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Rational points of the group of components of a Néron model

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Abstract. The structure of component groups of Néron models has been investigated on several occasions. Here we admit non-separably closed residue fields and are interested in the subgroup of rational points or, in other terms, in the subgroup of geometrically connected components of a Néron model. We consider Néron models of abelian varieties and of algebraic tori and give detailed computations in the case of Jacobians of curves.

Introduction

Let A_K be an abelian variety over a discrete valuation field K. Let A be the Néron model of A_K over the ring of integers $\mathcal{O}_K$ of K and A_k its special fibre over the residue field k of $\mathcal{O}_K$. Denote by A^0 and A_k^0 the corresponding identity components. Then we have an exact sequence

$$0 \to A_k^0 \to A_k \to \phi_A \to 0,$$

where ϕ_A is a finite étale group scheme over k. The latter is called the *group of components of* A. The group of rational points $\phi_A(k)$ counts the number of connected components of the special fibre A_k which are geometrically connected. In this paper we are interested in "computing" this group and the image of $A_K(K) \to \phi_A(k)$. The starting point of this work is an e-mail of E. Schaefer to the second author. He convinced us of the interest in computing $\phi_A(k)$. Actually, the order of the group $\phi_A(\mathbb{F}_p)$ is involved in the conjecture of Birch and Swinnerton-Dyer for abelian varieties over $\mathbb{Q}$. While a lot is known about $\phi_A(k)$ when k is separably closed, it is hard to find literature dealing with the general case.

This paper is organized as follows. Section 1 deals with the case where A_K is the Jacobian of a curve X_K over K. Let X be a regular model of X_K over $\mathcal{O}_K$. Then a modified intersection matrix gives an explicit subgroup of

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$\phi_A(k)$ and the quotient can be controlled by some cohomology groups. The main result of this section is Theorem 1.17 which determines $\phi_A(k)$ when k is finite.

In Section 2, we put together some classical results and general remarks about the canonical map $A_K(K) \to \phi_A(k)$.

In Sections 3 and 4, we assume that K is complete. First we consider algebraic tori T_K with multiplicative reduction (so T_K is not an abelian variety in this section). Let T be the Néron model of T_K. We show in 3.2 that $\phi_T(k)$ coincides with $\phi_{T_G}(k)$, where $T_{G,K}$ is the biggest split subtorus of T_K, and that $T(\mathcal{O}_K)/T^0(\mathcal{O}_K) \to \phi_T(k)$ is an isomorphism. If T_K does not admit multiplicative reduction, the same constructions lead to subgroups of finite index; cf. 3.3. Finally, we add some results on abelian varieties A_K with semi-stable reduction, which are more or less known. When the toric part of A_k is split, then ϕ_A is constant; cf. 4.3. In general, using data coming from the rigid uniformization of A_K, we are able to interpret the image of $A_K(K) \to \phi_A(k)$; see 4.4.

Throughout this paper, we fix a separable closure k^s of k, and we denote by G the absolute Galois group $\mathrm{Gal}(k^s/k)$ of k.

1. Component groups of Jacobians

In this section, we fix a proper flat and regular curve X over $\mathcal{O}_K$ whose generic fibre is geometrically irreducible. Let us start with some notations. In this section A_K will always denote the Jacobian of X_K. Let Γ_i, $i \in I$, be the irreducible components of the special fibre X_k. Denote by $\mathbb{Z}^I$ the free $\mathbb{Z}$-module generated by the Γ_i's. It can be identified canonically with the group of Weil divisors on X with support in X_k. We denote by d_i the multiplicity of Γ_i in X_k, e_i its geometric multiplicity (see [2], Def. 9.1.3), and let $r_i = [k(\Gamma_i) \cap k^s : k]$. The integer r_i is also the number of irreducible components of $(\Gamma_i)_{k^s}$. For two divisors V_1, V_2 on X, such that at least one of them, say V_1, is vertical (i.e. contained in X_k), we denote by $V_1 \cdot V_2$ their *intersection number* $\deg_k \mathcal{O}_X(V_2)|_{V_1}$. When it is necessary to refer to the ground field k, we denote this number by $\langle V_1, V_2\rangle_k$.

Now let us define two homomorphisms of $\mathbb{Z}$-modules which are essential for the computing of $\phi_A(k)$. First, $\alpha : \mathbb{Z}^I \to \mathbb{Z}^I$ is defined by

$$\alpha(V) = \sum_i r_i^{-1} e_i^{-1} \langle V, \Gamma_i\rangle_k \Gamma_i$$

for any $V \in \mathbb{Z}^I$ (see Lemma 1.4 which shows that α really takes values in $\mathbb{Z}^I$). Define $\beta : \mathbb{Z}^I \to \mathbb{Z}$ by $\beta(\Gamma_i) = r_i d_i e_i$. Note that α can be defined more canonically as a map $\mathbb{Z}^I \to (\mathbb{Z}^I)'$ using a suitable (not necessarily symmetric) bilinear form. But for our purpose, this does not seem to be useful.

Let $\mathcal{O}_K^{sh}$ denote a strict henselization of $\mathcal{O}_K$. The residue field of $\mathcal{O}_K^{sh}$ is k^s. The base change $X \times \operatorname{Spec} \mathcal{O}_K^{sh} \to \operatorname{Spec} \mathcal{O}_K^{sh}$ gives rise to a regular surface with special fibre X_{k^s}. Let $\overline{I}$ be a set indexing the irreducible components of X_{k^s}. We can define similarly $\overline{\alpha} : \mathbb{Z}^{\overline{I}} \to \mathbb{Z}^{\overline{I}}$ and $\overline{\beta} : \mathbb{Z}^{\overline{I}} \to \mathbb{Z}$. The Galois group G acts on $\mathbb{Z}^{\overline{I}}$ via its action on X_{k^s}. Moreover, it is not hard to check that the action of G commutes with $\overline{\alpha}$ and $\overline{\beta}$. Note that since ϕ_A is étale over k, $\phi_A(k^s) = \phi_A(k^{\mathrm{alg}})$.

Theorem 1.1 (Raynaud). *Let X be a proper flat and regular curve over $\mathcal{O}_K$, with geometrically irreducible generic fibre. Assume further that either k is perfect or X has an étale quasi-section. Let A be the Néron model of the Jacobian of X_K. Then there exists a canonical exact sequence of G-modules*

$$0 \to \operatorname{Im} \overline{\alpha} \to \operatorname{Ker} \overline{\beta} \to \phi_A(k^s) \to 0. \tag{1}$$

Proof. The existence and exactness of the complex as abstract groups are proved in [2], Theorem 9.6.1. Let us just explain quickly why the map $\operatorname{Ker} \overline{\beta} \to \phi_A(k^s)$ commutes with the natural action of G on both sides. To do this, let us go back to the construction of the map $\operatorname{Ker} \overline{\beta} \to \phi_A(k^s)$ as done in [2], Lemma 9.5.9.

Let P be the open subfunctor of $\operatorname{Pic}_{X/\mathcal{O}_K}$ corresponding to line bundles of total degree 0, then the Néron model A is a quotient of P ([2], Theorem 9.5.4). Since our assertion concerns only the special fibre and since the formation of Néron models commutes with étale base change, we can replace $\mathcal{O}_K$ by a henselization and thus assume that $\mathcal{O}_K$ is henselian. Then $\mathcal{O}_K^{sh}$ is Galois over $\mathcal{O}_K$ with group G. Let $S = \operatorname{Spec} \mathcal{O}_K^{sh}$, $Y = X \times S$, and let $\mathbb{Z}^{\overline{I}}$ be identified with D, group of Weil divisors of Y with support in X_{k^s}. Denote by Δ_j, $j \in \overline{I}$ the irreducible components of X_{k^s}. Consider the homomorphism $\rho : \operatorname{Pic}(Y) \to D$ defined by $\rho(\mathcal{L}) = \sum_j e_j^{-1}(\deg_{k^s} \mathcal{L}|_{\Delta_j})\Delta_j$. Then $\rho^{-1}(\operatorname{Ker} \overline{\beta}) = P(S)$, and ρ is a homomorphism of G-modules. Since S is strictly henselian, it turns out that $\rho|_{P(S)}$ induces an isomorphism $P(S)/P^0(S) \simeq \operatorname{Ker} \overline{\beta}/\operatorname{Im} \overline{\alpha}$, and the canonical map

$$P(S)/P^0(S) \to A(S)/A^0(S) = \phi_A(k^s)$$

is also an isomorphism. The composition of these two isomorphisms gives rise to the above exact sequence. Now it is clear that all the maps we considered are compatible with the natural actions of G. □

Corollary 1.2. *If $r_i = 1$ (i.e. Γ_i is geometrically irreducible) for all i, then ϕ_A is a constant algebraic group (or equivalently, $\phi_A(k) = \phi_A(k^s)$).*

Proof. Actually, in this case G acts trivially on $\operatorname{Ker} \overline{\beta}$. □

Example 1.3. It is known that for modular curves $X_0(N)$ over $\mathbb{Q}_p$, the multiplicities r_i are equal to 1 (at least when N is square-free). Thus the component group of the Jacobian $J_0(N)$ is constant.

Now we want to take the long exact sequence of Galois cohomology of (1). For this purpose, we need some informations on the action of G on $\mathbb{Z}^{\overline{I}}$. Let us first state a technical lemma.

Lemma 1.4. *Let Y be a proper regular scheme (of arbitrary dimension) over a discrete valuation ring $\mathcal{O}_K$. Let $\mathcal{O}_{K'}$ be a finite étale Galois extension of $\mathcal{O}_K$, with residue field k' at some maximal ideal of $\mathcal{O}_{K'}$. Let Γ be an irreducible component of the special fibre Y_k. Then for any irreducible component Γ' of $\Gamma_{k'}$, and for any curve C (i.e. closed subscheme of dimension 1) contained in Y, we have*

$$\langle \Gamma, C\rangle_k = [k' \cap k(\Gamma) : k]\langle \Gamma', p^*C\rangle_{k'}$$

where p is the canonical projection $Y_{\mathcal{O}_{K'}} \to Y$. Moreover, if Y has dimension 2, then the geometric multiplicity e of Γ divides $\langle \Gamma', p^*C\rangle_{k'}$.

Proof. We consider $Y_{\mathcal{O}_{K'}}$ as a regular scheme over $\mathcal{O}_K$. Let $r = [k' \cap k(\Gamma) : k]$. Then $p^*\Gamma = \Gamma_{k'}$ has r irreducible components. Since $Y_{\mathcal{O}_{K'}} \to Y$ is Galois, and $p^*\Gamma$ is invariant by $\mathrm{Gal}(K'/K)$, for any irreducible component Γ'' of $\Gamma_{k'}$ we have $\langle \Gamma', p^*C\rangle_{k'} = \langle \Gamma'', p^*C\rangle_{k'}$. Thus $r\langle \Gamma', p^*C\rangle_{k'} = \langle p^*\Gamma, p^*C\rangle_{k'} = \langle \Gamma, C\rangle_k$. The last equality holds because $\mathcal{O}_{K'}$ is étale over $\mathcal{O}_K$. This proves the first part of the lemma. This part can also be proved using the projection formula for p. For the second part, we notice that Γ' has the same geometric multiplicity as Γ since k'/k is separable. Thus e divides $\langle \Gamma', p^*C\rangle_{k'} = \deg_{k'} \mathcal{O}_{Y_{\mathcal{O}_{K'}}}(p^*C)|_{\Gamma'}$ according to [2], Cor. 9.1.8. □

Before going back to groups of components, let us derive the following consequence.

Corollary 1.5. *Let Y be a regular, proper scheme over a discrete valuation ring $\mathcal{O}_K$. Let Γ_i, $i \in I$, be the irreducible components of the special fibre Y_k. Denote by d_i the multiplicity of Γ_i in Y_k and set $r_i = [k^s \cap k(\Gamma_i) : k]$. Then for any closed point P of the generic fibre Y_K, the degree $[K(P) : K]$ is divisible by $\gcd\{r_i d_i \mid i \in I\}$. Furthermore, if $\dim Y = 2$, then $[K(P) : K]$ is divisible by $\gcd\{r_i d_i e_i \mid i \in I\}$, where e_i is the geometric multiplicity of Γ_i.*

Proof. Let $C := \overline{\{P\}}$ be the Zariski closure of $\{P\}$ in Y. Then we know that

$$[K(P) : K] = \langle Y_k, C\rangle_k = \sum_{i\in I} d_i \langle \Gamma_i, C\rangle_k \tag{2}$$

Let k'/k be a field extension that contains $k(\Gamma_i) \cap k^s$ for all $i \in I$, and such that there is an étale Galois extension $\mathcal{O}_{K'}/\mathcal{O}_K$ whose localization at some maximal ideal has k' as residue field. According to Lemma 1.4, this implies that r_i divides $\langle \Gamma_i, C\rangle_k$, and so does $r_i e_i$ if $\dim Y = 2$. Thus the corollary is proved. $\square$

Remark 1.6. This corollary confirms a prediction of Colliot-Thélène and Saito ([6], Remarque 3.2 (a)). Actually, let $I_3 = \gcd\{r_i d_i \mid i \in I\}$, and let I_2 be the g.c.d of $[K(P) : K]$, where P varies over the closed points of Y_K (see [6], Théorème 3.1). Then Corollary 1.5 is just the divisibility relation $I_3 \mid I_2$. We understood that in a forthcoming preprint, they will prove that $I_1 = I_2 = I_3$ for p-adic fields (where I_1 is some integer related to Brauer groups, see [6] loc. cit.). We think that Corollary 1.5 should still hold if one replaces r_i by $[k^{\mathrm{alg}} \cap k(\Gamma_i) : k] = r_i e_i$.

Corollary 1.7. *Let X be a connected, proper, flat and regular curve over $\mathcal{O}_K$. Let g be the genus of the generic fibre X_K and let $\delta' = \gcd\{r_i d_i e_i \mid i \in I\}$. Then $\delta' \mid 2g - 2$.*

Proof. Let $\omega_{X/\mathcal{O}_K}$ be the relative dualizing sheaf of X. Then

$$2g - 2 = \langle X_k, \omega_{X/\mathcal{O}_K}\rangle_k = \sum_i d_i \langle \Gamma_i, \omega_{X/\mathcal{O}_K}\rangle_k.$$

Using the bilinearity of the intersection forms and the same method as in the proof of Corollary 1.5, we see that $r_i e_i$ divides $\langle \Gamma_i, \omega_{X/\mathcal{O}_K}\rangle_k$. Thus δ' divides $2g - 2$. $\square$

Remark 1.8. It is known that $d \mid g - 1$ (apply the adjunction formula to $\frac{1}{d} X_k$). It should be noticed that, to the contrary, δ' does not divide $g - 1$ in general.

Now let us return to the Galois action. Consider the natural injective map $\lambda : \mathbb{Z}^I \to \mathbb{Z}^{\overline{I}}$ which sends Γ to Γ_{k^s}.

Proposition 1.9. *Let X be a proper flat and regular curve over $\mathcal{O}_K$ with geometrically connected generic fibre. Then we have $(\mathbb{Z}^{\overline{I}})^G = \mathbb{Z}^I$. Let $d = \gcd\{d_i \mid i \in I\}$ and $V_0 := \frac{1}{d} X_k$. Then we have the following commutative diagram of complexes:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_0\mathbb{Z} & \longrightarrow & \mathbb{Z}^{\overline{I}} & \xrightarrow{\overline{\alpha}} & \mathbb{Z}^{\overline{I}} & \xrightarrow{\overline{\beta}} & \mathbb{Z} \\
 & & \| & & \uparrow{\scriptstyle\lambda} & & \uparrow{\scriptstyle\lambda} & & \| \\
0 & \longrightarrow & V_0\mathbb{Z} & \longrightarrow & \mathbb{Z}^I & \xrightarrow{\alpha} & \mathbb{Z}^I & \xrightarrow{\beta} & \mathbb{Z}
\end{array}$$

Proof. The only non-trivial point is to check that $\overline{\alpha} \circ \lambda = \lambda \circ \alpha$. Let $V \in \mathbb{Z}^I$ be considered as a Weil divisor on X. Then $\alpha(V) = \sum_i r_i^{-1} e_i^{-1} \langle V, \Gamma_i \rangle_k \Gamma_i$. Denote by $\Gamma_{i,j}$ the irreducible components of X_{k^s} lying over Γ_i. Then $\lambda(\alpha(V)) = \sum_{i,j} r_i^{-1} e_i^{-1} \langle V, \Gamma_i \rangle_k \Gamma_{i,j}$. By Lemma 1.4,

$$\lambda(\alpha(V)) = \sum_{i,j} e_i^{-1} \langle V_{k^s}, \Gamma_{i,j} \rangle_{k^s} \Gamma_{i,j}.$$

(As in the proof of Corollary 1.5, one can reduce to a finite Galois extension before applying Lemma 1.4). Thus $\lambda(\alpha(V)) = \overline{\alpha}(V_{k^s})$. □

Corollary 1.10. *We have a canonical exact sequence of groups*

$$0 \to \operatorname{Im} \alpha \to \operatorname{Ker} \beta \to \phi_A(k) \to H^1(G, \operatorname{Im} \overline{\alpha}) \to H^1(G, \operatorname{Ker} \overline{\beta}) \quad (3)$$

Proof. It is clear that $(\operatorname{Ker} \overline{\beta})^G = \operatorname{Ker} \beta$. Let us show that $(\operatorname{Im} \overline{\alpha})^G = \operatorname{Im} \alpha$. Consider the exact sequence

$$0 \to V_0\mathbb{Z} \to \mathbb{Z}^{\overline{I}} \to \operatorname{Im} \overline{\alpha} \to 0, \quad (4)$$

and take the long exact sequence of cohomology. It is enough to see that $H^1(G, V_0\mathbb{Z}) = 0$. This follows immediately from the facts that G acts trivially on $V_0\mathbb{Z}$, G is profinite and that $V_0\mathbb{Z}$ has no torsion. Now we get the corollary just by taking Galois cohomology of the exact sequence (1) of Theorem 1.1. □

In next lemma we give some information on the last two terms of the exact sequence (3).

Lemma 1.11. *In the situation of 1.9, let* $d' = \gcd\{r_i d_i \mid i \in I\}$, $\delta = \gcd\{d_i e_i \mid i \in I\}$ *and let* $\delta' = \gcd\{r_i d_i e_i \mid i \in I\}$. *Then:*

(i) *There is an isomorphism* $H^1(G, \operatorname{Ker} \overline{\beta}) \simeq \delta\mathbb{Z}/\delta'\mathbb{Z}$*;*
(ii) *The group* $H^1(G, \operatorname{Im} \overline{\alpha})$ *is killed by* d'/d.

Proof. Let us first show that $H^1(G, \mathbb{Z}^{\overline{I}}) = 0$. Let J_i denote the set of irreducible components of $(\Gamma_i)_{k^s}$. Then $\mathbb{Z}^{\overline{I}} = \oplus_{i \in I} \mathbb{Z}^{J_i}$ as G-modules. Let $H \subseteq G$ be the stabilizer of some component $\Gamma_{i,0}$ of X_{k^s}. Then $[G : H] = r_i$ and $\mathbb{Z}^{J_i} = \operatorname{Ind}_H^G(\Gamma_{i,0}\mathbb{Z})$. It follows from Shapiro's lemma (see [5], III.6.2 and III.5.9) that $H^1(G, \mathbb{Z}^{J_i}) \simeq H^1(H, \Gamma_{i,0}\mathbb{Z}) = 0$.

(i) We have the exact sequence $0 \to \operatorname{Ker} \overline{\beta} \to \mathbb{Z}^{\overline{I}} \to \delta\mathbb{Z} \to 0$. Taking Galois cohomology we get

$$0 \to \operatorname{Im} \beta = \delta'\mathbb{Z} \to \delta\mathbb{Z} \to H^1(G, \operatorname{Ker} \overline{\beta}) \to H^1(G, \mathbb{Z}^{\overline{I}}) = 0.$$

(ii) Using the exact sequence (4) we get an exact sequence

$$0 \to H^1(G, \operatorname{Im}\overline{\alpha}) \to H^2(G, V_0\mathbb{Z}) \to H^2(G, \mathbb{Z}^{\overline{I}}). \tag{5}$$

Let $a \in H^1(G, \operatorname{Im}\overline{\alpha})$. It is enough to show that $d_i r_i/d$ kills a for all $i \in I$. Let $H \subseteq G$ be as above. Then $\mathbb{Z}^{J_i} = \Gamma_{i,0}\mathbb{Z} \oplus (\oplus_{j\neq 0}\Gamma_{i,j}\mathbb{Z})$ as H-modules. We have a commutative diagram

$$\begin{array}{ccccccc}
H^2(G, V_0\mathbb{Z}) & = & H^2(G, V_0\mathbb{Z}) & \xrightarrow{\operatorname{Res}^G_H} & H^2(H, V_0\mathbb{Z}) & = & H^2(H, V_0\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} \\
H^2(G, \mathbb{Z}^{\overline{I}}) & \xrightarrow{p} & H^2(G, \mathbb{Z}^{J_i}) & \xrightarrow{\operatorname{Res}^G_H} & H^2(H, \mathbb{Z}^{J_i}) & \xrightarrow{p} & H^2(H, \Gamma_{i,0}\mathbb{Z})
\end{array}$$

where Res^G_H are restrictions, the p's come from projections of Galois modules, and the vertical arrows are induced by $V_0\mathbb{Z} \to \mathbb{Z}^{\overline{I}} \to \mathbb{Z}^{J_i} \to \Gamma_{i,0}\mathbb{Z}$. The cokernel of the injective homomorphism $V_0\mathbb{Z} \to \Gamma_{i,0}\mathbb{Z}$ is $d\mathbb{Z}/d_i\mathbb{Z}$, thus one sees easily that $\ker(q)$ is killed by d_i/d. In particular $(d_i/d)\operatorname{Res}^G_H(a) = 0$. Finally $[G : H] = r_i$ implies that $r_i(d_i/d)a = 0$. $\square$

Corollary 1.12. *In the situation of 1.11, assume that $d' = d$. Then*

$$\operatorname{Ker}\beta/\operatorname{Im}\alpha \to \phi_A(k)$$

is an isomorphism. In particular, this isomorphism holds if $X_K(K)$ is not empty.

Proof. The first assertion follows from the exact sequence (3) and previous lemma. Assume that $X_K(K) \neq \emptyset$. It is enough to show that $d = d' = 1$. Let $P \in X_K(K)$, and let C be its Zariski closure in X. Then the equality (2) implies that there exists $i \in I$ such that $d_i\langle\Gamma_i, C\rangle_k = 1$. Thus $d_i = 1 = d$. Using the same argument as in the proof of Corollary 1.5, we get $r_i = 1$, so $d' = 1$. $\square$

Remark 1.13. The group $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ can be determined by means of elementary divisors of the matrix $(e_i^{-1}r_i^{-1}\langle\Gamma_i, \Gamma_j\rangle_k)_{i,j\in I}$ as in [2], Corollary 9.6.3.

Remark 1.14. Let X_K be an elliptic curve over K. Let X be its minimal regular model over $\mathcal{O}_K$. Then one can apply Corollary 1.12 and the previous remark to compute $\phi_A(k)$. But in some situations it is faster to directly determine $\phi_A(k)$ as a subset of ϕ_A using the interpretation of $\phi_A(k)$ given in the introduction (that is, $\phi_A(k)$ consists of those components of A_k which are geometrically connected). Note that the Néron model A of X_K is the smooth locus X' of X. Indeed, the canonical morphism $X' \to A$ is an isomorphism after the (faithfully flat and étale) base change to the strict henselization of $\mathcal{O}_K$ ([2], I.5.1), so $X' \to A$ is already an isomorphism.

Example 1.15. Assume that k is perfect and $\mathrm{char}(k) \neq 2$. Let $a, b \in \mathcal{O}_K$ be invertible and such that the class $\tilde{a} \in k$ is not a square. Let $n \geq 1$ be an integer. Consider the elliptic curve X_K given by the equation

$$y^2 = (x^2 - b\pi^{2n})(x + a)$$

where π is a uniformizing element of $\mathcal{O}_K$. Then the minimal regular model of X_K over $\mathcal{O}_K$ consists of a projective line Γ_1 over k, followed by a chain of $n-1$ projective lines over $k(\sqrt{\tilde{a}})$, and ends with the conic Γ_{2n} given by the equation $v^2 = (u^2 - \tilde{b})\tilde{a}$. So X_{k^s} is a $2n$-gone. Thus $\phi_A(k^s) = \mathbb{Z}/2n\mathbb{Z}$, $\phi_A(k) = \mathbb{Z}/2\mathbb{Z}$.

Remark 1.16. As D. Lorenzini pointed out to us, the order of $\mathrm{Ker}\,\beta/\mathrm{Im}\,\alpha$ is easy to compute. Let $M = (e_i^{-1}r_i^{-1}\langle\Gamma_i, \Gamma_j\rangle_k)_{i,j\in I}$, and let $M^* = (m_{ij}^*)_{i,j}$ be its adjoint. Then for all i, j, one has

$$\mathrm{ord}(\mathrm{Ker}\,\beta/\mathrm{Im}\,\alpha) = |m_{ij}^*|(e_jr_jd_jd_i)^{-1}d\delta',$$

where $\delta' = \gcd\{r_ie_id_i \mid i \in I\}$.

Proof. Let ${}^tD = (d_1d^{-1}, \ldots, d_nd^{-1}), {}^tD'' = (e_1r_1d_1\delta'^{-1}, \ldots, e_nr_nd_n\delta'^{-1})$. Then one checks easily that D is a generator of $\mathrm{Ker}(M : \mathbb{Z}^I \to \mathbb{Z}^I)$ and that D'' is a generator of $\mathrm{Ker}({}^tM : \mathbb{Z}^I \to \mathbb{Z}^I)$. Using a similar method as in [10], prop. 1.1, in conjunction with the relations $M \cdot M^* = 0, {}^tM \cdot {}^tM^* = 0$, we find that $M^* = \gamma D \cdot {}^tD''$ for some integer γ. Since the g.c.d of the entries of $D \cdot {}^tD''$ is 1, $|\gamma|$ is the greatest common divisor of all entries of M^* and thus equals $\mathrm{ord}(\mathrm{Ker}\,\beta/\mathrm{Im}\,\alpha)$ (see [10], 1.5). Finally the equality $|\gamma| = |m_{ij}^*|(d_ie_jr_jd_j/d\delta')^{-1}$ comes from $M^* = \gamma D \cdot {}^tD''$. □

Theorem 1.17. *Let X be a proper flat and regular curve over $\mathcal{O}_K$ with geometrically irreducible generic fibre X_K. Let $d = \gcd\{d_i \mid i \in I\}$ and $d' = \gcd\{r_id_i \mid i \in I\}$. Assume that $\mathrm{Gal}(k^s/k)$ is procyclic (i.e. any finite Galois extension k'/k is cyclic) and that k is perfect. Let A be the Néron model of the Jacobian of X_K. Then we have an exact sequence*

$$0 \to \mathrm{Ker}\,\beta/\mathrm{Im}\,\alpha \to \phi_A(k) \to (qd\mathbb{Z})/d'\mathbb{Z} \to 0$$

with $q = 1$ if $d' \mid g - 1$ and $q = 2$ otherwise.

Remark 1.18. The hypothesis $\mathrm{Gal}(k^s/k)$ procyclic in Theorem 1.17 is not optimal. As one can see in the proof below, it is enough to assume that the Galois closure over k of $\cup_i k(\Gamma_i) \cap k^s$ is a cyclic extension of k. We do not know whether the theorem is true without this condition. Note however that Corollary 1.12 provides some evidence for a positive response.

The remainder of the section is devoted to the proof of Theorem 1.17. Let k'/k be a finite Galois extension containing $k^s \cap k(\Gamma_i)$ for all $i \in I$. Then the components of $X_{k'}$ are geometrically irreducible. Thus the exact sequences (1) and (3) can be determined over k' (Corollary 1.2). For simplicity, *in the rest of the proof, we denote by G the group* $\mathrm{Gal}(k'/k)$. Since G is cyclic, we can determine explicitly each group of this exact sequence. Let us recall some notations and results of [13], VIII, §4. Fix a generator σ of G. Let $m = |G|$, $N = \sum_{0 \le j \le m-1} \sigma^j$ and $D = \sigma - 1$. Recall that for any G-module M we have the isomorphisms

$$H^1(G, M) \simeq {}_N M/DM, \quad H^2(G, M) \simeq M^G/NM$$

Moreover, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of G-modules, then the transition homomorphisms

$$\delta_1 : {}_N M''/DM'' \to M'^G/NM', \quad \delta_0 : M''^G \to {}_N M'/DM'$$

are given by

$$\delta_1([x]) = [Ny], \quad \delta_0([x]) = [Dy] \tag{6}$$

if $y \in M$ is in the preimage of $x \in M''$.

Lemma 1.19. *Recall that $V_0 = \frac{1}{d} X_k$. The following properties hold :*

(i) *The map $H^1(G, \mathrm{Im}\,\overline{\alpha}) \to \frac{md}{d'} V_0\mathbb{Z}/mV_0\mathbb{Z}$ defined by $[\overline{\alpha}(V)] \mapsto [N(V)]$ is an isomorphism.*

(ii) *Let $U \in \mathbb{Z}^{\overline{I}}$, then $DU \in {}_N\mathrm{Ker}\,\overline{\beta}$. The isomorphism $H^1(G, \mathrm{Ker}\,\overline{\beta}) \simeq d\mathbb{Z}/d'\mathbb{Z}$ (Lemma 1.11 (i)) is induced by the map $[DU] \mapsto [\overline{\beta}(U)]$.*

Proof. (i) Let J_i denote the set of irreducible components of $(\Gamma_i)_{k'}$. In the exact sequence (5), we have $H^2(G, V_0\mathbb{Z}) = V_0\mathbb{Z}/mV_0\mathbb{Z}$,

$$H^2(G, \mathbb{Z}^{\overline{I}}) = \oplus_{i\in I} H^2(G, \mathbb{Z}^{J_i}) = \oplus_{i \in I} \Gamma_i\mathbb{Z}/mr_i^{-1}\Gamma_i\mathbb{Z},$$

and the homomorphism $H^2(G, V_0\mathbb{Z}) \to H^2(G, \mathbb{Z}^{\overline{I}})$ sends $[V_0]$ to $([d_i d^{-1}\Gamma_i])_i$. Then it is not hard to check (i) using the definition of δ_1. (ii) Direct computation. □

Proof of Theorem 1.17. Let us first describe the map $\psi : H^1(G, \mathrm{Im}\,\overline{\alpha}) \to H^1(G, \mathrm{Ker}\,\overline{\beta})$ in the exact sequence (3). One should notice that while these groups are isomorphic by the previous lemma, ψ is not an isomorphism in general. Let $L : D\mathbb{Z}^{\overline{I}} \to \mathbb{Z}^{\overline{I}}$ be a section of $D : \mathbb{Z}^{\overline{I}} \to D\mathbb{Z}^{\overline{I}}$. Let $\overline{\alpha}(V) \in {}_N\mathrm{Im}\,\overline{\alpha}$. Since $H^1(G, \mathbb{Z}^{\overline{I}}) = 0$, one has $\overline{\alpha}(V) \in D\mathbb{Z}^{\overline{I}}$, and thus $\overline{\alpha}(V) = D(L \circ \overline{\alpha}(V))$. Hence using Lemma 1.19 (ii), we see that ψ is given by the formula

$$\psi([\overline{\alpha}(V)]) = [\overline{\beta}(L \circ \overline{\alpha}(V))] \in H^1(G, \mathrm{Ker}\,\overline{\beta}) \simeq d\mathbb{Z}/d'\mathbb{Z}.$$

Fix for each $i \in I$ an irreducible component $\Gamma_{i,0}$ of $(\Gamma_i)_{k'}$, and put $\Gamma_{i,j} := \sigma^j(\Gamma_{i,0})$. Let $V_1 := \sum_i \frac{r_i d_i}{d'} \Gamma_{i,0}$. Since $N(V_1) = \frac{md}{d'} V_0$, Lemma 1.19 (i) implies that $H^1(G, \operatorname{Im} \overline{\alpha}) = [\overline{\alpha}(V_1)] d\mathbb{Z}/d'\mathbb{Z}$. Put $n := \overline{\beta}(L \circ \overline{\alpha}(V_1)) \in \mathbb{Z}$. Then $\ker \psi$ is generated by $q[\overline{\alpha}(V_1)]$, where q is the smallest positive integer such that $d' \mid qn$. Using Corollary 1.7, we see that to prove the theorem, it is enough to show that $n \equiv g - 1 \bmod d'$.

Now let us construct a section of $D : \mathbb{Z}^{\overline{I}} \to D\mathbb{Z}^{\overline{I}}$. Since the set

$$\{\Gamma_{i,0}, D\Gamma_{i,j} \mid i \in I,\ 0 \le j \le r_i - 2\}$$

forms a basis of $\mathbb{Z}^{\overline{I}}$, we have a well-defined $\mathbb{Z}$-linear map $L' : \mathbb{Z}^{\overline{I}} \to \mathbb{Z}^{\overline{I}}$ given by $L'(\Gamma_{i,0}) = 0$, $L'(D\Gamma_{i,j}) = \Gamma_{i,j}$ for any $i \in I$ and $0 \le j \le r_i - 2$. By construction it is clear that $L := L'|_{D\mathbb{Z}^{\overline{I}}}$ is a section of $D : \mathbb{Z}^{\overline{I}} \to D\mathbb{Z}^{\overline{I}}$. Replacing $D\Gamma_{i,j}$ by $\Gamma_{i,j+1} - \Gamma_{i,j}$, we see that $L'(\Gamma_{i,j}) = \sum_{0 \le l \le j-1} \Gamma_{i,l}$ for any $i \in I$ and $0 \le j \le r_i - 1$.

Let us compute the integer n. Applying the definitions of $\overline{\alpha}$ and L, we get

$$\begin{aligned} n &= \sum_{i \in I, 0 \le j \le r_i - 1} e_i^{-1}(V_1 \cdot \Gamma_{i,j}) \overline{\beta} \circ L'(\Gamma_{i,j}) \\ &= \sum_{i,j} j d_i (V_1 \cdot \Gamma_{i,j}) = \sum_i d_i (V_1 \cdot U_i), \end{aligned} \tag{7}$$

where $U_i := \sum_{0 \le j \le r_i - 1} j \Gamma_{i,j}$. Consider $W_i := \sum_{0 \le j \le m-1} j \Gamma_{i,j}$. Since $\Gamma_{i,j} = \Gamma_{i,j'}$ if $j \equiv j'$ modulo r_i, we have (put $a = m r_i^{-1}$)

$$W_i = \sum_{0 \le l \le a-1} \sum_{0 \le h \le r_i - 1} (l r_i + h) \Gamma_{i,h} = \frac{a(a-1)}{2} r_i \Gamma_i + a U_i.$$

Since $N(V_1) = \frac{md}{d'} V_0 \in V_0 \mathbb{Q}$, we see that $V_1 \cdot \Gamma_i = m^{-1}(N(V_1) \cdot \Gamma_i) = 0$. So replacing in the equality (7) the divisor U_i by $r_i m^{-1} W_i$, and then V_1 by its definition, we get

$$n = \sum_{i,l \in I} \frac{r_i d_i r_l d_l}{md'} \sum_{1 \le j \le m-1} j (\Gamma_{i,j} \cdot \Gamma_{l,0}).$$

On the other hand, $\Gamma_{i,j} \cdot \Gamma_{l,0} = \sigma^{m-j}(\Gamma_{i,j}) \cdot \sigma^{m-j}(\Gamma_{l,0}) = \Gamma_{i,0} \cdot \Gamma_{l,m-j}$. So

$$n = \sum_{i,l \in I} \frac{r_i d_i r_l d_l}{md'} \sum_{1 \le j \le m-1} (m - j)(\Gamma_{i,0} \cdot \Gamma_{l,j}).$$

Adding these two equalities leads to

$$\begin{aligned}2n &= \sum_{i,l\in I} \frac{r_i d_i r_l d_l}{d'}(mr_i^{-1}\Gamma_i - \Gamma_{i,0}) \cdot \Gamma_{l,0} \\ &= \left(\sum_{l\in I} \frac{mr_l d_l}{d'} X_{k^s} \cdot \Gamma_{l,0}\right) - d'V_1^2 = -d'V_1^2.\end{aligned}$$

Let $\mathcal{O}_{K'}/\mathcal{O}_K$ be as in the proof of Corollary 1.5, and let $p : X_{\mathcal{O}_{K'}} \to X$ be the projection. Using the adjunction formula, we see that V_1^2 is congruent to $\langle V_1, p^*\omega_{X/\mathcal{O}_K}\rangle_{k'} \bmod 2$. By Lemma 1.4 and the bilinearity of the intersection forms we have

$$\langle V_1, p^*\omega_{X/\mathcal{O}_K}\rangle_{k'} = \sum_{i\in I} \frac{d_i}{d'}\langle \Gamma_i, \omega_{X/\mathcal{O}_K}\rangle_k = \frac{1}{d'}\langle X_k, \omega_{X/\mathcal{O}_K}\rangle_k = \frac{2g-2}{d'}.$$

Thus n is congruent to $g-1$ modulo d'. This achieves the proof of Theorem 1.17. $\square$

Example 1.20. Assume $\mathrm{char}(k) \neq 2$. Let $g \geq 1$, let X_K be the hyperelliptic curve defined by an equation $y^2 = a_0 \prod_{1\leq i\leq g+1}(x-a_i)^2 + \pi$, where $a_i \in \mathcal{O}_K$ are such that their images $\tilde{a}_i \in k$ are pairwise distinct and $\tilde{a}_0$ is not a square. Finally π is a uniformizing element of $\mathcal{O}_K$. Let X be the minimal regular model of X_K over $\mathcal{O}_K$. Then X_k is integral with $g+1$ ordinary double points. Over $k' = k[\sqrt{\tilde{a}_0}]$, $X_{k'}$ splits into two components isomorphic to $\mathbb{P}^1_{k'}$ intersecting transversally at $g+1$ points. Thus $d = 1$, $d' = 2$. Using Theorems 1.1, 1.17 (see Remark 1.18) and Remark 1.13, we see that $\phi_A(k^s) = \mathbb{Z}/(g+1)\mathbb{Z}$, and $\phi_A(k) = 0$ or $\mathbb{Z}/2\mathbb{Z}$ depending on g is even or odd.

2. The homomorphism $A_K(K) \to \phi_A(k)$

In this section, A_K is an abelian variety over K. Let A be the Néron model of A_K over $\mathcal{O}_K$. We would like to discuss some relationships between $A(K)$ and $\phi_A(k)$. By the properties of Néron models, $A(\mathcal{O}_K) = A_K(K)$. The specialization map gives rise to a homomorphism of groups $A_K(K) \to A_k(k)$. The second group maps canonically to $\phi_A(k)$. In general, the map $A_k(k) \to \phi_A(k)$ is not surjective. The reason is that $\phi_A(k)$ counts the number of geometrically connected components of A_k, while the image of $A_k(k)$ in $\phi_A(k)$ (which is isomorphic to $A_k(k)/A_k^0(k)$) parameterizes the components with rational points. Each geometrically connected component is a torsor under A_k^0. But such a torsor may be non-trivial (that is, without rational point).

Lemma 2.1. *Let A_K be an abelian variety over K.*
(i) *If K is henselian (e.g. complete), then $A_K(K) \to A_k(k)$ is surjective.*
(ii) *If k is finite, or if A_k^0 is an extension of a unipotent group by a split torus with k perfect, then $A_k(k) \to \phi_A(k)$ is surjective.*

Proof. (i) Since K is henselian and A is smooth, the map $A(\mathcal{O}_K) \to A_k(k)$ is surjective (see for instance [2], Prop. 2.3.5).
(ii) Let k'/k be a finite Galois extension of k such that $A_k(k') \to \phi_A(k')$ is surjective (such an extension exists because ϕ_A is finite). Then it is enough to show that $H^1(\mathrm{Gal}(k'/k), A_k^0(k')) = 0$. The case k finite is a theorem of Lang ([9], Theorem 2). The remaining case is Hilbert's 90th Theorem (see [13], Chap. X, §1) with induction on the dimension of A_k^0. $\square$

3. Algebraic tori

In this section we consider an algebraic torus T_K over K, its Néron model T over the ring of integers $\mathcal{O}_K$ of K, and the associated component group ϕ_T. As the formation of Néron models is compatible with passing from K to its completion by [2], 10.1.3, ϕ_T remains unchanged under this process, and we will assume in the following that $\mathcal{O}_K$ and K are *complete*. Writing $\mathcal{O}_K^{sh}$ for a strict henselization of $\mathcal{O}_K$ and K^{sh} for the field of fractions of $\mathcal{O}_K^{sh}$, we know then that the extension K^{sh}/K is Galois. The attached Galois group G is canonically identified with the one of k^s/k, the residue extension of K^{sh}/K.

Let us first assume that T_K has multiplicative reduction, so that the identity component T_k^0 of the special fibre T_k is a torus. Then T_K splits over K^{sh}, and we can view the group of characters X of T_K as a G-module. It is well-known that in this case we have an isomorphism of G-modules

$$\phi_T \simeq \mathrm{Hom}(X, \mathbb{Z});$$

see for example [14], 1.1. So we can identify the submodule of G-invariants $\mathrm{Hom}(X, \mathbb{Z})^G$ with the group of k-rational points of ϕ_T. In particular, if T_K is split over K, the action of G on X is trivial, and ϕ_T is isomorphic to the constant group $\mathbb{Z}^d$ with $d = \dim T_K$. Moreover, $T_K(K) \longrightarrow \phi_T(k)$ is surjective in this case, as is seen from the construction [2], 10.1.5.

Lemma 3.1. *Let X_G be the biggest $\mathbb{Z}$-free quotient of X which is fixed by G. Then the projection $X \longrightarrow X_G$ gives rise to an isomorphism*

$$\mathrm{Hom}(X_G, \mathbb{Z}) \longrightarrow \mathrm{Hom}(X, \mathbb{Z})^G.$$

Proof. The epimorphism $X \longrightarrow X_G$ induces injections

$$\mathrm{Hom}(X_G, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(X, \mathbb{Z})^G \hookrightarrow \mathrm{Hom}(X, \mathbb{Z}),$$

and we have to show that the left injection is, in fact, a bijection. To do this, consider a G-morphism $f: X \longrightarrow \mathbb{Z}$ which is fixed by G. Then f factors through a G-morphism $X/W \longrightarrow \mathbb{Z}$ where $W \subset X$ is the submodule generated by all elements of type $x - \sigma(x)$ with $x \in X$ and $\sigma \in G$. As X_G is obtained from X/W by dividing out its torsion part and as $\mathbb{Z}$ is torsion-free, we see that f must factor through X_G. Hence, the map $\mathrm{Hom}(X_G, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(X, \mathbb{Z})^G$ is bijective, as claimed. □

Now let $T_{G,K}$ be the torus with group of characters X_G. The projection $X \longrightarrow X_G$ defines $T_{G,K}$ as the biggest subtorus of T_K which is split over K, and we can identify the associated morphism $\mathrm{Hom}(X_G, \mathbb{Z}) \longrightarrow \mathrm{Hom}(X, \mathbb{Z})$ with the corresponding morphism of component groups $\phi_{T_G} \longrightarrow \phi_T$. Thereby we can conclude from 3.1:

Proposition 3.2. *Let T_K be a torus with multiplicative reduction, and let $T_{G,K}$ be the biggest subtorus which is split over K. Assume that K is complete. Then the injection $T_{G,K} \hookrightarrow T_K$ and the associated morphism of Néron models $T_G \longrightarrow T$ induce a monomorphism of component groups $\phi_{T_G} \hookrightarrow \phi_T$ and an isomorphism $\phi_{T_G}(k) \simeq \phi_T(k)$ between groups of k-rational points.*

Furthermore, the canonical map $T_K(K) \longrightarrow \phi_T(k)$ is surjective, as the same is true for the split torus $T_{G,K}$.

What can be said if, in the situation of 3.2, T_K does not have multiplicative reduction? In this case we can still view the group of characters X of T_K as a Galois module under the absolute Galois group of K. Similarly as above, we can use the inertia group I and look at the biggest subtorus $T_{I,K} \subset T_K$ which splits over the maximal unramified extension K^{sh} of K. We get an exact sequence of tori

$$0 \longrightarrow T_{I,K} \longrightarrow T_K \longrightarrow \tilde{T}_K \longrightarrow 0$$

with a torus $\tilde{T}_K$ such that $\tilde{T}_K \otimes_K K^{sh}$ does not admit a subgroup of type $\mathbb{G}_{\mathrm{m}}$. The Néron model $\tilde{T}$ of $\tilde{T}_K$ is quasi-compact by [2], 10.2.1, and, hence, the component group $\phi_{\tilde{T}}$ must be finite.

We view now Néron models as sheaves with respect to the étale (or smooth) topology on $\mathcal{O}_K$. Then the above exact sequence of tori induces a sequence of Néron models

$$0 \longrightarrow T_I \longrightarrow T \longrightarrow \tilde{T} \longrightarrow 0$$

which is exact by [4], 4.2. Furthermore, using the right exactness of the formation of component groups, see [4], 4.10, in conjunction with the facts that $T_{I,K}$ has multiplicative reduction and, hence, that the component group ϕ_{T_I} cannot have torsion, we get an exact sequence of component groups

$$0 \longrightarrow \phi_{T_I} \longrightarrow \phi_T \longrightarrow \phi_{\tilde{T}} \longrightarrow 0.$$

Restriction to k-rational points preserves the exactness,

$$0 \longrightarrow \phi_{T_I}(k) \longrightarrow \phi_T(k) \longrightarrow \phi_{\tilde{T}}(k) \longrightarrow 0,$$

as $H^1(G', \mathbb{Z}^d) = \mathrm{Hom}(G', \mathbb{Z}^d) = 0$ for any finite group G' acting trivially on $\mathbb{Z}^d$. Now, taking into account that $T_{I,K}$ has multiplicative reduction and that $\phi_{\tilde{T}}(k)$ is finite, we can conclude from 3.2:

Corollary 3.3. *Let T_K be an algebraic torus, let $T_{G,K}$ be the biggest subtorus which is split over K, and let $\tilde{T}_K$ be defined as above. Assume that K is complete. Then the canonical sequence*

$$0 \longrightarrow \phi_{T_G}(k) \longrightarrow \phi_T(k) \longrightarrow \phi_{\tilde{T}}(k) \longrightarrow 0,$$

is exact with $\phi_{T_G}(k)$ being free and $\phi_{\tilde{T}}(k)$ finite.

In particular, the image of $T_{G,K}(K)$ is of finite index in $\phi_T(k)$, and the same is true for the image of $T_K(K)$.

4. Abelian varieties with semi-stable reduction

Let A_K be an abelian variety over the base field K, which is assumed to be *complete*. We will view A_K as a rigid K-group and use its uniformization in the sense of rigid geometry; cf. [12] and [4], Sect. 1. So A_K can be expressed as a quotient E_K/M_K of rigid K-groups with the following properties:

(i) E_K is a semi-abelian variety sitting in a short exact sequence

$$0 \longrightarrow T_K \longrightarrow E_K \longrightarrow B_K \longrightarrow 0,$$

where T_K is an algebraic torus and B_K an abelian variety with potentially good reduction.

(ii) M_K is a lattice in E_K of maximal rank; i. e., a closed analytic subgroup of E_K which, after finite separable extension of K, becomes isomorphic to the constant group $\mathbb{Z}^d$ with $d = \dim T_K$.

Let A be the Néron model of A_K and A^0 its identity component. Recall that A_K is said to have *semi-stable reduction* if the special fibre A^0_k of A^0 is semi-abelian. Furthermore, let us talk about a *split* semi-stable reduction if the toric part of A^0_k is split over k. The property of semi-stable reduction is reflected on the uniformization of A_K in the following way:

Proposition 4.1. *The abelian variety A_K has semi-stable (resp. split semi-stable) reduction over K if and only if the following hold:*

(i) *The torus T_K splits over a finite unramified extension of K (resp. over K).*

(ii) *The abelian variety B_K has good reduction over K.*

If the above conditions are satisfied with T_K being split over K, the same is true for the lattice $M_K \subset E_K$; i. e., M_K is then isomorphic to the constant K-group $\mathbb{Z}^d$, where $d = \dim T_K$.

Proof. As any abelian variety with semi-stable reduction acquires split semi-stable reduction over a finite unramified extension of K, we need only to consider the case of split semi-stable reduction. So assume that A_K has split semi-stable reduction. Then we have an exact sequence

$$0 \longrightarrow T_k \longrightarrow A_k^0 \longrightarrow B_k \longrightarrow 0,$$

where T_k is a split torus and B_k an abelian variety over k. Let $\mathcal{A}$ be the formal completion of A along A_k and $\mathcal{A}^0$ its identity component. Using the infinitesimal lifting property of tori, see [8], exp. IX, 3.6, and working in terms of formal Néron models in the sense of [3], we see that T_k lifts to a split formal subgroup torus $\mathcal{T} \subset \mathcal{A}^0$ such that the quotient $\mathcal{B} = \mathcal{A}^0/\mathcal{T}$ is a formal abelian scheme lifting B_k. The theory of uniformizations, as explained for example in [1], Sect. 1, says now that the exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{B} \longrightarrow 0,$$

coincides with the one obtained from

$$0 \longrightarrow T_K \longrightarrow E_K \longrightarrow B_K \longrightarrow 0$$

by passing to identity components of associated formal Néron models. As the group of characters of T_K coincides with the one of $\mathcal{T}$, we see that T_K is a split torus. Furthermore, $\mathcal{B}$ is algebraizable with generic fibre B_K and, thus, B_K has good reduction over K.

Let us show that in this situation M_K will be constant. Indeed, writing K^s for a separable closure of K, we choose free generators of the group of characters of T_K and look at the associated "valuation"

$$\nu \colon E_K(K^s) \longrightarrow |K^s|^d \xrightarrow{-\log} \mathbb{R}^d,$$

where $d = \dim T_K$. One knows that M_K being a lattice (of maximal rank) in E_K means that M_K is of dimension zero and that $M_K(K^s)$ is mapped bijectively under ν onto a lattice (of maximal rank) in $\mathbb{R}^d$.

Now let us look at the action of the absolute Galois group $G_K := \mathrm{Gal}(K^s/K)$ of K on $M_K(K^s)$ and show that M_K is constant. As K is complete, the action of G_K is trivial on $|K^s|$. Hence, it respects the map ν. Therefore ν can only be injective if the action of G_K on $M_K(K^s)$ is trivial. However, then all points of M_K must be rational, and M_K is constant.

The converse, that conditions (i) and (ii) imply semi-stable reduction for A_K, follows from [4], 5.1. $\square$

Let us consider now an abelian variety A_K with semi-abelian reduction and with uniformization given by the exact sequence

$$0 \longrightarrow M_K \longrightarrow E_K \longrightarrow A_K \longrightarrow 0.$$

Then, by 4.1, M_K becomes constant over an unramified extension of K, and the associated sequence of formal Néron models

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{E} \longrightarrow \mathcal{A} \longrightarrow 0$$

is exact due to [4], 4.4. As the component group ϕ_M is torsion-free, and as the formation of component groups is right-exact, see [4], 4.10, the induced sequence

$$0 \longrightarrow \phi_M \longrightarrow \phi_E \longrightarrow \phi_A \longrightarrow 0 \qquad (*)$$

is exact, so that ϕ_A may be identified with the quotient ϕ_E/ϕ_M. Thus, if we view the objects of the latter sequence as Galois modules under $G = \mathrm{Gal}(K^{sh}/K)$ and apply Galois cohomology, we see:

Lemma 4.2. *As before, let A_K be an abelian variety with semi-stable reduction. Then the uniformization of A_K, in particular, the above sequence* $(*)$, *gives rise to an exact sequence*

$$0 \longrightarrow \phi_M(k) \longrightarrow \phi_E(k) \longrightarrow \phi_A(k) \longrightarrow H^1(G, M_K) \longrightarrow \dots$$

If A_K has split semi-stable reduction, M_K is constant and, hence, $H^1(G, M_K)$ is trivial.

To justify the latter statement, note that $H^1(G, M_K)$ equals the group of all continuous homomorphisms $G \longrightarrow M_K$ if the action of G on M_K is trivial; cf. [13], Chap. VII, §3, and Chap. X, §3. However, as M_K is torsion-free, all such homomorphisms must be trivial.

It follows from 4.2 that the quotient $\phi_E(k)/\phi_M(k)$ may be viewed as a subgroup of the group of k-rational points of ϕ_A, and it coincides with $\phi_A(k)$ in the case of split semi-stable reduction.

Let T_K be the toric and B_K the abelian part of E_K. Then we have an exact sequence

$$0 \longrightarrow T_K \longrightarrow E_K \longrightarrow B_K \longrightarrow 0$$

of algebraic K-groups and, associated to it, a sequence of Néron models

$$0 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0.$$

In terms of sheaves for the étale (or smooth) topology on $\mathcal{O}_K$, the latter is exact due to [4], 4.2, as A_K having semi-abelian reduction implies that T_K

splits over an unramified extension of K; use 4.1 and [4], 5.1. Similarly as before, we get an exact sequence of component groups

$$0 \longrightarrow \phi_T \longrightarrow \phi_E \longrightarrow \phi_B \longrightarrow 0,$$

where ϕ_B is trivial, since B_K has good reduction. Thus, the morphism $T \longrightarrow E$ induces an isomorphism $\phi_T \longrightarrow \phi_E$ and, using the above exact sequence $(*)$, we can view $\phi_A = \phi_E/\phi_M$ as a quotient ϕ_T/ϕ_M, although the morphism $M \longrightarrow E$ might not factor through T.

Proposition 4.3. *Let A_K be an abelian variety with split semi-stable reduction; i. e., we assume that the identity component A_k^0 of the special fibre of the Néron model A of A_K is extension of an abelian variety by a split algebraic torus. Then:*

(i) *The component group ϕ_A is constant (also valid if K is not necessarily complete).*
(ii) *The canonical map $A_K(K) \longrightarrow \phi_A(k)$ is surjective.*

Proof. It follows from 4.1 that M_K is constant and that T_K is split. Thus, the k-groups ϕ_T and ϕ_M are constant, and so is their quotient ϕ_A. If K is not complete, we may pass to the completion of K without changing the reduction of A_K and its type. This establishes assertion (i). Furthermore, assertion (ii) is due to the fact that the map $T_K(K) \longrightarrow \phi_A(k)$ is surjective, as T_K is a split torus. □

If the semi-stable reduction of A_K is not split, the component group ϕ_A is not necessarily constant, as can be seen from Example 1.15. Furthermore, the quotient $\phi_T(k)/\phi_M(k)$ will, in general, be a proper subgroup of $\phi_A(k)$; its index is controlled by the cohomology group $H^1(G, M_K)$. To make this subgroup more explicit, let X be the group of characters of the toric part T_K of E_K. As is explained in [1], Sect. 3 or [4], Sect. 5, we can evaluate characters of X on the lattice M_K, thereby obtaining a "bilinear map" $M_K \times X \longrightarrow P_K$ taking values in the Poincaré bundle P_K on $B_K \times B'_K$, the product of B_K with its dual. In fact, if the abelian part B_K of E_K is trivial, this pairing is just the evaluation of characters on the lattice M_K. Using the canonical valuation $P_K(K^{sh}) \longrightarrow \mathbb{Z}$, we get a non-degenerate pairing $M_K \times X \longrightarrow \mathbb{Z}$ of Galois modules with respect to the extension K^{sh}/K and from it an injection $i: M_K \hookrightarrow \mathrm{Hom}(X, \mathbb{Z})$ into the linear dual of X. Now, as a G-module, we can identify M_K with the component group ϕ_M. Furthermore, $\mathrm{Hom}(X, \mathbb{Z})$ can be viewed as the component group of T_K, and it follows from the discussion in [4], 5.2, that under this identification the inclusion map i corresponds to the canonical map $\phi_M \longrightarrow \phi_T$ as considered above.

In particular, we have a canonical commutative diagram of G-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \phi_M & \longrightarrow & \phi_T & \longrightarrow & \phi_A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M_K & \longrightarrow & \mathrm{Hom}(X, \mathbb{Z}) & \longrightarrow & \phi_A & \longrightarrow & 0
\end{array}$$

with exact rows and vertical isomorphisms. Restricting to G-invariants we get from 3.1:

Proposition 4.4. *Let A_K be an abelian variety with semi-stable reduction and with uniformization $A_K = E_K/M_K$. Let X be the group of characters of the toric part of E_K. Then $\Sigma = \mathrm{Hom}(X_G, \mathbb{Z})/M_K^G$ is a subgroup of $\phi_A(k)$, contained in the image of $A_K(K) \longrightarrow \phi_A(k)$, such that the quotient $\phi_A(k)/\Sigma$ is mapped injectively into $H^1(G, M_K)$. Furthermore, Σ coincides with $\phi_A(k)$ if A_K has split semi-stable reduction.*

If the abelian variety does not admit semi-stable reduction, we still have maps

$$\phi_{T_G} \longrightarrow \phi_{T_I} \longrightarrow \phi_T \longrightarrow \phi_E \longrightarrow \phi_A,$$

where $T_{G,K}$ stands for the maximal subtorus of T_K which is split over K and, likewise, $T_{I,K}$ for the maximal subtorus of T_K which splits over K^{sh}. The image in ϕ_A of each of these groups gives rise to a subgroup of ϕ_A, and we thereby get a filtration of ϕ_A. Up to the term ϕ_{T_G}, this filtration was dealt with in [4], Sect. 5; it goes back to Lorenzini [11]. Subsequent factors of the filtration are controlled by suitable first cohomology groups or by the component group of B_K; cf. [4], 5.5. So, to make general statements about k-rational points seems to be a little bit out of reach. However, the groups ϕ_{T_G} and ϕ_{T_I} are accessible, and this leads to the understanding of rational components in the semi-stable reduction case.

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