z_{\bar{K}}$ is flat, hence $Z_{\bar{K}}$ is regular. Finally, $Z \to Y$ can be identified to $Z \to Z^{(p^r)}$ by [Liu02, Proposition 7.4.21]. □

Note that f can also be decomposed into $X \to X^{(p^r)}$ followed by a separable morphism $X^{(p^r)} \to Y$.

3.6 Semi-stable models

LEMMA 3.7. *Let $f_i : X \to Y_i$, $i = 1, \dots, n$ be finite surjective morphisms of integral projective varieties over K and let $\mathcal{Y}_i$ be a model of Y_i over S. Then there exists a model $\mathcal{X}$ of X over S such that $\mathcal{X}$ dominates $\mathcal{Y}_i$ (that is, $X \to Y_i$ extends to $\mathcal{X} \to \mathcal{Y}_i$) for all i.*

Proof. For N large enough, we have a closed immersion $g : X \to \mathbb{P}^N_K \times_K (\prod_{K,i} Y_i)$ induced by the projective morphism $(f_i, \dots, f_n) : X \to \prod_{K,i} Y_i$. Now take $\mathcal{X}$ to be the Zariski closure of $g(X)$ in $\mathbb{P}^N_S \times_S (\prod_{S,i} \mathcal{Y}_i)$, endowed with the reduced structure. Note that if X is geometrically reduced, we can also use Lemma 4.3 with $\mathcal{X}_i = N(\mathcal{Y}_i, K(X))$. □

PROPOSITION 3.8. *Let S be a Dedekind scheme, and let $f : X \to Y$ be a finite morphism of smooth geometrically connected projective curves over $K := K(S)$. Let $\mathcal{X}, \mathcal{Y}$ be respective models of X and Y. Then there exists a finite separable extension K'/K such that over the normalization S' of S in K', the cover $X_{K'} \to Y_{K'}$ extends to a finite morphism $\mathcal{X}' \to \mathcal{Y}'$, where $\mathcal{X}'$ (respectively $\mathcal{Y}'$) is a semi-stable model of X (respectively Y) over S dominating $\mathcal{X}_{S'}$ (respectively, $\mathcal{Y}_{S'}$).*

Proof. Let $X \to Z \to Y$ be the decomposition given by Proposition 3.5. Let $\hat{X} \to Z$ be the Galois closure of $X \to Z$. After a finite separable extension of K, $\hat{X}$ is smooth over K. Let $\hat{\mathcal{X}}_0/S$ be a model of $\hat{X}$ dominating $\mathcal{X}$ and $\mathcal{Y}$ (3.7). Let $G := \mathrm{Gal}(K(\hat{X})/K(Z))$. By Corollary 2.5, after a finite separable extension of K, there exists a semi-stable model $\hat{\mathcal{X}}$ of $\hat{X}$ endowed with an action of G and dominating $\hat{\mathcal{X}}_0$. Let $\mathcal{X}' = \hat{\mathcal{X}}/H$ where $H = \mathrm{Gal}(K(\hat{X})/K(X))$, and $\mathcal{Z}' = \hat{\mathcal{X}}/G$. Then $\mathcal{X}' \to \mathcal{Z}'$ is a finite morphism of semi-stable models of X and Z, respectively [Ray90, Proposition 5]. Let $\mathcal{Y}' = \mathcal{Z}'^{(p^r)}$. Then the canonical map $\mathcal{Z}' \to \mathcal{Y}'$ is finite and $\mathcal{Y}'$ is semi-stable (or [Liu02, Exercise 10.3.19(a)]). Since $\hat{\mathcal{X}}$ dominates $\mathcal{Y}$ and is finite over $\mathcal{Y}'$, we see easily that $\mathcal{Y}'$ dominates $\mathcal{Y}$ (use, for instance, [LL99, Lemma 4.1]). Hence, the proposition is proved with f' equal to the composition $\mathcal{X}' \to \mathcal{Z}' \to \mathcal{Y}'$. □

Remark 3.9. If S is any Noetherian integral excellent scheme, then using the result of de Jong [Dej97] as quoted in Remark 2.9, we see that the proposition is still true.

The following corollary was known for separable morphisms (see [Col03] when K is complete; [LL99, Remark 4.6] when $g(X) \geqslant 1$).

COROLLARY 3.10. *Let $f : X \to Y$ be a finite morphism of smooth geometrically connected projective curves over K. Then after a finite separable extension of K, there exists a finite morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ of semi-stable models of X, Y, respectively.*

Proof. Apply Proposition 3.8 to any pair of models of X, Y. □

LEMMA 3.11. *Let S be local. Let F be a finite closed subset of X. Let $\mathcal{Z}$ be a semi-stable model of X over S. Then there exists an integer $d > 0$ such that for any finite extension $\mathcal{O}_{K'}/\mathcal{O}_S$ of discrete valuation rings with ramification index divisible by d, if $\widetilde{\mathcal{Z}}'$ denotes a desingularization of $\mathcal{Z}_{\mathcal{O}_{K'}}$, then the Zariski closure of $F_{K'}$ in $\widetilde{\mathcal{Z}}'$ is contained in the smooth locus of $\widetilde{\mathcal{Z}}'$.*

Proof. Let $\alpha \in F$, and let $x \in \mathcal{Z}_s$ be a singular specialization of α. The local ring of an étale neighborhood of $x \in \mathcal{Z}$ is isomorphic to $\mathcal{O}_K[[u, v]]/(uv - a)$, with a a power of a uniformizing element of $\mathcal{O}_K$. Let $u(\alpha)$ be the image of u in $K(\alpha)$. Then $|a| < |u(\alpha)| < 1$. After an extension of large enough ramification index, $|u(\alpha)|$ belongs to $|K|$. If this condition is satisfied for all $\alpha \in F$ and for all singular specializations of α, then it is easy to see that the specializations of F in $\widetilde{\mathcal{Z}}'$ are smooth points. Indeed, if $\mathcal{Z}$ is regular, the parameter a in the above local ring is a uniformizing element, hence $|a| < |u(\alpha)| < 1$ cannot hold in $|K|$, so F must specialize to smooth points. □

Proposition 3.12. *Let $f:(X,M)\to(Y,N)$ be a finite morphism of smooth geometrically connected marked projective curves over K, let $\mathcal{X},\mathcal{Y}$ be respective models of X,Y. Then after a finite separable extension of K, there exists a semi-stable marked model $\mathcal{X}'$ (respectively, $\mathcal{Y}'$) of (X,M) (respectively, (Y,N)) such that $\mathcal{X}'$ and $\mathcal{Y}'$ dominate $\mathcal{X}$ and $\mathcal{Y}$, respectively, and f extends to a finite morphism $\mathcal{X}'\to\mathcal{Y}'$.*

Proof. After enlarging K and replacing $\mathcal{X}$ and $\mathcal{Y}$ be their respective stable marked hull, we can suppose that $\mathcal{X}$, $\mathcal{Y}$ are semi-stable and that the Zariski closure $\overline{M}$ of M in $\mathcal{X}$ is a disjoint union of sections, and the same for N in $\mathcal{Y}$. Let $X\to Z\to Y$ be the decomposition as given by Proposition 3.5, and let $\hat{X}\to Z$ be the Galois closure of $X\to Z$. Let $f:\hat{X}\to X$ and $g:\hat{X}\to Y$ be the canonical morphisms. By Lemma 3.11, after a finite separable extension, and after replacing $\hat{\mathcal{X}}$ by its minimal desingularization (the group G still acts), we can suppose that the Zariski closure of $f^{-1}(M)\cup g^{-1}(N)$ in $\hat{\mathcal{X}}$ is contained in the smooth locus. Then the Zariski closure of M in $\mathcal{X}'$ is contained in the smooth locus because $\hat{\mathcal{X}}_{\mathrm{sm}}/H$ is smooth, and it is a disjoint union of sections because $\mathcal{X}'$ dominates $\mathcal{X}$ and $\overline{M}$ is already a disjoint union of sections. Hence, $\mathcal{X}'$ is semi-stable marked for (X,M). The same arguments hold for $\mathcal{Y}'$. □

4. Stable hull of a morphism

Definition 4.1. Let $f:X\to Y$ be a finite morphism of connected smooth projective curves over K. Let $\mathcal{X},\mathcal{Y}$ be respective models of X,Y over S. The *finite hull of* $\mathcal{X}\dashrightarrow\mathcal{Y}$ is a finite model $\mathcal{X}^{\mathrm{f}}\to\mathcal{Y}^{\mathrm{f}}$ of f over S, such that $\mathcal{X}^{\mathrm{f}}$ and $\mathcal{Y}^{\mathrm{f}}$ are normal models of X,Y dominating $\mathcal{X}$ and $\mathcal{Y}$, respectively, and which is minimal (for the domination relation) with respect to these properties.

Lemma 4.2. *Let $\mathcal{Y}$ be an integral scheme locally of finite type over S, let L be a finite extension of $K(\mathcal{Y})$, separable over $K=K(S)$, and let $\mathcal{X}$ be the normalization of $\mathcal{Y}$ in L. Then $\mathcal{X}$ is finite over $\mathcal{Y}$.*

Proof. The assertion is, of course, trivial if S is excellent. Let $y\in\mathcal{Y}_s$, let $A=\mathcal{O}_{\mathcal{Y},y}$ and let B be the integral closure of A in L. We have to show that B is finite over A. Let $\mathcal{O}_K=\mathcal{O}_{S,s}$. Let C be the integral closure of $A\otimes_{\mathcal{O}_K}\hat{\mathcal{O}}_K$ in $L\otimes_K\hat{K}$. The latter is reduced (because L is separable over K) and finite over $K(\mathcal{Y})\otimes_K\hat{K}$, the total ring of fractions of $A\otimes\hat{\mathcal{O}}_K$. Since $\hat{\mathcal{O}}_K$ is excellent, C (and, thus, $B\otimes\hat{\mathcal{O}}_K$) is finitely generated over $A\otimes\hat{\mathcal{O}}_K$. Then Nakayama's lemma implies that $B\otimes\hat{\mathcal{O}}_K$ is generated over $A\otimes\hat{\mathcal{O}}_K$ by finitely many elements of B. These elements generate B over A because $\mathcal{O}_K\to\hat{\mathcal{O}}_K$ is faithfully flat.

Note that the proof still work if S is any Noetherian integral scheme such that $\hat{\mathcal{O}}_{S,s}$ is reduced for all $s\in S$. □

Lemma 4.3. *Let X be an integral projective variety over K, and let $\mathcal{X}_1,\mathcal{X}_2,\dots,\mathcal{X}_n$ be models of X over S.*

(a) *There exists a smallest model $\mathcal{X}$ of X dominating $\mathcal{X}_i$ for all i. Let us denote $\mathcal{X}$ by $\mathcal{X}_1\vee\cdots\vee\mathcal{X}_n$.*
(b) *If X is geometrically integral, then for any flat morphism of Dedekind schemes $S'\to S$, we have $(\mathcal{X}_1\vee\cdots\vee\mathcal{X}_n)_{S'}=(\mathcal{X}_1)_{S'}\vee\cdots\vee(\mathcal{X}_n)_{S'}$.*
(c) *If $\dim X=1$, then every irreducible component of $(\mathcal{X}_1\vee\cdots\vee\mathcal{X}_n)_s$ dominates an irreducible component of $(\mathcal{X}_i)_s$ for some i.*

Proof. The proof is similar to that of Lemma 2.4. Let $\mathcal{P}$ be the fiber product $\prod_{S,i}\mathcal{X}_i$ over S. Then the diagonal map makes X a closed subscheme of $\mathcal{P}_K$. Let $\mathcal{X}$ be the Zariski closure of X in $\mathcal{P}$ endowed with the reduced structure. Then $\mathcal{X}$ is a model of X over S dominating the $\mathcal{X}_i$. Let $\mathcal{Z}$ be a model of X over S dominating the $\mathcal{X}_i$. Then we have a natural morphism $f:\mathcal{Z}\to\mathcal{P}$ whose image $f(\mathcal{Z})$ is irreducible, with generic fiber $\mathcal{X}_K$. Hence, $f(\mathcal{Z})=\mathcal{X}$ and f factorizes through $\mathcal{Z}\to\mathcal{X}\to\mathcal{P}$.

So $\mathcal{X}$ is minimal. If X is geometrically integral, then $\mathcal{X}_{S'}$ is an integral closed subscheme of $\mathcal{P}' := \prod_{S',i}(\mathcal{X}_i)_{S'}$, with generic fiber isomorphic to the diagonal of $\mathcal{P}'_{K'}$. By construction, $\mathcal{X}_{S'}$ is equal to $(\mathcal{X}_1)_{S'} \vee \cdots \vee (\mathcal{X}_n)_{S'}$.

Let $s \in S$. Then $\mathcal{X}_s$ is a closed subscheme of $\mathcal{P}_s = \prod_{k(s),i}(\mathcal{X}_i)_s$, pure of dimension $\dim X$. Let Γ be an irreducible component of $\mathcal{X}_s$. If $\dim X > 0$, then the image of Γ in $\mathcal{X}_{i_0}$ has positive dimension for some i_0. If $\dim X = 1$, then the image of Γ in $(\mathcal{X}_{i_0})_s$ is an irreducible component. $\square$

Proposition 4.4. *Let $\mathcal{X} \dashrightarrow \mathcal{Y}$ be a rational map. Suppose that either S is local and Henselian, or $\mathcal{X}$ and $\mathcal{Y}$ are semi-stable or regular. Then the finite hull of $\mathcal{X} \dashrightarrow \mathcal{Y}$ exists. It commutes with base changes in the following sense: let $S' \to S$ be a flat morphism of Dedekind schemes. Then the finite hull of $\mathcal{X}_{S'} \dashrightarrow \mathcal{Y}_{S'}$ exists and is equal to $((\mathcal{X}^{\mathrm{f}})_{S'})^{\sim} \to ((\mathcal{Y}^{\mathrm{f}})_{S'})^{\sim}$, where $^{\sim}$ means normalization.*

Proof. The rational map $\mathcal{X} \dashrightarrow \mathcal{Y}$ is defined and finite above an open dense subset of S. So we can suppose that $S = \operatorname{Spec} \mathcal{O}_K$ is local with closed point s. The case when $\mathcal{X}, \mathcal{Y}$ are semi-stable or regular is easily reduced to the Henselian case by passing to the completion of $\mathcal{O}_K$ and using Proposition 1.3. Suppose that $\mathcal{O}_K$ is Henselian. Let $\mathcal{X}_1$ be the normalization of $\mathcal{X} \vee N(\mathcal{Y}, K(X))$ (see Lemma 4.3). We have a morphism $\mathcal{X}_1 \to \mathcal{Y}$. By [LL99, Lemma 4.14], there exists a (unique) normal model $\mathcal{Y}^{\mathrm{f}}$ such that the rational map $\mathcal{X}_1 \dashrightarrow \mathcal{Y}^{\mathrm{f}}$ is quasi-finite and surjective in codimension 1 (in other words, if $\mathcal{U}$ is the domain of definition of $\mathcal{X}_1 \dashrightarrow \mathcal{Y}^{\mathrm{f}}$, then $\mathcal{U}_s$ is dense in $(\mathcal{X}_1)_s$, $\mathcal{U}_s \to \mathcal{Y}_s$ is quasi-finite and has dense image). Since $\mathcal{X}_1 \to \mathcal{Y}$ is a morphism, $\mathcal{Y}^{\mathrm{f}}$ dominates $\mathcal{Y}$. Let $\mathcal{X}^{\mathrm{f}}$ be the normalization of $\mathcal{Y}^{\mathrm{f}}$ in $K(X)$. Then it is easy to see that $\mathcal{X}^{\mathrm{f}} \to \mathcal{Y}^{\mathrm{f}}$ is the finite hull of $\mathcal{X} \dashrightarrow \mathcal{Y}$ (use [LL99, Lemma 4.1] for instance).

It remains to prove the base change property. The rational map $(\mathcal{X}_{S'})^{\sim} \dashrightarrow ((\mathcal{Y}^{\mathrm{f}})_{S'})^{\sim}$ is quasi-finite and surjective in codimension 1. By the above construction, $(\mathcal{Y}_{S'})^{\mathrm{f}} = ((\mathcal{Y}^{\mathrm{f}})_{S'})^{\sim}$ and $(\mathcal{X}_{S'})^{\mathrm{f}}$ is the normalization of $((\mathcal{Y}^{\mathrm{f}})_{S'})^{\sim}$ in $K(X_{K(S')})$. Since $((\mathcal{X}^{\mathrm{f}})_{S'})^{\sim} \to ((\mathcal{Y}^{\mathrm{f}})_{S'})^{\sim}$ is finite, $(\mathcal{X}_{S'})^{\mathrm{f}}$ is isomorphic to $((\mathcal{X}^{\mathrm{f}})_{S'})^{\sim}$. $\square$

Theorem 4.5. *Let S be a connected Noetherian regular scheme of dimension 1, let $f : X \to Y$ be a finite morphism of smooth geometrically connected projective curves over $K := K(S)$. Let $\mathcal{X}, \mathcal{Y}$ be respective models of X, Y over S. Then after a finite separable extension of K, $\mathcal{X} \dashrightarrow \mathcal{Y}$ admits a stable hull $\mathcal{X}' \to \mathcal{Y}'$. Moreover, the formation of $\mathcal{X}' \to \mathcal{Y}'$ commutes with flat base change.*

Proof. After a finite separable extension of K, we can suppose that there exists a semi-stable model $\mathcal{X}_\infty \to \mathcal{Y}_\infty$ of f dominating $\mathcal{X} \dashrightarrow \mathcal{Y}$ (Corollary 3.10). Let us show that $\mathcal{X} \dashrightarrow \mathcal{Y}$ then admits a stable hull over S. Consider the stable hull $\mathcal{Y}_1$ of $\mathcal{Y}$ and the stable hull $\mathcal{X}_1$ of $N(\mathcal{Y}_1, K(X))$. Then $\mathcal{X}_1$ and $\mathcal{Y}_1$ are dominated by $\mathcal{X}_\infty$ and $\mathcal{Y}_\infty$, respectively. Let $\mathcal{X}_2 \to \mathcal{Y}_2$ be the finite hull of $\mathcal{X}_1 \to \mathcal{Y}_1$. Then it is also dominated by $\mathcal{X}_\infty \to \mathcal{Y}_\infty$. Now restart again the process of taking stable hull and finite hull with $\mathcal{X}_2 \to \mathcal{Y}_2$. We construct in this way an increasing sequence of (normal) models $\mathcal{X}_n \to \mathcal{Y}_n$ of f over S which are dominated by $\mathcal{X}_\infty \to \mathcal{Y}_\infty$. This sequence is stationary at some rank n_0. Then $\mathcal{X}_{n_0} \to \mathcal{Y}_{n_0}$ is a semi-stable model of f. Note that the construction of $\mathcal{X}_n \to \mathcal{Y}_n$ does not depend on the choice of $\mathcal{X}_\infty \to \mathcal{Y}_\infty$. In particular, $\mathcal{X}_n \to \mathcal{Y}_n$ is dominated by any semi-stable model of f dominating $\mathcal{X} \dashrightarrow \mathcal{Y}$. Therefore, $\mathcal{X}_{n_0} \to \mathcal{Y}_{n_0}$ is the stable hull of $\mathcal{X} \dashrightarrow \mathcal{Y}$. Finally, the formation of the stable hull commutes with flat base change because the stable hull of a model and the finite hull of a morphism commute with flat base change (Theorem 2.3 and Proposition 4.4). $\square$

Corollary 4.6. *Suppose that either $g(X) \geqslant 2$, or $g(X) = 1$ and X has potentially good reduction. Then there exists a finite separable extension of K' of K such that $X_{K'} \to Y_{K'}$ admits a stable model $\mathcal{X}' \to \mathcal{Y}'$ over S', where S' is the integral closure of S in K'. Moreover, for any Dedekind scheme T dominating S', $\mathcal{X}'_T \to \mathcal{Y}'_T$ is the stable model of $X_{K(T)} \to Y_{K(T)}$.*

Proof. We can suppose that X has semi-stable reduction over S. The cover $X \to Y$ extends to a finite morphism of smooth projective models over a dense open subset of S. So we can suppose

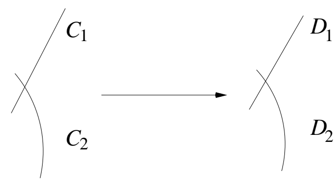

FIGURE 2. A stable model not coming from the Galois closure (see Remark 4.8).

that $S = \operatorname{Spec} \mathcal{O}_K$ is local. Let $\mathcal{X}^{\mathrm{st}}$ be the stable (respectively, smooth projective) model of X if $g(X) \geqslant 2$ (respectively, if $g(X) = 1$). Suppose first that $\mathcal{O}_K$ is complete (hence, Henselian). Let $\mathcal{X}^{\mathrm{st}} \dashrightarrow \mathcal{Y}''$ be the rational map extending $X \to Y$ and which is quasi-finite and surjective in codimension 1 (see [LL99, Lemma 4.14]). Then the stable hull $\mathcal{X} \to \mathcal{Y}$ of $\mathcal{X}^{\mathrm{st}} \dashrightarrow \mathcal{Y}''$ is clearly the stable model of $X \to Y$. The construction of $\mathcal{X} \to \mathcal{Y}$ commutes with flat base change because that of $\mathcal{Y}''$ and the stable hull commute with flat base change. If $\mathcal{O}_K$ is non-necessarily complete, we can use Corollary 1.6. □

Remark 4.7. Let $X \to Y$ be as above. If $X \to Y$ has a semi-stable model over S, then it has a stable model over S. This can be seen in the proof of Corollary 4.6. If $X \to Y$ is, moreover, Galois of group G, and if $\mathcal{X}$ is the stable model (or smooth model if $g(X) = 1$) of X over S, then the stable model of $X \to Y$ is equal to $\mathcal{X} \to \mathcal{X}/G$.

Remark 4.8. Suppose, moreover, that $X \to Y$ is separable, and that the Galois closure $\hat{X}$ of $X \to Y$ is smooth and geometrically connected over K (which is true after a finite separable extension of K). Let $\hat{\mathcal{X}}$ be the stable model of $\hat{X}$, and let $G = \mathrm{Gal}(K(\hat{X})/K(Y))$, $H = \mathrm{Gal}(K(\hat{X})/K(X))$ as in the proof of Proposition 3.8. Then we can ask whether $\hat{\mathcal{X}}/H \to \hat{\mathcal{X}}/G$ is the stable model of $X \to Y$. The answer is no in general. Let us give an example with X and Y having good reduction.

Let S be local, complete, with algebraically closed residue field k. Let $C_1 \to D_1$ be a finite separable morphism of degree $d \geqslant 3$ with $C_1, D_1 \simeq \mathbb{P}^1_k$, totally ramified above some point $y_1 \in D_1$, and such that the Galois closure E of $C_1 \to D_1$ is a curve of genus $g(E) \geqslant 1$. Let $C_2 \to D_2$ be a finite separable morphism of degree d of smooth connected projective curves over k, totally ramified above a point $y_2 \in D_2$ and such that $g(C_2) \geqslant 1$. Let D be the semi-stable curve over k obtained by identifying y_1 and y_2. Let C be the semi-stable curve defined in a similar way. Then we have a finite morphism $\rho : C \to D$, which is generically étale, and such that $C_1 \to D_1$, $C_2 \to D_2$ have the same ramification index at y_1 and y_2. (See Figure 2.) By [Liu03, Proposition 5.4], the cover $C \to D$ lifts to a finite morphism $\mathcal{C} \to \mathcal{D}$ over S with smooth generic fibers X, Y. Let $\mathcal{C} \to \mathcal{C}_2$ (respectively, $\mathcal{D} \to \mathcal{D}_2$) be the contraction of C_1 (respectively, of D_1). Then $\mathcal{C}_2$, $\mathcal{D}_2$ are smooth, and the canonical morphism $\mathcal{C}_2 \to \mathcal{D}_2$ is the stable model of $X \to Y$. Let $\mathcal{Z}$ be the normalization of $\mathcal{D}$ in $K(\hat{X})$. Let Θ be an irreducible component of $\mathcal{Z}_s$ lying over D_1. Then the separable closure of $k(D_1)$ in $k(\Theta)$ is Galois over $k(D_1)$ (see [Ser68, I, § 7, Proposition 20]) and contains $k(C_1)$. Thus, $k(\Theta)$ contains a subfield isomorphic to $k(E)$. In particular, $p_a(\Theta) \geqslant 1$. If $\mathcal{C}_2 \to \mathcal{D}_2$ is equal to $\hat{\mathcal{X}}/H \to \hat{\mathcal{X}}/G$, then $\mathcal{Z}$ dominates $\hat{\mathcal{X}}$ because $\mathcal{D}$ dominates $\mathcal{D}_2$. However, Θ maps to a closed point of $\hat{\mathcal{X}}_s$, thus $\hat{\mathcal{X}}$ cannot be semi-stable – contradiction.

Remark 4.9. If X has genus 1 and multiplicative reduction at some point of S, or if $g(X) = 0$, then over any ramified extension of S, there is no stable model of the identity map $X \to X$. The reason we take a ramified extension S'/S is that, when $g(X) = 1$, $X_{K'}$ has no minimal semi-stable model (see the proof of Lemma 1.5).

Let us give a characterization of the stable model.

Definition 4.10. Let $X_1 \to Y_1$, $X_2 \to Y_2$ be morphisms of schemes over some base scheme T. An *isomorphism* of the pairs $(X_1 \to Y_1) \to (X_2 \to Y_2)$ is given by T-isomorphisms $X_1 \to X_2$, $Y_1 \to Y_2$ such that the diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2 \end{array}$$

is commutative. We denote by $\mathrm{Isom}_T((X_1 \to Y_1), (X_2 \to Y_2))$ the set of these isomorphisms. Now $\mathrm{Aut}_T(X \to Y)$ has an obvious meaning.

Proposition 4.11. *Keep the hypothesis of Theorem 4.5 and suppose that $g(X) \geqslant 2$. Let $\mathcal{X} \to \mathcal{Y}$ be a semi-stable model of $X \to Y$. Consider the following properties.*

(i) *$\mathcal{X} \to \mathcal{Y}$ is the stable model of $X \to Y$ over S.*

(ii) *Let Γ be any irreducible component of $\mathcal{Y}_s$ such that $\deg \omega_{\mathcal{Y}_s/k(s)}|_\Gamma \leqslant 0$, then there exists an irreducible component Θ of $\mathcal{X}_s$ dominating Γ such that $\deg \omega_{\mathcal{X}_s/k(s)}|_\Theta > 0$.*

(iii) *$\mathrm{Aut}_{k(\bar{s})}(\mathcal{X}_{\bar{s}} \to \mathcal{Y}_{\bar{s}})$ is finite for all $s \in S$.*

Then (i) $\iff$ (ii) $\implies$ (iii).

Proof. Looking at the proofs of Theorem 4.5 and Corollary 4.6, we see that $X \to Y$ admits a stable model over S. Thus, $\mathcal{X} \to \mathcal{Y}$ is stable if and only if it is a relatively minimal semi-stable model. Hence the equivalence (i) $\iff$ (ii) is an immediate consequence of Lemma 2.13.

Suppose that condition (ii) is satisfied. Then the same condition holds over $k(\bar{s})$ (with dualizing sheaves on $\mathcal{X}_{\bar{s}}$ and $\mathcal{Y}_{\bar{s}}$). Consider the natural inclusion

$$\mathrm{Aut}_{k(\bar{s})}(\mathcal{X}_{\bar{s}} \to \mathcal{Y}_{\bar{s}}) \subseteq \mathrm{Aut}_{k(\bar{s})}(\mathcal{X}_{\bar{s}}) \times \mathrm{Aut}_{k(\bar{s})}(\mathcal{Y}_{\bar{s}}).$$

Note that the right-hand side is not a finite group in general. Let G be the subgroup (of finite index) of $\mathrm{Aut}_{k(\bar{s})}(\mathcal{X}_{\bar{s}})$ consisting in automorphisms which fix globally each irreducible component, and let H be the subgroup of $\mathrm{Aut}_{k(\bar{s})}(\mathcal{Y}_{\bar{s}})$ consisting of automorphisms τ such that $\tau|_\Gamma = \mathrm{Id}$ for every irreducible component Γ with $\deg \omega_{\mathcal{Y}_{k(\bar{s})/k(\bar{s})}}|_\Gamma > 0$. Then H is of finite index in $\mathrm{Aut}_{k(\bar{s})}(\mathcal{Y}_{\bar{s}})$ because $\{\tau \in \mathrm{Aut}_{k(\bar{s})}(\mathcal{Y}_{k(\bar{s})}) \mid \tau(\Gamma) = \Gamma\}$ is finite for any such Γ. Thus, it is enough to show that $G' := \mathrm{Aut}_{k(\bar{s})}(\mathcal{X}_{\bar{s}} \to \mathcal{Y}_{\bar{s}}) \cap (G \times H)$ is finite. Let I be the set of irreducible components Θ of $\mathcal{X}_{\bar{s}}$ such that $\deg \omega_{\mathcal{X}_{\bar{s}}/k(\bar{s})}|_\Theta > 0$. Then $G|_\Theta$ is finite for all $\Theta \in I$. Let (σ, τ) be an element of $\mathrm{Ker}(G' \to G \times H \to \prod_{\Theta \in I} G|_\Theta)$. Condition (ii) implies that $\tau = \mathrm{Id}$ on $\mathcal{Y}_{\bar{s}}$, hence $\sigma \in \mathrm{Aut}_{\mathcal{Y}_{\bar{s}}}(\mathcal{X}_{\bar{s}})$. The latter is a finite group because $\mathcal{X}_{\bar{s}} \to \mathcal{Y}_{\bar{s}}$ is a finite morphism. Since the projection map $\mathrm{Aut}_{k(\bar{s})}(\mathcal{X}_{\bar{s}} \to \mathcal{Y}_{\bar{s}}) \to \mathrm{Aut}_{k(\bar{s})}(\mathcal{X}_{\bar{s}})$ is injective, the above kernel is finite. Hence, G' is finite. □

Remark 4.12. In general, property (iii) does not imply property (i). Let S be local, complete with algebraically closed residue field k. Let $\pi : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be a finite separable cover with trivial automorphisms group $\mathrm{Aut}_k(\pi)$. Then π extends to a finite cover $\mathbb{P}^1_k \to \mathbb{P}^1_k$ totally ramified at ∞. We can glue π with a finite separable cover $C_2 \to D_2$ of smooth projective curves of genus $\geqslant$2 over k and obtain a finite cover $C \to D$ with finite automorphisms group (see the construction in Remark 4.8) which lifts to a finite morphism of semi-stable curves $\mathcal{C} \to \mathcal{D}$ over S. By the equivalence of properties (i) and (ii), $\mathcal{C} \to \mathcal{D}$ is not stable.

Remark 4.13. We can see that Theorem 4.5 holds for finite morphisms of smooth projective marked curves $(X, M) \to (Y, N)$: *let $\mathcal{X}, \mathcal{Y}$ be respective models of X, Y, then after finite separable extension, there exists a stable marked hull $\mathcal{X}' \to \mathcal{Y}'$ of $\mathcal{X} \dashrightarrow \mathcal{Y}$. Moreover, the formation of $\mathcal{X}' \to \mathcal{Y}'$ commutes with flat base change.* The proof is the same as for Theorem 4.5, except that we replace stable hull of a model by its stable marked hull, and we use Proposition 3.12 instead of Corollary 3.10.

The next lemma is a generalization of [LL99, Proposition 4.4(a)] (take $N = \emptyset$).

LEMMA 4.14. *Let $f : (X, M) \to (Y, N)$ be a finite morphism of connected smooth projective marked curves over K. Suppose that $2g(Y) - 2 + |N| > 1$ and $f^{-1}(N) = M$. If (X, M) has a stable marked model $\mathcal{X}$ over S, then (Y, N) has a stable marked model $\mathcal{Y}$, and $\mathcal{X} \dashrightarrow \mathcal{Y}$ is a morphism.*

Proof. The stable marked model exists over S when X has semi-stable reduction over S. So the existence of $\mathcal{X}$ implies that of $\mathcal{Y}$ (see [LL99, Remark 4.8], when $g(Y) \geqslant 1$, the case $g(Y) = 0$ is trivial). It remains to show that the rational map $\mathcal{X} \dashrightarrow \mathcal{Y}$ is defined everywhere. Since the stable marked model commutes with flat base change (2.21), we can suppose that S is local with algebraically closed residue field and that $\mathcal{W} := N(\mathcal{Y}, K(X))$ has a stable marked hull $\mathcal{Z}$ over S. By definition, $\mathcal{Z}$ dominates $\mathcal{X}$. We are going to show that $\mathcal{Z} \to \mathcal{X}$ is an isomorphism, or, equivalently, that $\mathcal{Z}$ is the stable marked hull of $\mathcal{X}$. Let $\overline{M}$ denote the Zariski closure of M in $\mathcal{Z}$. It is enough to show that $\deg \omega_{\mathcal{Z}/S}(\overline{M})|_\Theta > 0$ for all irreducible component Θ of $\mathcal{Z}_s$. If Θ is in the exceptional locus of $\mathcal{Z} \to \mathcal{W}$, then this is true because $\mathcal{Z} \to \mathcal{W}$ is the stable marked hull. Suppose that Θ is the strict transform of some irreducible component Γ of $\mathcal{W}_s$. Let Δ be the image of Γ in $\mathcal{Y}_s$. Then every point $y \in \Delta$ which is either a singular point of $\mathcal{Y}_s$ or a specialization of N lifts to a point of Θ which is either a singular point or a specialization of $f^{-1}(N) = M$ (see [LL99, Proposition 4.4(b)] for singular points; use going-down property as in [LL99, Lemma 4.3] for specializations of N). Since $\mathcal{Y}$ is stable marked, this implies that Θ contains at least three points of $(\mathcal{Z}_s)_{\mathrm{sing}} \cup \overline{M}_s$, hence $\deg \omega_{\mathcal{Z}/S}(\overline{M})|_\Theta > 0$. □

Remark 4.15. It is known that, in general, $f : X \to Y$ does not extend to a morphism of the respective minimal regular models of X and Y over S (see, e.g., [CES03, p. 333]). However, if $g(Y) > 0$, then f extends to a morphism of the respective minimal regular models with normal crossings, at least if $k(s)$ is perfect for all closed points $s \in S$.

COROLLARY 4.16. *Let $f : (X, M) \to (Y, N)$ be a finite morphism of geometrically connected smooth projective marked curves over K. Suppose that $2g(Y) - 2 + |N| > 1$ and $f^{-1}(N) = M$. Then, after a finite separable extension of K, f admits a stable marked model $\mathcal{X} \to \mathcal{Y}$ over S. This construction commutes with flat base change.*

Remark 4.17. Let $f : X \to Y$ be a finite *separable* morphism of smooth projective curves. A natural way to mark X and Y is to take M equal to the ramification locus of f and N equal to the branch locus. By definition $f^{-1}(N) = M$. Of course, in general, M is not contained in $X(K)$. However, if f is tamely ramified (e.g. if $\mathrm{char}(K) = 0$), then this becomes true over a finite separable extension of K (see [LL99, Lemma 3.3]). Moreover, if $g(Y) \geqslant 2$, or $g(Y) = 1$ and f is not étale, or if $g(Y) = 0$ and $g(X) \geqslant 1$, then $2g(Y) - 2 + |N| > 1$. So after again a finite separable extension of K, we have a canonical way to define a minimal semi-stable reduction of $X \to Y$ in which the (horizontal) ramification and branch loci are finite unions of sections contained in the smooth locus. If $(\deg f)!$ is invertible in $\mathcal{O}_S$, then $\mathcal{X}$ and $\mathcal{Y}$ are the respective stable marked models of X and Y (see [Moc95, § 3.11, second lemma]).

Remark 4.18. Let $\hat{X}$, G, H be as in Remark 4.8. Let $\hat{X}$ be marked with its ramification locus over X (rational over K after a finite extension of K). Let $\hat{\mathcal{X}}$ be its stable marked model. If G has order prime to $\mathrm{char}(k)$, then the stable marked model of $X \to Y$ is equal to $\hat{\mathcal{X}}/H \to \hat{\mathcal{X}}/G$ because both sides are, respectively, stable marked models of X and Y (see Remark 4.17). However, this is false if $\mathrm{char}(k)$ divides $|G|$. Let us go back to the example of Remark 4.8 and, moreover, let $C_1 \to D_1$ be étale outside of y_1. Let $\mathcal{C}' \to \mathcal{C}$, $\mathcal{D}' \to \mathcal{D}$ be the stable marked hulls of $\mathcal{C}$, $\mathcal{D}$ (marked with horizontal ramification/branch loci). Let $\mathcal{C}' \to \mathcal{C}''$, $\mathcal{D}' \to \mathcal{D}''$ be the contraction of (the strict transforms of) C_1, D_1, respectively. Then it is easy to see that the stable marked model (4.16) of $X \to Y$ is $\mathcal{C}'' \to \mathcal{D}''$. Similarly to Remark 4.8, we see that it is different from $\hat{\mathcal{X}}/H \to \hat{\mathcal{X}}/G$.

It remains to find a cover $C_1 \to D_1$ as above. Consider the cover[1] defined by the extension $k(D_1) = k(u) \to k(C_1) = k(u,v)$, with $v^{p^2+p} + v = u$, where $p = \mathrm{char}(k) > 0$. Then $C_1 \to D_1$ is étale outside of the pole y_1 of u. Let us show that the Galois closure $k(E)$ has positive genus. Let $t \in k(E)$ be such that $t^{p^2+p} + t = u$ and $t \neq v$. Let $w = t - v$. Then w satisfies the equation $(w^{p+1} - wt^p - w^p t)^p - w = 0$. Hence, $w = z^p$ where $z = w^{p+1} - wt^p - w^p t$. We have $z = z^{p^2+p} - z^p t^p - z^{p^2} t$, so

$$(t/z^p)^p + (t/z^p) = 1 - (1/z)^{p^2+p-1}.$$

This equation defines a p-cyclic cover $E' \to \mathbb{P}^1_k$, with conductor $m = p^2 + p - 1$ at $z = 0$, and étale elsewhere. Hence, $g(E) \geqslant g(E') = (p-1)(m-1)/2 \geqslant 2$.

Remark 4.19. If we restrict ourselves to the category of regular models of X, then Theorem 2.3 still holds. More precisely, given any model $\mathcal{X}$, there exists (after finite separable extension of K) a unique semi-stable and regular model dominating $\mathcal{X}$ and minimal for this property. This model is just the minimal desingularization of the stable hull of $\mathcal{X}$. However, it does not commute with base change except when the normalization of $\mathcal{X}$ is smooth. On the other hand, Theorem 4.5 is no longer true in the setting of regular models. In general, given a morphism of models $\mathcal{X} \to \mathcal{Y}$, there is no finite morphisms of regular models dominating $\mathcal{X} \to \mathcal{Y}$, even after any finite extension of K (see [LL99, Remark 6.5]).

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[1] This example is given by Michel Matignon.

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