HYPERSURFACES IN PROJECTIVE SCHEMES AND A MOVING LEMMA

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Abstract
Let X/S be a quasi-projective morphism over an affine base. We develop in this article a technique for proving the existence of closed subschemes H/S of X/S with various favorable properties. We offer several applications of this technique, including the existence of finite quasi-sections in certain projective morphisms, and the existence of hypersurfaces in X/S containing a given closed subscheme C and intersecting properly a closed set F.

Assume now that the base S is the spectrum of a ring R such that for any finite morphism $Z \to S$, $\mathrm{Pic}(Z)$ *is a torsion group. This condition is satisfied if R is the ring of integers of a number field or the ring of functions of a smooth affine curve over a finite field. We prove in this context a moving lemma pertaining to horizontal* 1*-cycles on a regular scheme X quasi-projective and flat over S. We also show the existence of a finite surjective S-morphism to $\mathbb{P}^d_S$ for any scheme X projective over S when X/S has all its fibers of a fixed dimension d.*

Introduction

Let $S = \operatorname{Spec} R$ be an affine scheme, and let X/S be a quasi-projective scheme. The core of this article is a method, summarized in Section 0.5 below, for proving the existence of closed subschemes of X with various favorable properties. As the technical details can be somewhat complicated, we start this introduction by discussing the applications of the method that the reader will find in this article.

Recall (Section 3.1) that a global section f of an invertible sheaf $\mathcal{L}$ on any scheme X defines a closed subset H_f of X, consisting of all points $x \in X$, where the stalk f_x does not generate $\mathcal{L}_x$. Since $\mathcal{O}_X f \subseteq \mathcal{L}$, the ideal sheaf $\mathcal{I} := \mathcal{O}_X f \otimes \mathcal{L}^{-1}$ endows H_f with the structure of closed subscheme of X. Let $X \to S$ be any morphism. We call the closed subscheme H_f of X a *hypersurface* (relative to $X \to S$) when no irreducible component of positive dimension of X_s is contained in H_f, for all $s \in S$. If, moreover, the ideal sheaf $\mathcal{I}$ is invertible, we say that the hypersurface

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H_f is *locally principal*. We remark that when a fiber X_s contains isolated points, it is possible for H_f (resp., $(H_f)_s$) to have codimension 0 in X (resp., in X_s), instead of the expected codimension 1.

(A) An avoidance lemma for families. It is classical that if X/k is a quasi-projective scheme over a field, $C \subsetneq X$ is a closed subset of positive codimension, and $\xi_1, \dots, \xi_r$ are points of X not contained in C, then there exists a hypersurface H in X such that $C \subseteq H$ and $\xi_1, \dots, \xi_r \notin H$. Such a statement is commonly referred to as an *avoidance lemma* (see, e.g., Lemma 4.6). Our next theorem establishes an *avoidance lemma for families*. As usual, when X is noetherian, $\operatorname{Ass}(X)$ denotes the finite *set of associated points* of X.

THEOREM 5.1
Let S be an affine scheme, and let $X \to S$ be a quasi-projective and finitely presented morphism. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to $X \to S$. Let
(i) *C be a closed subscheme of X, finitely presented over S;*
(ii) *$F_1, \dots, F_m$ be subschemes*[†] *of X of finite presentation over S;*
(iii) *A be a finite subset of X such that $A \cap C = \emptyset$.*
Assume that for all $s \in S$, C does not contain any irreducible component of positive dimension of $(F_i)_s$ and of X_s. Then there exists $n_0 > 0$ such that for all $n \geq n_0$, there exists a global section f of $\mathcal{O}_X(n)$ such that
(1) *the closed subscheme H_f of X is a hypersurface that contains C as a closed subscheme;*
(2) *for all $s \in S$ and for all $i \leq m$, H_f does not contain any irreducible component of positive dimension of $(F_i)_s$; and*
(3) *$H_f \cap A = \emptyset$.*
Assume in addition that S is noetherian and that $C \cap \operatorname{Ass}(X) = \emptyset$. Then there exists such a hypersurface H_f which is locally principal.

When H_f is locally principal, H_f is the support of an effective ample Cartier divisor on X. This divisor is *horizontal* in the sense that it does not contain in its support any irreducible component of fibers of $X \to S$ of positive dimension. In some instances, such as in Corollaries 5.5 and 5.6, we can show that H_f is a relative effective Cartier divisor, that is, that $H_f \to S$ is flat. Corollary 5.5 also includes a Bertini-type statement for $X \to S$ with Cohen–Macaulay fibers. We use Theorem 5.1 to establish in Theorem 6.3 the existence of finite quasi-sections in certain projective morphisms X/S, as we now discuss.

[†]Each F_i is a closed subscheme of an open subscheme of X (see [29, I.4.1.3]).

(B) Existence of finite quasi-sections. Let $X \to S$ be a surjective morphism. Following [29, Chapter IV, Section 14, p. 200] we define the following:

Definition 0.1
We call a closed subscheme C of X a *finite quasi-section* when $C \to S$ is finite and surjective. Some authors call *multisection* a finite quasi-section $C \to S$ which is also flat, with C irreducible.

When S is integral noetherian of dimension 1 and $X \to S$ is proper and surjective, the existence of a finite quasi-section C is well known and easy to establish. It suffices to take C to be the Zariski closure of a closed point of the generic fiber of $X \to S$. When $\dim S > 1$, the process of taking the closure of any closed point of the generic fiber does not always produce a closed subset *finite* over S (see Example 6.1).

THEOREM 6.3
Let S be an affine scheme, and let $X \to S$ be a projective, finitely presented morphism. Suppose that all fibers of $X \to S$ are of the same dimension $d \geq 0$. Let C be a finitely presented closed subscheme of X, with $C \to S$ finite but not necessarily surjective. Then there exists a finite quasi-section T of finite presentation which contains C. Moreover:

(1) *Assume that S is noetherian. If C and X are both irreducible, then there exists such a quasi-section with T irreducible.*
(2) *If $X \to S$ is flat with Cohen–Macaulay fibers (e.g., if S is regular and X is Cohen–Macaulay and equidimensional over S), then there exists such a quasi-section with $T \to S$ flat.*
(3) *If $X \to S$ is flat and a local complete intersection morphism, then there exists such a quasi-section with $T \to S$ flat and a local complete intersection morphism.*
(4) *Assume that S is noetherian. Suppose that $\pi : X \to S$ has equidimensional fibers and that $C \to S$ is unramified. Let Z be a finite subset of S (such as the set of generic points of $\pi(C)$), and suppose that there exists an open subset U of S containing Z such that $X \times_S U \to U$ is smooth. Then there exists such a quasi-section T of $X \to S$ and an open set $V \subseteq U$ containing Z such that $T \times_S V \to V$ is étale.*

As an application of Theorem 6.3, we obtain a strengthening in the affine case of the classical splitting lemma for vector bundles.

PROPOSITION 6.10
Let A be a commutative ring. Let M be a projective A-module of finite presentation with constant rank $r > 1$. Then there exists an A-algebra B, finite, and faithfully flat over A, with B a local complete intersection over A, such that $M \otimes_A B$ is isomorphic to a direct sum of projective B-modules of constant rank 1.

Another application of Theorem 6.3 to the problem of extending a given family of stable curves $D \to Z$ after a finite surjective base change is found in Proposition 6.12. It is natural to wonder whether Theorems 5.1 and 6.3 hold for more general bases S which are not affine, such as a noetherian base S having an ample invertible sheaf. It is also natural to wonder if the existence of finite quasi-sections in Theorem 6.3 holds for proper morphisms.

(C) Existence of integral points. Let R be a Dedekind domain,[†] and let $S = \operatorname{Spec} R$. When $X \to S$ is quasi-projective, an integral finite quasi-section is also called an *integral point* in [50, 1.4]. The existence of a finite quasi-section in the quasi-projective case over $S = \operatorname{Spec} \mathbb{Z}$ when the generic fiber is geometrically irreducible is Rumely's famous local-global principle [63]. This existence result was extended in [50, 1.6], as follows. As in [50, 1.5], we make the following definition.

Definition 0.2
We say that a Dedekind scheme S satisfies *condition* (T) if:
(a) for any finite extension L of the field of fractions K of R, the normalization S' of S in $\operatorname{Spec} L$ has torsion Picard group $\operatorname{Pic}(S')$;
(b) the residue fields at all closed points of S are algebraic extensions of finite fields.

For example, S satisfies condition (T) if S is an affine integral smooth curve over a finite field, or if S is the spectrum of the ring of P-integers in a number field K, where P is a finite set of finite places of K.

Our next theorem is only a mild sharpening of the local-global principle in [50, 1.7]: we show in Theorem 7.9 that the hypothesis in [50, 1.7] that the base scheme S is excellent and can be removed.

THEOREM 7.9
Let S be a Dedekind scheme satisfying condition (T). Let $X \to S$ be a separated surjective morphism of finite type. Assume that X is irreducible and that the generic fiber of $X \to S$ is geometrically irreducible. Then $X \to S$ has a finite quasi-section.

[†]A Dedekind domain in this article has dimension 1, and a Dedekind scheme is the spectrum of a Dedekind domain.

Condition (T)(a) is necessary in the local-global principle, but it is not sufficient, as shown by an example of Raynaud over $S = \operatorname{Spec} \overline{\mathbb{Q}}[x]_{(x)}$ (see [49, 3.2] and [13, 5.5]). In [49], 3.2, we use the fact that if F in an algebraically closed field which is not the algebraic closure of a finite field, then the group of F-rational points of a non-zero abelian variety has infinite rank ([62], Corollary, page 106, or [22], 10.1). The following weaker condition is needed for our next two theorems.

Definition 0.3
Let R be any commutative ring, and let $S = \operatorname{Spec} R$. We say that R or S is *pictorsion* if $\operatorname{Pic}(Z)$ is a torsion group for any finite morphism $Z \to S$.

Any semilocal ring R is pictorsion. A Dedekind domain satisfying condition (T) is pictorsion (see [50, 2.3]; see also Lemma 8.10(2)). Rings which satisfy the primitive criterion (see Proposition 8.13) are pictorsion and only have infinite residue fields.

(D) A moving lemma. Let S be a Dedekind scheme, and let X be a noetherian scheme over S. An integral closed subscheme C of X finite and surjective over S is called an *irreducible horizontal* 1*-cycle* on X. A *horizontal* 1*-cycle* on X is an element of the free abelian group generated by the irreducible horizontal 1-cycles. Our next application of the method developed in this article is a moving lemma for horizontal 1-cycles.

THEOREM 7.2
Let R be a Dedekind domain, and let $S := \operatorname{Spec} R$. Let $X \to S$ be a flat and quasi-projective morphism, with X integral. Let C be a horizontal 1*-cycle on X. Let F be a closed subset of X. Assume that for all $s \in S$, $F \cap X_s$ and $\operatorname{Supp}(C) \cap X_s$ have positive codimension*† *in X_s. Assume in addition either that*
(a) *R is pictorsion and the support of C is contained in the regular locus of X, or*
(b) *R satisfies condition (T).*

Then some positive multiple mC of C is rationally equivalent to a horizontal 1*-cycle C' on X whose support does not meet F. Under the assumption (a), if furthermore R is semilocal, then we can take $m = 1$.*

Moreover, if $Y \to S$ is any separated morphism of finite type and $h : X \to Y$ is any S-morphism, then $h_*(mC)$ is rationally equivalent to $h_*(C')$ on Y.

Example 7.11 shows that the condition (T)(a) is necessary for Theorem 7.2 to hold. A different proof of Theorem 7.2 when S is semilocal, X is regular, and $X \to S$

†The definition of codimension in [29, Chapter 0, 14.2.1] implies that the codimension of the empty set in X_s is $+\infty$, which we consider to be positive.

is quasi-projective is given in [25, 2.3], where the result is then used to prove a formula for the index of an algebraic variety over a Henselian field (see [25, 8.2]).

It follows from [25, 6.5] that for each $s \in S$, a multiple $m_s C_s$ of the 0-cycle C_s is rationally equivalent on X_s to a 0-cycle whose support is disjoint from F_s. Theorem 6.5 in [25] expresses such an integer m_s in terms of Hilbert–Samuel multiplicities. The 1-cycle C in X can be thought of as a family of 0-cycles, and Theorem 7.2 may be considered as a moving lemma for 0-cycles in families.

Even for schemes of finite type over a finite field, Theorem 7.2 is not a consequence of the classical Chow's moving lemma. Indeed, let X be a smooth quasi-projective variety over a field k. The classical Chow's moving lemma (see [61]) immediately implies the following statement.

0.4
Let Z be a 1-cycle on X. Let F be a closed subset of X of codimension at least 2 in X. Then there exists a 1-cycle Z' on X, rationally equivalent to Z, and such that $\operatorname{Supp}(Z') \cap F = \emptyset$.

Consider a morphism $X \to S$ as in Theorem 7.2, and assume in addition that S is a smooth affine curve over a finite field k. Let F be a closed subset as in Theorem 7.2. Such a subset may be of codimension 1 in X. Thus, Theorem 7.2 is not a consequence of Chow's moving lemma for 1-cycles just recalled, since Section 0.4 can only be applied to $X \to S$ when F is a closed subset of codimension at least 2 in X.

(E) Existence of finite morphisms to $\mathbb{P}^d_S$. Let k be a field. A strong form of the normalization theorem of E. Noether that applies to graded rings (see, e.g., [17, 13.3]) implies that every projective variety X/k of dimension d admits a finite k-morphism $X \to \mathbb{P}^d_k$. Our next theorem guarantees the existence of a finite S-morphism $X \to \mathbb{P}^d_S$ when $X \to S$ is projective with R pictorsion and $d := \max\{\dim X_s, s \in S\}$.

THEOREM 8.1
Let R be a pictorsion ring, and let $S := \operatorname{Spec} R$. Let $X \to S$ be a projective morphism, and set $d := \max\{\dim X_s, s \in S\}$. Then there exists a finite S-morphism $X \to \mathbb{P}^d_S$. If we assume in addition that $\dim X_s = d$ for all $s \in S$, then there exists a finite surjective S-morphism $X \to \mathbb{P}^d_S$.

The above theorem generalizes to schemes X/S of any dimension the results of [28, Theorem 2] and [10, Theorem 1.2], which apply to morphisms of relative dimension 1. After this article was written, we became aware of a preprint [11], where the general case is also discussed. We also prove the converse of this theorem.

PROPOSITION (see Proposition 8.7)
Let R be any commutative ring, and let $S := \operatorname{Spec} R$. Suppose that for any $d \geq 0$, and for any projective morphism $X \to S$ such that $\dim X_s = d$ for all $s \in S$, there exists a finite surjective S-morphism $X \to \mathbb{P}^d_S$. Then R is pictorsion.

(F) Method of proof. Now that the main applications of our method for proving the existence of hypersurfaces H_f in projective schemes X/S with certain desired properties have been discussed, let us summarize the method.

0.5
Let $X \to S$ be a projective morphism with $S = \operatorname{Spec} R$ affine and noetherian. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to $X \to S$. Let $C \subseteq X$ be a closed subscheme defined by an ideal $\mathcal{I}$, and set $\mathcal{I}(n) := \mathcal{I} \otimes \mathcal{O}_X(n)$. Our goal is to show the existence, for all n large enough, of a global section f of $\mathcal{I}(n)$ such that the associated subscheme H_f has the desired properties.

To do so, we fix a system of generators $f_1, \dots, f_N$ of $H^0(X, \mathcal{I}(n))$, and we consider for each $s \in S$ a subset $\Sigma(s) \subset \mathbb{A}^N(k(s))$ consisting of all the vectors $(\alpha_1, \dots, \alpha_N)$ such that $\sum_i \alpha_i f_{i|X_s}$ does not have the desired properties. We show then that all these subsets $\Sigma(s)$, $s \in S$, are the rational points of a single pro-constructible subset T of $\mathbb{A}^N_S$ (which depends on n). To find a desired global section $f := \sum_i a_i f_i$ with $a_i \in R$ which avoids the subset T of "bad" sections, we show that for some n large enough the pro-constructible subset T satisfies the hypotheses of the following theorem. The section σ whose existence follows from Theorem 2.1 provides the desired vector $(a_1, \dots, a_N) \in R^N$.

THEOREM (see Theorem 2.1)
Let S be a noetherian affine scheme. Let $T := T_1 \cup \cdots \cup T_m$ be a finite union of pro-constructible subsets of $\mathbb{A}^N_S$. Suppose that
(1) *for all $i \leq m$, $\dim T_i < N$, and $(T_i)_s$ is constructible in $\mathbb{A}^N_{k(s)}$ for all $s \in S$;*
(2) *for all $s \in S$, there exists a $k(s)$-rational point in $\mathbb{A}^N_{k(s)}$ which does not belong to T_s.*
Then there exists a section σ of $\pi : \mathbb{A}^N_S \to S$ such that $\sigma(S) \cap T = \emptyset$.

To explain the phrasing of (1) in the above theorem, note that the union $T_1 \cup \cdots \cup T_m =: T$ is proconstructible since each T_i is. However, it may happen that $\dim T > \max_i(\dim T_i)$. This can be seen already on the spectrum T of a discrete valuation ring, which is the union of two (constructible) points, each of dimension 0. The proof of Theorem 2.1 is given in Section 2. The construction alluded to in Section 0.5 of the pro-constructible subset T whose rational points are in bijection with $\Sigma(s)$ is done in Proposition 1.10.

We present our next theorem as a final illustration of the strength of the method. This theorem, stated in a slightly stronger form in Section 3, is the key to the proof of Theorem 7.2(a), as it allows for a reduction to the case of relative dimension 1. Note that in this theorem, S is not assumed to be pictorsion.

THEOREM (see Theorem 3.4)
Let S be an affine noetherian scheme of finite dimension, and let $X \to S$ be a quasi-projective morphism. Let C be a closed irreducible subscheme of X of codimension $d > \dim S$ in X. Assume that $C \to S$ is finite and surjective and that $C \to X$ is a regular immersion. Let F be a closed subset of X. Fix a very ample sheaf $\mathcal{O}_X(1)$ relative to $X \to S$. Then there exists $n_0 > 0$ such that for all $n \geq n_0$, there exists a global section f of $\mathcal{O}_X(n)$ such that

(1) *C is a closed subscheme of codimension $d-1$ in H_f, and $C \to H_f$ is a regular immersion;*
(2) *for all $s \in S$, H_f does not contain any irreducible component of positive dimension of F_s.*

The proof of Theorem 3.4 is quite subtle and spans Sections 3 and 4. In Theorem 7.2(a), we start with the hypothesis that C is contained in the regular locus of X. It is not possible in general to expect that a hypersurface H_f containing C can be chosen so that C is again contained in the regular locus of H_f. Thus, when no regularity conditions can be expected on the total space, we impose regularity conditions by assuming that C is regularly immersed in X. Great care is then needed in the proof of Theorem 3.4 to ensure that a hypersurface H_f can be found with the property that C is regularly immersed in H_f.

Section 3 contains most of the proof of Theorem 3.4. Several lemmas needed in the proof of Theorem 3.4 are discussed separately in Section 4. Sections 5, 6, 7, and 8 contain the proofs of the applications of our method.

Contents

1. Zero locus of sections of a quasi-coherent sheaf

We start this section by reviewing basic facts on constructible subsets, a concept introduced by Chevalley [9]. We follow the exposition in [29]. We introduce the zero-locus $\mathcal{Z}(\mathcal{F}, f)$ of a global section f of a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X, and we show in Lemma 1.4 that this subset is locally constructible in X. Given a finitely presented morphism $\pi : X \to Y$, we further define a subset $T_{\mathcal{F},f,\pi}$ in Y and show in Proposition 1.6 that it is locally constructible in Y. The main result in this section is Proposition 1.10, which is a key ingredient in the proofs of Theorems 3.3 and 3.4.

Let X be a topological space. A subset T of X is *constructible*† if it is a finite union of subsets of the form $U \cap (X \setminus V)$, where U and V are open and retrocompact†† in X. A subset T of X is *locally constructible* if for any point $t \in T$, there exists an open neighborhood V of t in X such that $T \cap V$ is constructible in V (see [29, Chapter 0, 9.1.3, 9.1.11]). When X is a quasi-compact and quasi-separated scheme††† (e.g., if X is noetherian, or affine), then any quasi-compact open subset is retrocompact and any locally constructible subset is constructible (see [29, IV.1.8.1]).

When T is a subset of a topological space X, we endow T with the induced topology, and we define the *dimension of* T to be the Krull dimension of the topological space T. As usual, $\dim T < 0$ if and only if $T = \emptyset$. Let $\pi : X \to S$ be a morphism, and let $T \subseteq X$ be any subset. For any $s \in S$, we will denote by T_s the subset $\pi^{-1}(s) \cap T$.

1.1

Recall that a *pro-constructible* subset in a *noetherian* scheme X is an (possibly infinite) intersection of constructible subsets of X (see [29, IV.1.9.4]). Clearly, the constructible subsets of X are pro-constructible in X, and so are the finite subsets of X (see [29, IV.1.9.6]), and the constructible subsets of any fiber of a morphism of schemes $X \to Y$ (see [29, IV.1.9.5(vi)]). The complement in X of a pro-constructible subset is called an *ind-constructible* of X. Equivalently, an ind-constructible subset of X is any union of constructible subsets of X.

† See [29, Chapter 0, 9.1.2]. Beware that in the second edition [30, Chapter 0, 2.3], a *globally constructible* subset now refers to what is called a *constructible* subset in [29].

†† A topological space X is *quasi-compact* if every open covering of X has a finite refinement. A continuous map $f : X \to Y$ is *quasi-compact* if the inverse image $f^{-1}(V)$ of every quasi-compact open V of Y is quasi-compact. A subset Z of X is *retrocompact* if the inclusion map $Z \to X$ is quasi-compact.

††† X is *quasi-separated* if and only if the intersection of any two quasi-compact open subsets of X is quasi-compact (see [29, IV.1.2.7]).

LEMMA 1.2

(a) *Let X/k be a scheme of finite type over a field k, and let $T \subseteq X$ be a constructible subset, with closure $\overline{T}$ in X. Then* $\dim T = \dim \overline{T}$.

(b) *Let X/k be as in (a). Let k'/k be a finite extension, and denote by $T_{k'}$ the preimage of T under the map $X \times_k k' \to X$. Then* $\dim T_{k'} = \dim T$.

(c) *Let Y be any noetherian scheme, and let $\pi : X \to Y$ be a morphism of finite type. Let $T \subseteq X$ be constructible. Assume that for each $y \in Y$,* $\dim T_y \leq d$. *Then* $\dim T \leq \dim Y + d$.

(d) *Suppose that X is a noetherian scheme. Let T be a pro-constructible subset of X. Then T has finitely many irreducible components and each of them has a generic point.*

Proof

(a)–(b) Let Γ be an irreducible component of $\overline{T}$ of dimension $\dim \overline{T}$. As T is dense in $\overline{T}$, $T \cap \Gamma$ is dense in Γ. As $T \cap \Gamma$ is constructible and dense in Γ, it contains a dense open subset U of Γ. Therefore, because Γ is integral of finite type over k, $\dim \Gamma = \dim U \leq \dim T \leq \dim \overline{T} = \dim \Gamma$ and $\dim T = \dim \overline{T}$. This proves (a).

We also have

$$\dim \Gamma = \dim \Gamma_{k'} = \dim U_{k'} \leq \dim T_{k'} \leq \dim \overline{T}_{k'} = \dim \overline{T} = \dim \Gamma.$$

This proves (b).

(c) Let $\{\Gamma_i\}_i$ be the irreducible components of T. They are closed in T, and thus constructible in X. As $\dim T = \max_i\{\dim \Gamma_i\}$ and the fibers of $\Gamma_i \to Y$ all have dimension bounded by d, it is enough to prove the statement when T itself is irreducible. Replacing X with the Zariski closure of T in X with reduced scheme structure, we can suppose that X is integral and T is dense in X. Let ξ be the generic point of X, and let $\eta = \pi(\xi)$. As T is constructible and dense in X, it contains a dense open subset U of X. Then U_η is dense in X_η. Hence $\dim X_\eta = \dim U_\eta \leq \dim T_\eta \leq d$. Therefore

$$\dim T \leq \dim X \leq \dim \overline{\pi(X)} + d \leq \dim Y + d,$$

where the middle inequality is given by [29, IV.5.6.5].

(d) The subspace T of X is noetherian and, hence, it has finitely many irreducible components (see [7, II, Section 4.2, Proposition 8(i), Proposition 10]). Let Γ be an irreducible component of T. Let $\overline{\Gamma}$ be its closure in X. Since $\overline{\Gamma}$ is also irreducible, it has a generic point $\xi \in X$. We claim that $\xi \in \Gamma$, so that ξ is also the generic point of Γ. Indeed, suppose that Γ is contained in a constructible $W := \bigcup_{i=1}^m U_i \cap F_i$, with U_i open and F_i closed in X, and such that $(U_i \cap F_i) \cap \Gamma \neq \emptyset$ for all $i = 1, \ldots, m$. Then there exists j such that $\overline{\Gamma} \subset F_j$. Since U_j contains an element of Γ by hypothesis, we find that it must also contain ξ, so that $\xi \in W$. The subset Γ is pro-constructible in X since it is closed in the pro-constructible T. Hence, by definition, Γ is the intersection of constructible subsets, which all contain ξ. Hence, $\xi \in \Gamma$. $\square$

1.3

Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module, such as a finitely presented $\mathcal{O}_X$-module (see [29, Chapter 0, (5.2.5)]). Fix a section $f \in H^0(X, \mathcal{F})$. For $x \in X$, denote by $f(x)$ the canonical image of f in the fiber $\mathcal{F}(x) := \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. We say that f *vanishes at x if* $f(x) = 0$ (in $\mathcal{F}(x)$). Define

$$Z(\mathcal{F}, f) := \{x \in X \mid f(x) = 0\}$$

to be the *zero-locus of f*.

Let $q : X' \to X$ be any morphism of schemes. Let $\mathcal{F}' := q^*\mathcal{F}$, and let $f' \in H^0(X', \mathcal{F}')$ be the canonical image of f. Then

$$Z(\mathcal{F}', f') = q^{-1}\big(Z(\mathcal{F}, f)\big).$$

Indeed, for any $x' \in q^{-1}(x)$, the natural morphism $\mathcal{F}(x) \to \mathcal{F}'(x') = \mathcal{F}(x) \otimes_{k(x)} k(x')$ is injective.

When $\mathcal{F}$ is invertible or, more generally, locally free, then $Z(\mathcal{F}, f)$ is closed in X. As our next lemma shows, $Z(\mathcal{F}, f)$ in general is locally constructible. When X is noetherian, this is proved for instance in [57, Proposition 5.3]. We give here a different proof.

LEMMA 1.4

Let X be a scheme, and let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Then the set $Z(\mathcal{F}, f)$ is locally constructible in X.

Proof

Since the statement is local on X, it suffices to prove the lemma when $X = \operatorname{Spec} A$ is affine. We can use the stratification $X = \bigcup_{1 \le i \le n} X_i$ of X described in [29, IV.8.9.5]: each X_i is a quasi-compact subscheme of X, and $\mathcal{F}_i := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i}$ is flat on X_i. Let f_i be the canonical image of f in $H^0(X_i, \mathcal{F}_i)$. Then

$$Z(\mathcal{F}, f) = \bigcup_{1 \le i \le n} Z(\mathcal{F}_i, f_i).$$

Since $\mathcal{F}_i$ is finitely presented and flat, it is projective (see [44, 1.4]) and, hence, locally free. So $Z(\mathcal{F}_i, f_i)$ is closed in X_i. $\square$

1.5
Let $\pi : X \to Y$ be a finitely presented[†] morphism of schemes. Let T be a locally constructible subset of X. Set

$$T_\pi := \{y \in Y \mid T_y \text{ contains a generic point of } X_y\},$$

where a *generic point* of a scheme X is the generic point of an irreducible component of X. Such a point is called a *maximal point* in [29], just before IV.1.1.5. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module, and fix a global section $f \in \mathcal{F}(X)$. Set

$$T_{\mathcal{F},f,\pi} := \{y \in Y \mid f \text{ vanishes at a generic point of } X_y\}.$$

For future use, let us note the following equivalent expression for $T_{\mathcal{F},f,\pi}$. For any $y \in Y$, let

$$\mathcal{F}_y := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_y} = \mathcal{F} \otimes_{\mathcal{O}_Y} k(y),$$

and let f_y be the canonical image of f in $H^0(X_y, \mathcal{F}_y)$. Let $x \in X_y$. Since the canonical map $\mathcal{F}(x) \to \mathcal{F}_y(x)$ of fibers at x is an isomorphism, f_y vanishes at x if and only if f vanishes at x. Thus

$$T_{\mathcal{F},f,\pi} = \{y \in Y \mid f_y \text{ vanishes at a generic point of } X_y\}.$$

When the morphism π is understood, we may denote $T_{\mathcal{F},f,\pi}$ simply by $T_{\mathcal{F},f}$. Note that when $\pi = \mathrm{id} : X \to X$, the set $T_{\mathcal{F},f,\mathrm{id}}$ is equal to the zero locus $Z(\mathcal{F}, f)$ of f introduced in Section 1.3.

PROPOSITION 1.6
Let $\pi : X \to Y$ be a finitely presented morphism of schemes. Let T be a locally constructible subset of X. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module, and fix a section $f \in H^0(X, \mathcal{F})$. Then the subsets T_π and $T_{\mathcal{F},f,\pi}$ are both locally constructible in Y.

Proof
Let us start by showing that T_π is locally constructible in Y. By definition of *locally constructible*, the statement is local on Y, and it suffices to prove the statement when Y is affine. Assume then that from now on Y is affine. Since π is quasi-compact and quasi-separated and Y is affine, X is also quasi-compact and quasi-separated (see [29, IV.1.2.6]). Hence, T is constructible, and we can write it as a finite union of locally closed subsets $T_i := U_i \cap (X \setminus V_i)$ with U_i and V_i open and retrocompact. Then T_y contains a generic point of X_y if and only if $(T_i)_y$ contains a generic point of X_y for some i. Therefore, it suffices to prove the statement when $T = U \cap (X \setminus V)$ with U and V open and retrocompact.

[†]A morphism $\pi : X \to Y$ is *finitely presented* (or *of finite presentation*) if it is locally of finite presentation, quasi-compact, and quasi-separated (see [29, IV.1.6.1]).

We therefore assume now that $T = U \cap Z$, with U and $X \setminus Z$ open and retrocompact. Fix $y \in Y$. We claim that T_y *contains a generic point of* X_y *if and only if there exists* $x \in T_y$ *such that* $\operatorname{codim}_x(Z_y, X_y) = 0$.

To justify this claim, let us recall the following. Let $\Gamma_1, \ldots, \Gamma_n$ be the irreducible components of Z_y passing through x (X_y is noetherian). Then

$$\operatorname{codim}_x(Z_y, X_y) = \min_{1 \le i \le n} \{\operatorname{codim}(\Gamma_i, X_y)\}$$

(see [29, 0.14.2.6(i)]). So $\operatorname{codim}_x(Z_y, X_y) = 0$ if and only if Z_y contains an irreducible component of X_y passing through x. Now, if T_y contains a generic point ξ of X_y, then Z_y contains the irreducible component $\overline{\{\xi\}} \ni \xi$ of X_y, and $\operatorname{codim}_\xi(Z_y, X_y) = 0$. Conversely, if $\operatorname{codim}_x(Z_y, X_y) = 0$ for some $x \in T_y$, then Z_y contains an irreducible component Γ of X_y passing through x. As T_y is open in Z_y, $T_y \cap \Gamma$ is open in Γ and nonempty, so T_y contains the generic point of Γ.

Since Z is closed, we can apply [29, IV.9.9.1(ii)] and find that the set

$$X_0 := \{x \in X \mid \operatorname{codim}_x(Z_{\pi(x)}, X_{\pi(x)}) = 0\}$$

is locally constructible in X. It is easy to check that

$$T_\pi = \pi(T \cap X_0).$$

Since $T \cap X_0$ is locally constructible, it follows then from Chevalley's theorem [29, IV.1.8.4] that T_π is locally constructible in Y.

Let us now show that $T_{\mathcal{F},f,\pi}$ is locally constructible in Y. Set T to be the zero-locus $Z(\mathcal{F}, f)$ of f in X, which is locally constructible in X by Lemma 1.4. Then $T_{\mathcal{F},f,\pi}$ is nothing but the associated subset T_π which was shown to be locally constructible in Y in the first part of the proposition. $\square$

The formation of $T_{\mathcal{F},f,\pi}$ is compatible with base changes $Y' \to Y$, as our next lemma shows.

LEMMA 1.7
Let $\pi : X \to Y$ *be a finitely presented morphism of schemes. Let* $q : Y' \to Y$ *be any morphism of schemes. Let* $X' := X \times_Y Y'$ *and* $\pi' : X' \to Y'$. *Let* $\mathcal{F}$ *be a finitely presented* $\mathcal{O}_X$*-module, and fix a section* $f \in H^0(X, \mathcal{F})$. *Let* $\mathcal{F}' := \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}$ *and let* f' *be the image of* f *in* $H^0(X', \mathcal{F}')$. *Then* $T_{\mathcal{F}',f',\pi'} = q^{-1}(T_{\mathcal{F},f,\pi})$.

Proof
For any $y' \in Y'$, we have a natural $k(y')$-isomorphism $X'_{y'} \to (X_y)_{k(y')}$. Any generic point ξ' of $X'_{y'}$ maps to a generic point ξ of X_y, and any generic point of X_y is the

image of a generic point of $X'_{y'}$. Moreover, $f'(\xi')$ is identified with the image of $f(\xi)$ under the natural injection $\mathcal{F}_y(\xi) \to \mathcal{F}'_{y'}(\xi') = \mathcal{F}_y(\xi) \otimes k(\xi')$. $\square$

1.8
Let $X \to Y$ be a finitely presented morphism of schemes, and let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $N \geq 1$, and let $f_1, \dots, f_N \in H^0(X, \mathcal{F})$. For each $y \in Y$, define

$$\Sigma(y) := \Big\{(\alpha_1, \dots, \alpha_N) \in k(y)^N \;\Big|\; \sum_i \alpha_i f_{i,y} \text{ vanishes at some generic point of } X_y\Big\}.$$

When $X_y = \emptyset$, we set $\Sigma(y) := \emptyset$. The subset $\Sigma(y)$ depends on the data $X \to Y$, $\mathcal{F}$, and $\{f_1, \dots, f_N\}$.

Example 1.9
Consider the special case in Section 1.8 where $Y = \operatorname{Spec} k = \{y\}$, with k a field. For each generic point ξ of $X = X_y$, consider the k-linear map

$$k^N \to \mathcal{F} \otimes k(\xi), \qquad (\alpha_1, \dots, \alpha_N) \mapsto \sum \alpha_i f_i(\xi).$$

The kernel $K(\xi)$ of this map is a linear subspace of k^N and, hence, can be defined by a system of homogeneous polynomials of degree 1. The same equations define a closed subscheme $T(\xi)$ of $\mathbb{A}^N_y$. Then the set $\Sigma(y)$ is the union of the sets $K(\xi)$, where the union is taken over all the generic points of X, and $\Sigma(y)$ is the subset of $k(y)$-rational points of the closed scheme $T := \bigcup_\xi T(\xi)$ of $\mathbb{A}^N_Y$. This latter statement is generalized to any base Y in our next proposition.

PROPOSITION 1.10
Let $X \to Y$, $\mathcal{F}$, and $\{f_1, \dots, f_N\} \subset H^0(X, \mathcal{F})$ be as in Section 1.8. Then there exists a locally constructible subset T of $\mathbb{A}^N_Y$ such that for all $y \in Y$, the set of $k(y)$-rational points of $\mathbb{A}^N_{k(y)}$ contained in T_y is equal to $\Sigma(y)$. Moreover:

(a) *The set T satisfies the following natural compatibility with respect to base change. Let $Y' \to Y$ be any morphism of schemes, and denote by $q : \mathbb{A}^N_{Y'} \to \mathbb{A}^N_Y$ the associated morphism. Let $X' := X \times_Y Y' \to Y'$, and let $\mathcal{F}' := \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}$. Let $f'_1, \dots, f'_N$ be the images of $f_1, \dots, f_N$ in $H^0(X', \mathcal{F}')$. Then the constructible set T' associated with the data $X' \to Y'$, $\mathcal{F}'$, and $f'_1, \dots, f'_N$ is equal to $q^{-1}(T)$. In particular, for all $y' \in Y'$, the set of $k(y')$-rational points of $\mathbb{A}^N_{k(y')}$ contained in $(q^{-1}(T))_{y'}$ is equal to the set $\Sigma(y')$ associated with $f'_1, \dots, f'_N \in H^0(X', \mathcal{F}')$.*

(b) *We have $\dim T \leq \dim Y + \sup_{y \in Y} \dim T_y$ when Y is noetherian. In general for each $y \in Y$, $\dim T_y$ is the maximum of the dimension over $k(y)$ of the* *kernels of the $k(y)$-linear maps*

$$k(y)^N \to \mathcal{F} \otimes k(\xi), \qquad (\alpha_1, \dots, \alpha_N) \mapsto \sum_i \alpha_i f_i(\xi),$$

for each generic point ξ of X_y.

Proof

Let $\pi : \mathbb{A}^N_X \to \mathbb{A}^N_Y$ be the finitely presented morphism induced by the given morphism $X \to Y$. Let $p : \mathbb{A}^N_X \to X$ be the natural projection, and consider the finitely presented sheaf $p^*\mathcal{F}$ on $\mathbb{A}^N_X$ induced by $\mathcal{F}$.

Write $\mathbb{A}^N_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[u_1, \dots, u_N]$, and identify $H^0(\mathbb{A}^N_X, p^*\mathcal{F})$ with $H^0(X, \mathcal{F}) \otimes_{\mathbb{Z}} \mathbb{Z}[u_1, \dots, u_N]$. Using this identification, let $f \in H^0(\mathbb{A}^N_X, p^*\mathcal{F})$ denote the section corresponding to $\sum_{1 \le i \le N} f_i \otimes u_i$. Apply now Proposition 1.6 to the data $\pi : \mathbb{A}^N_X \to \mathbb{A}^N_Y$, $p^*\mathcal{F}$, and f to obtain the locally constructible subset $T := T_{p^*\mathcal{F}, f}$ of $\mathbb{A}^N_Y$.

Fix $y \in Y$, and let z be a $k(y)$-rational point in $\mathbb{A}^N_Y$ above y. We may write $z = (\alpha_1, \dots, \alpha_N) \in k(y)^N$. The fiber of π above z is isomorphic to X_y, and the section $f_z \in (p^*\mathcal{F})_z$ is identified with $\sum_i \alpha_i f_i \in H^0(X_y, \mathcal{F}_y)$. Therefore, it follows from the definitions that $z \in T_y$ if and only if $z \in \Sigma(y)$, and the first part of the proposition is proved.

(a) The compatibility of T with respect to a base change $Y' \to Y$ results from Lemma 1.7.

(b) The inequality on the dimensions follows from Lemma 1.2(c). By the compatibility described in (a), we are immediately reduced to the case $Y = \operatorname{Spec} k$ for a field k, which is discussed in Example 1.9. □

2. Sections in an affine space avoiding pro-constructible subsets

The following theorem is an essential part of our method for producing interesting closed subschemes of a scheme X when $X \to S$ is projective and S is affine.

THEOREM 2.1

Let $S = \operatorname{Spec} R$ be a noetherian affine scheme. Let $T := T_1 \cup \cdots \cup T_m$ be a finite union of pro-constructible subsets of $\mathbb{A}^N_S$. Suppose that

(1) *there exists an open subset $V \subseteq S$ with zero-dimensional complement such that for all $i \le m$, $\dim(T_i \cap \mathbb{A}^N_V) < N$ and $(T_i)_s$ is constructible in $\mathbb{A}^N_{k(s)}$ for all $s \in V$;*

(2) *for all $s \in S$, there exists a $k(s)$-rational point in $\mathbb{A}^N_{k(s)}$ which does not belong to T_s.*

Then there exists a section σ of $\pi : \mathbb{A}^N_S \to S$ such that $\sigma(S) \cap T = \emptyset$.

Proof
We proceed by induction on N, using Claims (A) and (B) below.

CLAIM (A)
There exists $\delta \geq 1$ such that for all $s \in V$, T_s is contained in a hypersurface in $\mathbb{A}^N_{k(s)}$ of degree at most δ.

Proof
It is enough to prove the claim for each T_i. So to lighten the notation we set $T := T_i$ in this proof. Thus, by hypothesis, $\dim T \cap \mathbb{A}^N_V < N$. We start by proving that for each $s \in V$, there exist a positive integer δ_s and an ind-constructible subset (see Section 1.1) W_s of V containing s, such that for each $s' \in W_s$, $T_{s'}$ is contained in a hypersurface of degree δ_s in $\mathbb{A}^N_{k(s')}$. Indeed, let $s \in V$. As $\dim T_s \leq \dim T \cap \mathbb{A}^N_V < N$, and T_s is constructible, T_s is not dense in $\mathbb{A}^N_{k(s)}$ (see Lemma 1.2(a)). Thus, there exists some polynomial f_s of degree $\delta_s > 0$ whose zero-locus contains T_s. Hence, for some affine open neighborhood V_s of s, we can find a polynomial $f \in \mathcal{O}_V(V_s)[t_1, \ldots, t_N]$ of degree δ_s, lifting f_s and defining a closed subscheme $V(f)$ of $\mathbb{A}^N_{V_s}$.

Let $W_1 := \pi(\mathbb{A}^N_{V_s} \setminus V(f))$, which is constructible in V by Chevalley's theorem. Let W_2 be the complement in V of $\pi(T \cap (\mathbb{A}^N_{V_s} \setminus V(f)))$, which is ind-constructible in V since $\pi(T \cap (\mathbb{A}^N_{V_s} \setminus V(f)))$ is pro-constructible in V (see [29, IV.1.9.5(vii)]). Hence, both W_1 and W_2 are ind-constructible and contain s. The intersection $W_s := W_1 \cap W_2$ is the desired ind-constructible subset containing s. Since V is quasi-compact because it is noetherian, and since each W_s is ind-constructible, it follows from [29, IV.1.9.9] that there exist finitely many points $s_1, \ldots, s_n$ of V such that $V = W_{s_1} \cup \cdots \cup W_{s_n}$. We can take $\delta := \max_i \{\delta_{s_i}\}$, and Claim (A) is proved. □

A proof of the following lemma in the affine case is given in [64, Proposition 13]. We provide here an alternate proof.

LEMMA 2.2
Let S be any scheme. Let $c \in \mathbb{N}$. Then the subset $\{s \in S \mid \operatorname{Card}(k(s)) \leq c\}$ is closed in S and has dimension at most 0. When S is noetherian, this subset is then finite.

Proof
It is enough to prove that when S is a scheme over a finite prime field $\mathbb{F}_p$, and q is a power of p, the set $\{s \in S \mid \operatorname{Card}(k(s)) = q\}$ is closed of dimension ≤ 0.

Let $\mathbb{F}_q$ be a field with q elements. Then any point $s \in S$ with $\operatorname{Card}(k(s)) = q$ is the image by the projection $S_{\mathbb{F}_q} \to S$ of a rational point of $S_{\mathbb{F}_q}$. Therefore we can suppose that S is a $\mathbb{F}_q$-scheme, and we have to show that $S(\mathbb{F}_q)$ is closed of dimension 0. Let Z be the Zariski closure of $S(\mathbb{F}_q)$ in S, endowed with the reduced structure. Let U be an affine open subset of Z. Let $f \in \mathcal{O}_Z(U)$. For any $x \in U(\mathbb{F}_q)$, $(f^q - f)(x) = 0$ in $k(x)$; hence $x \in V(f^q - f)$. As $U(\mathbb{F}_q)$ is dense in U and U is reduced, we have $f^q - f = 0$. For any irreducible component Γ of U, this identity then holds on $\mathcal{O}(\Gamma)$, so Γ is just a rational point. Hence $U = U(\mathbb{F}_q)$ and $\dim U = 0$. Consequently, $Z = S(\mathbb{F}_q)$ is closed and has dimension 0. $\square$

The key to the proof of Theorem 2.1 is the following assertion.

CLAIM (B)
Suppose that $N \geq 1$. Then there exist $t := t_1 + a_1 \in R[t_1, \dots, t_N]$ with $a_1 \in R$ and an open subset $U \subseteq S$ with zero-dimensional complement, such that $H := V(t)$ is S-isomorphic to $\mathbb{A}_S^{N-1}$ and the pro-constructible subsets $T_1 \cap H, \dots, T_m \cap H$ of H satisfy the following:

(i) *For all $i \leq m$, $\dim(T_i \cap H_U) < N - 1$, and $(T_i \cap H)_s$ is constructible in H_s for all $s \in U$.*

(ii) *For all $s \in S$, there exists a $k(s)$-rational point in H_s which does not belong to $T_s \cap H_s$.*

Using Claim (B), we conclude the proof of Theorem 2.1 as follows. First, note that when $N = 0$, condition Theorem 2.1(2) implies that $T = \emptyset$, and the theorem trivially holds true. When $N \geq 1$, we apply Claim (B) repeatedly to obtain a sequence of closed sets

$$\mathbb{A}_S^N \supset V(t_1 + a_1) \supset \cdots \supset V(t_1 + a_1, t_2 + a_2, \dots, t_N + a_N).$$

The latter set is the image of the desired section, as we saw in the case $N = 0$.

Proof of Claim (B)
Let $\{\xi_1, \dots, \xi_\rho\}$ be the set of generic points of all the irreducible components of the pro-constructible sets $T_i \cap \mathbb{A}_V^N$, $i = 1, \dots, m$ (see Lemma 1.2(d) for the existence of generic points). Upon renumbering these points if necessary, we can assume that for some $r \leq \rho$, the image of ξ_i under $\pi : \mathbb{A}_S^N \to S$ has finite residue field if and only if $i > r$. Let $\delta > 0$. Let Z be the union of $S \setminus V$ with $\{\pi(\xi_{r+1}), \dots, \pi(\xi_\rho)\}$ and with the finite subset of the closed points s of S satisfying $\mathrm{Card}(k(s)) \leq \delta$ (that this set is finite follows from Lemma 2.2). We will later set δ appropriately to be able to use Claim (A). For each $s \in Z$, we can use Theorem 2.1(2), and fix a $k(s)$-rational point $x_s \in \mathbb{A}_{k(s)}^N \setminus T_s$.

We now construct a closed subset $V(t) \subset \mathbb{A}_S^N$ which contains x_s for all $s \in Z$ and does not contain any ξ_i with $i \leq r$. Since every point of Z is closed in S, the Chinese remainder theorem implies that the canonical map $R \to \prod_{s \in Z} k(s)$ is surjective. Let $a \in R$ be such that $a \equiv t_1(x_s)$ in $k(s)$, for all $s \in Z$. Replacing t_1 by $t_1 - a$, we can assume that $t_1(x_s) = 0$ for all $s \in Z$. Let $\mathfrak{p}_j \subset R[t_1, \dots, t_N]$ be the prime ideal corresponding to ξ_j. Let $\mathfrak{m}_s \subset R$ denote the maximal ideal of R corresponding to $s \in Z$. Let $I := \bigcap_{s \in Z} \mathfrak{m}_s$, and in case $Z = \emptyset$, we let $I := R$. For $t \in R[t_1, \dots, t_N]$, let $I + t := \{a + t \mid a \in I\}$. *We claim that*

$$I + t_1 \not\subseteq \bigcup_{1 \le j \le r} \mathfrak{p}_j.$$

Indeed, the intersection $(I + t_1) \cap \mathfrak{p}_j$ is either empty or contains $a_j + t_1$ for some $a_j \in I$. In the latter case, $(I + t_1) \cap \mathfrak{p}_j = t_1 + a_j + (\mathfrak{p}_j \cap I)$. If $I + t_1 \subseteq \bigcup_{1 \le j \le r} \mathfrak{p}_j$, then every $t_1 + a$ with $a \in I$ belongs to some $t_1 + a_j + (\mathfrak{p}_j \cap I)$. Let $\mathfrak{q}_j := R \cap \mathfrak{p}_j$. It follows that

$$I \subseteq \bigcup_j (a_j + \mathfrak{q}_j),$$

where the union runs over a subset of $\{1, \dots, r\}$. Since the domains $R/\mathfrak{q}_j$ are all infinite when $j \le r$, Lemma 2.3 below implies that I is contained in some $\mathfrak{q}_{j_0}$ for $1 \le j_0 \le r$. As $I = \bigcap_{s \in Z} \mathfrak{m}_s$, we find that $\mathfrak{q}_{j_0} = \mathfrak{m}_s$ for some $s \in Z$. This is a contradiction, since for $j \le r$, $\pi(\xi_j)$ does not belong to Z because the residue field of $\pi(\xi_j)$ is infinite and $\pi(\xi_j) \notin S \setminus V$. This proves our claim.

Now that the claim is proven, we can choose $t \in (I + t_1) \setminus \bigcup_{1 \le j \le r} \mathfrak{p}_j$. Clearly, the closed subset $H := V(t) \subset \mathbb{A}^N_S$ does not contain any ξ_i with $i \le r$. Since t has the form $t = t_1 + a_1$ for some $a_1 \in I = \bigcap_{s \in Z} \mathfrak{m}_s$, we find that $V(t)$ contains x_s for all $s \in Z$. Let $U := S \setminus Z$. The complement of U in S is a finite set of closed points of S. It is clear that $H := V(t)$ is S-isomorphic to $\mathbb{A}^{N-1}_S$ and that for each i, the fibers of $T_i \cap H \to S$ are constructible.

Let us now prove (i), that is, that $\dim(T_i \cap H_U) < N - 1$ for all $i \le m$. Let Γ be an irreducible component of some $T_i \cap \mathbb{A}^N_V$, with generic point ξ_j for some j. If $j > r$, then $\pi(\xi_j) \in Z$ and $\Gamma \cap \mathbb{A}^N_U = \emptyset$. Suppose now that $j \le r$. Then $\Gamma \cap \mathbb{A}^N_U$ is nonempty and open in Γ, and hence irreducible. By construction, H does not contain ξ_j since $j \le r$. So $\Gamma \cap H_U$ is a proper closed subset of the irreducible space $\Gamma \cap \mathbb{A}^N_U$. Thus

$$\dim(\Gamma \cap H_U) < \dim(\Gamma \cap \mathbb{A}^N_U) \le \dim \Gamma < N.$$

As $T_i \cap H_U$ is the finite union of its various closed subsets $\Gamma \cap H_U$, this implies that $\dim(T_i \cap H_U) < N - 1$.

Let us now prove (ii), that is, that for all $s \in S$, H_s contains a $k(s)$-rational point that does not belong to T_s. When $s \in Z$, H contains the $k(s)$-rational point x_s and this point does not belong to T_s. Let now $s \notin Z$. Then $|k(s)| \ge \delta + 1$ by construction. Choose now δ so that the conclusion of Claim (A) holds: for all $s \in V$, T_s is contained in a hypersurface in $\mathbb{A}^N_{k(s)}$ of degree at most δ. Then, since t has degree 1, we find that $H_s \cap T$ is contained in a hypersurface $V(f)$ of H_s with $\deg(f) \leq \delta$.

We conclude that H_s contains a $k(s)$-rational point that does not belong to T_s by using the following claim. *Assume that k is either an infinite field or that $|k| = q \geq \delta + 1$. Let $f \in k[T_1, \ldots, T_\ell]$ with* $\deg(f) \leq \delta$, $f \neq 0$. *Then* $V(f)(k) \subsetneq \mathbb{A}^\ell(k)$. When k is a finite field, we use the bound $|V(f)(k)| \leq \delta q^{\ell-1} + (q^{\ell-1} - 1)/(q-1) < q^\ell$ found in [68]. When k is infinite, we can use induction on ℓ to prove the claim. □

Our next lemma follows from [47, Theorem 5]. We provide here a more direct proof by using the earlier reference [55].

LEMMA 2.3

Let R be a commutative ring, and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be (not necessarily distinct) prime ideals of R with infinite quotients $R/\mathfrak{q}_i$ for all $i = 1, \ldots, r$. Let I be an ideal of R, and suppose that there exist $a_1, \ldots, a_r \in R$ such that

$$I \subseteq \bigcup_{1 \leq i \leq r} (a_i + \mathfrak{q}_i).$$

Then I is contained in the union of those $a_i + \mathfrak{q}_i$ with $I \subseteq \mathfrak{q}_i$. In particular, I is contained in at least one $\mathfrak{q}_i$.

Proof

We have $I = \bigcup_i ((a_i + \mathfrak{q}_i) \cap I)$. If $(a_i + \mathfrak{q}_i) \cap I \neq \emptyset$, then it is equal to $\alpha_i + (\mathfrak{q}_i \cap I)$ for some $\alpha_i \in I$. Hence

$$I = \bigcup_i \big(\alpha_i + (\mathfrak{q}_i \cap I)\big),$$

where the union runs on part of $\{1, \ldots, r\}$. By [55, 4.4], I is the union of those $\alpha_i + (\mathfrak{q}_i \cap I)$ with $I/(\mathfrak{q}_i \cap I)$ finite. For any such i, the ideal $(I + \mathfrak{q}_i)/\mathfrak{q}_i$ of $R/\mathfrak{q}_i$ is finite and, hence, equal to (0) because $R/\mathfrak{q}_i$ is an infinite domain. □

Remark 2.4

One can show that the conclusion of Theorem 2.1 holds without assuming in Theorem 2.1(1) that T_s is constructible in $\mathbb{A}^N_{k(s)}$ for all $s \in V$. Since we will not need this statement, let us only note that *when S has only finitely many points with finite residue field, then the conclusion of Theorem 2.1 holds if in Theorem 2.1(1) we remove the hypothesis that T_s is constructible for all $s \in V$.* Indeed, with this hypothesis, we do not need to use Claim (A). First shrink V so that $k(s)$ is infinite for all $s \in V$, and then proceed to construct the closed subset $H = V(t)$ discussed in Claim (B). The use of Claim (A) in the proof of Claim (B)(ii) can be avoided by using our next lemma.

Let k be an infinite field, and let $V \subset \mathbb{A}_k^N$ be a closed subset. The property that *if* $\dim(V) < N$, *then* $V(k) \neq \mathbb{A}_k^N(k)$ can be generalized as follows.

LEMMA 2.5
Let k be an infinite field. Let $T \subset \mathbb{A}_k^N$ be a pro-constructible subset with $\dim(T) < N$. *Then T does not contain all k-rational points of $\mathbb{A}_k^N$.*

Proof
Assume that T contains all k-rational points of $\mathbb{A}_k^N$. We claim first that T is irreducible. Indeed, if $T = (V(f) \cap T) \cup (V(g) \cap T)$, then $V(fg) = V(f) \cup V(g)$ contains all k-rational points of $\mathbb{A}_k^N$. Thus $V(fg) = \mathbb{A}_k^N$ and either $V(f) \cap T = T$ or $V(g) \cap T = T$. Since T is irreducible, it has a generic point ξ (see Lemma 1.2(d)), and the closure F of ξ in $\mathbb{A}_k^N$ contains all k-rational points of $\mathbb{A}_k^N$. Hence, $F = \mathbb{A}_k^N$, so that T then contains the generic point of $\mathbb{A}_k^N$. Consider now an increasing sequence of closed linear subspaces $F_0 \subset F_1 \subset \cdots \subset F_N$ contained in $\mathbb{A}_k^N$, with $F_i \cong \mathbb{A}_k^i$. Then $T \cap F_i$ contains all k-rational points of F_i by hypothesis, and the discussion above shows that it then contains the generic point of F_i. It follows that $\dim(T) = N$. $\square$

Remark 2.6
The hypothesis in Theorem 2.1(1) on the dimension of T is needed. Indeed, let $S = \operatorname{Spec} \mathbb{Z}$, and let $N = 1$. Consider the closed subset $V(t^3 - t)$ of $\operatorname{Spec} \mathbb{Z}[t] = \mathbb{A}_{\mathbb{Z}}^1$. Let T be the constructible subset of $\mathbb{A}_{\mathbb{Z}}^1$ obtained by removing from $V(t^3 - t)$ the maximal ideals $(2, t-1)$ and $(3, t-1)$. Then, for all $s \in S$, the fiber T_s is distinct from $\mathbb{A}_{k(s)}^1(k(s))$, and $\dim T_s = 0$. However, $\dim T = 1$, and we note now that there exists no section of $\mathbb{A}_{\mathbb{Z}}^1$ disjoint from T. Indeed, let $V(t-a)$ be a section. If it is disjoint from T, then $a \neq 0, 1, -1$, and $6 \mid a - 1$. So there exists a prime $p > 3$ with $p \mid a$, and $V(t-a)$ meets T at the point (p, t).

For a more geometric example, let k be any infinite field. Let $S = \operatorname{Spec} k[u]$ and $\mathbb{A}_S^1 = \operatorname{Spec} k[u, t]$. When $T := V(t^2 - u) \subset \mathbb{A}_S^1$, then $\mathbb{A}_S^1 \setminus T$ does not contain any section $V(t - g(u))$ of $\mathbb{A}_S^1$. Indeed, otherwise $(t^2 - u, t - g(u)) = (1)$, and $g(u)^2 - u$ would be an element of k^*.

3. Existence of hypersurfaces

Let us start by introducing the terminology needed to state the main results of this section.

3.1
Let X be any scheme. A global section f of an invertible sheaf $\mathcal{L}$ on X defines a closed subset H_f of X, consisting of all points $x \in X$ where the stalk f_x does not generate $\mathcal{L}_x$. Since $\mathcal{O}_X f \subseteq \mathcal{L}$, the ideal sheaf $\mathcal{I} := (\mathcal{O}_X f) \otimes \mathcal{L}^{-1}$ endows H_f with

the structure of a closed subscheme of X. When X is noetherian and $H_f \neq \emptyset$, it follows from Krull's principal ideal theorem that any irreducible component Γ of H_f has codimension at most 1 in X.

Assume now that $X \to \operatorname{Spec} R$ is a projective morphism, and write $X = \operatorname{Proj} A$, where A is the quotient of a polynomial ring $R[T_0, \dots, T_N]$ by a homogeneous ideal I. Let $\mathcal{O}_X(1)$ denote the very ample sheaf arising from this presentation of X. Let $f \in A$ be a homogeneous element of degree n. Then f can be identified with a global section $f \in H^0(X, \mathcal{O}_X(n))$, and H_f is the closed subscheme $V_+(f)$ of X defined by the homogeneous ideal fA. When $X \to S$ is a quasi-projective morphism and f is a global section of a very ample invertible sheaf $\mathcal{L}$ relative to $X \to S$, we may also sometimes denote the closed subset H_f of X by $V_+(f)$.

Let now S be any affine scheme and $X \to S$ any morphism. We call the closed subscheme H_f of X a *hypersurface* (relative to $X \to S$) when no irreducible component of *positive dimension* of X_s is contained in H_f, for all $s \in S$. If, moreover, the ideal sheaf $\mathcal{I}$ is invertible, we say that the hypersurface H_f is *locally principal*. Note that in this case, H_f is the support of an effective Cartier divisor on X. Hypersurfaces satisfy the following elementary properties.

LEMMA 3.2

Let S be affine. Let $X \to S$ be a finitely presented morphism. Let $\mathcal{L}$ be an invertible sheaf on X, and let $f \in H^0(X, \mathcal{L})$ be such that $H := H_f$ is a hypersurface on X relative to $X \to S$.

(1) *If* $\dim X_s \geq 1$, *then* $\dim H_s \leq \dim X_s - 1$. *If, moreover, $X \to S$ is projective, $\mathcal{L}$ is ample, and $H \neq \emptyset$, then H_s meets every irreducible component of positive dimension of X_s, and in particular* $\dim H_s = \dim X_s - 1$.

(2) *The morphism $H \to S$ is finitely presented.*

(3) *Assume that $X \to S$ is flat of finite presentation. Then H is locally principal and flat over S if and only if for all $s \in S$, H does not contain any associated point of X_s.*

(4) *Assume that S noetherian. If H does not contain any associated point of X, then H is locally principal.*

Proof

(1) Recall that by convention, if H_s is empty, then $\dim H_s < 0$, and the inequality is satisfied. Assume now that H_s is not empty. By hypothesis, H_s does not contain any irreducible component of X_s of positive dimension. Since H_s is locally defined by one equation, we obtain that $\dim H_s \leq \dim X_s - 1$. The strict inequality may occur for instance in case $\dim X_s \geq 2$, and H_s does not meet any component of X_s of maximal dimension.

Consider now the open set $X_f := X \setminus H$. Under our additional hypotheses, for any $s \in S$, $X_f \cap X_s = (X_s)_{f_s}$ is affine and, thus, can only contain irreducible components of dimension 0 of the projective scheme X_s.

(2) This results from the fact that H is locally defined by a single equation in X.

(3) See [29, IV.11.3.8], (c) $\Leftrightarrow$ (a). Each fiber X_s is noetherian. Use the fact that in a noetherian ring, an element is regular if and only if it is not contained in any associated prime.

(4) The property is local on X, so we can suppose that $X = \operatorname{Spec} A$ is affine and $\mathcal{L} = \mathcal{O}_X \cdot e$ is free. So $f = he$ for some $h \in A$. The hypothesis $H \cap \operatorname{Ass}(X) = \emptyset$ implies that h is a regular element of A. So the ideal sheaf $\mathcal{I} = (\mathcal{O}_X f) \otimes \mathcal{L}^{-1}$ is invertible. $\square$

We can now state the main results of this section.

THEOREM 3.3

Let S be an affine noetherian scheme of finite dimension, and let $X \to S$ be a quasi-projective morphism with a given very ample invertible sheaf $\mathcal{O}_X(1)$.

(i) *Let C be a closed subscheme of X.*

(ii) *Let $F_1, \ldots, F_m$ be locally closed subsets*† *of X such that for all $s \in S$ and for all $i \leq m$, C_s does not contain any irreducible component of positive dimension of $(F_i)_s$.*

(iii) *Let A be a finite subset of X such that $A \cap C = \emptyset$.*

Then there exists $n_0 > 0$ such that for all $n \geq n_0$, there exists a global section f of $\mathcal{O}_X(n)$ such that

(1) *C is a closed subscheme of H_f,*

(2) *for all $s \in S$ and for all $i \leq m$, H_f does not contain any irreducible component of positive dimension of $F_i \cap X_s$, and*

(3) *$H_f \cap A = \emptyset$. Moreover,*

(4) *if, for all $s \in S$, C does not contain any irreducible component of positive dimension of X_s, then there exists f as above such that H_f is a hypersurface relative to $X \to S$. If in addition $C \cap \operatorname{Ass}(X) = \emptyset$, then there exists f as above such that H_f is a locally principal hypersurface.*

We will first give a complete proof of Theorem 3.3 in the case where $X \to S$ is projective in Section 3.11, after a series of technical lemmas. The proof of Theo-

†Recall that a *locally closed* subset F of a topological space X is the intersection of an open subset U of X with a closed subset Z of X. When X is a scheme, we can endow F with the structure of a *subscheme* of X by considering U as an open subscheme of X and F as the closed subscheme $Z \cap U$ of U endowed with the reduced induced structure.

rem 3.3 when $X \to S$ is only assumed to be quasi-projective is given in Section 3.13. Theorem 3.3 will be generalized to the case where S is not noetherian in Theorem 5.1.

Theorem 3.4 below is the key to reducing the proof of Theorem 7.2(a) to the case of relative dimension 1. This theorem is stated in a slightly different form in the introduction, and we note in Lemma 3.5(3) that the two versions are compatible.

THEOREM 3.4

Let S be an affine noetherian scheme of finite dimension, and let $X \to S$ be a quasi-projective morphism with a given very ample invertible sheaf $\mathcal{O}_X(1)$ relative to $X \to S$. Assume that the hypotheses (i), (ii), and (iii) in Theorem 3.3 hold. Suppose further that

(a) *$C \to S$ is finite,*

(b) *$C \to X$ is a regular immersion and C has pure codimension*† *$d > \dim S$ in X, and*

(c) *for all $s \in S$,* $\operatorname{codim}(C_s, X_s) \geq d$.

Then there exists $n_0 > 0$ such that for all $n \geq n_0$ there exists a global section f of $\mathcal{O}_X(n)$ such that H_f satisfies (1), (2), and (3) in Theorem 3.3, and such that H_f is a locally principal hypersurface, $C \to H_f$ is a regular immersion, and C has pure codimension $d-1$ in H_f.

Suppose now that $\dim S = 1$. *Then there exists a closed subscheme Y of X such that C is a closed subscheme of Y defined by an invertible sheaf of ideals of Y (i.e., C corresponds to an effective Cartier divisor on Y). Moreover, for all $s \in S$ and all $i \leq m$, any irreducible component Γ of $F_i \cap X_s$ is such that* $\dim(\Gamma \cap Y_s) \leq \max(\dim(\Gamma) - (d-1), 0)$. *In particular, if $(F_i)_s$ has positive codimension in X_s in a neighborhood of C_s, then $F_i \cap Y_s$ has dimension at most* 0 *in a neighborhood of C_s.*

Proof

The main part of Theorem 3.4 is given a complete proof in the case where $X \to S$ is projective in Section 3.12. The proof when $X \to S$ is only assumed to be quasi-projective is given in Section 3.13. We prove here the end of the statement of Theorem 3.4, where we assume that $\dim S = 1$. Apply Theorem 3.4 $(d-1)$ times, starting with $X' := V_+(f)$, $F_i' := F_i \cap V_+(f)$, and $C \subseteq X'$. Note that at each step condition Theorem 3.4(c) holds by Lemma 3.5(2). □

LEMMA 3.5

Let S be a noetherian scheme, and let $\pi : X \to S$ be a morphism of finite type. Let C be a closed subset of X, with $C \to S$ finite.

(1) *Let $s \in S$ be such that C_s is not empty. Then the following are equivalent:*

†By *pure* codimension d, we mean that every irreducible component of C has codimension d in X.

(a) $\operatorname{codim}(C_s, X_s) \geq d$.

(b) *Every point x of C_s is contained in an irreducible component of X_s of dimension at least equal to d (equivalently,* $\dim_x X_s \geq d$ *for all $x \in C_s$).*†

(2) *Let $\mathcal{L}$ be a line bundle on X with a global section f defining a closed subscheme H_f which contains C. Let $s \in S$. Suppose that* $\operatorname{codim}(C_s, X_s) \geq d$. *Then* $\operatorname{codim}(C_s, (H_f)_s) \geq d - 1$.

(3) *Assume that C has codimension $d \geq 0$ in X and that each irreducible component of C dominates an irreducible component of S (e.g., when $C \to S$ is flat). Then for all $s \in S$,* $\operatorname{codim}(C_s, X_s) \geq d$. *In particular, if X/S and C satisfy the hypotheses of the version of Theorem 3.4 given in the introduction, then they satisfy the hypotheses of Theorem 3.4 as stated above.*

Proof

(1) This is immediate since, X_s being of finite type over $k(s)$, C_s is the union of finitely many closed points of X_s.

(2) We can suppose that C_s is not empty. Let $x \in C_s$. Then x is contained in an irreducible component Γ' of X_s of dimension at least equal to d. Consider an irreducible component Γ of $\Gamma' \cap (H_f)_s$ which contains x. Since $\Gamma' \cap (H_f)_s$ is defined in Γ' by a single equation, we find that $\dim(\Gamma) \geq \dim(\Gamma') - 1 \geq d - 1$, as desired.

(3) Let ξ be a generic point of C. By hypothesis, $\pi(\xi)$ is a generic point of S and ξ is closed in $X_{\pi(\xi)}$. So

$$\dim_\xi X_{\pi(\xi)} = \dim \mathcal{O}_{X_{\pi(\xi)},\xi} = \dim \mathcal{O}_{X,\xi} \geq \operatorname{codim}_\xi(C, X) \geq d.$$

The set $\{x \in X \mid \dim_x X_{\pi(x)} \geq d\}$ is closed (see [29, IV.13.1.3]). Since this set contains the generic points of C, it contains C. Hence, when C_s is not empty, $\operatorname{codim}(C_s, X_s) \geq d$ by (1)(b). When $C_s = \emptyset$, $\operatorname{codim}(C_s, X_s) = +\infty$ by definition and the statement of (3) also holds.

In the version of Theorem 3.4 given in the introduction, we assume that C is irreducible, that $C \to S$ is finite and surjective, and that C has codimension $d > \dim S$ in X. It follows then from (3) that (c) in Theorem 3.4 is automatically satisfied. □

Notation 3.6

We fix here some notation needed in the proofs of Theorems 3.3 and 3.4. Let $S = \operatorname{Spec} R$ be a noetherian affine scheme. Consider a projective morphism $X \to S$. Fix a very ample sheaf $\mathcal{O}_X(1)$ on X relative to S. As usual, if $\mathcal{F}$ is any quasi-coherent

†Recall that $\dim_x X_s$ is the infimum of $\dim U$, where U runs through the open neighborhoods of x in X_s.

sheaf on X and $s \in S$, let $\mathcal{F}_s$ denote the pullback of $\mathcal{F}$ to the fiber X_s and, if $x \in X$, $\mathcal{F}(x) := \mathcal{F}_x \otimes k(x)$ (see Section 1.3).

Let $C \subseteq X$ be a closed subscheme defined by an ideal sheaf $\mathcal{J}$. For $n \geq 1$, set $\mathcal{J}(n) := \mathcal{J} \otimes \mathcal{O}_X(n)$, and for $s \in S$, let $\mathcal{J}_s(n) := \mathcal{J}_s \otimes \mathcal{O}_{X_s}(n) = \mathcal{J}(n)_s$. Let $\overline{\mathcal{J}}_s$ denote the image of $\mathcal{J}_s \to \mathcal{O}_{X_s}$. When $x \in C \cap X_s$, we note the following natural isomorphisms of $k(x)$-vector spaces:

$$\big(\mathcal{J}(n)|_C\big)_s(x) \to \mathcal{J}_s(n)/\mathcal{J}_s^2(n) \otimes k(x) \to \mathcal{J}_s(n) \otimes k(x) \to \mathcal{J}(n) \otimes k(x)$$

and

$$\overline{\mathcal{J}}_s(n)/\overline{\mathcal{J}}_s^2(n) \otimes k(x) \to \overline{\mathcal{J}}_s(n) \otimes k(x).$$

To prove Theorems 3.3 and 3.4, we will show the existence of $f \in H^0(X, \mathcal{J}(n))$, for all n sufficiently large, such that the associated closed subscheme $H_f \subset X$ satisfies the conclusions of the theorems. To enable us to use the results of the previous section to produce the desired f, we define the following sets.

Let n be big enough such that $\mathcal{J}(n)$ is generated by its global sections. Fix a system of generators $f_1, \dots, f_N$ of $H^0(X, \mathcal{J}(n))$. Let $s \in S$. Denote by $\overline{f}_{i,s}$ the image of f_i in $H^0(X_s, \overline{\mathcal{J}}_s(n))$. Let F be a locally closed subset of X.

- Let $\Sigma_F(s)$ denote the set of $(\alpha_1, \dots, \alpha_N) \in k(s)^N$ such that the closed subset $V_+(\sum_{i=1}^N \alpha_i \overline{f}_{i,s})$ in X_s, defined by the global section $\sum_{i=1}^N \alpha_i \overline{f}_{i,s}$ of $\mathcal{O}_{X_s}(n)$, contains at least one irreducible component of F_s of positive dimension.

For the purpose of Theorem 3.4, we will also consider the following set.

- Let $\Sigma_C(s)$ denote the set of $(\alpha_1, \dots, \alpha_N) \in k(s)^N$ for which there exists $x \in C \cap X_s$ such that the image of $\sum_{i=1}^N \alpha_i (f_i|_{X_s}) \in H^0(X_s, \mathcal{J}_s(n))$ vanishes in $\mathcal{J}_s(n) \otimes k(x)$.

To lighten the notation, we will not always explicitly use symbols to make it clear that indeed the sets $\Sigma_C(s)$ and $\Sigma_F(s)$ depend on n and on $f_1, \dots, f_N$. We will use the fact that if $f \in H^0(X, \mathcal{J}(n))$ and $\overline{f}_s$ is its image in $\overline{\mathcal{J}}_s(n)$, then $V_+(f) \cap X_s = V_+(\overline{f}_s)$.

LEMMA 3.7

Let S be an affine noetherian scheme, and let $X \to S$ be a morphism of finite type. Let F be a locally closed subset of X. Let $\mathbf{F}$ be the union of the irreducible components of positive dimension of F_s, when s runs over all points of S. Then $\mathbf{F}$ is closed in F.

Assume now that $X \to S$, n, $\mathcal{J}(n)$, and $\{f_1, \dots, f_N\}$ are as above in Notation 3.6. Then

(1) $\Sigma_F(s) = \Sigma_{\mathbf{F}}(s)$ *for all $s \in S$.*

(2) *There exists a natural constructible subset T_F of $\mathbb{A}^N_S$ such that for all $s \in S$, $\Sigma_F(s)$ is exactly the set of $k(s)$-rational points of $\mathbb{A}^N_{k(s)}$ contained in $(T_F)_s$.*

Proof
Endow F with the structure of a reduced subscheme of X, and consider the induced morphism $g : F \to S$. Then the set of $x \in F$ such that x is isolated in $g^{-1}(g(x))$ is open in F (see [29, IV.13.1.4]). Thus, $\mathbf{F}$ is closed in F.

(1) By construction, for all $s \in S$, F_s and $\mathbf{F}_s$ have the same irreducible components of positive dimension, so $\Sigma_F(s) = \Sigma_{\mathbf{F}}(s)$ for all $s \in S$.

(2) By (1) we can replace F by $\mathbf{F}$ and suppose that for all $s \in S$, F_s contains no isolated point. Endow F with the structure of a reduced subscheme of X. Let $\mathcal{O}_F(1) := \mathcal{O}_X(1)|_F$. Consider the following data: the morphism of finite type $F \to S$, the sheaf $\mathcal{O}_F(n)$, and the sections $h_1, \dots, h_N$ in $H^0(F, \mathcal{O}_F(n))$, with $h_i := f_i|_F$. We associate to these data, for each $s \in S$, the subset $\Sigma(s)$ as in Section 1.8. We claim that for each $s \in S$, we have $\Sigma_F(s) = \Sigma(s)$. For convenience, recall that

$$\Sigma(s) = \Big\{(\alpha_1, \dots, \alpha_N) \in k(s)^N \;\Big|\; \sum_{i=1}^N \alpha_i h_{i,s} \text{ vanishes at some generic point of } F_s\Big\}.$$

Let $f \in H^0(X, \mathcal{J}(n)) \subseteq H^0(X, \mathcal{O}_X(n))$, and let $h = f|_F \in H^0(F, \mathcal{O}_F(n))$. Recall that $\overline{f}_s$ denotes the image of f_s under the natural map $\mathcal{J}_s(n) \to \mathcal{O}_{X_s}(n)$. Thus, $\overline{f}_s$ is nothing but the image of $f \in H^0(X, \mathcal{O}_X(n))$ under the natural map $H^0(X, \mathcal{O}_X(n)) \to H^0(X_s, \mathcal{O}_{X_s}(n))$. For any $s \in S$ and for any $x \in F_s$, we have

$$x \in V_+(\overline{f}_s) \iff \overline{f}_s(x) = 0 \in \mathcal{O}_X(n) \otimes k(x) \iff h_s(x) = 0 \in \mathcal{O}_F(n) \otimes k(x).$$

Since we are assuming that F_s does not have any irreducible component of dimension 0, $\Sigma_F(s)$ is equal to

$$\Big\{(\alpha_1, \dots, \alpha_N) \in k(s)^N \;\Big|\; \sum_{i=1}^N \alpha_i \overline{f}_{i,s} \text{ vanishes at some generic point of } F_s\Big\}.$$

Therefore, $\Sigma_F(s) = \Sigma(s)$. We can thus apply Proposition 1.10 to the above data $F \to S$, $\mathcal{O}_F(n)$, and the sections $h_1, \dots, h_N$ to obtain a natural constructible subset T_F of $\mathbb{A}^N_S$ such that for all $s \in S$, $\Sigma_F(s)$ is exactly the set of $k(s)$-rational points of $\mathbb{A}^N_{k(s)}$ contained in $(T_F)_s$. $\square$

Our goal now is to bound the dimension of $(T_F)_s$ so that Theorem 2.1 can be used to produce the desired $f \in H^0(X, \mathcal{J}(n))$. Let V/k be a projective variety over a field k, endowed with a very ample invertible sheaf $\mathcal{O}_V(1)$. Recall that the *Hilbert polynomial* $P_V(t) \in \mathbb{Q}[t]$ is the unique polynomial such that $P_V(n) = \chi(\mathcal{O}_V(n))$ for all integers n (where $\chi(\mathcal{G})$ denotes as usual the Euler–Poincaré characteristic of a coherent sheaf $\mathcal{G}$). A finiteness result for the Hilbert polynomials of the fibers of a projective morphism, needed in the final step of the proof of our next lemma, is recalled in Facts 4.1.

LEMMA 3.8
Let $S = \operatorname{Spec} R$ be an affine noetherian scheme, and let $X \to S$ be a projective morphism. Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf relative to $X \to S$. Let C be a closed subscheme of X with ideal sheaf $\mathcal{J}$, and let F be a locally closed subset of X. Assume that for all $s \in S$, no irreducible component of F_s of positive dimension is contained in C_s.

Let $c \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for any choice $\{f_1, \ldots, f_N\}$ of generators of $H^0(X, \mathcal{J}(n))$, the constructible subset $T_F \subseteq \mathbb{A}^N_S$ introduced in Lemma 3.7(2) satisfies $\dim(T_F)_s \leq N - c$ for all $s \in S$.

Proof
Lemma 3.7(1) shows that we can suppose that the locally closed subset F is such that for all $s \in S$, F_s has no isolated point. We now further reduce to the case where F is open and dense in X.

Let Z be the Zariski closure of F in X. Then F is open and dense in Z. Endow Z with the induced structure of reduced closed subscheme. Denote by $\mathcal{J}\mathcal{O}_Z$ the image of $\mathcal{J}$ under the natural homomorphism $\mathcal{O}_X \to \mathcal{O}_Z$. This sheaf is the sheaf of ideals associated with the image of the closed immersion $C \times_X Z \to Z$. The morphism of $\mathcal{O}_X$-modules $\mathcal{J} \to \mathcal{J}\mathcal{O}_Z$ is surjective with kernel $\mathcal{K}$. Since $\mathcal{O}_X(1)$ is very ample, we find that there exists $n_0 > 0$ such that for all $n \geq n_0$, $H^1(X, \mathcal{K}(n)) = (0)$, so that the natural map

$$H^0\big(X, \mathcal{J}(n)\big) \to H^0\big(Z, \mathcal{J}\mathcal{O}_Z(n)\big)$$

is surjective. Fix $n \geq n_0$, and fix a system of generators $\{f_1, \ldots, f_N\}$ of $H^0(X, \mathcal{J}(n))$. It follows that the images of $f_1, \ldots, f_N$ generate the R-module $H^0(Z, \mathcal{J}\mathcal{O}_Z(n))$. Note that $C \times_X Z$ does not contain any irreducible component of F_s for all s. It follows that it suffices to prove the bound on the dimension of $(T_F)_s$ when $Z = X$, that is, when F is open and dense in X. We need the following fact.

LEMMA 3.9
Let S be a noetherian scheme. Let $X \to S$ be a morphism of finite type. Then there exist an affine scheme S' and a quasi-finite surjective morphism of finite type $S' \to S$ with the following properties:
(a) *S' is reduced and is the disjoint union of its irreducible components.*

(b) *Let $X' := X \times_S S'$, and let $\Gamma_1, \dots, \Gamma_m$ be the irreducible components of X' endowed with the induced structure of reduced closed subschemes. Then for $i = 1, \dots, m$, Γ_i is flat with geometrically integral fibers over an irreducible component of S'.*

(c) *For each $s' \in S'$, the irreducible components of $X'_{s'}$ are exactly (without repetition) the irreducible components of the nonempty $(\Gamma_i)_{s'}$, $i = 1, \dots, m$.*

Proof
We proceed by noetherian induction on S. We can suppose that S is reduced. First consider the case $S = \operatorname{Spec} K$ for some field K. Then there exists a finite extension L/K such that each irreducible component of X_L, endowed with the structure of reduced closed subscheme, is geometrically integral (see [29, IV.4.5.11, IV.4.6.6]). The lemma is proven with $S' = \operatorname{Spec} L$.

Suppose now that the property is true for any strict closed subscheme Z of S and for the scheme of finite type $X \times_S Z \to Z$. If S is reducible with irreducible components $S_1, \dots, S_\ell$, then by the induction hypothesis we can find $S'_i \to S_i$ with the desired properties (a)–(c). Then it is enough to take S' equal to the disjoint union of the S'_i. Now we are reduced to the case S is integral. Let η be the generic point of S and let $K = k(\eta)$. Let L/K be a finite extension defined as in the zero-dimensional case above. Restricting S to a dense open subset V if necessary, we can find a finite surjective morphism $\pi : U \to V$ with U integral that extends $\operatorname{Spec} L \to \operatorname{Spec} K$. Let $X_1, \dots, X_r$ be the (integral) irreducible components of $X \times_V U$. Their generic fibers over U are geometrically integral or empty, and we can make V smaller so that they are nonempty.

It follows from [29, IV.6.9.1, IV.9.7.7], that there exists a dense open subset U' of U such that $X_i \times_V U' \to U'$ is flat with geometrically integral fibers for all $i = 1, \dots, r$. Restricting U' further if necessary, we can suppose that the number of geometric irreducible components in the fibers of $X \times_V U' \to U'$ is constant (see [29, IV.9.7.8]). Note now that for each $y \in U'$, the irreducible components of $(X \times_V U')_y$ are exactly the fibers $(X_i)_y$, $i = 1, \dots, r$. As $S \setminus \pi(U \setminus U')$ is open and dense in S, it contains a dense affine open subset V' of S. By induction hypothesis, there exists $T' \to (S \setminus V')_{\mathrm{red}}$ with the desired properties (a)–(c). Let S' be the disjoint union of $\pi^{-1}(V')$ with T'. It is clear that S' satisfies the properties (a)–(c). $\square$

Let us return to the proof of Lemma 3.8. We now proceed to prove that *it suffices to bound the dimension of $(T_F)_s$ for all $s \in S$ when all fibers of $X \to S$ are integral.* To prove this reduction, we use the fact that the formation of T_F is compatible with any base change $S' \to S$ as in Proposition 1.10(a), and the fact that the dimension of a fiber $(T_F)_s$ is invariant by finite field extensions in the sense of Lemma 1.2(b).

Finally, the conditions that C_s does not contain any irreducible component of F_s is also preserved by base change. While making these reductions, care will be needed to keep track of the hypothesis that $f_1, \dots, f_N$ generate $H^0(X, \mathcal{J}(n))$.

Let $g : S' \to S$ be as in Lemma 3.9 with natural morphism $g' : X \times_S S' \to X$. Let F' be the preimage of F in $X \times_S S'$. For any $s \in S$ and $s' \in S'$ lying over s, $\dim(T_F)_s = \dim(T_F)_{s'}$ and $(T_F)_{s'} = (T_{F'})_{s'}$ is the finite union of the $(T_{\Gamma_i \cap F'})_{s'}$. Increasing n_0 if necessary, we find by using Fact 4.1(i) that the natural map

$$H^0\big(X, \mathcal{J}(n)\big) \otimes \mathcal{O}(S') \to H^0\big(X', g'^* \mathcal{J}(n)\big)$$

is an isomorphism. Denote now by $\mathcal{J}\mathcal{O}_{\Gamma_i}$ the image of $g'^*\mathcal{J}$ under the natural map $g'^*\mathcal{J} \to \mathcal{O}_{X'} \to \mathcal{O}_{\Gamma_i}$. The morphism $g'^*\mathcal{J} \to \mathcal{J}\mathcal{O}_{\Gamma_i}$ of $\mathcal{O}_{X'}$-modules is surjective. Increasing n_0 further if necessary, we find that

$$H^0\big(X', g'^* \mathcal{J}(n)\big) \to H^0\big(\Gamma_i, \mathcal{J}\mathcal{O}_{\Gamma_i}(n)\big)$$

is surjective for all $i = 1, \dots, m$, where the twisting is done with the very ample sheaf $\mathcal{O}_{X'}(1) := g'^*\mathcal{O}_X(1)$ relative to $X' \to S'$. It follows that the images of $f_1, \dots, f_N$ in $H^0(\Gamma_i, \mathcal{J}\mathcal{O}_{\Gamma_i}(n))$ also form a system of generators of $H^0(\Gamma_i, \mathcal{J}\mathcal{O}_{\Gamma_i}(n))$. Therefore, we can replace $X \to S$ with $\Gamma \to S'$ for Γ equal to some Γ_i. Now we are in the situation where all fibers of $X \to S$ are integral.

By Proposition 1.10(b), if $F_s \neq \emptyset$, $\dim(T_F)_s$ is the dimension of the kernel of the natural map

$$k(s)^N \to \mathcal{O}_{F_s}(n) \otimes k(\xi) = \mathcal{O}_{X_s}(n) \otimes k(\xi)$$

defined by the $\overline{f}_{i,s}$ and where ξ is the generic point of X_s. This map is given by sections in $H^0(X, \mathcal{J}(n))$, so it factorizes into a sequence of linear maps

$$\begin{aligned} k(s)^N \to H^0\big(X, \mathcal{J}(n)\big) \otimes k(s) &\to H^0\big(X_s, \mathcal{J}_s(n)\big) \\ &\to H^0\big(X_s, \overline{\mathcal{J}}_s(n)\big) \to \mathcal{O}_{X_s}(n) \otimes k(\xi), \end{aligned}$$

where the first one is surjective because $f_1, \dots, f_N$ generate $H^0(X, \mathcal{J}(n))$, the composition of the second and the third is surjective (independently of s) by Lemma 4.4(a) (after increasing n_0 if necessary so that Lemma 4.4(a) can be applied), and the last one is injective because X_s is integral. If $F_s = \emptyset$, then $(T_F)_s = \emptyset$. Therefore, in any case

$$\dim(T_F)_s \leq N - \dim_{k(s)} H^0\big(X_s, \overline{\mathcal{J}}_s(n)\big).$$

We now end the proof of Lemma 3.8 by showing that after increasing n_0 if necessary, we have $\dim_{k(s)} H^0(X_s, \overline{\mathcal{J}}_s(n)) \geq c$ for all $s \in S_F := \{s \in S \mid F_s \neq \emptyset\}$. We note that for all $s \in S_F$, $\dim F_s > 0$ (since F_s has no isolated point), so C_s does not contain F_s and, thus, $C_s \neq X_s$. As X_s is irreducible, we have $\dim C_s < \dim X_s$. It follows that the Hilbert polynomial $P_{C_s}(t)$ of C_s satisfies $\deg P_{C_s}(t) < \deg P_{X_s}(t)$. Since the set of all Hilbert polynomials $P_{X_s}(t)$ and $P_{C_s}(t)$ with $s \in S$ is finite (Fact 4.1(iii)), and since such polynomials have positive leading coefficient (see [33, III.9.10]), we can assume, increasing n_0 if necessary, that

$$P_{X_s}(n) - P_{C_s}(n) \geq c$$

for all $s \in S_F$. Using Fact 4.1(ii), and increasing n_0 further if necessary, we find that

$$H^i\big(X_s, \mathcal{O}_{X_s}(n)\big) = (0) = H^i\big(C_s, \mathcal{O}_{C_s}(n)\big)$$

for all $i \geq 1$ and for all $s \in S$. We have $P_{X_s}(n) = \chi(\mathcal{O}_{X_s}(n))$, and $P_{C_s}(n) = \chi(\mathcal{O}_{C_s}(n))$ for all $n \geq 1$. Therefore, using the above vanishings for $i > 0$, we find that for all $s \in S$,

$$P_{X_s}(n) - P_{C_s}(n) = \dim H^0\big(X_s, \mathcal{O}_{X_s}(n)\big) - \dim H^0\big(C_s, \mathcal{O}_{C_s}(n)\big).$$

Hence, for all $s \in S_F$,

$$\dim H^0\big(X_s, \overline{\mathcal{J}}_s(n)\big) \geq \dim H^0\big(X_s, \mathcal{O}_{X_s}(n)\big) - \dim H^0\big(C_s, \mathcal{O}_{C_s}(n)\big) \geq c,$$

and the lemma is proved. □

Assume now that $C \to S$ is as in Theorem 3.4, and let $\mathcal{J}$ denote the ideal sheaf of C in X, as in Notation 3.6. In particular, $C \to S$ is finite, $C \to X$ is a regular immersion, C has pure codimension d in X, and for all $s \in S$, $\operatorname{codim}(C_s, X_s) \geq d$. This latter hypothesis and Lemma 3.5(1.b) imply that C_s does not contain any isolated point of X_s. Therefore, for any $x \in C_s$, $(\overline{\mathcal{J}}_s)_x \neq 0$ and, hence, both $\overline{\mathcal{J}}_s(n)/\overline{\mathcal{J}}_s^2(n) \otimes k(x)$ and $\mathcal{J}_s(n)/\mathcal{J}_s^2(n) \otimes k(x)$ are nonzero. In fact, as $C \to X$ is a regular immersion, $\mathcal{J}(n)/\mathcal{J}^2(n)$ is a rank d vector bundle on C.

LEMMA 3.10

Assume that $C \to S$ is as in Theorem 3.4, with C of pure codimension d in X and suppose that $X \to S$ is projective. Keep Notation 3.6. Let $n_0 > 0$ be an integer such that for all $n \geq n_0$, $\mathcal{J}(n)$ is generated by its global sections, and $H^1(X, \mathcal{J}^2(n)) = (0)$. Then for all $n \geq n_0$ and for any system of generators $f_1, \dots, f_N$ of $H^0(X, \mathcal{J}(n))$, there exists a constructible subset T_C of $\mathbb{A}^N_S$ such that

(i) *for all $s \in S$, $\Sigma_C(s)$ is exactly the set of $k(s)$-rational points of $\mathbb{A}^N_{k(s)}$ contained in $(T_C)_s$, and*

(ii) $\dim(T_C)_s \leq N - d$.

Proof
Consider the data consisting of the morphism $C \to S$, the sheaf $\mathcal{F} := \mathcal{J}(n)|_C$ on C, and the images of $f_1, \dots, f_N$ under the natural map $H^0(X, \mathcal{J}(n)) \to H^0(C, \mathcal{F})$. To these data are associated in Section 1.8 a set $\Sigma(s)$ for any $s \in S$. As C_s is finite, $\Sigma_C(s)$ is nothing but the set $\Sigma(s)$. We thus apply Proposition 1.10 to the above data to obtain a constructible set T_C of $\mathbb{A}^N_S$ such that for all $s \in S$, $\Sigma_C(s)$ is exactly the set of $k(s)$-rational points of $\mathbb{A}^N_{k(s)}$ contained in $(T_C)_s$.

Our additional hypothesis implies that the images of the sections $f_1, \dots, f_N$ generate $H^0(X, \mathcal{J}(n)/\mathcal{J}^2(n))$, which we identify with $H^0(C, \mathcal{F})$. Since $C \to S$ and S are affine, and C_s is finite for each $s \in S$, we have isomorphisms

$$H^0(C, \mathcal{F}) \otimes k(s) \simeq H^0(C_s, \mathcal{F}_s) \simeq \bigoplus_{x \in C_s} (\mathcal{F}_s)_x.$$

It follows that for each $x \in C_s$, the natural map $H^0(C_s, \mathcal{F}_s) \to (\mathcal{F}_s)_x$ is surjective. As $(\mathcal{F}_s)_x$ is free of rank d and the images of $f_1, \dots, f_N$ generate $H^0(C_s, \mathcal{F}_s)$, we find that the linear maps $k(s)^N \to \mathcal{F}(x)$ in Proposition 1.10(b) are surjective for all $s \in S$. It follows that $\dim(T_C)_s \le N - d$ (the equality holds if $C_s \neq \emptyset$). □

3.11

Proof of Theorem 3.3 when $\pi : X \to S$ is projective
Let $\{F_1, \dots, F_m\}$ be the locally closed subsets of X given in (ii) of the statement of the theorem. When C does not contain any irreducible component of positive dimension of X_s for all $s \in S$, we set $F_0 := X$ and argue below using the set $\{F_0, F_1, \dots, F_m\}$. Let A denote the finite set given in (iii). When $C \cap \mathrm{Ass}(X) = \emptyset$, we enlarge A if necessary by adjoining to it the finite set $\mathrm{Ass}(X)$.

Let $A_0 \subset X$ be the union of A with the set of the generic points of the irreducible components of positive dimension of $(F_1)_s, \dots, (F_m)_s$, for all $s \in S$. When relevant, we also add to A_0 the generic points of the irreducible components of positive dimension of $(F_0)_s$, for all $s \in S$. Using [29, IV.9.7.8], we see that the number of points in $A_0 \cap X_s$ is bounded when s varies in S. We are thus in a position to apply Lemma 4.8(a) with the set A_0. Let $c := 1 + \dim S$. Let n_0 be an integer satisfying simultaneously the conclusion of Lemma 4.8(a) for A_0, and of Lemma 3.8 for c and for each locally closed subset $F = F_i$, with $i = 1, \dots, m$, and $i = 0$ when relevant.

Fix now $n \ge n_0$, and fix $f_1, \dots, f_N$ a system of generators of $H^0(X, \mathcal{J}(n))$. Increasing n_0 if necessary, we can assume by using Lemma 4.4 that for all $s \in S$, the composition of the canonical maps

$$H^0\big(X, \mathcal{J}(n)\big) \otimes k(s) \to H^0\big(X_s, \mathcal{J}_s(n)\big) \to H^0\big(X_s, \overline{\mathcal{J}}_s(n)\big)$$

is surjective. Let $T_{F_1}, \dots, T_{F_m}$ be the constructible subsets of $\mathbb{A}_S^N$ pertaining to $\Sigma_{F_1}(s), \dots, \Sigma_{F_m}(s)$ and whose existence is proven in Lemma 3.7(2). When relevant, we also consider T_{F_0} and $\Sigma_{F_0}(s)$. Since Lemma 3.8 is applicable for c and for each $F = F_i$, we find that for all $s \in S$ we have

$$\dim(T_{F_i})_s \leq N - c = N - 1 - \dim S.$$

It follows from Lemma 1.2(c) that $\dim T_{F_i} \leq N - 1$.

Let $\pi(A) := \{s_1, \dots, s_r\} \subseteq S$. Fix $s_j \in \pi(A)$, and for each $x \in A \cap X_{s_j}$, consider the hyperplane of $\mathbb{A}^N_{k(s_j)}$ defined by $\sum_i \alpha_i f_i(x) = 0$. This is indeed a hyperplane because otherwise $f_i(x) = 0$ for all $i \leq N$ at x, which would imply that $x \in C$, but $A \cap C = \emptyset$ by hypothesis. Denote by T_{A_j} the finite union of all such hyperplanes of $\mathbb{A}^N_{k(s_j)}$, for each $x \in A \cap X_{s_j}$. The subset T_{A_j} is pro-constructible in $\mathbb{A}_S^N$ (see Section 1.1). It has dimension $N-1$, and its fibers $(T_{A_j})_s$ are constructible for each $s \in S$ (and $(T_{A_j})_s$ is empty if $s \neq s_j$).

We now apply Theorem 2.1 to the set of pro-constructible subsets T_{A_j}, $j = 1, \dots, r$ and T_{F_i}, $i = 1, \dots, m$, and $i = 0$ when relevant. Our discussion so far implies that these pro-constructible subsets all satisfy condition (1) in Theorem 2.1 with $V = S$. Let $T = (\bigcup_j T_{A_j}) \cup (\bigcup_i T_{F_i})$. For each $s \in S$, the element $f_{s,n} \in H^0(X_s, \mathcal{J}_s(n))$ exhibited in Lemma 4.8(a) gives rise to a $k(s)$-rational point of $\mathbb{A}^N_{k(s)}$ not contained in T_s. So condition (2) in Theorem 2.1 is also satisfied by T. We can thus apply Theorem 2.1 to find a section $(a_1, \dots, a_N) \in R^N = \mathbb{A}_S^N(S)$ such that for all $s \in S$, $(a_1(s), \dots, a_N(s))$ is a $k(s)$-rational point of $\mathbb{A}^N_{k(s)}$ that is not contained in T_s.

Let $f := \sum_{i=1}^N a_i f_i$ and consider the closed subscheme $H_f \subset X$. As $f \in H^0(X, \mathcal{J}(n))$, C is a closed subscheme of H_f. By definition of T_{F_i} and T_{A_j}, for all $s \in S$ and for all $0 \leq i \leq m$, H_f does not contain any irreducible component of $(F_i)_s$ of positive dimension and $H_f \cap A = \emptyset$. This proves the conclusions (1), (2), and (3) of Theorem 3.3.

When the hypothesis of (4) is satisfied, we have included in our proof above conditions pertaining to $F_0 = X$, and we find then that H_f contains no irreducible component of X_s. It is thus by definition a hypersurface relative to $X \to S$. If furthermore $C \cap \mathrm{Ass}(X) = \emptyset$, as we enlarged A to include $\mathrm{Ass}(X)$, we have $H_f \cap \mathrm{Ass}(X) = \emptyset$. Hence, it follows from Lemma 3.2(4) that H_f is locally principal. This proves (4), and completes the proof of Theorem 3.3 when $X \to S$ is projective. $\square$

3.12

Proof of Theorem 3.4 when $\pi : X \to S$ is projective

We assume now that $C \to S$ is finite. Thus C_s is finite for each $s \in S$, and we find that C_s does not contain any irreducible component of positive dimension of X_s.

Let $\{F_1, \dots, F_m\}$ be the locally closed subsets of X given in (ii) of Theorem 3.3. We set $F_0 := X$ and argue as in the proof of Theorem 3.3 above by using the set $\{F_0, F_1, \dots, F_m\}$. Let A denote the finite set given in (iii) of Theorem 3.3. We have that $C \cap \mathrm{Ass}(X) = \emptyset$: indeed, for all $x \in C$, $\mathrm{depth}(\mathcal{O}_{X,x}) \geq d > 0$, so that $x \notin \mathrm{Ass}(X)$. We therefore enlarge A if necessary by adjoining to it the finite set $\mathrm{Ass}(X)$. We define A_0 and $c := 1 + \dim S$ exactly as in the proof of Theorem 3.3 in Section 3.11. Let n_0 be an integer satisfying simultaneously the conclusion of Lemma 4.8(b) for A_0, of Lemma 3.10, and of Lemma 3.8 for c and for each locally closed subset $F = F_i$, with $i = 0, 1, \dots, m$.

Fix now $n \geq n_0$, and fix $f_1, \dots, f_N$ a system of generators of $H^0(X, \mathcal{J}(n))$. Increasing n_0 if necessary, we can assume by using Lemma 4.4 that for all $s \in S$, the composition of the canonical maps

$$H^0\big(X, \mathcal{J}(n)\big) \otimes k(s) \to H^0\big(X_s, \mathcal{J}_s(n)\big) \to H^0\big(X_s, \overline{\mathcal{J}}_s(n)\big)$$

is surjective. Let $T_{F_0}, T_{F_1}, \dots, T_{F_m}$ be the constructible subsets of $\mathbb{A}^N_S$ pertaining to $\Sigma_{F_0}(s)$, $\Sigma_{F_1}(s), \dots, \Sigma_{F_m}(s)$, and whose existence is proved in Lemma 3.7(2). As in Section 3.11, we find that $\dim T_{F_i} \leq N - 1$ for each $i = 0, \dots, m$. Define now T_{A_j}, $j = 1, \dots, r$ as in Section 3.11. Again, T_{A_j} is pro-constructible in $\mathbb{A}^N_S$, it has dimension $N - 1$, and its fibers $(T_{A_j})_s$ are constructible for each $s \in S$.

Since Lemma 3.10 is applicable, we can also consider the constructible subset T_C of $\mathbb{A}^N_S$ pertaining to $\Sigma_C(s)$. Since we assume that $d > \dim S$, we find from Lemma 3.10 that

$$\dim(T_C)_s \leq N - d \leq N - (\dim S + 1)$$

for all $s \in S$. Thus, it follows from Lemma 1.2(c) that $\dim T_C \leq N - 1$.

As in Section 3.11, we set T to be the union of the sets T_{F_i}, $i = 0, \dots, m$, and T_{A_j}, $j = 1, \dots, r$. Lemma 4.8(b) implies that $\mathbb{A}^N_{k(s)}$ is not contained in $(T_C \cup T)_s$ because $(\overline{\mathcal{J}}_s(n)/\overline{\mathcal{J}}_s^2(n)) \otimes k(x) \neq 0$ for all $x \in C_s$ (see the paragraph before Lemma 3.10). Applying Theorem 2.1 to the pro-constructible subsets T_C, T_{F_i}, $i = 0, \dots, m$, and T_{A_j}, $j = 1, \dots, r$, we find $(a_1, \dots, a_N) \in \mathbb{A}^N_S(S)$ such that for all $s \in S$, $(a_1(s), \dots, a_N(s))$ is a $k(s)$-rational point of $\mathbb{A}^N_{k(s)}$ that is not contained in $(T_C \cup T)_s$.

Let $f := \sum_{i=1}^N a_i f_i$ and consider the closed subscheme $H_f \subset X$. As in Section 3.11, we find that H_f satisfies the conclusions (1), (2), and (3) of Theorem 3.3, and that H_f is a locally principal hypersurface.

It remains to use the properties of the set T_C to show that C is regularly immersed in H_f, and that C is pure of codimension $d-1$ in H_f. Indeed, this is a local question. Fix $x \in C$. Let $I := \mathcal{J}_x \subset \mathcal{O}_{X,x}$ and let $g \in I$ correspond to the section f. Since the image of g in $I/I^2 \otimes k(x)$ is nonzero by the definition of T_C, the image of g in the free $\mathcal{O}_{C,x}$-module I/I^2 can be completed into a basis of I/I^2, and it is then well known that g belongs to a regular sequence generating I. This concludes the proof of Theorem 3.4 when $X \to S$ is projective. $\square$

3.13

Proof of Theorems 3.3 and 3.4 when $X \to S$ is quasi-projective
Since $\mathcal{O}_X(1)$ is assumed to be very ample relative to $X \to S$, there exists a projective morphism $\overline{X} \to S$ with an open immersion $X \to \overline{X}$ of S-schemes, and a very ample sheaf $\mathcal{O}_{\overline{X}}(1)$ relative to $\overline{X} \to S$ which restricts on X to the given sheaf $\mathcal{O}_X(1)$.

Let us first prove Theorem 3.3. We are given in Theorem 3.3(i) a closed subscheme C of X. Let $\overline{C}$ be the scheme-theoretical closure of C in $\overline{X}$. We are given in Theorem 3.3(ii) m locally closed subsets $F_1, \dots, F_m$ of X. Since X is open in $\overline{X}$, each set F_i is again locally closed in $\overline{X}$. It is clear that the finite subset $A \subset X$ given in Theorem 3.3(iii) which does not intersect C is such that $A \subset \overline{X}$ does not intersect $\overline{C}$.

We are thus in the position to apply Theorem 3.3 to the projective morphism $\overline{X} \to S$ with the data $\overline{C}$, $F_1, \dots, F_m$, and A. When C satisfies the first hypothesis of Theorem 3.3(4), we set $F_0 := X$ and add the locally closed subset F_0 to the list $F_1, \dots, F_m$, as in Section 3.11. When $C \cap \mathrm{Ass}(X) = \emptyset$, we replace A by $A \cup \mathrm{Ass}(X)$. We can then conclude that there exists $n_0 > 0$ such that for any $n \geq n_0$, there exists a global section f of $\mathcal{O}_{\overline{X}}(n)$ such that the closed subscheme H_f in $\overline{X}$ contains $\overline{C}$ as a closed subscheme and satisfies the conclusions (2), (3), and, when relevant, (4), of Theorem 3.3 for $\overline{X} \to S$. The restriction of f to $\mathcal{O}_X(n)$ defines the desired closed subscheme $H_f \cap X$ satisfying the conclusions of Theorem 3.3 for $X \to S$.

Let us now prove Theorem 3.4, where we assume that $C \to S$ is finite and, hence, proper. It follows that $\overline{C} = C$. We apply Theorem 3.4 to the projective morphism $\overline{X} \to S$, and the data C, $F_1, \dots, F_m$, and A. We can then conclude that there exists $n_0 > 0$ such that for any $n \geq n_0$, there exists a global section f of $\mathcal{O}_{\overline{X}}(n)$ such that the closed subscheme H_f in $\overline{X}$ contains $\overline{C}$ as a closed subscheme and satisfies the conclusions (2), (3), and (4), of Theorem 3.3 for $\overline{X} \to S$. The restriction of f to $\mathcal{O}_X(n)$ defines the desired closed subscheme $H_f \cap X$ satisfying the conclusions of Theorem 3.4 for $X \to S$. $\square$

4. Variations on the classical avoidance lemma
In this section, we prove various assertions used in the proofs of Theorem 3.3 and Theorem 3.4. The main result in this section is Lemma 4.8.

Facts 4.1
Let S be a noetherian scheme, and let $\pi : X \to S$ be a projective morphism. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to π, and let $\mathcal{F}$ be any coherent sheaf on X.

(i) Let $g : S' \to S$ be a morphism of finite type, and consider the cartesian square

$$\begin{tikzcd} X' := X \times_S S' \arrow[r, "g'"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then there exists a positive integer n_0 such that for all $n \geq n_0$, the canonical morphism $g^*\pi_*(\mathcal{F}(n)) \to \pi'_* g'^*(\mathcal{F}(n))$ is an isomorphism.

(ii) There exists a positive integer n_0 such that for all $n \geq n_0$ and for all $s \in S$, $H^i(X_s, \mathcal{F}_s(n)) = (0)$ for all $i > 0$, and $\pi_*\mathcal{F}(n) \otimes k(s) \to H^0(X_s, \mathcal{F}_s(n))$ is an isomorphism.

(iii) The set of Hilbert polynomials $\{P_{X_s}(t) \in \mathbb{Q}[t] \mid s \in S\}$ is finite.

Proof

The properties in the statements are local on the base, and we may thus assume that S is affine. In this case, there is no ambiguity in the definition of a projective morphism, as all standard definitions coincide when the target is affine (see [29, II.5.5.4(ii)]). The proofs of (i) and (ii) when $X = \mathbb{P}^d_S$ and any coherent sheaf $\mathcal{F}$ can be found, for instance, in [51, p. 50(i)] and [51, p. 58(i)] (see also [66], step 3 in the proof of Theorem 4.2.11). The statement (iii) follows from [51, p. 58(ii)]. The general case follows immediately by using the closed S-immersion $i : X \to \mathbb{P}^d_S$ defining $\mathcal{O}_X(1)$. □

4.2

Let us recall the definition and properties of m-regular sheaves needed in our next lemmas. Let X be a projective variety over a field k, with a fixed very ample sheaf $\mathcal{O}_X(1)$. Let $\mathcal{F}$ be a coherent sheaf on X, and let $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$. Let $m \in \mathbb{Z}$. Recall in [51, Lecture 14, p. 99] that $\mathcal{F}$ is called *m-regular* if $H^i(X, \mathcal{F}(m-i)) = 0$ for all $i \geq 1$.

Assume that $\mathcal{F}$ is m-regular. Then it is known (see, e.g., [66, Proposition 4.1.1]) that for all $n \geq m$,

(a) $\mathcal{F}$ is n-regular,

(b) $H^i(X, \mathcal{F}(n)) = 0$ for all $i \geq 1$,

(c) $\mathcal{F}(n)$ is generated by its global sections, and

(d) the canonical homomorphism

$$H^0\big(X, \mathcal{F}(n)\big) \otimes H^0\big(X, \mathcal{O}_X(1)\big) \to H^0\big(X, \mathcal{F}(n+1)\big)$$

is surjective.

LEMMA 4.3
Let S be a noetherian scheme, and let $\pi : X \to S$ be a projective morphism. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to π, and let $\mathcal{F}$ be any coherent sheaf on X. Then there exists a positive integer n_0 such that for all $n \geq n_0$ and all $s \in S$, the sheaf $\mathcal{F}_s$ is n-regular on X_s.

Proof
Let r denote the maximum of $\dim X_s$, $s \in S$, and 0. This maximum exists (see [29, IV.13.1.7]). Then $H^i(X_s, \mathcal{F}_s(n)) = (0)$ for all $i \geq r+1$ and for any n. Using Fact 4.1(ii), there exists $n_1 > 0$ such that $H^i(X_s, \mathcal{F}_s(n)) = (0)$ for all $s \in S$, for all $n \geq n_1$, and for all $i > 0$. It follows that $\mathcal{F}_s$ is n-regular for all $s \in S$ and for all $n \geq n_0 := r + n_1$. □

We now discuss a series of lemmas needed in the proof of Lemma 4.8.

LEMMA 4.4
Let $\pi : X \to S$ be a projective scheme over a noetherian scheme S. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to π. Let C be a closed subscheme of X, with sheaf of ideals $\mathcal{J}$ in $\mathcal{O}_X$. Let $\overline{\mathcal{J}}_s$ denote the image of $\mathcal{J}_s$ in $\mathcal{O}_{X_s}$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $s \in S$,
(a) *the canonical map $\pi_* \mathcal{J}(n) \otimes k(s) \to H^0(X_s, \overline{\mathcal{J}}_s(n))$ is surjective;*
(b) *the sheaf $\overline{\mathcal{J}}_s$ is n-regular.*

Proof
By the generic flatness theorem [29, IV.6.9.3], there exist finitely many locally closed subsets U_i of S such that $S = \bigcup_i U_i$ (set theoretically), and such that when each U_i is endowed with the structure of reduced subscheme of S, then $C_{U_i} := C \times_S U_i \to U_i$ is flat. Refining each U_i by an affine covering, we can suppose that U_i is affine.

Denote by U one of these affine schemes U_i. Denote by $\mathcal{K}$ and $\mathcal{J}'$ the kernel and image of the natural morphism $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_U} \to \mathcal{O}_{X_U}$, with associated exact sequence of sheaves on X_U,

$$0 \to \mathcal{K} \to \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_U} \to \mathcal{J}' \to 0.$$

For all $n \in \mathbb{Z}$, we then have the exact sequence

$$0 \to \mathcal{K}(n) \to \mathcal{J}(n) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_U} \to \mathcal{J}'(n) \to 0.$$

Since $X_U \to U$ is projective, we can find n_1 such that $H^1(X_U, \mathcal{K}(n)) = (0)$ for all $n \geq n_1$ (Serre vanishing). Using Fact 4.1(ii), we find that by increasing n_1 if necessary, we can assume that for all $n \geq n_1$ and for all $s \in U$,

$$H^0\big(X_U, \mathcal{J}'(n)\big) \otimes k(s) \to H^0\big(X_s, \mathcal{J}'(n)_s\big) \tag{4.4.1}$$

is an isomorphism. The exact sequence $0 \to \mathcal{J}' \to \mathcal{O}_{X_U} \to \mathcal{O}_{C_U} \to 0$ induces an exact sequence $0 \to \mathcal{J}'_s \to \mathcal{O}_{X_s} \to \mathcal{O}_{C_s} \to 0$ for all $s \in U$ because $C \times_S U \to U$ is flat. It follows that $\mathcal{J}'_s = \overline{\mathcal{J}}_s$. We can thus apply Lemma 4.3 to the morphism $X_U \to U$ and the sheaf $\mathcal{J}'$ to obtain that $\overline{\mathcal{J}}_s$ is n-regular for all $n \geq n_1$ and all $s \in U$ (after increasing n_1 further if necessary).

For any $s \in U$ and for $n \geq n_1$, consider the commutative diagram

$$\begin{tikzcd}
H^0\big(X_U, \mathcal{J}(n) \otimes \mathcal{O}_U\big) \otimes k(s) \arrow[r, two heads] \arrow[d] \arrow[dr] & H^0\big(X_U, \mathcal{J}'(n)\big) \otimes k(s) \arrow[d] \\
H^0\big(X_s, \mathcal{J}(n)_s\big) \arrow[r] & H^0\big(X_s, \overline{\mathcal{J}}_s(n)\big)
\end{tikzcd}$$

The top horizontal map is surjective because $H^1(X_U, \mathcal{K}(n)) = (0)$, and the right vertical arrow is an isomorphism by the isomorphism (4.4.1) above. Thus, the bottom arrow $H^0(X_s, \mathcal{J}(n)_s) \to H^0(X_s, \overline{\mathcal{J}}_s(n))$ is surjective for all $n \geq n_1$ and all $s \in U$.

To complete the proof of (b), it suffices to choose n_0 to be the maximum in the set of integers n_1 associated with each U_i in the stratification. To complete the proof of (a), we further increase n_0 if necessary to be able to use the isomorphism in Fact 4.1(ii) applied to $\mathcal{F} = \mathcal{J}$ on $X \to S$. $\square$

LEMMA 4.5

Let X be a projective variety over a field k with a fixed very ample sheaf $\mathcal{O}_X(1)$. Let C be a closed subscheme of X. Let $\mathcal{J}$ denote the ideal sheaf of C in X, and assume that $\mathcal{J}$ is m_0-regular for some $m_0 \geq 0$. Let D be a finite set of closed points of C. Let $\{\xi_1, \dots, \xi_r\}$ be a finite subset of X disjoint from C.

(a) *If* $\operatorname{Card}(k) \geq r + \operatorname{Card}(D)$, *then for all $n \geq m_0$, there exists a section $f_n \in H^0(X, \mathcal{J}(n))$ such that $V_+(f_n)$ does not contain any ξ_i, and such that, for all $x \in D$ such that $(\mathcal{J}(n)/\mathcal{J}^2(n)) \otimes k(x) \neq (0)$, the image of f_n in $(\mathcal{J}(n)/\mathcal{J}^2(n)) \otimes k(x)$ is nonzero.*

(b) *There exists an integer $n_0 > 0$ such that for all $n \geq n_0$, there exists a section $f_n \in H^0(X, \mathcal{J}(n))$ as in (a).*

Proof

It suffices to prove the lemma for the subset of D obtained by removing from D all points x such that $(\mathcal{J}(n)/\mathcal{J}(n)^2) \otimes k(x) = 0$. We thus suppose now that $(\mathcal{J}(n)/\mathcal{J}(n)^2) \otimes k(x) \neq 0$ for all $x \in D$. Note also that the natural map $(\mathcal{J}(n)/\mathcal{J}(n)^2) \otimes k(x) \to \mathcal{J}(n) \otimes k(x)$ is an isomorphism for all $x \in D$, and we will use the latter expression.

(a) Let $x \in D$ and $n \geq m_0$. Consider the k-linear map

$$H^0\big(X, \mathcal{J}(n)\big) \to \mathcal{J}(n) \otimes k(x),$$

and denote by H_x its kernel. Since $\mathcal{J}(n)$ is generated by its global sections (Section 4.2(c)); this map is nonzero and $H_x \neq H^0(X, \mathcal{J}(n))$.

Let $B = \bigoplus_{j \geq 0} H^0(X, \mathcal{O}_X(j))$. This is a graded k-algebra and $X \simeq \operatorname{Proj} B$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the homogeneous prime ideals of B defining $\xi_1, \ldots, \xi_r$. Let J be the homogeneous ideal $\bigoplus_{j \geq 0} H^0(X, \mathcal{J}(j))$ of B. Then C is the closed subscheme of X defined by J. By hypothesis, for each $i \leq r$, $\mathfrak{p}_i$ neither contains J nor $B(1)$. Let $J(n) := H^0(X, \mathcal{J}(n))$. We claim that for each $i \leq r$, $J(n) \cap \mathfrak{p}_i$ is a proper subspace of $J(n)$. Indeed, if $J(n_0) \cap \mathfrak{p}_i = J(n_0)$ for some $n_0 \geq m_0$, then the surjectivity of the map in Section 4.2(d) implies that $J(n) \cap \mathfrak{p}_i = J(n)$ for all $n \geq n_0$. This would imply $C = V_+(J) \supseteq V_+(\mathfrak{p}_i) \ni \xi_i$.

We have constructed above at most $r + \operatorname{Card}(D)$ proper subspaces of $H^0(X, \mathcal{J}(n))$. Since $r + \operatorname{Card}(D) \leq \operatorname{Card}(k)$ by hypothesis, the union of these proper subspaces is not equal to $H^0(X, \mathcal{J}(n))$ (Lemma 4.7). Since any element f_n in the complement of the union of these subspaces satisfies the desired properties, (a) follows.

(b) Let $\mathcal{J}_D$ be the ideal sheaf on X defining the structure of reduced closed subscheme on D. Choose $m \geq 0$ large enough such that both $\mathcal{J}$ and $\mathcal{J}\mathcal{J}_D$ are m-regular. As $H^1(X, (\mathcal{J}\mathcal{J}_D)(n)) = (0)$ for $n \geq m$ by Section 4.2(b), the map

$$H^0\big(X, \mathcal{J}(n)\big) \to H^0\big(X, \mathcal{J}(n)|_D\big) = H^0\big(X, \mathcal{J}(n)/\mathcal{J}\mathcal{J}_D(n)\big)$$

is surjective for all $n \geq m$. Note now the isomorphisms

$$H^0\big(X, \mathcal{J}(n)|_D\big) \to \bigoplus_{x \in D} \big(\mathcal{J}(n)|_D\big)_x \to \bigoplus_{x \in D} \mathcal{J}(n) \otimes k(x).$$

Let then $f \in H^0(X, \mathcal{J}(n))$ be a section such that its image in $\mathcal{J}(n) \otimes k(x)$ is nonzero for each $x \in D$. Keep the notation introduced in (a). Then $I := \bigoplus_{n \geq 0} H^0(X, \mathcal{J}^2(n))$ is a homogeneous ideal of B and $J^2 \subseteq I \subseteq J$. Hence $I \not\subseteq \mathfrak{p}_i$ for all $i \leq r$, since otherwise $J \subseteq \mathfrak{p}_i$, which contradicts the hypothesis that $\xi_i \notin C$.

Lemma 4.6(a) below implies then the existence of $n_0 \geq 0$ such that for all $n \geq n_0$, there exists $x_n \in I(n)$ such that $f_n := f + x_n \notin \bigcup_{1 \leq i \leq r} \mathfrak{p}_i$. We have $f_n \in J(n) = H^0(X, \mathcal{J}(n))$ and for all $x \in D$, f_n is nonzero in $\mathcal{J}(n) \otimes k(x)$. $\square$

The following prime avoidance lemma for graded rings is needed in the proof of Lemma 4.5. This lemma is slightly stronger than in [25, 4.11]. For related statements, see [71, Theorem A.1.2] or [7, III, 1.4, Proposition 8, p. 161]. We do not use the statement Lemma 4.6(b) in this article.

LEMMA 4.6

Let $B = \bigoplus_{n \geq 0} B(n)$ *be a graded ring. Let* $I = \bigoplus_{n \geq 0} I(n)$ *be a homogeneous ideal of* B*. Let* $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ *be homogeneous prime ideals of* B *not containing* $B(1)$ *and not containing* I*.*

(a) *Then there exists an integer* $n_0 \geq 0$ *such that for all* $n \geq n_0$ *and for all* $f \in B(n)$*, we have*

$$f + I(n) \not\subseteq \bigcup_{1 \leq i \leq r} \mathfrak{p}_i .$$

(b) *Let* k *be a field with* $\mathrm{Card}(k) > r$*, and assume that* B *is a* k*-algebra. If* I *can be generated by elements of degree at most* d*, then in (a) we can take* $n_0 = d$*.*

Proof

We can suppose that there are no inclusion relations between $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$.

(a) Let $i \leq r$, and set $I_i := I \cap (\bigcap_{j \neq i} \mathfrak{p}_j)$. We first observe that there exists $n_i \geq 0$ such that for all $n \geq n_i$, we have $I_i(n) \not\subseteq \mathfrak{p}_i$. Indeed, as $I_i \not\subseteq \mathfrak{p}_i$ and I_i is homogeneous, we can find a homogeneous element α in $I_i \setminus \mathfrak{p}_i$. Let $t \in B(1) \setminus \mathfrak{p}_i$. Set $n_i := \deg \alpha$. Then for all $n \geq n_i$, we have $t^{n-n_i}\alpha \in I_i(n) \setminus \mathfrak{p}_i$.

Let $n_0 := \max_{1 \leq i \leq r}\{n_i\}$. Let $n \geq n_0$, and let $f \in B(n)$. If $f \notin \bigcup_i \mathfrak{p}_i$, then clearly $f + I(n) \not\subseteq \bigcup_{1 \leq i \leq r} \mathfrak{p}_i$. Assume now that $f \in \bigcup_i \mathfrak{p}_i$, and for each j such that $f \in \mathfrak{p}_j$, choose $t_j \in I_j(n) \setminus \mathfrak{p}_j$. Then we easily verify that

$$f + \sum_{\mathfrak{p}_j \ni f} t_j \in \big(f + I(n)\big) \setminus \bigcup_{1 \leq i \leq r} \mathfrak{p}_i .$$

(b) Let $n \geq d$. For each $j \leq r$, let us show that $I(n) \not\subseteq \mathfrak{p}_j$. Suppose by contradiction that $I(n) \subseteq \mathfrak{p}_j$, and choose $t \in B(1) \setminus \mathfrak{p}_j$. Then $t^{n-e} I(e) \subseteq I(n) \subseteq \mathfrak{p}_j$ for all $1 \leq e \leq d$. Hence, $I(e) \subseteq \mathfrak{p}_j$, and then $I \subseteq \mathfrak{p}_j$ because I can be generated by the union of the $I(e)$, $1 \leq e \leq d$, which is a contradiction.

Let $f \in B(n)$, and suppose that $f + I(n) \subseteq \bigcup_{1 \leq i \leq r} \mathfrak{p}_i$. Then

$$I(n) = \bigcup_{1 \leq i \leq r} \big((-f + \mathfrak{p}_i) \cap I(n)\big).$$

If $(-f + \mathfrak{p}_i) \cap I(n)$ is not empty, pick $c_i \in ((-f + \mathfrak{p}_i) \cap I(n))$, and let $W_i := -c_i + ((-f + \mathfrak{p}_i) \cap I(n))$. The reader may easily check that W_i is a k-subspace of the k-vector space $I(n)$. Moreover, we claim that $W_i \neq I(n)$. Indeed, if $W_i = I(n)$, then $I(n) = c_i + W_i = (-f + \mathfrak{p}_i) \cap I(n)$. But then $f \in \mathfrak{p}_i$, which implies that $I(n) \subseteq \mathfrak{p}_i$, a contradiction. Therefore, the k-vector space $I(n)$ is a finite union of at most r proper k-affine subspaces, and this is also a contradiction (Lemma 4.7). $\square$

LEMMA 4.7

Let V be a vector space over a field k. For $i = 1, \dots, m$, let $v_i \in V$, and let V_i be a proper subspace of V. If $\mathrm{Card}(k) \geq m+1$, *then*

$$V \neq (v_1 + V_1) \cup \cdots \cup (v_m + V_m).$$

If $\mathrm{Card}(k) \geq m$, *then* $V \neq V_1 \cup \cdots \cup V_m$.

Proof

Assume that $\mathrm{Card}(k) \geq m+1$, and that $V = (v_1 + V_1) \cup \cdots \cup (v_m + V_m)$. We claim then that $V = V_1 \cup \cdots \cup V_m$. Indeed, fix $x \in V_1$, and let $y \in V \setminus V_1$. Since $\mathrm{Card}(k^*) \geq m$, we can find at least m elements of the form $v_1 + (x + \lambda y)$ with $\lambda \in k^*$, and

$$v_1 + (x + \lambda y) \in V \setminus (v_1 + V_1) \subseteq \bigcup_{2 \leq i \leq m} (v_i + V_i).$$

Thus there exist an index i and distinct λ_1, λ_2 in k^* such that $v_1 + x + \lambda_1 y$ and $v_1 + x + \lambda_2 y$ both belong to $v_i + V_i$. It follows that $(\lambda_1 - \lambda_2) y \in V_i$ and, thus, $y \in V_i$. Hence, $V = V_1 \cup \cdots \cup V_m$. The second statement of the lemma is well known and can be found, for instance, in [5, Lemma 2]. $\square$

Our final lemma in this section is a key ingredient in the proofs of Theorems 3.3 and 3.4 and is used to ensure that condition (2) in Theorem 2.1 holds for n big enough uniformly in $s \in S$.

LEMMA 4.8

Let S be a noetherian affine scheme, and let $X \to S$ be projective. Let $\mathcal{O}_X(1)$ be a very ample sheaf on X relative to $X \to S$. Let $C := V(\mathcal{J})$ be a closed subscheme of X. Let A_0 be a subset of X disjoint from C and such that there exists $c_0 \in \mathbb{N}$ with $\mathrm{Card}(A_0 \cap X_s) \leq c_0$ *for all $s \in S$. Let $\overline{\mathcal{J}}_s$ denote the image of $\mathcal{J}_s$ in $\mathcal{O}_{X_s}$. Then there exists $n_0 \geq 0$ such that for all $s \in S$ and for all $n \geq n_0$:*

(a) *There exists $f_{s,n} \in H^0(X_s, \overline{\mathcal{J}}_s(n))$ whose zero locus $V_+(f_{s,n})$ in X_s (when viewed as a section of $\mathcal{O}_{X_s}(n)$) does not contain any point of A_0.*

(b) *Suppose that $C \to S$ is finite. Then there exists $f_{s,n} \in H^0(X_s, \overline{\mathcal{J}}_s(n))$ as in (a) such that the image of $f_{s,n}$ in $(\overline{\mathcal{J}}_s(n)/\overline{\mathcal{J}}_s^2(n)) \otimes k(x)$ is nonzero for all $x \in C$ with $(\overline{\mathcal{J}}_s(n)/\overline{\mathcal{J}}_s^2(n)) \otimes k(x) \neq (0)$.*

Proof

When $C \to S$ is finite, we increase c_0 if necessary so we can assume that $\mathrm{Card}(C_s) \leq c_0$ for all $s \in S$. Let

$$Z_0 := \{ s \in S \mid \mathrm{Card}(k(s)) \leq 2c_0 \}.$$

Lemma 2.2 shows that Z_0 is a finite set.

Let n_0 be such that Lemma 4.4 applies. Fix $n > n_0$. It follows from Lemma 4.4(b) that $\overline{\mathcal{J}}_s$ is n-regular for all $s \in S$. Let $s \in S \setminus Z_0$. Then $\mathrm{Card}(k(s)) > \mathrm{Card}(A_0) + \mathrm{Card}(C_s)$. Parts (a) and (b) both follow from Lemma 4.5(a) applied to $\overline{\mathcal{J}}_s$, with D empty in the proof of (a), and $D = C_s$ in the proof of (b). For the remaining finitely many points $s \in Z_0$, we increase n_0 if necessary so that we can use Lemma 4.5(b) for each $s \in Z_0$. □

5. Avoidance lemma for families

We present in this section further applications of our method. Our first result below is a generalization of Theorem 3.3, where the noetherian hypothesis on the base has been removed.

THEOREM 5.1

Let S be an affine scheme, and let $X \to S$ be a quasi-projective and finitely presented morphism. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to $X \to S$. Let

(i) *C be a closed subscheme of X, finitely presented over S;*

(ii) *$F_1, \dots, F_m$ be subschemes of X of finite presentation over S;*

(iii) *A be a finite subset of X such that $A \cap C = \emptyset$.*

Assume that for all $s \in S$, C does not contain any irreducible component of positive dimension of $(F_i)_s$ and of X_s. Then there exists $n_0 > 0$ such that for all $n \geq n_0$, there exists a global section f of $\mathcal{O}_X(n)$ such that

(1) *the closed subscheme H_f of X is a hypersurface that contains C as a closed subscheme;*

(2) *for all $s \in S$ and for all $i \leq m$, H_f does not contain any irreducible component of positive dimension of $(F_i)_s$; and*

(3) *$H_f \cap A = \emptyset$.*

Assume in addition that S is noetherian, and that $C \cap \mathrm{Ass}(X) = \emptyset$. Then there exists such a hypersurface H_f which is locally principal.

Proof

The last statement when S is noetherian is immediate from the main statement of the theorem: simply apply the main statement of the theorem with A replaced by $A \cup \mathrm{Ass}(X)$. The fact that H_f is locally principal when $H_f \cap \mathrm{Ass}(X) = \emptyset$ is noted in Lemma 3.2(4).

Let us now prove the main statement of the theorem. First we add X to the set of subschemes F_i. Then the property of H_f being a hypersurface results from Theorem 5.1(2). Our main task is to reduce to the case S is noetherian and of finite dimension, in order to then apply Theorem 3.3. Using [29, IV.8.9.1, IV.8.10.5], we find the existence of an affine scheme S_0 of finite type over $\mathbb{Z}$, and of a morphism $S \to S_0$ such that all the objects of Theorem 5.1 descend to S_0. More precisely, there exists a quasi-projective scheme $X_0 \to S_0$ such that X is isomorphic to $X_0 \times_{S_0} S$. We will denote by $p: X \to X_0$ the associated "first projection" morphism. There also exists a very ample sheaf $\mathcal{O}_{X_0}(1)$ relative to $X_0 \to S_0$ whose pullback to X is $\mathcal{O}_X(1)$. There exists a closed subscheme C_0 of X_0 such that C is isomorphic to $C_0 \times_{S_0} S$. Finally, there exist subschemes $F_{1,0}, \dots, F_{m,0}$ of X_0 such that F_i is isomorphic to $F_{i,0} \times_{S_0} S$. Let $A_0 := p(A)$.

Since S_0 is of finite type over $\mathbb{Z}$, S_0 is noetherian and of finite dimension. The data $X_0, C_0, \{(F_{1,0} \setminus C_0), \dots, (F_{m,0} \setminus C_0)\}, A_0$ satisfy the hypothesis of Theorem 3.3. Let $n_0 > 0$ and let, for all $n \geq n_0$, an $f_0 \in H^0(X_0, \mathcal{O}_{X_0}(n))$ be given by Theorem 3.3 with respect to these data. Let H_f be the closed subscheme of X defined by the canonical image of f_0 in $H^0(X, \mathcal{O}_X(n))$. Then $H_f = H_{f_0} \times_{S_0} S$ contains C as a closed subscheme and $H_f \cap A = \emptyset$. It remains to check condition (2) of Theorem 5.1. Let ξ be the generic point of an irreducible component of positive dimension of $F_{i,s}$. Let $s = p(s_0)$, and let $\xi_0 = p(\xi)$. Then an open neighborhood of ξ in $(F_i)_s$ has empty intersection with C. As $C = C_0 \times_{S_0} S$, this implies that the same is true for ξ_0 in $(F_{i,0})_s$. Hence ξ_0 is the generic point of an irreducible component of $(F_{i,0} \setminus C_0)_s$ of positive dimension. Thus $\xi_0 \notin H_{f_0}$ and $\xi \notin H_f$. $\square$

Remark 5.2

The classical avoidance lemma states that if X/k is a quasi-projective scheme over a field, $C \subsetneq X$ is a closed subset of positive codimension, and $\xi_1, \dots, \xi_r$ are points of X not contained in C, then there exists a hypersurface H in X such that $C \subseteq H$ and $\xi_1, \dots, \xi_r \notin H$.

Let S be a noetherian scheme, and let X/S be a quasi-projective scheme. One may wonder whether it is possible to strengthen Theorem 5.1, the avoidance lemma for families, by strengthening its condition (2). The following example shows that Theorem 5.1 does not hold if condition (2) is replaced by the stronger condition (2′): For all $s \in S$, H_f does not contain any irreducible component of F_s.

Let $S = \operatorname{Spec} R$ be a Dedekind scheme such that $\operatorname{Pic}(S)$ is not a torsion group (see, e.g., [27, Corollary 2]). Let $\mathcal{L}$ be an invertible sheaf on S of infinite order. Consider as in Proposition 7.10 the scheme $X = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$ with its natural projective morphism $X \to S$. Let F be the union of the two horizontal sections C_0 and C_∞. If Theorem 5.1 with condition (2′) holds, then there exists a hypersurface H_f which is a finite quasi-section of $X \to S$ (as defined in Definition 0.1), and which does not meet F. Proposition 7.10 shows that this can only happen when $\mathcal{L}$ has finite order.

Remark 5.3

Let k be any field, and let X/k be an irreducible proper scheme over k. Let $C \subsetneq X$ be a closed subscheme, and let $\xi_1, \dots, \xi_r$ be points of X not contained in C. We may ask whether an avoidance lemma holds for X/k in the following senses: (1) Does there exist a line bundle $\mathcal{L}$ on X and a section $f \in \mathcal{L}(X)$ such that the closed subscheme H_f contains C and $\xi_1, \dots, \xi_r \notin H_f$? We may also ask (2) whether there exists a codimension 1 subscheme H of X such that H contains C and $\xi_1, \dots, \xi_r \notin H$.

The answer to the first question is negative, as there exist proper schemes X/k with $\mathrm{Pic}(X) = (0)$. For instance, a normal proper surface X/k over an uncountable field k with $\mathrm{Pic}(X) = (0)$ is constructed in [65, Section 3].

The answer to the second question is also negative when X/k is not smooth. Recall the example of Nagata–Mumford (see [4, pp. 32–33]). Consider the projective plane $\mathbb{P}^2_k/k$, and fix an elliptic curve E/k in it, with origin O. Assume that $E(k)$ contains a point x of infinite order. Fix ten distinct multiples $n_i x$, $i = 1, \dots, 10$. Blow up $\mathbb{P}^2_k$ at these ten points to get a scheme Y/k. Since x has infinite order, any codimension 1 closed subset of Y intersects the strict transform F of E in Y. Now F has negative self-intersection on Y by construction, and so there exists an algebraic space Z and a morphism $Y \to Z$ which contracts F. The algebraic space Z is not a scheme, since the image z of F in Z cannot be contained in an open affine of Z. It follows from [43, 16.6.2], that there exists a scheme X with a finite surjective morphism $X \to Z$. The finite set consisting of the preimage of z in X meets every codimension 1 subscheme H of X.

Remark 5.4

Let S be an affine scheme, and let $X \to S$ be projective and smooth. Fix a very ample invertible sheaf $\mathcal{O}_X(1)$ relative to $X \to S$ as in Theorem 5.1. It is not possible in general to find $n > 0$ and a global section $f \in \mathcal{O}_X(n)$ such that $H_f \to S$ is smooth. Examples of N. Fakhruddin illustrating this point can be found in [59, 5.14, 5.15]. M. Nishi ([56, Section 2] and [16, Remarks (b), p. 80]) gave an example of a nonsingular cubic surface C in $\mathbb{P}^4_k$ which is not contained in any nonsingular hypersurface of $\mathbb{P}^4_k$. Our next two corollaries are examples of weaker "theorems of Bertini type for families," where, for instance, smooth is replaced by Cohen–Macaulay.

Recall that a locally noetherian scheme Z is (S_ℓ) for some integer $\ell \geq 0$ if for all $z \in Z$, the depth of $\mathcal{O}_{Z,z}$ is at least equal to $\min\{\ell, \dim \mathcal{O}_{Z,z}\}$ (see [29], IV.5.7.2).

COROLLARY 5.5

Let S be an affine scheme, and let $X \to S$ be a quasi-projective and finitely presented morphism. Let C be a closed subscheme of X finitely presented over S. Assume that for all $s \in S$, C does not contain any irreducible component of positive dimension of X_s. Suppose that for some $\ell \geq 1$, X_s is (S_ℓ) for all $s \in S$. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to $X \to S$.

Then there exists $n_0 > 0$ such that for all $n \geq n_0$, there exists a global section f of $\mathcal{O}_X(n)$ such that $H_f \subset X$ is a hypersurface containing C as a closed subscheme, and the fibers of $H_f \to S$ are $(\mathrm{S}_{\ell-1})$. In particular, if the fibers of $X \to S$ are Cohen–Macaulay, then the same is true of the fibers of $H_f \to S$. Moreover:

(a) *Assume that X_s has no isolated point for all $s \in S$. If $X \to S$ is flat, then $H_f \to S$ can be assumed to be flat and locally principal.*

(b) *Assume that S is noetherian and that $\mathrm{Ass}(X) \cap C = \emptyset$. Then $H_f \to S$ can be assumed to be locally principal.*

Proof

We apply Theorem 5.1 to $X \to S$ and C, with $F_1 = X$, $m = 1$ and with $A = \emptyset$. Let $H := H_f$ be a hypersurface in X as given by Theorem 5.1. For all $s \in S$, H_s does not contain any irreducible component of X_s of positive dimension. Since X_s is (S_ℓ) with $\ell \geq 1$, X_s has no embedded points. At an isolated point of X_s contained in H, H_s is trivially (S_k) for any $k \geq 0$. At all other points, it follows that H_s is locally generated everywhere by a regular element and, thus, H_s is $(\mathrm{S}_{\ell-1})$.

The statement (a) follows from Lemma 3.2(3). For (b), we apply Theorem 5.1 and Lemma 3.2(4) to $X \to S$ and C, with $F_1 = X$ and with $A = \mathrm{Ass}(X)$. ☐

COROLLARY 5.6

Let S be an affine irreducible scheme of dimension 1. Let $X \to S$ be quasi-projective and flat of finite presentation. Assume that its generic fiber is (S_1), and that it does not contain any isolated point. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to $X \to S$. Then there exists $n_0 > 0$ such that for all $n \geq n_0$, there exists a global section f of $\mathcal{O}_X(n)$ such that $H_f \subset X$ is a locally principal hypersurface, flat over S.

Proof

Consider the set M of all $x \in X$ such that X_s is not (S_1) at x (or, equivalently, the set of all $x \in X$ such that x is contained in an embedded component of X_s). Then this set is constructible (see [29, IV.9.9.2(viii)]), and even closed since $X \to S$ is flat (see [29, IV.12.1.1(iii)]). Since the generic fiber is (S_1), the image of M in S must be finite because S has dimension 1. Therefore, there are only finitely many $s \in S$ such that the fiber X_s is not (S_1). Let M' denote the set of associated points in the fibers that are not (S_1). This set is finite. Apply now Theorem 5.1 with the very ample sheaf $\mathcal{O}_X(1)$ and with $F_i = C = \emptyset$ and $A = M'$, to find a hypersurface H_f which does not intersect M'. This hypersurface is locally principal and flat over S by Lemma 3.2(3). Indeed, by construction, $(H_f)_s$ does not contain any irreducible component of X_s of positive dimension. Our hypothesis on the generic fiber having no isolated point implies that X_s has no isolated point for all s, since the set of all points that are isolated in their fibers is open (see [29, IV.13.1.4]). ☐

We discuss below one additional application of Theorem 2.1.

PROPOSITION 5.7

Let $S = \operatorname{Spec} R$ be an affine scheme, and let $\pi : X \to S$ be projective and finitely presented. Let C be a closed subscheme of X, finitely presented over S. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to $X \to S$. Let $Z \subset S$ be a finite subset. Suppose that

(i) *$\pi : X \to S$ is smooth at every point of $\pi^{-1}(Z)$;*

(ii) *for all $s \in Z$, C_s is smooth and $\operatorname{codim}_x(C_s, X_s) > \frac{1}{2}\dim_x X_s$ for all $x \in C_s$;*

(iii) *for all $s \in S$, C_s does not contain any irreducible component of positive dimension of X_s, and for all $s \in Z$, X_s has no isolated point.*

Then there exists an integer n_0 such that for all $n \geq n_0$, there exists a global section f of $\mathcal{O}_X(n)$ such that H_f is a hypersurface containing C as a closed subscheme and such that $H_f \to S$ is smooth in an open neighborhood of $\pi^{-1}(Z) \cap H_f$.

Proof

By arguing as in the proof of Theorem 5.1 (using also [29, IV.11.2.6] or [29, IV.17.7.8]), we find that it suffices to prove the proposition in the case where S is noetherian and has finite dimension.

Let $\mathcal{J}$ be the ideal sheaf defining C. As in the proof of Theorem 3.3 in Section 3.11, when $m = 1$ and $F_1 = X$, there exists n_0 such that for any $n \geq n_0$ and for any choice of generators $f_1, \dots, f_N$ of $H^0(X, \mathcal{J}(n))$, the associated constructible subset $T_X \subseteq \mathbb{A}^N_S$ (see Lemma 3.7) has dimension at most $N - 1$. For each $s \in Z$, we can apply Lemma 5.8 to the following data: the morphism $X_s \to \operatorname{Spec} k(s)$, $C_s \subseteq X_s$ defined by the ideal sheaf $\overline{\mathcal{J}}_s \subseteq \mathcal{O}_{X_s}$ (as in Notation 3.6), and the sections $\overline{f}_{i,s}$, image of f_i in $H^0(X_s, \overline{\mathcal{J}}_s(n))$, $i = 1, \dots, N$. We denote by $T_{Z,s}$ the constructible subset of $\mathbb{A}^N_{k(s)}$ associated in Lemma 5.8(1) with these data. Note that $T_{Z,s}$ is then a pro-constructible subset of $\mathbb{A}^N_S$.

Let n and $f_1, \dots, f_N$ be as above. Consider the finite union T of T_X and the pro-constructible subsets $T_{Z,s}$, $s \in Z$. It is pro-constructible with constructible fibers over S. Lemma 5.8 shows that conditions (1) and (2) in Theorem 2.1 (with $V = S$) are satisfied, after increasing n_0 if necessary. Then by Theorem 2.1, there exists a section $(a_1, \dots, a_N) \in R^N = \mathbb{A}^N_S(S)$ such that for all $s \in S$, $(a_1(s), \dots, a_N(s))$ is a $k(s)$-rational point of $\mathbb{A}^N_{k(s)}$ that is not contained in T_s. Let $f := \sum_{i=1}^N a_i f_i$ and consider the closed subscheme $H_f \subset X$. As $f \in H^0(X, \mathcal{J}(n))$, C is a closed subscheme of H_f. By definition of T_X, we find that for all $s \in S$, H_f does not contain any irreducible component of X_s of positive dimension, so that H_f is a hypersurface relative to $X \to S$. By definition of $T_{Z,s}$, we find that $(H_f)_s$ is smooth for all $s \in Z$. Since $X \to S$ is flat in a neighborhood U of $\pi^{-1}(Z)$, and since X_s does not contain any isolated point when $s \in Z$, we find that $U \cap H_f \to S$ is flat at every point of

$\pi^{-1}(Z)$ (Lemma 3.2(3)). So $H_f \to S$ is smooth in a neighborhood of $\pi^{-1}(Z)$ by the openness of the smooth locus. $\square$

LEMMA 5.8

Let S be a noetherian scheme. Let $X \to S$ be a quasi-projective morphism. Let $\mathcal{O}_X(1)$ be a very ample sheaf relative to $X \to S$. Let C be a closed subscheme of X defined by a sheaf of ideals $\mathcal{J}$. Let $n \geq 1$, and let $f_1, \ldots, f_N \in H^0(X, \mathcal{J}(n))$.

(1) *Define for any $s \in S$*

$$\Sigma_{\text{sing}}(s) := \Big\{(a_1, \ldots, a_N) \in k(s)^N \;\Big|\; V_+\Big(\sum_i a_i \overline{f}_{i,s}\Big) \subseteq X_s \text{ is not smooth over } k(s)\Big\}.$$

(Here $\sum_i a_i \overline{f}_{i,s}$ is viewed as a section of $\mathcal{O}_{X_s}(n)$ and $V_+(\sum_i a_i \overline{f}_{i,s})$ denotes its zero locus in X_s.) Then there exists a constructible subset T_{sing} of $\mathbb{A}^N_S$ such that for all $s \in S$, the set of $k(s)$-rational points of $\mathbb{A}^N_S$ contained in T is equal to $\Sigma_{\text{sing}}(s)$. Moreover, T_{sing} is compatible with base changes $S' \to S$ as in Proposition 1.10(a).

(2) *Let k be a field and assume that $S = \operatorname{Spec} k$. Suppose also that C and X are projective and smooth over k, and that $\operatorname{codim}_x(C, X) > \frac{1}{2} \dim_x X$ for all $x \in C$.*
Then there exists n_0 such that for any $n \geq n_0$ and for any choice of a system of generators $f_1, \ldots, f_N$ of $H^0(X, \mathcal{J}(n))$, we have $\dim T_{\text{sing}} \leq N - 1$, and there exists $(a_1, \ldots, a_N) \in k^N$ such that $V_+(\sum_i a_i f_i)$ is smooth and does not contain any irreducible component of X.

Proof

(1) Write $\mathbb{A}^N_S = \operatorname{Spec} \mathcal{O}_S[u_1, \ldots, u_N]$. Consider the natural projections

$$p : \mathbb{A}^N_X \to X, \qquad q : \mathbb{A}^N_X \to \mathbb{A}^N_S.$$

With the appropriate identifications, consider the global section $\sum_{1 \leq i \leq N} u_i f_i$ of $p^*(\mathcal{J}(n))$, which defines the closed subscheme $Y := V_+(\sum_{1 \leq i \leq N} u_i f_i)$ of $\mathbb{A}^N_X$. Consider

$$\mathcal{S} := \big\{y \in Y \;\big|\; Y_{q(y)} \text{ is not smooth at } y \text{ over } k\big(q(y)\big)\big\}.$$

Set $T_{\text{sing}} := q(\mathcal{S})$. Clearly T_{sing} satisfies all the requirements of the lemma except for the constructibility, which we now prove. By Chevalley's theorem, it is enough to show that $\mathcal{S}$ is constructible. The subset $\mathcal{S}$ is the union for $0 \leq d \leq \max_{z \in \mathbb{A}^N_S} \dim Y_z$ (see [29, IV.13.1.7]) of the subsets

$$\mathcal{S}_d := \big\{y \in Y \;\big|\; \dim_y Y_{q(y)} \leq d < \dim_{k(y)}\big(\Omega^1_{Y/\mathbb{A}^N_S} \otimes k(y)\big)\big\}.$$

Thus it is enough to show that $\mathcal{S}_d$ is constructible. The set $\{y \in Y \mid \dim_y Y_{q(y)} \leq d\}$ is open by Chevalley's semicontinuity theorem (see [29, IV.13.1.3]). On the other hand, for any coherent sheaf $\mathcal{F}$ on Y, the subset $\{y \in Y \mid \dim_{k(y)}(\mathcal{F} \otimes k(y)) \geq d+1\}$ is closed in Y. So $\mathcal{S}_d$ is constructible and (1) is proved.

(2) By the compatibility with base changes, the dimension of T_{sing} can be computed over an algebraic closure of k. Now over an infinite field, [40, Theorem (7), p. 787] implies that the generic point of $\mathbb{A}_k^N$ is not contained in T_{sing} for all n big enough and for all systems of generators $\{f_1, \ldots, f_N\}$ of $H^0(X, \mathcal{J}(n))$. As T_{sing} is constructible, we have $\dim T_{\mathrm{sing}} \leq N-1$.

Let $\Gamma_1, \ldots, \Gamma_m$ be the connected components of X. They are irreducible and

$$H^0\big(X, \mathcal{J}(n)\big) = \bigoplus_{1 \leq i \leq m} H^0\big(\Gamma_i, \mathcal{J}(n)|_{\Gamma_i}\big).$$

By hypothesis, $\dim C \cap \Gamma_i < \frac{1}{2} \dim \Gamma_i$. So by [40, Theorem (7)], when k is infinite, and by [60, Theorem 1.1(i)] when k is finite, increasing n_0 if necessary, for any $n \geq n_0$, there exists $g_i \in H^0(\Gamma_i, \mathcal{J}(n)|_{\Gamma_i})$ such that $V_+(g_i) \subset \Gamma_i$ is smooth and of dimension $\dim \Gamma_i - 1$. Let $f := g_1 \oplus \cdots \oplus g_m \in H^0(X, \mathcal{J}(n))$. Then $V_+(f)$ is a smooth subvariety of X not containing any irreducible component of X. Let $f_1, \ldots, f_N$ be any system of generators of $H^0(X, \mathcal{J}(n))$, and write $f = \sum_i a_i f_i$ with $a_i \in k$. Then $(a_1, \ldots, a_N) \in k^N$ is the desired point. $\square$

6. Finite quasi-sections

Let $X \to S$ be a surjective morphism. We call a closed subscheme T of X a *finite quasi-section* when $T \to S$ is finite and surjective (Definition 0.1). We establish in Theorem 6.3 the existence of a finite quasi-section for certain types of projective morphisms. The existence of quasi-finite quasi-sections locally on S for flat or smooth morphisms is discussed in [29, IV.17.16].

When S is integral noetherian of dimension 1 and $X \to S$ is proper, the existence of a finite quasi-section T is well known and easy to establish. It suffices to take T to be the Zariski closure of a closed point of the generic fiber of $X \to S$. Then $T \to S$ have fibers of dimension 0 (see, e.g., [45, 8.2.5]), so it is quasi-finite and proper and, hence, finite. When $\dim S > 1$, the process of taking the closure of a closed point of the generic fiber does not always produce a closed subset *finite* over S, as the simple example below shows.

Example 6.1

Let $S = \operatorname{Spec} A$ with A a noetherian integral domain, and let $K = \operatorname{Frac}(A)$. Let $X = \mathbb{P}^1_A$. Choose coordinates and write $X = \operatorname{Proj} A[t_0, t_1]$. Let $P \in X_K(K)$ be given as $(a : b)$, with $a, b \in A \setminus \{0\}$. When $(bt_0 - at_1)$ is a prime ideal in $A[t_0, t_1]$, then $T := V_+(bt_0 - at_1)$ is the Zariski closure of P in X. When in addition $aA + bA \neq A$,

T is not finite over S. For a concrete example with S regular of dimension 2, take k a field and $A = k[t,s]$, with $a = t$, and $b = s$. (Note that when $\dim(A) = 1$ and $aA + bA \neq A$, the ideal $(bt_0 - at_1)$ is never prime in $A[t_0, t_1]$.)

More generally, to produce K-rational points on the generic fiber of $\mathbb{P}^n_S \to S$ for some $n > 1$ whose closure is not finite over S, we can proceed as follows. Let $T \to S$ be the blowing-up of S with respect to a coherent sheaf of ideals I, and choose I so that $T \to S$ is not finite. Then $T \to S$ is a projective morphism, and we can choose $T \to \mathbb{P}^n_S$ to be a closed immersion over S for some $n > 0$. Let ξ denote the generic point of the image of T in $X := \mathbb{P}^n_S$. Then ξ is a closed point of the generic fiber of $X \to S$, and the closure of ξ in X is not finite over S.

The composition $\mathbb{P}^d_T = \mathbb{P}^d_S \times T \to T \to S$ is an example of a projective morphism which does not have any finite quasi-section. In this example, one irreducible fiber has dimension greater than d.

Before turning to the main theorem of this section, let us note here an instance of interest in arithmetic geometry where the closure of a rational point of the generic fiber is a section.

PROPOSITION 6.2

Let S be a noetherian regular integral scheme, with function field K. Let $X \to S$ be a proper morphism such that no geometric fiber $X_{\bar{s}}$ contains a rational curve. Then any K-rational point of the generic fiber of $X \to S$ extends to a section over S.

Proof

Let T be the (reduced) Zariski closure of a rational point of the generic fiber of $X \to S$. Consider the proper birational morphism $f : T \to S$. Denote by $E := E(f)$ the exceptional set of f, that is, the set of points $x \in T$ such that f is not a local isomorphism at x. Suppose $E \neq \emptyset$. Since S is regular, by van der Waerden's purity theorem (see [29, IV.21.12.12] or [45, 7.2.22]), E has pure codimension 1 in T. Let ξ be a generic point of E, and let $s = f(\xi)$. Using the dimension formula (see [29, IV.5.5.8], [45, 8.2.5]) and because S is regular and hence universally catenary (see [29, IV.5.6.4]), we find

$$\operatorname{trdeg}_{k(s)} k(\xi) = \dim \mathcal{O}_{S,s} - 1.$$

Let $\overline{T} \to T$ be the normalization of T, and let η be a point of $\overline{T}$ lying over ξ. Then by Krull–Akizuki [17, 6.3.23], $\mathcal{O}_v := \mathcal{O}_{\overline{T},\eta}$ is a discrete valuation ring. It has center s in S. As $k(\eta)$ is algebraic (even finite) over $k(\xi)$, we have $\operatorname{trdeg}_{k(s)} k(\eta) = \dim \mathcal{O}_{S,s} - 1$. So $\mathcal{O}_v$ is a prime divisor of $K(S)$ in the sense of [1, Definition 1]. It follows from a theorem of Abhyankar [1, Proposition 3] that $k(\eta)$ is the function field of a ruled variety of positive dimension over $k(s)$. One can also prove this result in a more geometric flavor as in [45, Exercise 8.3.14, (a)–(b)]. (The hypothesis that the base scheme is Nagata is not needed in our situation as the local rings which intervene are all regular.) So $\overline{T}_{\overline{s}}$ contains a rational curve. As $\overline{T}_{\overline{s}} \to T_{\overline{s}}$ is integral, the image of such a curve is a rational curve in $T_{\overline{s}}$. It follows that $X_{\overline{s}}$ contains a rational curve, and this is a contradiction. So E is empty, and $T \to S$ is an isomorphism. $\square$

THEOREM 6.3

Let S be an affine scheme, and let $\pi : X \to S$ be a projective, finitely presented morphism. Suppose that all fibers of $X \to S$ are of the same dimension $d \geq 0$. Let C be a finitely presented closed subscheme of X, with $C \to S$ finite but not necessarily surjective. Then there exists a finite quasi-section $T \to S$ of finite presentation which contains C. Moreover:

(1) *Assume that S is noetherian. If C and X are both irreducible, then there exists such a quasi-section with T irreducible.*

(2) *If $X \to S$ is flat with Cohen–Macaulay fibers (e.g., if S is regular and X is Cohen–Macaulay and equidimensional over S), then there exists such a quasi-section with $T \to S$ flat.*

(3) *If $X \to S$ is flat and a local complete intersection morphism,*† *then there exists such a quasi-section with $T \to S$ flat and a local complete intersection morphism.*

(4) *Assume that S is noetherian. Suppose that $\pi : X \to S$ has equidimensional fibers and that $C \to S$ is unramified. Let Z be a finite subset of S (such as the set of generic points of $\pi(C)$), and suppose that there exists an open subset U of S containing Z such that $X \times_S U \to U$ is smooth. Then there exists such a quasi-section T of $X \to S$ and an open set $V \subseteq U$ containing Z such that $T \times_S V \to V$ is étale.*

Proof

To prove the first conclusion of the theorem, it suffices to show that X/S has a finite quasi-section T of finite presentation. Then $T \cup C$ is a finite quasi-section which contains C. If $d = 0$, then $X \to S$ itself is finite. Suppose $d \geq 1$. It follows from Theorem 5.1, with $A = \emptyset$ and $F = \emptyset$, that there exists a hypersurface H in X. By definition of a hypersurface, for all $s \in S$, H_s does not contain any irreducible component of X_s of positive dimension. Lemma 3.2(1) and our hypotheses show that every fiber H_s has dimension $d - 1$. Lemma 3.2(2) shows that H/S is also finitely

†Since the morphism $X \to S$ is flat, it is a local complete intersection (l.c.i.) morphism if and only if every fiber is l.c.i. (see, e.g., [45, 6.3.23]).

presented. Repeating this process another $d-1$ times produces the desired quasi-section.

(1) Since X is assumed irreducible and since the fibers of $X \to S$ are all not empty by hypothesis, we find that $X \to S$ is surjective and that S is irreducible. When $d=0$, $X \to S$ is then an irreducible finite quasi-section and contains C as a closed subscheme. Assume now that $d \geq 1$. Then we can find a hypersurface H_f which contains C as a closed subscheme (Theorem 5.1). Since S is noetherian, we can use Lemma 6.4 below and the assumption that C is irreducible to find an irreducible component Γ of H_f which contains (set-theoretically) C, dominates S, and such that all fibers of $\Gamma \to S$ have dimension $d-1$. Let $\mathcal{J}_C$ and $\mathcal{J}_\Gamma$ denote the sheaves of ideals in $\mathcal{O}_X$ defining C and Γ, respectively. Then some positive power $\mathcal{J}_\Gamma^m$ is contained in $\mathcal{J}_C$, and we endow the irreducible closed set Γ with the structure of scheme given by the structure sheaf $\mathcal{O}_X/\mathcal{J}_\Gamma^m$. By construction, the scheme Γ is irreducible and contains C as a closed subscheme. If $d-1>0$, we repeat the process with $\Gamma \to S$.

(2) When $d=0$, the statement is obvious. Assume now that $d>0$. Since X_s has no embedded point for all $s \in S$, we find that for each $i \geq 0$, the set X_i of all $x \in X$ such that every irreducible component of $X_{\pi(x)}$ passing through x has dimension i is open in X (see [29, IV.12.1.1(ii)], using here that $X \to S$ is flat). Moreover, since X_s is Cohen–Macaulay for all s, the irreducible components of X_s passing through a given point x have the same dimension. We find that X is the disjoint union of the open sets X_i. Each $X_i \to S$ is of finite presentation, since each X_i is open and closed in X (see [29, IV.1.6.2(i)]).

Consider now $X_0 \to S$, which is clearly quasi-finite of finite presentation and flat. Since $X \to S$ is projective, $X_0 \to S$ is then also finite (see [29, IV.8.11.1]). We apply Corollary 5.5(a) to the finitely presented scheme $X' := (X \setminus X_0) \to S$ and the finite quasi-section $C' := C \times_X X'$. We obtain a hypersurface H' containing C', and using the same method as in the proof of the first statement of the theorem, we obtain a finite flat quasi-section T' of $X' \to S$ containing C'. Then $T := T' \cup X_0$ is the desired finite flat quasi-section.

To prove (3), we proceed as in (2), and remark that the hypersurface H' obtained from Corollary 5.5(a) is flat and locally principal, so that its fiber H'_s is l.c.i. over $k(s)$ when X_s is. By hypothesis, X_0/S has only l.c.i. fibers, and (3) follows.

(4) When $d=0$, $X \to S$ is the desired finite quasi-section, since it is étale over the given open subset U of S. Assume now that $d>0$. By hypothesis, $C \to S$ is finite and unramified, so that for each $s \in S$, $C_s \to \operatorname{Spec} k(s)$ is smooth. Moreover, since we are assuming that the fibers are pure of dimension d, condition (iii) in Proposition 5.7 is satisfied. We can therefore apply Proposition 5.7 with Z, to find a hypersurface H_f of $X \to S$ containing C as a closed subscheme, with H_f smooth over an open neighborhood W of Z in S. For all $s \in S$, X_s is pure of dimension d and $(H_f)_s$ is

a hypersurface in X_s. Thus, $(H_f)_s$ is pure of dimension $d-1$ for all $s \in S$. Therefore, the above discussion can be applied to the morphism $H_f \to S$, which induces a smooth morphism $H_f \times_S W \to W$, to produce a hypersurface H_{f_2} of $H_f \to S$ containing C as a closed subscheme, with H_{f_2} smooth over an open neighborhood W_2 of Z in S. Thus, we obtain the desired finite quasi-section after d such steps. □

LEMMA 6.4

Let S be affine, noetherian, and irreducible, with generic point η. Let $\pi : X \to S$ be a morphism of finite type. For each irreducible component Δ of X, suppose that $\Delta \to S$ has generic fiber of positive dimension. Let $\mathcal{L}$ be an invertible sheaf on X with a global section f, and assume that $H := H_f \subset X$ is a hypersurface relative to $X \to S$. Then:

(1) *Each irreducible component Γ of H dominates S.*

(2) *Assume in addition that for some $d \geq 1$, the fibers of each morphism $\Delta \to S$ all have dimension d. Then $X \to S$ is equidimensional of dimension d and $H \to S$ is equidimensional of dimension $d-1$.*

Proof

(1) Apply [29, IV.13.1.1] to each morphism $\Delta \to S$ to find that for all $s \in S$, all irreducible components of X_s have positive dimension. Let Γ be an irreducible component of H. Let Z denote the Zariski closure of $\pi(\Gamma)$ in S. We need to show that $Z = S$. Let us first show by contradiction that $\operatorname{codim}(\Gamma, X_Z) > 0$. Otherwise, Γ contains an irreducible component T of X_Z. Let t be the generic point of T. Since $T_{\pi(t)}$ is irreducible and dense in T, it is an irreducible component of $X_{\pi(t)}$. In particular, $T_{\pi(t)}$ has positive dimension, and is contained in Γ. This contradicts our hypothesis that H is a hypersurface.

Every irreducible component of X_Z is contained in an irreducible component of X, and every irreducible component of X has nonempty generic fiber. Thus, if $Z \neq S$, then $\operatorname{codim}(X_Z, X) > 0$, and

$$\operatorname{codim}(\Gamma, X) \geq \operatorname{codim}(\Gamma, X_Z) + \operatorname{codim}(X_Z, X) \geq 2.$$

This is a contradiction with the inequality $\operatorname{codim}(\Gamma, X) \leq 1$, which follows from Krull's principal ideal theorem. Hence, $Z = S$.

(2) Let us first show that $X \to S$ is equidimensional of dimension $d \geq 1$. The definition of *equidimensional* is found in [29, IV.13.3.2]. We use [29, IV.13.3.3] to prove our claim. Indeed, our hypotheses imply that the image under $X \to S$ of each irreducible component Δ of X is S, and that the generic fibers of all induced morphisms $\Delta \to S$ have equal dimension d. Let $x \in \Delta$, and let $s \in S$ be its image. We have $\dim_x \Delta_s \leq \dim \Delta_s = d$. We find by using [29, IV.13.1.6] that $\dim_x \Delta_s \geq \dim \Delta_\eta = d$. Our claim follows immediately.

Let now Γ be an irreducible component of H. We know from (1) that $\Gamma \to S$ is dominant. Thus, by using [29, IV.13.3.3] to show that $H \to S$ is equidimensional of dimension $d-1$, it suffices to show that $\Gamma \to S$ is equidimensional of dimension $d-1$. Since Γ_η is an irreducible component of the hypersurface $H_\eta \cap \Delta_\eta$ of Δ_η, we have $\dim \Gamma_\eta = \dim \Delta_\eta - 1 = \dim X_\eta - 1$.

Let $s \in S$ be such that Γ_s is not empty. Our hypothesis that H is a hypersurface implies that Γ_s does not contain any irreducible component of X_s of positive dimension. Thus, $\dim \Gamma_s \leq \dim X_s - 1 = d-1$. By [29, IV.13.1.6], Γ_s is equidimensional of dimension $d-1$. It follows that $\Gamma \to S$ is equidimensional of dimension $d-1$. $\square$

Remark 6.5

Let S be an affine integral scheme. The scheme $X := \mathbb{P}^1_S \sqcup S$ is an S-scheme in a natural way, and every irreducible component of X dominates S. Any proper closed subset of S defined by a principal ideal is a hypersurface H_0 of S. Thus, there exist hypersurfaces $H := H_1 \sqcup H_0$ of X such that the irreducible component H_0 of H does not dominate S. As Lemma 6.4(a) shows, this cannot happen when every irreducible component of X has a generic fiber of positive dimension.

Remark 6.6

Let S be a noetherian affine scheme. A variant of Theorem 6.3 can be obtained when the morphism $\pi : X \to S$ is only assumed to be quasi-projective, but satisfies the following additional condition: there exists a scheme $\overline{X}$ with a projective morphism $\pi' : \overline{X} \to S$ having all fibers of dimension $d > 0$, and an open S-immersion $X \to \overline{X}$ with dense image and $\dim(\overline{X} \setminus X) < d$. Keeping all other hypotheses of Theorem 6.3 in place, its conclusions then also hold under the above weaker hypotheses on $\pi : X \to S$. The proof of this variant is similar to the proof of Theorem 6.3, and consists in applying Theorem 5.1 d times, starting with the data $\overline{X}$, C, $F := \overline{X} \setminus X$, and the finite set A containing the generic points of F.

Remark 6.7

Let S be an affine scheme, and let $X \to S$ be a *smooth*, projective, and surjective, morphism. We may ask whether $X \to S$ always admits a *finite étale* quasi-section. (The existence of a quasi-finite étale quasi-section is proved in [29, IV.17.16.3(ii)].) The answer to the above question is known in two cases of arithmetic interest.

First, let S be a smooth affine geometrically irreducible curve over a finite field. Let $X \to S$ be a smooth and surjective morphism, with geometrically irreducible generic fiber. Then X/S has a finite étale quasi-section (see [73, Theorem (0.1)]).

Let now $S = \operatorname{Spec} \mathbb{Z}$. The answer to this question in this case is negative, as examples of K. Buzzard [8] show. Indeed, a positive answer to this question over $S =$

$\operatorname{Spec}\mathbb{Z}$ would imply that any smooth, projective, surjective, morphism $X \to \operatorname{Spec}\mathbb{Z}$ has a generic fiber which has a $\mathbb{Q}$-rational point. The hypersurface X/S in $\mathbb{P}^7_{\mathbb{Z}}$ defined by the quadratic form $f(x_1, \ldots, x_8)$ associated with the E_8-lattice is smooth over S because the determinant of the associated symmetric matrix is ± 1, and the generic fiber of X/S has no $\mathbb{R}$-points because f is positive definite.

Let $S = \operatorname{Spec}\mathcal{O}_K$, where K is a number field. Let L/K denote the extension maximal with the property that the integral closure $\mathcal{O}_L$ of $\mathcal{O}_K$ in L is unramified over $\mathcal{O}_K$. Does the above question have a positive answer if L/K is infinite? Obviously, if it is possible to find such a K and L where $L \subset \mathbb{R}$, then the example of Buzzard would still show that the answer is negative. We do not know if examples of such K exist.

Some conditions on the dimension of the fibers of a projective morphism $X \to S$ are indeed necessary for a finite quasi-section to exist, as the following proposition shows.

PROPOSITION 6.8
Let X and S be irreducible noetherian schemes. Let $\pi : X \to S$ be a proper morphism, and suppose that π has a finite quasi-section T.

(a) *Assume that $\pi : X \to S$ is generically finite. Then π is finite.*

(b) *Assume that the generic fiber of $X \to S$ has dimension* 1. *If X is regular, then for all $s \in S$, X_s has an irreducible component of dimension* 1.

Proof
(a) Since π is generically finite and X is irreducible, the generic fiber of $X \to S$ is reduced to one point, namely, the generic point of X. Since $T \to S$ is surjective, T meets the generic fiber of $X \to S$, and so it contains the generic point of X. Thus, $T = X$ set-theoretically. Since $X_{\mathrm{red}} \subseteq T$, we find that X_{red} is finite over S. Since X is then quasi-finite and proper, it is finite over S.

(b) Let Γ be an irreducible component of T which surjects onto S. Let us first show that $\operatorname{codim}(\Gamma, X) = 1$. Let Y be an irreducible closed subset of X of codimension 1 which contains Γ. Since the generic fiber of $X \to S$ has dimension 1, the generic fibers of $\Gamma \to S$ and $Y \to S$ are both irreducible and 0-dimensional. Hence, these generic fibers are equal. Therefore, $\Gamma = Y$ and Γ has codimension 1 in X. Since X is regular, Γ is then the support of a Cartier divisor on X. By hypothesis, for all $s \in S$, Γ_s is not empty, and has dimension 0. Thus, for all $s \in S$ and all $t \in \Gamma_s$, we have $0 = \dim_t \Gamma_s \geq \dim_t X_s - 1$. It follows that the irreducible components of X_s which intersect Γ_s all have dimension at most 1. Since every irreducible component of X_s has dimension at least 1 (see [29, IV.13.1.1]), (b) follows. □

Example 6.9
As the following example shows, it is not true in general in Proposition 6.8(b) that for all $s \in S$, all irreducible components of X_s have dimension 1. Let S be regular of dimension $d \geq 2$. Fix a section T of $\mathbb{P}^1_S \to S$. Let x_0 be a closed point of $\mathbb{P}^1_S$ not contained in T and lying over a point $s \in S$ with $\dim_s S = d$. Let $X \to \mathbb{P}^1_S$ be the blowing-up of x_0. Then X is regular, and $X \to S$ has the preimage of T as a section (and thus it has a finite quasi-section). However, X_s consists of the union of a projective line and the exceptional divisor E of $X \to \mathbb{P}^1_S$, which has dimension d. So $\dim X_s = d \geq 2$.

Our next example shows that some regularity assumption on X is necessary in Proposition 6.8(b). Let k be any field, $R := k[t_1, t_2]$, and let $B := R[u_0, u_1, u_2]/(t_1 u_2 - t_2 u_1)$. Consider the induced projective morphism

$$X := \mathrm{Proj}(B) \to S := \mathrm{Spec}\, R = \mathbb{A}^2_k.$$

The scheme X is singular at the point P corresponding to the homogeneous ideal (t_1, t_2, u_1, u_2) of B. The fibers of $X \to S$ are isomorphic to $\mathbb{P}^1_{k(s)}$ if $s \neq (0,0)$. When $s = (0,0)$, then X_s is isomorphic to $\mathbb{P}^2_{k(s)}$. The morphism $X \to S$ has a finite section T, corresponding to the homogeneous ideal (u_1, u_2). As expected in view of the proof of Proposition 6.8(b), any section of $X \to S$, and in particular the section T, contains the singular point P.

We conclude this section with two applications of Theorem 6.3.

PROPOSITION 6.10
Let A be a commutative ring. Let M be a projective A-module of finite presentation with constant rank $r > 1$. Then there exists an A-algebra B, finite and faithfully flat over A, with B a local complete intersection over A, such that $M \otimes_A B$ is isomorphic to a direct sum of projective B-modules of constant rank 1.

Proof
Let $S := \mathrm{Spec}\, A$. Let $\mathcal{M}$ denote the locally free $\mathcal{O}_S$-module of rank r associated with M. Let $X := \mathbb{P}(\mathcal{M})$. Then the natural map $X \to S$ is projective, smooth, and its fibers all have dimension $r - 1$. We are thus in a position to apply Theorem 6.3(3) to obtain the existence of a finite flat quasi-section $f : T \to S$ as in Theorem 6.3(3). In particular, $T = \mathrm{Spec}\, B$ for some finite and faithfully flat A-algebra B, with B a local complete intersection over A. Moreover, the existence of an S-morphism $g : T \to X$ corresponds to the existence of an $\mathcal{O}_T$-invertible sheaf $\mathcal{L}_1$ and of a surjective morphism $f^*\mathcal{M} \to \mathcal{L}_1$. Let $\mathcal{M}_1$ denote the kernel of this morphism. The $\mathcal{O}_T$-module $\mathcal{M}_1$ is locally free of rank $r - 1$, and $f^*\mathcal{M} \cong \mathcal{L}_1 \oplus \mathcal{M}_1$. We may thus proceed as

above and use Theorem 6.3(3) another $r-2$ times to obtain the conclusion of the corollary. □

Remark 6.11
The proposition strengthens, in the affine case, the classical splitting lemma for vector bundles (see [24, V.2.7]). When A is of finite type over an algebraically closed field k and is regular, it is shown in [69, 3.1] that it is possible to find a finite faithfully flat *regular* A-algebra B over which M splits.

We provide now an example of a commutative ring A with a finitely generated projective module M which is not free and such that it is not possible to find a finite *étale* A-algebra B which splits M into a direct sum of rank 1 projective modules. For this, we exhibit a ring A such that the étale fundamental group of $\operatorname{Spec} A$ is trivial and such that $\operatorname{Pic}(A)=(0)$. Then, if a projective module M of finite rank is split over a finite *étale* A-algebra B, it must be split over A. Since $\operatorname{Pic}(A)=(0)$, we find then that M is a free module. Let $n>2$, and consider the algebra

$$A:=\mathbb{C}[x_1,\dots,x_{2n}]/(x_1^2+\cdots+x_{2n}^2-1).$$

This ring is regular, and it is well known that it is a unique factorization and domain (UFD), so that $\operatorname{Pic}(A)=(0)$ (see, e.g., [70, Theorem 5]). It is shown in [72, Theorem 3.1], (use $p=2$), that for each $n>2$, there exists a projective module M of rank $n-1$ which is not free. Let now $X:=\operatorname{Spec} A$. The étale fundamental group of X is trivial if the topological fundamental group of $X(\mathbb{C})$ is trivial (use [31, XII, Corollaire 5.2]). The topological fundamental group of $X(\mathbb{C})$ is trivial because there exists a retraction $X(\mathbb{C})\to S^{2n-1}$, where S^{2n-1} is the real sphere in $\mathbb{R}^{2n}$ given by the equation $x_1^2+\cdots+x_{2n}^2=1$ (see, e.g., [76, Section 2]). It is well known that the fundamental group of S^{2n-1} is trivial for all $n\geq 2$. Hence, the module M cannot be split after a finite étale base change.

Let S be a scheme, and let $U\subseteq S$ be an open subset. Given a family $C\to U$ of stable curves over U, conditions are known (see, e.g., [14]) to ensure that this family extends to a family of stable curves over S. It is natural to consider the analogous problem of extending a given family $D\to Z$ of stable curves over a closed subset Z of S. For this, we may use the existence of finite quasi-sections in appropriate moduli spaces, as in the proposition below.

Let $\overline{\mathcal{M}}:=\overline{\mathcal{M}}_{g,S}$ be the proper Deligne–Mumford stack of stable curves of genus g over S (see [12, 5.1]). Our next proposition uses the following statement: *Over* $S=\operatorname{Spec}\mathbb{Z}$*, the stack* $\overline{\mathcal{M}}_{g,S}$ *admits a coarse moduli space* $\overline{M}_{g,\mathbb{Z}}$ *which is a projective scheme over* $\operatorname{Spec}\mathbb{Z}$. Such a statement is found in an appendix [52, p. 228], with a sketch of proof. See also [41, 5.1] for another brief proof.

PROPOSITION 6.12

Let S be a noetherian affine scheme. Let Z be a closed subscheme of S, and let $D \to Z$ be a stable curve of genus $g \geq 2$. Then there exist a finite surjective morphism $S' \to S$ mapping each irreducible component of S' onto an irreducible component of S, a finite surjective morphism $Z' \to Z$, a closed S-immersion $Z' \to S'$, and a stable curve $\mathcal{D} \to S'$ of genus g with a morphism $D \times_Z Z' \to \mathcal{D}$ such that the diagram below commutes and the top square in the diagram is cartesian:

$$\begin{tikzcd} D \times_Z Z' \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[d] \\ Z' \arrow[r, hook] \arrow[d] & S' \arrow[d] \\ Z \arrow[r, hook] & S \end{tikzcd}$$

Proof

Let $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{g,S}$ be the proper Deligne–Mumford stack of stable curves of genus g over S (see [12, 5.1]). We first construct a finite surjective morphism $X \to \overline{\mathcal{M}}$ such that X is a scheme, projective over S and with constant fiber dimensions over S. It is known that over $\mathbb{Z}$, the coarse moduli space $\overline{M}_{g,\mathbb{Z}}$ of $\overline{\mathcal{M}}$ is a projective scheme and that its fibers over $\operatorname{Spec}\mathbb{Z}$ are all geometrically irreducible of the same dimension $3g-3$. Let $\overline{M} := \overline{M}_{g,\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$. Then we have a canonical morphism $\overline{\mathcal{M}} \to \overline{M}$ which is proper and a universal homeomorphism (hence quasi-finite). By construction, the S-scheme $\overline{M}$ is projective with constant fiber dimension.

Since $\overline{\mathcal{M}}$ is a noetherian separated Deligne–Mumford stack, there exists a (representable) finite surjective morphism from a scheme X to $\overline{\mathcal{M}}$ (see [43, 16.6]). The composition $X \to \overline{\mathcal{M}} \to \overline{M}$ is a finite (because proper and quasi-finite) surjective morphism of schemes. Thus $X \to S$ is projective since S is affine and $\overline{M} \to S$ is projective. So $X \to S$ is projective and all its fibers have the same dimension.

The curve $D \to Z$ corresponds to an element in the set $\overline{\mathcal{M}}(Z)$, which in turn corresponds to a finite morphism $Z \to \overline{\mathcal{M}}$. So $Z' := Z \times_{\overline{\mathcal{M}}} X$ is a scheme, finite surjective over Z and finite over X. Let Z_0 denote the schematic image of Z' in X. It is finite over S.

To be able to apply Theorem 6.3(1), we note the following. Let T be the disjoint union of the reduced irreducible components of S. Replacing if necessary S with T and $D \to Z$ with $D \times_S T \to Z \times_S T$, we easily reduce the proof of the proposition to the case where S is irreducible. Once S is assumed irreducible, we use the fact that $\overline{M} \to S$ is proper with irreducible fibers to find that $\overline{M}$ is also irreducible. Replacing

X by an irreducible component of X which dominates $\overline{M}$, we can suppose that X is irreducible.

Theorem 6.3(1) can then be applied to the morphism $X \to S$ and to each irreducible component of Z_0. We obtain a finite quasi-section S_0 of X/S containing (set-theoretically) Z_0 and such that each irreducible component of S_0 maps onto S. Modifying the structure of closed subscheme on S_0 as in the proof of Theorem 6.3(1), we can suppose that Z_0 is a subscheme of S_0.

Because S_0 is affine, it is clear that there exists a scheme S', finite and faithfully flat (and even l.c.i.) over S_0, and a closed immersion $Z' \to S'$ making the following diagram commute:

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & S' \arrow[d] \\ Z_0 \arrow[r, hook] & S_0 \end{tikzcd}$$

As $S' \to S_0$ is flat, each irreducible component of S' maps onto an irreducible component of S_0, and hence onto S.

The stable curve $\mathcal{D} \to S'$, whose existence is asserted in the statement of Proposition 6.12, corresponds to the element of $\overline{\mathcal{M}}(S')$ given by the composition of the finite morphisms $S' \to S_0 \to X \to \overline{\mathcal{M}}$. □

Remark 6.13

Consider the finite surjective S-morphism $X \to \overline{\mathcal{M}}$ introduced at the beginning of the proof of Proposition 6.12 above. If we can find such a cover $X \to \overline{\mathcal{M}}$ such that $X \to S$ is flat with Cohen–Macaulay fibers (resp., with l.c.i. fibers), then using Theorem 6.3(2) and (3), we can further require in the statement of Proposition 6.12 that $S' \to S$ be finite and faithfully flat (resp., l.c.i.).

When some prime number p is invertible in $\mathcal{O}_S(S)$, then it is proved in [15, 2.3.6(1), 2.3.7] that there exists such an X which is even smooth over S. Therefore, in this case, we can find a morphism $S' \to S$ which is finite, faithfully flat, and l.c.i.

7. Moving lemma for 1-cycles

We review below the basic notation needed to state our moving lemma. Let X be a noetherian scheme. Let $\mathcal{Z}(X)$ denote the free abelian group on the set of closed integral subschemes of X. An element of $\mathcal{Z}(X)$ is called a *cycle*, and if Y is an integral closed subscheme of X, we denote by $[Y]$ the associated element in $\mathcal{Z}(X)$.

Let $\mathcal{K}_X$ denote the sheaf of meromorphic functions on X (see [39, p. 204] or [45, Definition 7.1.13]). Let $f \in \mathcal{K}_X^*(X)$. Its associated principal Cartier divisor is denoted by $\operatorname{div}(f)$ and defines a cycle on X:

$$\big[\mathrm{div}(f)\big] = \sum_x \mathrm{ord}_x(f_x)\big[\overline{\{x\}}\big],$$

where x ranges through the points of codimension 1 in X, and $\mathrm{ord}_x : \mathcal{K}^*_{X,x} \to \mathbb{Z}$ is defined, for a regular element of $g \in \mathcal{O}_{X,x}$, to be the length of the $\mathcal{O}_{X,x}$-module $\mathcal{O}_{X,x}/(g)$.

A cycle Z is *rationally equivalent to* 0 or *rationally trivial*, if there are finitely many integral closed subschemes Y_i and nonzero rational functions f_i on Y_i such that $Z = \sum_i [\mathrm{div}(f_i)]$. Two cycles Z and Z' are *rationally equivalent* in X if $Z - Z'$ is rationally equivalent to 0. We denote by $\mathcal{A}(X)$ the quotient of $\mathcal{Z}(X)$ by the subgroup of rationally trivial cycles.

A morphism of schemes of finite type $\pi : X \to Y$ induces by *pushforward of cycles* a group homomorphism $\pi_* : \mathcal{Z}(X) \to \mathcal{Z}(Y)$. If Z is any closed integral subscheme of X, then $\pi_*([Z]) := [k(Z) : k(\overline{\pi(Z)})][\overline{\pi(Z)}]$, with the convention that $[k(Z) : k(\overline{\pi(Z)})] = 0$ if the extension $k(Z)/k(\overline{\pi(Z)})$ is not finite.

7.1

Let S be a noetherian scheme which is universally catenary and equidimensional at every point (e.g., S is regular). Assume that both $X \to S$ and $Y \to S$ are morphisms of finite type, and let $\pi : X \to Y$ be a *proper* morphism of S-schemes. Let C and C' be two cycles on X which are rationally equivalent. Then $\pi_*(C)$ and $\pi_*(C')$ are rationally equivalent on Y (see [74, Note 6.7 or Proposition 6.5 and 3.11]). We denote by $\pi_* : \mathcal{A}(X) \to \mathcal{A}(Y)$ the induced morphism. For an example showing that the hypotheses on S are needed for $\pi_* : \mathcal{A}(X) \to \mathcal{A}(Y)$ to be well defined (see [25, 1.3]).

We are now ready to state the main theorem of this section. Recall that the support of a horizontal 1-cycle C in a scheme X over a Dedekind scheme S is a finite quasi-section (Definition 0.3). The definitions of *condition* (T) and of *pictorsion* are given in Definition 0.2 and Definition 0.3, respectively.

THEOREM 7.2

Let R be a Dedekind domain, and let $S :=$ Spec R. Let $X \to S$ be a flat and quasi-projective morphism, with X integral. Let C be a horizontal 1-cycle on X. Let F be a closed subset of X. Assume that for all $s \in S$, $F \cap X_s$ and Supp$(C) \cap X_s$ *have positive codimension in X_s. Assume in addition that either*

(a) *R is pictorsion and the support of C is contained in the regular locus of X or*

(b) *R satisfies condition (T).*

Then some positive multiple mC of C is rationally equivalent to a horizontal 1-cycle C' on X whose support does not meet F. Under the assumption (a), if furthermore R is semilocal, then we can take $m = 1$.

Moreover, if $Y \to S$ is any separated morphism of finite type and $h : X \to Y$ is any S-morphism, then $h_*(mC)$ is rationally equivalent to $h_*(C')$ on Y.

The proof of Theorem 7.2 is postponed to Section 7.7. We first briefly introduce below needed facts about contraction morphisms. We then discuss several statements needed in the proof of Theorem 7.2(b) when S is not excellent.

PROPOSITION 7.3

Let R be a Dedekind domain, and $S := \operatorname{Spec} R$. Let $X \to S$ be a projective morphism of relative dimension 1, with X integral. Let C be an effective Cartier divisor on X, flat over S. Then:

(a) *There exists $m_0 \geq 0$ such that the invertible sheaf $\mathcal{O}_X(mC)$ is generated by its global sections for all $m \geq m_0$.*

(b) *The morphism $X' := \operatorname{Proj}(\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mC))) \to S$ is projective, with X' integral, and the canonical morphism $u : X \to X'$ is projective, with $u_*\mathcal{O}_X = \mathcal{O}_{X'}$ and connected fibers.*

(c) *For any vertical prime divisor Γ on X, $u|_\Gamma$ is constant if $\Gamma \cap \operatorname{Supp} C = \emptyset$ and is finite otherwise.*

(d) *Let Z be the union of the vertical prime divisors of X disjoint from $\operatorname{Supp} C$. Then u induces an isomorphism $X \setminus Z \to X' \setminus u(Z)$.*

Proof

In [6, Theorem 1, 6.7], a similar statement is proved, with R local, and X normal. (The normality is not assumed in [18] and [58]. A global base is considered in [45, 8.3.30].) We leave it to the reader to check that the proof of [6, 6.7/1] can be used mutatis mutandis to prove Proposition 7.3. Part (a) follows from the first part of the proof of 6.7/1. Part (b) follows from 6.7/2. Part (c) follows from the second part of the proof of 6.7/1. We now give a proof of (d). The morphism u is birational because it induces an isomorphism $X_\eta \to X'_\eta$ over the generic point η of S, since C_η is ample, being effective of positive degree. It follows that Z is the union of finitely many prime divisors of X. As u has connected fibers, it follows from (c) that $Z = u^{-1}(u(Z))$. The restriction $v : X \setminus Z \to X' \setminus u(Z)$ of u is thus projective and quasi-finite. Therefore, v is finite and, hence, affine. As $\mathcal{O}_{X'\setminus\pi(Z)} = v_*\mathcal{O}_{X\setminus Z}$, v is an isomorphism. $\square$

Let K be a field of characteristic $p > 0$. Let $K' := K^{p^{-\infty}}$ be the perfect closure of K. Let $n \geq 0$, and set $q := p^n$. Let $K^{1/q}$ denote the extension of K in K' generated by the qth roots of all elements of K. Let $i : K \to K^{1/q}$ denote the natural inclusion, and let $\rho : K^{1/q} \to K$ be defined by $\lambda \mapsto \lambda^q$. The composition $F := \rho \circ i : K \to K$ is the qth Frobenius morphism of K. By definition, given a morphism $Y \to \operatorname{Spec} K$, the morphism $Y^{(q)} \to \operatorname{Spec} K$ is the base change $(Y \times_{\operatorname{Spec} K, F^*} \operatorname{Spec} K) \to \operatorname{Spec} K$. It follows that we have a natural isomorphism of K-schemes:

$$Y^{(q)} \simeq \big(Y \times_{\operatorname{Spec} K, i^*} \operatorname{Spec}(K^{1/q})\big) \times_{\operatorname{Spec}(K^{1/q}), \rho^*} \operatorname{Spec} K. \tag{7.3.1}$$

LEMMA 7.4
Let K be a field of characteristic $p > 0$. Let $Y \to \operatorname{Spec} K$ be a morphism of finite type, with Y integral of dimension 1. Then there exists $n \geq 0$ such that the normalization of $(Y^{(p^n)})_{\mathrm{red}}$ is smooth over K.

Proof
The normalization Z of $(Y_{K'})_{\mathrm{red}}$ is regular and, hence, smooth over the perfect closure K'. There is a finite subextension L/K of K' such that the curve Z and the morphism $Z \to (Y_{K'})_{\mathrm{red}}$ are defined over L. This implies that the normalization of $(Y_L)_{\mathrm{red}}$ is $Z_{/L}$, and hence smooth over L. Let $q = p^n$ be such that $L \subseteq K^{1/q}$. As $Z_{/L} \to Y_L$ is finite and induces an isomorphism on the residue fields at the generic points, the same is true for

$$Z_{K^{1/q}} \to (Y_L)_{K^{1/q}} = Y_{K^{1/q}}.$$

Using $\rho : K^{1/q} \to K$ and (7.3.1),

$$(Z_{K^{1/q}})_K \to (Y_{K^{1/q}})_K \simeq Y^{(q)}$$

is finite and induces an isomorphism on the residue fields at the generic points. As the left-hand side is smooth, this morphism is the normalization of $(Y^{(q)})_{\mathrm{red}}$. $\square$

LEMMA 7.5
Let S be a universally catenary noetherian scheme which is equidimensional at every point. Let $\pi : X \to X_0$ be a finite surjective morphism of S-schemes of finite type, with induced homomorphism of Chow groups $\pi_* : \mathcal{A}(X) \to \mathcal{A}(X_0)$. Then:
(1) *The cokernel of π_* is a torsion group.*
(2) *If π is a homeomorphism, then the kernel of π_* is also a torsion group.*

Proof
Our hypotheses on S allow us to use Section 7.1, so that the morphism $\pi_* : \mathcal{A}(X) \to \mathcal{A}(X_0)$ is well defined.

(1) Let Z_0 be an integral closed subscheme on X_0, and let Z be an irreducible component of $\pi^{-1}(Z_0)$ whose image in X_0 is Z_0. When Z is endowed with the reduced induced structure, $Z \to Z_0$ is finite and surjective, and $\pi_*[Z] = [k(Z) : k(Z_0)][Z_0]$. Hence, the cokernel of $\mathcal{Z}(X) \to \mathcal{Z}(X_0)$ is torsion, and the same holds for the corresponding homomorphism of Chow groups.

(2) Let W_0 be an integral closed subscheme of X_0. Since π is a homeomorphism, $W := \pi^{-1}(W_0)$ is irreducible, and we endow it with the reduced induced structure. The induced morphism $\pi : W \to W_0$ is finite and surjective between integral noetherian schemes. Let $f \in k(W_0)$ be a nonzero rational function. Using, for instance, [45, 7.1.38], we find that

$$\pi_*\big(\big[\mathrm{div}_W(\pi^* f)\big]\big) = \big[k(W) : k(W_0)\big]\big[\mathrm{div}_{W_0}(f)\big].$$

This implies that for every integer multiple r of $[k(W) : k(W_0)]$, $r[\mathrm{div}_{W_0}(f)] = \pi_*(D_r)$ for some principal cycle D_r on X.

Now let Z be any cycle on X such that $\pi_* Z$ is principal on X_0. Then for a suitable integer N, $N\pi_* Z = \pi_*(D)$ for some principal cycle D on X. Since π is a homeomorphism, $\pi_* : \mathcal{Z}(X) \to \mathcal{Z}(X_0)$ is injective. Therefore, $NZ = D$ in $\mathcal{Z}(X)$, and the class of NZ is trivial in $\mathcal{A}(X)$. □

For our next proposition, recall that a normal scheme X is called $\mathbb{Q}$-*factorial* if every Weil divisor D on X is such that some positive integer multiple of D is the cycle associated with a Cartier divisor on X.

PROPOSITION 7.6

Let S be a Dedekind scheme with generic point η. Let $X \to S$ be a dominant morphism of finite type, with X integral. Suppose that the normalization of X_η is smooth over $k(\eta)$. Then:

(a) *The normalization morphism $\pi : X' \to X$ is finite.*

(b) *If X is normal, then the following properties are true.*

  (1) *The completion $\widehat{\mathcal{O}}_{X,x}$ is normal for all $x \in X$.*

  (2) *The locus* $\mathrm{Reg}(X)$ *of regular points of X is open in X.*

  (3) *If* $\dim X_\eta = 1$ *and S satisfies* condition *(T), then X is $\mathbb{Q}$-factorial.*

Proof

When S is assumed to be excellent, then X is also excellent and most of the statements in the proposition follow from this property. The statement Proposition 7.6(b)(3) can be found in [50, 3.3]. We now give a proof of Proposition 7.6 without assuming that S is excellent.

(a) We can and will assume that X is affine. As $\pi_\eta : X'_\eta \to X_\eta$ is finite, there exists a factorization $X' \to X'' \to X$ with $X'' \to X$ finite and birational, and such that $X'_\eta \to X''_\eta$ is an isomorphism (simply take generators of $\mathcal{O}_{X'_\eta}(X'_\eta)$ which belong to $\mathcal{O}_{X'}(X')$). Replacing X with X'', we can suppose that X_η is smooth. The smooth locus of $X \to S$ is open and contains X_η, so it contains an open set of the form $X_V := X \times_S V$ for some dense open subset V of S. So $X'_V = X_V$ and we find that $\pi_*\mathcal{O}_{X'}/\mathcal{O}_X$ is supported on finitely many closed fibers $X_{s_1}, \dots, X_{s_n}$.

To show that π is finite, it is enough to show that the normalization morphism of $X \times_S \operatorname{Spec}(\mathcal{O}_{S,s_i})$ is finite for all s_i. Therefore, we can suppose that $S = \operatorname{Spec} R$ for some discrete valuation ring R. Let $\widehat{R}$ be the completion of R. As X_η is smooth, the normalization morphism $\widehat{\pi} : (X_{\widehat{R}})' \to X_{\widehat{R}}$ is an isomorphism on the generic fiber. It is finite because $\widehat{R}$ is excellent. By [45, 8.3.47, 8.3.48], $\widehat{\pi}$ descends to a finite morphism $Z \to X$ over R. By faithfully flat descent, this implies that Z is normal and, thus, isomorphic to X', and that $X' \to X$ is finite and $X'_{\widehat{R}} = (X_{\widehat{R}})'$ is normal.

(b) Suppose now that X is normal with smooth generic fiber. To prove (1), let $x \in X$ with image $s \in S$. Then $\mathcal{O}_{X,x}$ is also the local ring of $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ at x. To prove that its completion is normal, we can thus suppose that S is local. We can even restrict to s closed in S as X_V is regular. Let $R = \mathcal{O}_{S,s}$. We saw above that $X_{\widehat{R}}$ is normal. As $\widehat{\mathcal{O}}_{X,x}$ is also the completion of $\mathcal{O}_{X_{\widehat{R}},x}$ (see, e.g., [45, 8.3.49(b)]), it is normal because $X_{\widehat{R}}$ is excellent (see [29, IV.7.8.3(vii)]).

(2) We have $\operatorname{Reg}(X) \supseteq X_V$ and $\operatorname{Reg}(X) \cap X_s = \operatorname{Reg}(X \times_S \operatorname{Spec} \mathcal{O}_{S,s})$ for all $s \in S \setminus V$. As $S \setminus V$ consists of finitely many closed points of S, $\operatorname{Reg}(X)$ is open by [29, IV.6.12.6(ii)].

(3) The statement of (3) is proven in [50, Lemme 3.3], provided that the singular points of X are isolated, and that [50, Théorème 2.8] holds when $A = \mathcal{O}_{X,x}$. In our case, the singular points of X are isolated by (2). Théorème 2.8 in [50] is proved under the hypothesis that A is excellent, but the proof in [50] only uses the fact that the completion of A is normal (in step 2.10). So in our case, this property is satisfied by (1). $\square$

7.7

Proof of Theorem 7.2 when (a) holds

It suffices to prove the theorem in the case where the given 1-cycle is the cycle associated with an integral closed subscheme of X finite over S. We will denote again by C this integral closed subscheme. As in the proof of in [25, Theorem 2.3], we reduce the proof of Theorem 7.2 to the case where $C \to X$ is a regular immersion[†] as follows.

Proposition 3.2 in [25] shows the existence of a finite birational morphism $D \to C$ such that the composition $D \to C \to S$ is an l.c.i. morphism. Since C is affine, there exists for some $N \in \mathbb{N}$ a closed immersion $D \to C \times_S \mathbb{P}^N_S \subseteq X \times_S \mathbb{P}^N_S$. Note that since C is contained in the regular locus of X, then D is contained in the regular locus of $X \times_S \mathbb{P}^N_S$. We claim that it suffices to prove the theorem for the 1-cycle D and the closed subset $\mathbf{F} := F \times_S \mathbb{P}^N_S$ in the scheme $X \times_S \mathbb{P}^N_S$. Indeed, let D' be a horizontal 1-cycle whose existence is asserted by the theorem in this case, with mD ratio-

[†]The hypothesis that $C \to X$ is a regular immersion is equivalent to the condition that $C \to X$ is a local complete intersection morphism (see, e.g., [45, 6.3.21]).

nally equivalent to D'. In particular, $\mathrm{Supp}(D') \cap \mathbf{F} = \emptyset$. Consider the projection $p : X \times_S \mathbb{P}^N_S \to X$, which is a projective morphism. Then $p_*(D) = C$ because $D \to C$ is birational. It follows from Section 7.1 that $mC = p_*(mD)$ is rationally equivalent to the horizontal 1-cycle $C' := p_*(D')$ on X. Moreover, $\mathrm{Supp}(C') \cap F = \emptyset$. Since D/S is l.c.i., each local ring $\mathcal{O}_{D,x}$, $x \in D$, is an absolute complete intersection ring, and the closed immersion $D \to X \times_S \mathbb{P}^N_S$ is a regular immersion (see [29, IV.19.3.2]). Finally, consider a morphism $h : X \to Y$ as in the last statement of the theorem. Apply this statement to mD, D', and to the associated morphism $h' : X \times_S \mathbb{P}^N_S \to Y \times_S \mathbb{P}^N_S$. Since the projection $Y \times_S \mathbb{P}^N_S \to Y$ is proper, we find as desired that $h_*(mC)$ is rationally equivalent to $h_*(C')$ on Y.

Let us now assume that $C \to X$ is a regular immersion. Let d denote the codimension of C in X. If $d > 1$, we can apply Theorem 3.4 (as stated in the introduction since C is integral) and obtain a closed subscheme Y of X such that C is the support of a Cartier divisor on Y and such that $F \cap Y_s$ is finite for all $s \in S$. Clearly, C is also the support of a Cartier divisor on Y_{red}, and on any irreducible component of Y_{red} passing through C. Thus, we are reduced to proving the theorem when X is integral of dimension 2 and F is quasi-finite over S. Note that after this reduction process, we cannot and do not assume anymore that C is contained in the regular locus of X.

When $d = 1$, we do not apply Theorem 3.4, but we note that in this case too $F \cap X_s$ is finite for all $s \in S$. Indeed, since $C \to S$ is finite, the generic point of C is a closed point in the generic fiber X_η of $X \to S$. Since the codimension of C in X is $d = 1$, and since the generic fiber is a scheme of finite type over a field, we find that one irreducible component of X_η has dimension 1. Since S is a Dedekind scheme and $X \to S$ is flat with X integral, we find that all fibers are equidimensional of dimension $\dim X_\eta = 1$ (see [45, 4.4.16]). Hence, our hypothesis on F implies that $F \to S$ is quasi-finite.

7.8

Since X/S is quasi-projective and X is integral, there exists an integral scheme $\overline{X}$ with a projective morphism $\overline{X} \to S$ and an S-morphism $X \to \overline{X}$ which is an open immersion. Let $\overline{F}$ be the Zariski closure of F in $\overline{X}$. The closed subscheme $\overline{F}$ is finite over S because $F \to S$ is quasi-finite and S has dimension 1. Recall that by definition, a horizontal 1-cycle on X is finite over S. Hence, C is closed in $\overline{X}$. Since C is the support of a Cartier divisor on X, we find that C is also the support of a Cartier divisor on $\overline{X}$. We are thus in a situation where we can consider the contraction morphism $u : \overline{X} \to X'$ associated to C in Proposition 7.3. Let Z denote the union of the irreducible components E of the fibers of $\overline{X} \to S$ such that $E \cap \mathrm{Supp}(C) = \emptyset$. Let $U = X \setminus (Z \cap X)$. Then $\mathrm{Supp}\, C \subseteq U$, and $u|_U$ is an isomorphism onto its image.

Let $F' = u(\overline{F} \cup Z \cup (\overline{X} \setminus X)) \cup u(\mathrm{Supp}(C))$. Then $X' \setminus F' \subseteq u(U)$, and F' is finite over S. We endow F' with the structure of a reduced closed subscheme of X'.

Now suppose that R is pictorsion. Then $\mathrm{Pic}(F')$ is a torsion group by hypothesis. So, fix $n > 0$ such that $\mathcal{O}_{X'}(nC)|_{F'}$ is trivial. Since C meets every irreducible component of every fiber of $X' \to S$, the sheaf $\mathcal{O}_{X'}(C)$ is relatively ample for $X' \to S$ (see [29, III.4.7.1]). Let $\mathcal{I}$ denote the ideal sheaf of F' in X'. Then there exists a multiple m of n such that $H^1(X', \mathcal{I} \otimes \mathcal{O}_{X'}(mC)) = (0)$. It follows that a trivialization of $\mathcal{O}_{X'}(mC)|_{F'}$ lifts to a section $f \in H^0(X', \mathcal{O}_{X'}(mC))$.

Recall that by definition, $\mathcal{O}_{X'}(mC)$ is a subsheaf of $\mathcal{K}_{X'}$. We thus consider $f \in H^0(X', \mathcal{O}_{X'}(mC)) \subseteq \mathcal{K}_{X'}(X')$ as a rational function. The support of the divisor $\mathrm{div}_{X'}(f) + mC$ is disjoint from F' by construction. In particular, it is contained in $u(U)$ and is horizontal, and $\mathrm{div}_{X'}(f)$ has also its support contained in $u(U)$. Considering the pullback of the divisors under $\overline{X} \to X'$ shows that the divisor $C' := \mathrm{div}_{\overline{X}}(f) + mC$ is contained in U, disjoint from F, horizontal and linearly equivalent to mC on $\overline{X}$.

When R is semi-local, the set $F' \subset X'$ is a finite set of points. Thus we may apply [25, Proposition 6.2] directly to the Cartier divisor whose support is $u(C)$ to find a Cartier divisor D linearly equivalent to $u(C)$ and whose support does not meet F'.

It remains to prove the last statement of the theorem, which pertains to the morphism $h : X \to Y$. To summarize, in the situation of Theorem 7.2(a), when C is integral, we found a closed integral subscheme W of X containing C, a projective scheme $\overline{W}/S$ containing W as a dense open subset, and $m \geq 1$ (with $m = 1$ when R is semi-local) such that mC is rationally equivalent on $\overline{W}$ to some horizontal 1-cycle C' contained in W. The morphism $h : X \to Y$ in the statement of Theorem 7.2 induces an S-morphism $h : W \to Y$. Our proof now proceeds as in [25, proof of Proposition 2.4(2)]. For the convenience of the reader, we recall the main ideas of that proof here.

Let g be the function on W such that $[\mathrm{div}_W(g)] = mC - C'$. Let $\Gamma \subseteq \overline{W} \times_S Y$ be the schematic closure of the graph of the rational map $\overline{W} \dashrightarrow Y$ induced by $h : W \to Y$. Let $p : \Gamma \to \overline{W}$ and $q : \Gamma \to Y$ be the associated projection maps over S. Since Γ is integral and its generic point maps to the generic point of W, the rational function g on W induces a rational function, again denoted by g, on Γ. As $p : p^{-1}(W) \to W$ is an isomorphism, we let $p^*(C)$ and $p^*(C')$ denote the preimages of C and C' in $p^{-1}(W)$; they are closed subschemes of Γ. Since g is an invertible function in a neighborhood of $\overline{W} \setminus W$, $[\mathrm{div}_\Gamma(g)] = mp^*(C) - p^*(C')$, and $p^*(mC)$ and $p^*(C')$ are rationally equivalent on Γ. Then, as q is proper and S is universally catenary, $q_*p^*(mC)$ and q_*p^*C' are rationally equivalent in Y. Since $h_*C = q_*p^*C$ and $h_*C' = q_*p^*C'$, we find that $h_*(mC)$ is rationally equivalent to $h_*(C')$ in Y. ☐

Proof of Theorem 7.2 when (b) holds
It suffices to prove the theorem in the case where the given 1-cycle is the cycle associated with an integral closed subscheme of X finite over S. We will denote again by C this integral closed subscheme. By hypothesis, $X \to S$ is flat, so all its fibers are of the same dimension d.

Since C is not empty, C_s is not empty, and thus has positive codimension in X_s by hypothesis. Therefore, we find that $d \geq 1$. Moreover, C does not contain any irreducible component of positive dimension of F_s and of X_s. If $d > 1$, we fix a very ample invertible sheaf $\mathcal{O}_X(1)$ on X and apply Theorem 3.3 to $X \to S$, C, and F, to find that there exists $n > 0$ and a global section f of $\mathcal{O}_X(n)$ such that the closed subscheme H_f of X is a hypersurface that contains C as a closed subscheme, and such that for all $s \in S$, H_f does not contain any irreducible component of positive dimension of F_s. Using Lemma 6.4 and the assumption that C is integral, we can find an irreducible component Γ of H_f which contains C, and such that all fibers of $\Gamma \to S$ have dimension $d - 1$. If $d - 1 > 1$, we repeat the process with Γ endowed with the reduced induced structure, C, and $F \cap \Gamma$.

It follows that we are reduced to proving the theorem when $X \to S$ has fibers of dimension 1 and X is integral. In this case, $F \to S$ is quasi-finite. We now reduce to the case where X is normal and $X \to S$ has a smooth generic fiber. Let K denote the function field of S. When K has positive characteristic $p > 0$, consider the homeomorphism $\pi : X \to X^{(p^n)}$ with n as in Lemma 7.4, so that the normalization of the reduced generic fiber of $X^{(p^n)}$ is smooth over K. Applying Lemma 7.5 to $\pi_* : \mathcal{A}(X) \to \mathcal{A}(X^{(p^n)})$, we find that it suffices to prove Theorem 7.2 for $X^{(p^n)}$, $\pi(C)$, and $\pi(F)$. So we can suppose that the normalization of the reduced generic fiber X_K is smooth.

Let $\pi : X' \to X$ be the normalization morphism. By Proposition 7.6(a), this morphism is finite. Using Lemma 7.5 applied to $\pi_* : \mathcal{A}(X') \to \mathcal{A}(X)$, we see that it is enough to prove Theorem 7.2 for X', $\pi^{-1}(C)$, and $\pi^{-1}(F)$. Replacing X with X' if necessary, we can now suppose that X is normal, and that X_K is smooth over K.

We can now apply Proposition 7.6(3) and we find that X is $\mathbb{Q}$-factorial. So there exists an integer $n > 0$ such that the effective Weil divisor nC is associated to a Cartier divisor on X. We are thus reduced to the case where C is a Cartier divisor on X, and the statement then follows from the end of the proof in Section 7.8 of Case (a). □

We show in our next theorem that in Rumely's local-global principle as formulated in [50, 1.7], the hypothesis that the base scheme S is excellent can be removed.

THEOREM 7.9
Let S be a Dedekind scheme satisfying condition (T). Let $X \to S$ be a separated

surjective morphism of finite type. Assume that X is irreducible and that the generic fiber of $X \to S$ is geometrically irreducible. Then $X \to S$ has a finite quasi-section.

Proof
In [50], the hypothesis that S is excellent is only used in 3.3 (which relies on 2.8) and, implicitly, in 2.5. The removal of the hypothesis that S is excellent in 2.5 is addressed in Lemma 8.10(2). To prove the local-global principle, it is enough to prove it for integral quasi-projective schemes of relative dimension 1 over S (see [50, 3.1]).

Assume that S is not excellent. Consider a finite S-morphism $X \to X^{(p^n)}$ such that the normalization of the reduced generic fiber of $X^{(p^n)} \to S$ is smooth (Lemma 7.4). Clearly, $X^{(p^n)} \to S$ has a finite quasi-section if and only if $X \to S$ has one. Similarly, since $(X^{(p^n)})_{\mathrm{red}} \to X^{(p^n)}$ is a finite S-morphism, $(X^{(p^n)})_{\mathrm{red}} \to S$ has a finite quasi-section if and only if $X^{(p^n)} \to S$ has one. We also find from Proposition 7.6(a) that the normalization morphism $X' \to (X^{(p^n)})_{\mathrm{red}}$ is finite and again $(X^{(p^n)})_{\mathrm{red}} \to S$ has a finite quasi-section if and only if $X' \to S$ has one. Thus we are reduced to the case where X is normal and the generic fiber of $X \to S$ is smooth. We now proceed as in the proof of Proposition 7.6(3) to remove the "excellent" hypothesis in [50, 2.8] and in [50, 3.3]. □

The following proposition is needed to produce the examples below which conclude this section.

PROPOSITION 7.10
Let S be a noetherian irreducible scheme. Let $\mathcal{L}$ be an invertible sheaf over S, and consider the scheme $X := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$, with the associated projective† *morphism $\pi : X \to S$. Denote by C_0 and C_∞ the images of the two natural sections of π obtained from the projections $\mathcal{O}_S \oplus \mathcal{L} \to \mathcal{O}_S$ and $\mathcal{O}_S \oplus \mathcal{L} \to \mathcal{L}$. Suppose that there exists a finite flat quasi-section $g : Y \to S$ of $X \to S$ of degree d which does not meet $F := C_0 \cup C_\infty$. Then $\mathcal{L}^{\otimes d}$ is trivial in* $\mathrm{Pic}(S)$.

Proof
Let $X' := X \times_S Y$, with projection $\pi' : X' \to Y$. Clearly, π' corresponds to the natural projection $\mathbb{P}(\mathcal{O}_Y \oplus g^*\mathcal{L}) \to Y$. We find that the morphism π' has now three pairwise disjoint sections, corresponding to three homomorphisms from $\mathcal{O}_Y \oplus g^*\mathcal{L}$ to lines bundles, two of them being the obvious projection maps.

We claim that three such pairwise disjoint sections can exist only if $\mathcal{L}' := g^*\mathcal{L}$ is the trivial invertible sheaf. Let $\mathcal{N} \subset \mathcal{O}_Y \oplus \mathcal{L}'$ be the submodule corresponding to the third section (see [29, II.4.2.4]). For any $y \in Y$, $\mathcal{N} \otimes k(y)$ is different from

†This morphism is projective by definition (see [29, II.5.5.2]). It is then also proper (see [29, II.5.5.3]).

$\mathcal{L}' \otimes k(y)$ (viewed as a submodule of $(\mathcal{O}_Y \oplus \mathcal{L}') \otimes k(y)$) because in the fiber above y, the section defined by $\mathcal{N}$ is disjoint from the section defined by the projection to $\mathcal{O}_Y$, so the image of $\mathcal{N} \otimes k(y)$ in the quotient $k(y)$ is nonzero. Therefore the canonical map $\mathcal{N} \to \mathcal{O}_Y \oplus \mathcal{L}' \to \mathcal{O}_Y$ is surjective and, hence, it is an isomorphism. Similarly, the canonical map $\mathcal{N} \to \mathcal{L}'$ is an isomorphism. Therefore $\mathcal{L}' \simeq \mathcal{O}_Y$. It is known (see, e.g., [32, 2.1]) that since $Y \to S$ is finite and flat, the kernel of the induced map $\operatorname{Pic}(S) \to \operatorname{Pic}(Y)$ is killed by d. □

Example 7.11
Let R be any Dedekind domain, and let $S = \operatorname{Spec} R$. Our next example shows that Theorem 7.2 can hold only if R has the property that $\operatorname{Pic}(R')$ is a torsion group for all Dedekind domains R' finite over R.

Indeed, choose an invertible sheaf $\mathcal{L}$ over S, and consider the scheme $X := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$, with the associated smooth projective morphism $\pi : X \to S$. Let C_0 and C_∞ be as in Proposition 7.10, and let $C := C_0 + C_\infty$. Let $F := \operatorname{Supp}(C)$.

If Theorem 7.2 holds, then a multiple of C can be moved, and there exists a horizontal 1-cycle C' of X such that $\operatorname{Supp}(C') \cap F = \emptyset$. Hence, we find the existence of an integral subscheme Y of X, finite and flat over S, and disjoint from F. Thus, Proposition 7.10 implies that $\mathcal{L}$ is a torsion element in $\operatorname{Pic}(S)$, and for Theorem 7.2 to hold, it is necessary that $\operatorname{Pic}(S)$ be a torsion group. Repeating the same argument starting with any invertible sheaf $\mathcal{L}'$ over any S' (which is regular, and finite and flat over S) and considering the map $\mathbb{P}(\mathcal{O}_{S'} \oplus \mathcal{L}') \to S' \to S$, we find that for Theorem 7.2 to hold, it is necessary that $\operatorname{Pic}(S')$ be a torsion group.

Remark 7.12
An analogue of Theorem 7.2 cannot be expected to hold when S is assumed to be a smooth *proper* curve over a field k, even when k is a finite field. Indeed, suppose that $X \to S$ is given as in Theorem 7.2 with both X/k and S/k smooth and proper. Then any ample divisor C on X will have positive intersection number $(C \cdot D)_X$ with any curve D on X. Such an ample divisor then cannot be contained in a fiber of $X \to S$, and thus must be finite over S. Set $F = \operatorname{Supp}(C)$. The conclusion of Theorem 7.2 cannot hold in this case: it is not possible to find on X a divisor rationally equivalent to C which does not meet the closed set $F = \operatorname{Supp}(C)$.

Example 7.13
Keep the notation introduced in Example 7.11, and choose a nontrivial line bundle $\mathcal{L}$ of finite order $d > 1$ in $\operatorname{Pic}(S)$. Let $X := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$. Let $F := C_0 \cup C_\infty$. Theorem 7.2 (with the appropriate hypotheses on S) implies that a multiple mC_0 of C_0 can be moved away from F. We claim that C_0 itself cannot be moved away from F.

Indeed, otherwise, there exist finitely many finite quasi-sections $Y_i \to S$ in $X \setminus F$ such that the greatest common divisor of the degrees d_i of $Y_i \to S$ is 1 (because $C_0 \to S$ has degree 1). Hence, as in Example 7.11, we find that the order of $\mathcal{L}$ divides d_i for all i. Since $\mathcal{L}$ has order $d > 1$ by construction, we have obtained a contradiction. In fact, we find that dC_0 is the smallest positive multiple of C_0 that could possibly be moved away from F.

8. Finite morphisms to $\mathbb{P}^d_S$

Let $X \to S$ be an affine morphism of finite type, with $S = \operatorname{Spec} R$. Assume that S is irreducible with generic point η, and let $d := \dim X_\eta$. When $R = k$ is a field, the normalization theorem of E. Noether states that there exists a finite k-morphism $X \to \mathbb{A}^d_k$. When R is not a field, no *finite* S-morphism $X \to \mathbb{A}^d_S$ may exist in general, even when $X \to S$ is surjective and S is noetherian.

When $R = k$ is a field, a stronger form of the normalization theorem that applies to graded rings (see, e.g., [17, 13.3]) implies that every projective variety X/k of dimension d admits a finite k-morphism $X \to \mathbb{P}^d_k$. Our main theorem in this section, Theorem 8.1 below, guarantees the existence of a finite S-morphism $X \to \mathbb{P}^d_S$ when $X \to S$ is projective with R pictorsion (Definition 0.3), and $d := \max\{\dim X_s, s \in S\}$. A converse to this statement is given in Proposition 8.7. We end this section with some remarks and examples of pictorsion rings.

THEOREM 8.1

Let R be a pictorsion ring, and let $S :=$ Spec R. Let $X \to S$ be a projective morphism, and set $d := \max\{\dim X_s, s \in S\}$. Then there exists a finite S-morphism $r : X \to \mathbb{P}^d_S$. If we assume in addition that $\dim X_s = d$ for all $s \in S$, then r is surjective.

Proof

Identify X with a closed subscheme of a projective space $P := \mathbb{P}^N_S$. Assume first that $X \to S$ is of finite presentation. We first apply Theorem 5.1 to the projective scheme $P \to S$ with $\mathcal{O}_P(1)$, $C = \emptyset$, $m = 1$, and $F_1 = X$, to find $n_0 > 0$ and $f_0 \in H^0(P, \mathcal{O}_P(n_0))$ such that $X \cap H_{f_0} \to S$ has all its fibers of dimension $\leq d - 1$ (use Lemma 3.2(1)). We apply again Theorem 5.1, this time to $P \to S$ and $\mathcal{O}_P(n_0)$, $C = \emptyset$, $m = 1$, and $F_1 = X \cap H_{f_0}$. We find an integer n_1 and a section $f_1 \in H^0(P, \mathcal{O}_P(n_0 n_1))$ such that $(X \cap H_{f_0}) \cap H_{f_1}$ has fibers over S of maximal dimension $d - 2$. We continue this process $d - 2$ additional times, to find a sequence of homogeneous polynomials $f_0, \dots, f_{d-1}$ such that the closed subscheme $Y := X \cap H_{f_{d-1}} \cap \dots \cap H_{f_0}$ has all its fibers of dimension at most 0 and, hence, is finite over S since it is projective (see [29, IV.8.11.1]). Note that replacing f_i by a positive power of f_i

does not change the topological properties of the closed set H_{f_i}. So we can suppose $f_1, \dots, f_{d-1} \in H^0(P, \mathcal{O}_P(n))$ for some $n > 0$.

Since S is pictorsion, $\mathrm{Pic}(Y)$ is a torsion group. So there exists $j \geq 1$ such that $\mathcal{O}_P(nj)|_Y \simeq \mathcal{O}_Y$. Let $e \in H^0(Y, \mathcal{O}_P(nj)|_Y)$ be a basis. As Y is finitely presented over S, both Y and e can be defined on some noetherian subring R_0 of R (see [29, IV.8.9.1(iii)]). By Serre's vanishing theorem on $\mathbb{P}^N_{R_0}$ applied with the very ample sheaf $\mathcal{O}_{\mathbb{P}^N_{R_0}}(nj)$, we can find $k > 0$ such that $e^{\otimes k}$ lifts to a section $f_d \in H^0(P, \mathcal{O}_P(njk))$. It follows that $H_{f_d} \cap Y = \emptyset$.

We have constructed $d+1$ sections $f_0^{jk}, \dots, f_{d-1}^{jk}, f_d$ in $H^0(P, \mathcal{O}_P(njk))$, whose zero loci on X have empty intersection. The restrictions to X of these sections define a morphism $r : X \to \mathbb{P}^d_S$. Since $X \to S$ is of finite presentation and $\mathbb{P}^d_S \to S$ is separated of finite type, the morphism $r : X \to \mathbb{P}^d_S$ is also of finite presentation (see [29, IV.1.6.2(v)] or [30, I.6.3.8(v)]). By a standard argument (see, e.g., [38, Lemma 3]), the morphism $r : X \to \mathbb{P}^d_S$ is finite. When $\dim X_s = d$, as $X_s \to \mathbb{P}^d_{k(s)}$ is finite, it is also surjective.

Let us consider now the general case where $X \to S$ is not assumed to be of finite presentation. The scheme X, as a closed subscheme of $\mathbb{P}^N_S$, corresponds to a graded ideal $J \subset R[T_0, \dots, T_N]$. Then X is a filtered intersection of subschemes $X_\lambda \subset \mathbb{P}^N_S$ defined by finitely generated graded subideals of J. Thus each natural morphism $f_\lambda : X_\lambda \to S$ is of finite presentation. The points x of X_λ where the fiber dimension $\dim_x(f_\lambda^{-1}(f_\lambda(x))$ is greater than d form a closed subset E_λ of X_λ (see [29, IV.13.1.3]). Since the fibers of $Z \to S$ are of finite type over a field, the corresponding set for $X \to S$ is nothing but $\bigcap_\lambda E_\lambda$, and by hypothesis, the former set is empty. Since $\bigcap_\lambda E_\lambda$ is a filtered intersection of closed subsets in the quasi-compact space $\mathbb{P}^N_S$, we find that E_{λ_0} is empty for some λ_0. We can thus apply the statement of the theorem to the morphism of finite presentation $f_{\lambda_0} : X_{\lambda_0} \to S$ and find a finite S-morphism $X_{\lambda_0} \to \mathbb{P}^d_S$. Composing with the closed immersion $X \to X_{\lambda_0}$ produces the desired finite morphism $X \to \mathbb{P}^d_S$. □

Remark 8.2

Assume that the morphism $r : X \to \mathbb{P}^d_S$ obtained in the above theorem is finite and surjective. When S is a noetherian regular scheme and X is Cohen–Macaulay and irreducible, then r is also flat. Indeed, $\mathbb{P}^d_S$ is regular since S is also regular. Since r is finite and surjective, X is irreducible, and $\mathbb{P}^d_S$ is universally catenary, we find that $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{\mathbb{P}^d_S, r(x)}$ for all $x \in X$ (see [45, 8.2.6]). We can then use [2, V.3.5] to show that r is flat.

Let us note here one class of projective morphisms $X \to \operatorname{Spec} R$ which satisfy the conclusion of Theorem 8.1 without a pictorsion hypothesis on R.

PROPOSITION 8.3

Let R be a noetherian ring of dimension 1 with $S := \operatorname{Spec} R$ connected. Let $\mathcal{E}$ be any locally free $\mathcal{O}_S$-module of rank $r \geq 2$. Then there exists a finite S-morphism $\mathbb{P}(\mathcal{E}) \to \mathbb{P}_S^{r-1}$ of degree r^{r-1}.

Proof

Let S be any scheme. Recall that given any locally free sheaf of rank r of the form $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, with $\mathcal{L}_i$ invertible for $i = 1, \ldots, r$, there exists a finite S-morphism

$$\mathbb{P}(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r) \to \mathbb{P}(\mathcal{L}_1^{\otimes d} \oplus \cdots \oplus \mathcal{L}_r^{\otimes d})$$

of degree d^{r-1}, defined on local trivializations by raising the coordinates to the dth tensor power.

Assume now that $S = \operatorname{Spec} R$ is connected, and recall that when R is noetherian of dimension 1, any locally free $\mathcal{O}_S$-module $\mathcal{E}$ of rank r is isomorphic to a locally free $\mathcal{O}_S$-module of the form $\mathcal{O}_S^{r-1} \oplus \mathcal{L}$, where $\mathcal{L}$ is some invertible sheaf on S (see [67, Proposition 7]),[†] and $\wedge^r \mathcal{E} \cong \mathcal{L}$. Consider the morphism $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{O}_S^{r-1} \oplus \mathcal{L}^{\otimes r})$ of degree r^{r-1} described above. We claim that $\mathbb{P}(\mathcal{O}_S^{r-1} \oplus \mathcal{L}^{\otimes r})$ is isomorphic to $\mathbb{P}_S^{r-1}$. Indeed, we find that $\mathcal{L}^{\oplus r}$ is isomorphic to $\mathcal{O}_S^{r-1} \oplus \mathcal{L}^{\otimes r}$ using the result quoted above, and $\mathbb{P}(\mathcal{L}^{\oplus r})$ is S-isomorphic to $\mathbb{P}(\mathcal{O}_S^r) = \mathbb{P}_S^{r-1}$ (see [29, II.4.1.4]). $\square$

To prove a converse to Theorem 8.1 in Proposition 8.7, we will need the following proposition.

PROPOSITION 8.4

Let S be a connected noetherian scheme. Let $\mathcal{E}$ be a locally free sheaf of rank $n+1$. Consider the natural projection morphism $\pi : \mathbb{P}(\mathcal{E}) \to S$.

(a) *Any invertible sheaf on $\mathbb{P}(\mathcal{E})$ is isomorphic to a sheaf of the form $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m) \otimes \pi^*(\mathcal{L})$, where $m \in \mathbb{Z}$ and $\mathcal{L}$ is an invertible sheaf on S.*

(b) *Assume that $S = \operatorname{Spec} R$ is affine, and let $f : \mathbb{P}_S^n \to \mathbb{P}_S^n$ be a finite morphism. Then $f^*(\mathcal{O}_{\mathbb{P}_S^n}(1))$ is isomorphic to a sheaf of the form $\mathcal{O}_{\mathbb{P}_S^n}(m) \otimes \pi^*(\mathcal{L})$, where $\mathcal{L}$ is an invertible sheaf on S of finite order and $m > 0$.*

Proof

(a) This statement is well known (it is found for instance in [52, p. 20], or in [29, II.4.2.7]). Since we did not find a complete proof in the literature, let us sketch a proof here. Let $\mathcal{M}$ be an invertible sheaf on $\mathbb{P}(\mathcal{E})$. For each $s \in S$, the pullback $\mathcal{M}_s$

[†]When k is an algebraically closed field and R is a finitely generated regular k-algebra of dimension 2, conditions on R are given in [53, Theorem 2], which ensure that such an isomorphism also exists for any locally free sheaf $\mathcal{E}$ on $\operatorname{Spec} R$.

of $\mathcal{M}$ to the fiber over s is of the form $\mathcal{O}_{\mathbb{P}^n_{k(s)}}(m_s)$ for some integer m_s. Using the fact that the Euler characteristic of $\mathcal{M}_s$ is locally constant on S ([46, 4.2(b)]), we find that m_s is locally constant on S. Since S is connected, $m_s = m$ for all $s \in S$. The conclusion follows as in [46, 5.1].

(b) Let $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^n_S}(1)$. Using (a), we find that $f^*(\mathcal{O}(1))$ is isomorphic to a sheaf of the form $\mathcal{O}(m) \otimes \pi^*(\mathcal{L})$, where $\mathcal{L}$ is an invertible sheaf on S. We have $m > 0$ because over each point s, $f^*(\mathcal{O}(1))_s$ is ample, being the pullback by a finite morphism of the ample sheaf $\mathcal{O}(1)_s$, and is isomorphic to $\mathcal{O}(m)_s$.

Write $M := H^0(S, \mathcal{L})$, and identify $H^0(\mathbb{P}^n_S, (\pi^*\mathcal{L})(m))$ with $M \otimes_R R[x_0, x_1, \dots, x_n]_m$, where $R[x_0, x_1, \dots, x_n]_m$ denotes the set of homogeneous polynomials of degree m. The section of $\mathcal{O}(1)$ corresponding to $x_i \in R[x_0, x_1, \dots, x_n]_1$ pulls back to a section of $f^*(\mathcal{O}(1))$ which we identify with an element $F_i \in M \otimes_R R[x_0, x_1, \dots, x_n]_m$.

Since M is locally free of rank 1, there is a cover $\bigcup_{j=1}^t D(s_j)$ of S by special affine open subsets such that $M \otimes_R R[1/s_j]$ has a basis t_j. Hence, for each $i \leq n$, we can write $F_i = t_j \otimes G_{ij}$ with $G_{ij} \in R[1/s_j][x_0, \dots, x_n]_m$. Denote the resultant of $G_{0j}, \dots, G_{nj}$ by $\operatorname{Res}(G_{0j}, \dots, G_{nj})$ (see [36, 2.3]). We claim that $\operatorname{Res}(G_{0j}, \dots, G_{nj})$ is a unit in $R[1/s_j]$. Indeed, over $D(s_j) := \operatorname{Spec} R[1/s_j]$, the restricted morphism $f_{S_j} : \mathbb{P}^n_{S_j} \to \mathbb{P}^n_{S_j}$ is given by the global sections of $\mathcal{O}(m)_{|D(s_j)}$ corresponding to $G_{0j}, \dots, G_{jn} \in R[1/s_j][x_0, \dots, x_n]_m$. Since these global sections generate the sheaf $\mathcal{O}(m)_{|D(s_j)}$, we find that the hypersurfaces $G_{0j} = 0, \dots, G_{nj} = 0$ cannot have a common point and, thus, that $\operatorname{Res}(G_{0j}, \dots, G_{nj}) \in R[1/s_j]^*$.

For $j = 1, \dots, t$, consider now

$$r_j := \operatorname{Res}(G_{0j}, \dots, G_{nj}) t_j^{\otimes(n+1)m^n} \in M^{\otimes(n+1)m^n} \otimes R[1/s_j].$$

Since $\operatorname{Res}(G_{0j}, \dots, G_{nj}) \in R[1/s_j]^*$, the element r_j is a basis for $M^{\otimes(n+1)m^n} \otimes R[1/s_j]$. We show now that $M^{\otimes(n+1)m^n}$ is a free R-module of rank 1 by showing that the elements r_j can be glued to produce a basis r of $M^{\otimes(n+1)m^n}$ over R. Indeed, over $D(s_j) \cap D(s_k)$, we note that there exists $a \in R[1/s_j, 1/s_k]$ such that $at_j = t_k$. Then from $F_i = t_j \otimes G_{ij} = t_k \otimes G_{ik}$, we conclude that $G_{ij} = aG_{ik}$, so that

$$\operatorname{Res}(G_{0j}, \dots, G_{nj}) = a^{(n+1)m^n} \operatorname{Res}(G_{0k}, \dots, G_{nk})$$

(see [36, 5.11.2]). We thus find that r_j is equal to r_k when restricted to $D(s_j) \cap D(s_k)$, as desired. □

Example 8.5

Let S be a connected noetherian affine scheme. Assume that $\operatorname{Pic}(S)$ contains an element $\mathcal{L}$ of infinite order. Suppose that $\mathcal{L}$ can be generated by $d + 1$ sections for some $d \geq 0$. We construct in this example a projective morphism $X_{\mathcal{L}} \to S$, with fibers of dimension d, and such that there exists no finite S-morphism $X_{\mathcal{L}} \to \mathbb{P}^d_S$. Let $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^d_S}(1)$.

Using $d + 1$ global sections in $\mathcal{L}(S)$ which generate $\mathcal{L}$, define a closed S-immersion $i_1 : S \to \mathbb{P}^d_S$, with $i_1^*(\mathcal{O}(1)) = \mathcal{L}$. Consider also the closed S-immersion $i_0 : S \to \mathbb{P}^d_S$ given by $(1 : 0 : \ldots : 0) \in \mathbb{P}^d_S(S)$, so that $i_0^*(\mathcal{O}(1)) = \mathcal{O}_S$. Consider now the scheme $X_{\mathcal{L}}$ obtained by gluing two copies of $\mathbb{P}^d_S$ over the closed subschemes $\mathrm{Im}(i_0)$ and $\mathrm{Im}(i_1)$ ([3, 1.1.1]). It is noted in [3, 1.1.5], that under our hypotheses, the resulting gluing is endowed with a natural morphism $\pi : X_{\mathcal{L}} \to S$ which is separated and of finite type. Recall that the scheme $X_{\mathcal{L}}$ is endowed with two natural closed immersions $\varphi_0 : X_0 = \mathbb{P}^d_S \to X_{\mathcal{L}}$ and $\varphi_1 : X_1 = \mathbb{P}^d_S \to X_{\mathcal{L}}$ such that $\varphi_0 \circ i_0 = \varphi_1 \circ i_1$. Moreover, the S-morphism $(\varphi_0, \varphi_1) : \mathbb{P}^d_S \sqcup \mathbb{P}^d_S \to X_{\mathcal{L}}$ is finite and surjective. Since $\mathbb{P}^d_S \to S$ is proper, we find that $X_{\mathcal{L}} \to S$ is also proper.

Suppose that there exists a finite S-morphism $f : X_{\mathcal{L}} \to \mathbb{P}^d_S$. Then, using Proposition 8.4(b), we find that there exist two torsion invertible sheaves $\mathcal{G}_0$ and $\mathcal{G}_1$ on S and $m \geq 0$ such that $(f \circ \varphi_0 \circ i_0)^*(\mathcal{O}_{\mathbb{P}^d_S}(1)) = \mathcal{G}_0$, and $(f \circ \varphi_1 \circ i_1)^*(\mathcal{O}_{\mathbb{P}^d_S}(1)) = \mathcal{L}^{\otimes m} \otimes \mathcal{G}_1$. Since we must have then $\mathcal{G}_0$ isomorphic to $\mathcal{L}^{\otimes m} \otimes \mathcal{G}_1$ and since $\mathcal{L}$ is not torsion, we find that such a morphism f cannot exist.

To conclude this example, it remains to show that $\pi : X_{\mathcal{L}} \to S$ is a projective[†] morphism. For this, we exhibit an ample sheaf on $X_{\mathcal{L}}$ as follows. Consider the sheaf $\mathcal{F}_0 := (\pi \circ \varphi_0)^*(\mathcal{L})(1)$ on $X_0 = \mathbb{P}^d_S$ and the sheaf $\mathcal{F}_1 := \mathcal{O}(1)$ on $X_1 = \mathbb{P}^d_S$. We clearly have a natural isomorphism of sheaves $i_0^*(\mathcal{F}_0) \to i_1^*(\mathcal{F}_1)$ on S. Thus, we can glue the sheaves $\mathcal{F}_0$ and $\mathcal{F}_1$ to obtain a sheaf $\mathcal{F}$ on $X_{\mathcal{L}}$. Since both $\mathcal{F}_0$ and $\mathcal{F}_1$ are invertible, we find that $\mathcal{F}$ is also invertible on $X_{\mathcal{L}}$ (such a statement in the affine case can be found in [21, 2.2]). Under the finite S-morphism $(\varphi_0, \varphi_1) : \mathbb{P}^d_S \sqcup \mathbb{P}^d_S \to X_{\mathcal{L}}$, the sheaf $\mathcal{F}$ pulls back to the sheaf restricting to $\mathcal{F}_0$ on X_0 and $\mathcal{F}_1$ on X_1. In particular, the pullback is ample (since $\mathcal{L}$ is generated by its global sections), and since $X_{\mathcal{L}} \to S$ is proper, we can apply [29, III.2.6.2], to find that $\mathcal{F}$ is also ample.

Remark 8.6

Let R be a Dedekind domain, and let $S := \operatorname{Spec} R$. Let $X \to S$ be a projective morphism with fibers of dimension 1. When R is pictorsion, Theorem 8.1 shows that there exists a finite S-morphism $X \to \mathbb{P}^1_S$. It is natural to wonder, when R is not assumed to be pictorsion, whether it would still be possible to find a locally free $\mathcal{O}_S$-module $\mathcal{E}$ of rank 2 and a finite S-morphism $X \to \mathbb{P}(\mathcal{E})$. The answer to this question is negative, as the following example shows.

Assume that $\mathrm{Pic}(S)$ contains an element $\mathcal{L}$ of infinite order. Then $\mathcal{L}$ can be generated by two sections. Consider the projective morphism $X_{\mathcal{L}} \to S$ constructed in

[†]For an example where the gluing of two projective spaces over a "common" closed subscheme is not projective, see [21, 6.3].

Example 8.5. Suppose that there exists a locally free $\mathcal{O}_S$-module $\mathcal{E}$ of rank 2 and a finite S-morphism $X_{\mathcal{L}} \to \mathbb{P}(\mathcal{E})$. Proposition 8.3 shows that there exists then a finite S-morphism $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1_S$. We would then obtain by composition a finite S-morphism $X_{\mathcal{L}} \to \mathbb{P}^1_S$, which is a contradiction. We thank Pascal Autissier for bringing this question to our attention.

We are now ready to prove a converse to Theorem 8.1.

PROPOSITION 8.7
Let R be any nonzero commutative ring, and let $S := \operatorname{Spec} R$. Suppose that for any $d \geq 0$, and for any projective morphism $X \to S$ such that $\dim X_s = d$ for all $s \in S$, there exists a finite surjective S-morphism $X \to \mathbb{P}^d_S$. Then R is pictorsion.

When R is noetherian of finite Krull dimension $\dim R$, then R is pictorsion if for all projective morphisms $X \to S$ such that $\dim X_s \leq \dim R$ for all $s \in S$, there exists a finite S-morphism $X \to \mathbb{P}^{\dim R}_S$.

Proof
Let R' be a finite extension of R, and let $\mathcal{L}' \in \operatorname{Pic}(\operatorname{Spec} R')$. The sheaf $\mathcal{L}'$ descends to an element $\mathcal{L}$ of $\operatorname{Pic}(\operatorname{Spec} R_0)$ for some noetherian subring R_0 of R'. For each connected component S_i of $\operatorname{Spec} R_0$, let d_i be such that $\mathcal{L}_{|S_i}$ can be generated by $d_i + 1$ global sections. Let $d := \max(d_i)$. When R is noetherian we take $R = R_0$, and when in addition $\dim(R) < \infty$, we can always choose $d \leq \dim(R)$.

Assume now that $\mathcal{L}'$ is of infinite order. It follows that $\mathcal{L}$ is of infinite order on some connected component of $\operatorname{Spec} R_0$. Apply the construction of Example 8.5 to each connected component S_i of $\operatorname{Spec} R_0$ where $\mathcal{L}$ has infinite order, with a choice of d global sections of $\mathcal{L}_{|S_i}$ which generate $\mathcal{L}_{|S_i}$. We obtain a projective scheme $X_i \to S_i$ with fibers of dimension d and which does not admit a finite morphism to $X_i \to \mathbb{P}^d_{S_i}$. If S_j is a connected component of $\operatorname{Spec} R_0$ such that $\mathcal{L}$ has finite order, we set $X_j \to S_j$ to be $\mathbb{P}^d_{S_j} \to S_j$. We let X denote the disjoint union of the schemes X_i. The natural morphism $X \to \operatorname{Spec} R_0$ has fibers of dimension d and does not admit a finite morphism $X \to \mathbb{P}^d_{R_0}$.

Let R_1 be any noetherian ring such that $R_0 \subset R_1 \subset R'$. By construction, the pullback of $\mathcal{L}$ to $\operatorname{Spec} R_1$ has infinite order on some connected component of $\operatorname{Spec} R_1$. Since the construction in Example 8.5 is compatible with pullbacks, we conclude that $X \times_{R_0} R_1$ does not admit a finite morphism to $\mathbb{P}^d_{R_1}$. It follows then from [29, IV.8.8.2, IV.8.10.5] that there is no finite morphism $X \times_R R' \to \mathbb{P}^d_{R'}$. Hence, there is no finite R-morphism $X \times_R R' \to \mathbb{P}^d_R$. Replacing $X \times_R R'$ with its disjoint union with $\mathbb{P}^d_R$ if necessary, we obtain a projective morphism to S with fibers of dimension d and which does not factor through a finite morphism to $\mathbb{P}^d_S$. $\square$

We present below an example of an affine regular scheme S of dimension 3 with a locally free sheaf $\mathcal{E}$ of rank 2 of the form $\mathcal{E} = \mathcal{O}_S \oplus \mathcal{L}$ such that $\mathbb{P}(\mathcal{E})$ does not admit a finite S-morphism to $\mathbb{P}^1_S$.

Example 8.8
Let V be any smooth connected quasi-projective variety over a field k. Let $\mathcal{E}$ be a locally free sheaf of rank r on V. Let $p : \mathbb{P}(\mathcal{E}) \to V$ denote the associated projective bundle. Denote by $A(V)$ the Chow ring of algebraic cycles on V modulo rational equivalence. Let ξ denote the class in $A(\mathbb{P}(\mathcal{E}))$ of the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Then p induces a ring homomorphism $p^* : A(V) \to A(\mathbb{P}(\mathcal{E}))$, and $A(\mathbb{P}(\mathcal{E}))$ is a free $A(V)$-module generated by $1, \xi, \dots, \xi^{r-1}$. For $i = 0, 1, \dots, r$, one defines (see, e.g., [33, p. 429]) the ith Chern class of $\mathcal{E}$, $c_i(\mathcal{E}) \in A^i(V)$, and these classes satisfy the requirements that $c_0(\mathcal{E}) = 1$ and

$$\sum_{i=0}^{r} (-1)^i p^*\big(c_i(\mathcal{E})\big)\xi^{r-i} = 0$$

in $A^r(\mathbb{P}(\mathcal{E}))$. When $\mathcal{E} = \mathcal{O}_V \oplus \mathcal{L}$ for some invertible sheaf $\mathcal{L}$, we find that $c_2(\mathcal{E}) = 0$.

Consider now the case where $\mathcal{E}$ has rank 2, and suppose that there exists a finite V-morphism $f : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1_V$. Then $f^*(\mathcal{O}_{\mathbb{P}^1_V}(1))$ is isomorphic to a locally free sheaf of the form $p^*\mathcal{M} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)$ for some $m > 0$ and some invertible sheaf $\mathcal{M}$ on V (Proposition 8.4(a)). Consider the ring homomorphism

$$f^* : A(\mathbb{P}^1_V) = A(V)[h]/(h^2) \to A\big(\mathbb{P}(\mathcal{E})\big) = A(V)[\xi]/\big(\xi^2 - c_1(\mathcal{E})\xi + c_2(\mathcal{E})\big),$$

where h denote the class in $A(\mathbb{P}^1_V)$ of the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_V}(1)$. It follows that in $A(\mathbb{P}(\mathcal{E}))$, $f^*(h) = a + m\xi$ with $a \in A^1(V)$, and $(a + m\xi)^2 = 0$. Hence, $m^4(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E})) = 0$ in $A^2(V)$. Thus, in $A^2(V)_{\mathbb{Q}}$, $c_1(\mathcal{E})^2 = 4c_2(\mathcal{E})$. Choose now $\mathcal{E} = \mathcal{O}_V \oplus \mathcal{L}$ for some invertible sheaf $\mathcal{L}$. Then $0 = c_2(\mathcal{E}) = c_1(\mathcal{E})^2 = c_1(\mathcal{L})^2$ in $A^2(V)_{\mathbb{Q}}$.

We are now ready to construct our example. Recall that under our hypotheses on V, there exists an affine variety S and a surjective morphism $\pi : S \to V$ such that π is a torsor under a vector bundle (see Jouanolou's device, [34, 1.5]). We will use only the simplest case of this construction, when $V = \mathbb{P}^2_k$. In this case, S is the affine variety formed by all (3×3)-matrices which are idempotent and have rank 1. We claim that we have an isomorphism

$$\pi^* : A(V)_{\mathbb{Q}} \to A(S)_{\mathbb{Q}}.$$

Indeed, this statement with the Chow rings replaced by K-groups is proven in [35, 1.1]. Then we use the fact that the Chern character determines an isomorphism of $\mathbb{Q}$-algebras $ch : K^0(X)_{\mathbb{Q}} \to A(X)_{\mathbb{Q}}$ where $K^0(X)$ denotes the Grothendieck group of algebraic vector bundles on a smooth quasi-projective variety X over a field (see [23, 15.2.16(b)]).

Choose on S the line bundle $\mathcal{L} := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then since $A(V)_{\mathbb{Q}} = \mathbb{Q}[h]/(h^3)$, we find that $h^2 \neq 0$, so that $c_1(\mathcal{L})^2 \neq 0$ in $A(S)_{\mathbb{Q}}$. Hence, we have produced a smooth affine variety S of dimension 3, and a locally free sheaf $\mathcal{E} := \mathcal{O}_S \oplus \mathcal{L}$ such that $\mathbb{P}(\mathcal{E})$ does not admit a finite surjective S-morphism to $\mathbb{P}^1_S$.

We conclude this section with some remarks and examples of pictorsion rings (Definition 0.3). We first note the following.

LEMMA 8.9
Let R be any commutative ring. Denote by R^{red} the quotient of R by its nilradical. Then R is pictorsion if and only if R^{red} is pictorsion.

Proof
Since $R \to R^{\text{red}}$ is a finite homomorphism, it is clear that if R is pictorsion, then so is R^{red}. Assume now that R^{red} is pictorsion, and let $R \to R'$ be a finite homomorphism. Then $R^{\text{red}} \to (R')^{\text{red}}$ is a finite homomorphism. Thus $\text{Pic}((R')^{\text{red}})$ is a torsion group. As we can see using Nakayama's lemma, $\text{Pic}(R') \to \text{Pic}((R')^{\text{red}})$ is injective, so $\text{Pic}(R')$ is a torsion group. □

A pictorsion Dedekind domain R satisfies condition (T)(a) in Definition 0.2; that is, $\text{Pic}(R_L)$ is a torsion group for any Dedekind domain R_L obtained as the integral closure of R in a finite extension L/K. This statement is obvious when R is excellent; when R_L is not finite over R, we use the fact that $\text{Pic}(R_L) = \varinjlim \text{Pic}(R')$, with the direct limit taken over all finite extensions R' of R contained in L. The statement of (2) below is found in [50, 2.3], when R is excellent. We follow the proof given in [50], modifying it only in 2.5 to also treat the case where R is not excellent. We do not know of an example of a Dedekind domain which satisfies condition (T)(a) and which is not pictorsion.

LEMMA 8.10
Let R be a Dedekind domain with field of fractions K.
(1) *Let L/K be a finite extension of degree d, and let R' denote a sub-R-algebra of L, integral over R. Then the kernel of* $\text{Pic}(R) \to \text{Pic}(R')$ *is killed by d.*
(2) *If R satisfies condition (T) in Definition 0.2, then R is pictorsion.*

Proof
(1) When R' is finite and flat over R, this is well known (see, e.g., [32, 2.1]). (The hypothesis that R is Dedekind is used here to ensure that the ring R' is flat over R.) In

general, let M be a locally free R-module of rank 1 such that $M \otimes_R R'$ is isomorphic as R'-module to R'. Then there exist a finite R-algebra A contained in R' such that $M \otimes_R A$ is isomorphic as A-module to A. It follows that M^d is trivial in $\mathrm{Pic}(R)$, since A/R is finite.

(2) Let $S = \operatorname{Spec} R$. Let Z be a finite S-scheme. We need to show that $\mathrm{Pic}(Z)$ is torsion. The proof in [50, 2.3–2.6] is complete when R is excellent. When R is not necessarily excellent, only 2.5 needs to be modified as follows. Assume that Z is reduced. Let $Z' \to Z$ be the normalization morphism, which need not be finite. Then Z' is a finite disjoint union of Dedekind schemes, and the hypothesis that R satisfies condition (T)(a) implies that $\mathrm{Pic}(Z')$ is a torsion group. Let $\mathcal{L} \in \mathrm{Pic}(Z)$. Then there exists $n \geq 1$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{O}_{Z'} \simeq \mathcal{O}_{Z'}$. This isomorphism descends to some Z-scheme Z_α with $\pi : Z_\alpha \to Z$ finite and birational. We now use the proof of 2.5 in [50] applied to $Z_1 = Z_\alpha$ (instead of the normalization which is not necessarily finite), to find that the kernel of $\mathrm{Pic}(Z) \to \mathrm{Pic}(Z_\alpha)$ is torsion. Hence, $\mathcal{L}$ is torsion. □

PROPOSITION 8.11

Let R be a Dedekind domain with field of fractions K. Let $\overline{R}$ denote the integral closure of R in an algebraic closure $\overline{K}$ of K. The following are equivalent:

(1) *Condition (T)(a) in Definition 0.2 holds.*

(2) *$\overline{R}$ is a Bézout domain (i.e., all finitely generated ideals of $\overline{R}$ are principal).*

Proof

That (1) implies (2) is the content of [37, Theorem 102]. Assume that (2) holds, and let R_L be the integral closure of R in a finite extension L/K. Let I be a nonzero ideal in R_L. Then $I\overline{R}$ is principal. Hence, there exists a finite extension F/L such that in the integral closure R_F of R in F, IR_F is principal. Since the kernel of $\mathrm{Pic}(R_L) \to \mathrm{Pic}(R_F)$ is killed by $[F : L]$ (Lemma 8.10), we find that I has finite order in $\mathrm{Pic}(R_L)$. □

Remark 8.12

Keep the notation of Proposition 8.11, and denote by R_F the integral closure of R in any algebraic extension F/K. Then condition (T)(a) in Definition 0.2 implies that $\mathrm{Pic}(R_F)$ is a torsion group. Indeed, one finds that $\mathrm{Pic}(R_F) = \varinjlim \mathrm{Pic}(R_L)$, with the direct limit taken over all finite extensions L/K contained in F.

Condition (2) in Proposition 8.11 is equivalent to $\mathrm{Pic}(\overline{R}) = (0)$. Indeed, the ring $\overline{R}$ is a Prüfer domain (see [37, Theorem 101]), and a Prüfer domain D is a Bézout domain if and only if $\mathrm{Pic}(D) = (0)$.

We now recall two properties of commutative rings and relate them to the notion of pictorsion introduced in this article. A *local-global* ring R is a commutative ring where the following property holds: whenever $f \in R[x_1, \dots, x_n]$ is such that the ideal of values $(f(r), r \in R^n)$ is equal to the full ring R, then there exists $r \in R^n$ such that $f(r) \in R^*$. A commutative ring R *satisfies the primitive criterion* if, whenever $f(x) = a_n x^n + \cdots + a_0$ is such that $(a_n, \dots, a_0) = R$ (such f is called *primitive*), there exists $r \in R$ such that $f(r) \in R^*$. A ring R satisfies the primitive criterion if and only if it is local-global and for each maximal ideal M of R, the residue field R/M is infinite (see [48, Proposition, bottom of p. 456] or [19, 3.5]).

PROPOSITION 8.13
Let R be a local-global commutative ring. Then every finite R-algebra R' has $\mathrm{Pic}(R') = (1)$. *In particular, R is pictorsion.*

Proof
The ring R' is also a local-global ring (see [19, 2.3]). In a local-global ring, every finitely generated projective R-module of constant rank is free (see [48, Theorem, p. 457]). It follows that $\mathrm{Pic}(R') = (1)$. $\square$

Example 8.14
Rings which satisfy the primitive criterion can be constructed as follows (see, e.g., [75, 1.13] and also [19, Section 5]). Let R be any commutative ring, and consider the multiplicative subset S of $R[x]$ consisting of all primitive polynomials. Then the ring $R(x) := S^{-1}R[x]$ satisfies the primitive criterion.† Indeed, suppose that $g(y) \in R(x)[y]$ is primitive. Then write $g(y) = \sum_{i=0}^{n} f_i(x) y^i$, with $f_i(x) \in R(x)$. It is easy to reduce to the case where $f_i(x) \in R[x]$ for all i. Since $g(y)$ is primitive, we find that the ideal generated by the coefficients of the polynomials $f_0(x), \dots, f_n(x)$ is the unit ideal of R. Hence, choosing $y := x^t$ for t large enough, we find that $g(x^t)$ is a primitive polynomial in $R[x]$ and thus is a unit in $R(x)$.

Example 8.15
We have seen already in this article examples of commutative rings R such that for every finite morphism $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$, $\mathrm{Pic}(R')$ is trivial (Example 8.14), or $\mathrm{Pic}(R')$ is finite but not necessarily trivial (take $R = \mathbb{Z}$ or $\mathbb{F}_p[x]$). When needed, such rings could be called *pictrivial* and *picfinite*, respectively.

†The ring $R(x)$ is considered already in [42, p. 535], after Hilfssatz 1. The notation $R(x)$ was introduced by Nagata (see the historical remark in [54, p. 213]). When R is a local ring, the extension $R \to R(x)$ is used to reduce some considerations to the case of local rings with infinite residue fields (see, e.g., [71, 8.4, p. 159]). Let X be any scheme with an ample invertible sheaf. An affine scheme X' with a faithfully flat morphism $X' \to X$ is constructed in [20, 4.3], in analogy with the purely affine situation $\mathrm{Spec}\, R(x) \to \mathrm{Spec}\, R$.

Let us note in this example a ring R which is pictorsion and such that at least one of the groups $\operatorname{Pic}(R')$ is not finite. Consider the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$, and let $R := \overline{\mathbb{F}}_p[x]$. Then R is pictorsion because it satisfies condition (T) (Lemma 8.10(2)). Indeed, let R' be the integral closure of R in a finite extension of $\overline{\mathbb{F}}_p(x)$. Then $U := \operatorname{Spec}(R')$ is a dense open subset of a smooth connected projective curve $X/\overline{\mathbb{F}}_p$. One shows that the natural restriction map $\operatorname{Pic}(X) \to \operatorname{Pic}(U)$ induces a surjective map $\operatorname{Pic}^0(X) \to \operatorname{Pic}(U)$ with finite kernel. When the genus of X is bigger than 0, it is known that $\operatorname{Pic}^0(X)$, which is isomorphic to the $\overline{\mathbb{F}}_p$-points of the Jacobian of X, is an infinite torsion group.

We also note that the ring $R := \overline{\mathbb{F}}_p[x, y]$ is not pictorsion. Indeed, let $X/\overline{\mathbb{F}}_p$ be a smooth projective surface over $\overline{\mathbb{F}}_p$ such that its Néron–Severi group $\operatorname{NS}(X)$ has rank greater than 1. (For instance, X could be the product of two smooth projective curves.) Let $D \subset X$ be an irreducible divisor whose complement $V := X \setminus D$ is affine. Write $V = \operatorname{Spec} A$, and use Noether's normalization lemma to view A as a finite $\overline{\mathbb{F}}_p[x, y]$-algebra. We claim that $\operatorname{Pic}(V)$ is not a torsion group. Indeed, the natural restriction map $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is surjective, with kernel generated by the class of D. If $\operatorname{Pic}(V)$ is torsion, then the quotient of $\operatorname{NS}(X)$ by the subgroup generated by image of D is torsion. This contradicts the hypothesis on the rank of $\operatorname{NS}(X)$.

Example 8.16

Robert Varley suggested the following example of a Dedekind domain A, which is pictorsion with infinitely many maximal ideals, each having residue field which is not an algebraic extension of a finite field, that is, such that A does not satisfies condition (T)(b) in Definition 0.2. Rather than providing a direct proof that the ring below is pictorsion, we interpret the example in light of the above definitions:

Let Z denote a countable subset of $\mathbb{C}$. Consider the polynomial ring $\mathbb{C}[x]$, and let T denote the multiplicative subset of all polynomials which do not vanish on Z. Then $A := T^{-1}(\mathbb{C}[x])$ satisfies the primitive criterion, and is thus pictorsion by Proposition 8.13.

Indeed, let $F(y) \in A[y]$ be a primitive polynomial. Up to multiplication by elements of T, we can assume that $F(y) = f_n(x)y^n + \cdots + f_0(x)$ with $f_i \in \mathbb{C}[x]$ for all i, and that $x - z$ does not divide $\gcd(f_i(x), i = 0, \ldots, n)$ for all $z \in Z$. We claim that there exists $a \in \mathbb{C}$ such that $F(a) \in \mathbb{C}[x]$ is coprime to $x - z$ for all $z \in Z$. This shows that $F(a)$ is invertible in $A := T^{-1}(\mathbb{C}[x])$, and thus A satisfies the primitive criterion. To prove this claim, let us think of $F(y)$ as a polynomial $F(x, y)$ in two variables, and let us first note that the curve $F(x, y) = 0$ intersects the line $x - z = 0$ in at most $\deg(F)$ places. Thus, there are only countably many points in the plane $\mathbb{C}^2$ of the form (z, v) with $z \in Z$ and $F(z, v) = 0$. Therefore, it is possible to choose $a \in \mathbb{C}$ such that $F(x, a) = 0$ does not contain any of these countably many points.

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$. In view of the above example, it is natural to wonder whether there exists a multiplicative subset T of $\overline{\mathbb{Q}}[x]$ such that $R := T^{-1}(\overline{\mathbb{Q}}[x])$ is pictorsion and Spec R is infinite. Clearly, the integral closure $\tilde{R}$ of R in the algebraic closure of $\overline{\mathbb{Q}}(x)$ must be Bézout (Proposition 8.11). A related question is addressed in [13, Section 5] and in [26].

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