Convergence of a Discrete-Velocity Model for the Boltzmann-BGK Equation

Luc Mieussens[1,2]
[1] *Mathématiques Appliquées de Bordeaux, Université Bordeaux I, 33405 Talence Cedex, France*

[2] *CEA-CESTA/DEV/SIS, BP2 33114 Le Barp, France*
`Luc.Mieussens@math.u-bordeaux.fr`

Abstract

We prove the convergence of a conservative and entropic discrete-velocity model for the Bathnagar-Gross-Krook (BGK) equation. In this model, the approximation of the Maxwellian is based on a discrete entropy minimization principle. The main difficulty, due to its implicit definition, is to prove that this approximation is consistent. We also demonstrate the existence and uniqueness of a solution to the discrete-velocity model, by using a fixed point theorem. Finally, the model is written in a continuous equation form, and we prove the convergence of its solution toward a solution of the BGK equation.

keywords: kinetic theory – discrete-velocity models – Boltzmann equation – BGK model – convergence of numerical schemes

1. Introduction

In rarefied gas dynamics, the Boltzmann equation is commonly used to describe rarefied flows. However, this equation is very complex and the simplified model introduced by Bathnagar, Gross, and Krook (BGK) has been used in numerous qualitative and quantitative studies (see for instance [1, 2]). The BGK model describes the evolution of the distribution $f(t,x,v)$ of molecules, which at time $t \geq 0$ are at the position $x \in \mathbb{R}^D$ with the velocity $v \in \mathbb{R}^D$:

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f = \frac{1}{\tau}(M[f] - f) \\ &f(0,x,v) = f^0(x,v) \end{aligned} \tag{1.1}$$

The collisions of the molecules in the gas are modeled by the relaxation of f to the local Maxwellian equilibrium state $M[f]$ (see [3]). This distribution is an isotropic

Gaussian function of v, which depends only on the density ρ, the macroscopic velocity $u = (u^{(1)}, \dots, u^{(D)})$, and the temperature θ of the gas

$$M[f] = \frac{\rho}{(2\pi\theta)^{D/2}} \exp\Big(-\frac{|v-u|^2}{2\theta}\Big). \tag{1.2}$$

These fluid quantities ρ, u, θ are defined through the first $D+2$ moments of f:

$$\rho = \langle f \rangle, \qquad \rho u = \langle v f \rangle, \qquad E = \langle \tfrac{1}{2}|v|^2 f \rangle = \tfrac{1}{2}\rho|u|^2 + \tfrac{D}{2}\rho\theta, \tag{1.3}$$

where we denote by $\langle g \rangle$ the integral of any function g on $\mathbb{R}^D_v$, *i.e.*

$$\langle g \rangle = \int g(v)\, dv. \tag{1.4}$$

These moments are called density, momentum, and total energy of the gas. We denote by $\boldsymbol{m}(v) = (1, v, \frac{1}{2}|v|^2)^T$ the vector of microscopic quantities mass, momentum and kinetic energy (normalized by the mass). Similarly we denote by $\boldsymbol{\rho} = (\rho, \rho u, E)^T$ the vector of $D+2$ first moments of f. These notations yield a more compact definition of the moments

$$\boldsymbol{\rho} = \langle \boldsymbol{m} f \rangle. \tag{1.5}$$

Note that throughout this paper, bold symbols are only used for vectors of $\mathbb{R}^{D+2}$ such as, for example, $\boldsymbol{\rho}$ and $\boldsymbol{m}(v)$.

A fundamental property of the Maxwellian state $M[f] = M[\boldsymbol{\rho}]$ is that it is the unique solution of the *entropy minimization problem*

$$(\mathcal{P}) \qquad H(M[\boldsymbol{\rho}]) = \min\Big\{ H(g) = \langle g \log g \rangle,\ g \geq 0,\ \langle \boldsymbol{m} g \rangle = \boldsymbol{\rho} \Big\}. \tag{1.6}$$

This means that $M[\boldsymbol{\rho}]$ minimizes the entropy of all the possible states leading to the same macroscopic properties. The problem $(\mathcal{P})$ may be solved by a Lagrange multiplier method (if $\rho, \theta > 0$). This yields the following expression, which is equivalent to (1.2)

$$M[\boldsymbol{\rho}] = \exp(\boldsymbol{\alpha} \cdot \boldsymbol{m}(v)), \tag{1.7}$$

where $\boldsymbol{\alpha}$ is defined through the invertible relation

$$\boldsymbol{\alpha} = \left(\log\left(\frac{\rho}{(2\pi\theta)^{D/2}} \right) - \frac{|u|^2}{2\theta}, \frac{u}{\theta}, -\frac{1}{\theta} \right)^T. \tag{1.8}$$

Hence, it may be seen that the BGK model possesses the main properties of the Boltzmann equation: positive solutions, conservation of moments, and dissipation of entropy:

$$\partial_t \langle \boldsymbol{m} f \rangle + \nabla_x \cdot \langle v \boldsymbol{m} f \rangle = 0, \tag{1.9}$$

$$\partial_t \langle f \log f \rangle + \nabla_x \cdot \langle v f \log f \rangle \leq 0. \tag{1.10}$$

Some important mathematical results about the BGK equation have been obtained in the past decade. For instance Perthame has proved in [4] an existence and stability result of a distribution solution in the whole space. This result has been extended to bounded domain with various boundary conditions by Ringeisen [5]. More recently, Perthame and Pulvirenti have proved in [6] the existence and uniqueness of a mild solution in the case of the flat torus with weighted L^∞ estimates. This result has been generalized to $\mathbb{R}^D$ by Mischler in [7]. We also mention the result of Issautier [8] who has proved that the mild solution of Perthame and Pulvirenti is in fact a strong one if some regularity assumptions on the initial condition are made. However, it is important to note that in all these results, the authors assume a constant relaxation time (*i.e.* $\tau = 1$). This is physically not very realistic because τ is rather a function of ρ and θ. To our knowledge, no global existence result exists for realistic relaxation time, and thus the usual assumption $\tau = 1$ will be made in this paper.

For the purpose of numerical simulations of aerodynamical flows, we have introduced in previous papers [9, 10] a discrete-velocity approximation of the BGK equation. Like the discrete-velocity models of Rogier and Schneider [11], Buet [12], and Heintz and Panferov [13] for the Boltzmann equation, our approximation has the same conservation and entropy properties as the continuous BGK model. This model allows efficient computations of complex flows [10]. In such an approximation, which may be viewed as a simplification of the physical description of the gas, the molecules are assumed to move with a finite number of velocities v_k, $k \in \mathcal{K} = \{1, ..., N\}$. The gas is described by a discrete distribution function $f_\mathcal{K}(t,x) = (f_k(t,x))_{k\in\mathcal{K}}$ which solves the system of discrete kinetic equations

$$\begin{aligned} &\partial_t f_k + v_k \cdot \nabla_x f_k = \mathcal{E}_k - f_k, \qquad \forall k \in \mathcal{K}, \\ &f_k(0,x) = f_k^0(x). \end{aligned} \tag{1.11}$$

The approximation $\mathcal{E}_\mathcal{K} = (\mathcal{E}_k)_{k\in\mathcal{K}}$ of the Maxwellian distribution is the essential point for the definition of the model. In our approach, $\mathcal{E}_\mathcal{K}$ is defined as the discrete equilibrium function, *i.e.* it solves the discrete version of the entropy minimization problem $(\mathcal{P})$

$$(\mathcal{P}_\mathcal{K}) \qquad H_\mathcal{K}(\mathcal{E}_\mathcal{K}) = \min\Big\{ H_\mathcal{K}(g) = \langle g \log g\rangle_\mathcal{K},\ g \geq 0 \in \mathbb{R}^N,\ \langle \boldsymbol{m} g\rangle_\mathcal{K} = \boldsymbol{\rho}_\mathcal{K} \Big\}. \tag{1.12}$$

The notations introduced in (1.12) follow logically from (1.4) and (1.5)

$$\langle g\rangle_\mathcal{K} = \sum_{k\in\mathcal{K}} g_k\,\lambda_k, \qquad \boldsymbol{\rho}_\mathcal{K} = \langle \boldsymbol{m} f_\mathcal{K}\rangle_\mathcal{K} = \sum_{k\in\mathcal{K}} \boldsymbol{m}(v_k) f_k\,\lambda_k, \tag{1.13}$$

where λ_k is the measure of a cell around v_k. An existence and uniqueness result for the problem $(\mathcal{P}_\mathcal{K})$ has been proved in [9, 10] and is recalled in the next section.

It is now mathematically interesting to investigate the convergence of such an approximation to the continuous BGK equation. The procedure for this kind of result is quite general. See for instance the proof of Mischler for the convergence of a discrete-velocity model for the Boltzmann equation [14]. There are essentially three distinct points to be proved:

- convergence of the approximation of the source term, which is local in t, x.
- existence and uniqueness of a solution to the discrete-velocity model.
- convergence of the discrete kinetic equation to the continuous one.

The first point is strongly dependent on the problem. It has been proved by Schneider et al. [15] for their quadrature of the Boltzmann collision operator (see also [13]), but it is completely different in our case. For our model, the difficulty is that the discrete equilibrium is implicitly defined by $f_{\mathcal{K}}$. Then, we define some derived minimization problems to obtain uniform coercivity.

For the second point, as noted by Mischler, such a result is not known in general for discrete-velocity models of the Boltzmann equation. However, due to both the particular structure of the BGK collision term and the bounded and discrete-velocity set, we are able to prove existence and uniqueness of a global solution for our model (1.11). We only use the assumption of Perthame [4] for the initial condition.

For the last point, Mischler has proposed a quite elegant approach for the Boltzmann equation. He has defined a continuous formulation of its discrete-velocity model, and he has proved the convergence as $N \to \infty$ using the stability result of DiPerna-Lions [16]. The main difficulty of this method is the validity of the averaging lemma in the context of discrete velocities, but it has been proved by Mischler. We follow a similar approach by defining a continuous form of our discrete velocity BGK model, and we use the stability proof of Perthame [4] for the BGK equation.

It is also important to note the recent result of Issautier [8] who has proved a convergence result for a particular method for the BGK equation in which both time, space and velocity are discretized. The advantage of his approach is the explicit approximation of the Maxwellian (he takes $\mathcal{E}_k = M[\boldsymbol{\rho}_{\mathcal{K}}](v_k)$) which allows to derive error estimates. But as opposed to our model, the discretization of Issautier neither is conservative nor entropic. For our method, the price to be paid for these properties is the implicit definition of $\mathcal{E}_{\mathcal{K}}$, that makes the derivation of error estimates difficult.

The remainder of this paper is organized as follows. In Section 2, our discrete-velocity model is rigorously derived from the continuous BGK equation. The existence result of a discrete equilibrium is recalled, and we give all the results proved in the following sections. Section 3 is devoted to the convergence of the approximation of the Maxwellian. In Section 4, an existence and uniqueness result for the discrete-velocity model is proved. In Section 5, we prove the convergence of this solution to a solution of the BGK equation. Most of the results presented here have been announced in a previous paper [17].

2. Notations and main results

Let Δv_n and B_n be two sequences of real numbers such that

$$\Delta v_n \underset{n\to\infty}{\longrightarrow} 0, \qquad \Delta v_n B_n \underset{n\to\infty}{\longrightarrow} +\infty. \tag{2.14}$$

Let $\mathcal{V}^n$ be a grid of N_n velocities defined by

$$\mathcal{V}^n = \{v_k^n = k\Delta v_n, \quad k \in \mathcal{K}^n\}, \tag{2.15}$$

where $\mathcal{K}^n$ is the set of multi-indexes $\mathcal{K}^n = \{k \in \mathbb{Z}^D, \quad |k| \le B_n\}$. We also define the velocity cells Λ_k^n by

$$\Lambda_k^n = [v_{k,1}^n - \tfrac{1}{2}\Delta v_n, v_{k,1}^n + \tfrac{1}{2}\Delta v_n[\times \ldots \times [v_{k,D}^n - \tfrac{1}{2}\Delta v_n, v_{k,D}^n + \tfrac{1}{2}\Delta v_n[. \tag{2.16}$$

A discrete distribution function $g = (g_k)_{k\in\mathcal{K}^n}$ on $\mathcal{V}^n$ is a vector of $\mathbb{R}^{N_n}$. By analogy with (1.4) and (1.5), we set

$$\langle g\rangle_n = \sum_{k\in\mathcal{K}^n} g_k\, \Delta v_n^D, \qquad \langle \boldsymbol{m} g\rangle_n = \sum_{k\in\mathcal{K}^n} \boldsymbol{m}(v_k^n) g_k\, \Delta v_n^D. \tag{2.17}$$

The Maxwellian $M[\boldsymbol{\rho}]$ associated to a given vector of moments $\boldsymbol{\rho}$ is approximated on $\mathcal{V}^n$ by $\mathcal{E}_{\mathcal{K}}^n = (\mathcal{E}_k^n)_{k\in\mathcal{K}^n}$. This approximation is defined by the discrete version of the entropy minimization problem

$$(\mathcal{P}_n) \qquad H^n(\mathcal{E}_{\mathcal{K}}^n) = \min_{\mathcal{X}_{\boldsymbol{\rho}_n}} \Big\{ H^n(g) = \langle g \log g\rangle_n \Big\}, \\ \text{with } \mathcal{X}_{\boldsymbol{\rho}_n} = \big\{ g \ge 0 \in \mathbb{R}^{N_n}, \langle \boldsymbol{m} g\rangle_n = \boldsymbol{\rho}_n \big\}, \tag{2.18}$$

where $\boldsymbol{\rho}_n$ is some approximation of $\boldsymbol{\rho}$. It may easily be proved that $(\mathcal{P}_n)$ has a unique solution, provided that $\mathcal{X}_{\boldsymbol{\rho}_n} \neq \emptyset$. Moreover, the following result (proved in [9, 10]) shows that $\mathcal{E}_{\mathcal{K}}^n$ has an exponential form, provided that a necessary and sufficient condition on $\boldsymbol{\rho}_n$ is fulfilled.

Proposition 1 ([9, 10]). *If $\mathcal{X}_{\boldsymbol{\rho}_n} \neq \emptyset$ then $(\mathcal{P}_n)$ has a unique solution $\mathcal{E}_{\mathcal{K}}^n$. Moreover, if $B_n \ge 1$, then there exists a unique $\boldsymbol{\alpha}_n \in \mathbb{R}^{D+2}$ such that*

$$\mathcal{E}_k^n = \exp(\boldsymbol{\alpha}_n \cdot \boldsymbol{m}(v_k^n)) \qquad \forall k \in \mathcal{K}^n, \tag{2.19}$$

if, and only if, $\boldsymbol{\rho}_n$ is strictly realizable on $\mathcal{V}^n$, i.e.

$$\exists g > 0 \in \mathcal{X}_{\boldsymbol{\rho}_n}. \tag{2.20}$$

Consequently, note that $\boldsymbol{\alpha}_n$ is the unique solution of the set of $D+2$ non-linear equations

$$\langle \boldsymbol{m} \exp(\boldsymbol{\alpha}_n \cdot \boldsymbol{m})\rangle_n = \boldsymbol{\rho}_n. \tag{2.21}$$

This vector must be compared to the vector $\boldsymbol{\alpha}$ defined by (1.8), which uniquely solves the following set of $D+2$ non-linear equations

$$\langle \boldsymbol{m} \exp(\boldsymbol{\alpha} \cdot \boldsymbol{m})\rangle = \boldsymbol{\rho}. \tag{2.22}$$

The first result of this paper, crucial for the convergence of our discrete-velocity model, shows that the approximation of $M[\boldsymbol{\rho}]$ by $\mathcal{E}^n_{\mathcal{K}}$ is consistent if $\boldsymbol{\rho}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{\rho}$.

Theorem 1. *Let $\{\boldsymbol{\rho}_n\}_{n\geq 0}$ be a sequence of $\mathbb{R}^{D+2}$ strictly realizable on $\mathcal{V}^n$ for all n (in the sense of (2.20)). Let $\boldsymbol{\rho} \in \mathbb{R}^{D+2}$ be such that $\rho, \theta > 0$. If $\boldsymbol{\rho}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{\rho}$, then the vector $\boldsymbol{\alpha}_n$, given by proposition 1, converges to $\boldsymbol{\alpha}$ defined by (1.8).*

We can now define our discrete-velocity approximation of the BGK equation. Assume that the initial condition f^0 is non-negative and satisfies the classical estimates

$$\int_{\mathbb{R}^D} \langle (1+|x|^2+|v|^2+|\log f^0|)f^0\rangle \, dx = \Gamma_0 < +\infty. \tag{2.23}$$

Note that from the result of Perthame [4], these estimates guarantee the existence of a distribution solution of (1.1). Then, we define the following approximation of f^0

$$f^{0,n}_k(x) = \min\left(n, \frac{1}{\Delta v_n^D} \int_{\Lambda^n_k} (f^0(x,v) + \frac{1}{n}\exp(-|x|^2-|v|^2))\, dv\right), \tag{2.24}$$

so that $f^{0,n}_{\mathcal{K}} = (f^{0,n}_k)_{k\in\mathcal{K}}$ satisfies the estimates

$$\delta^n_0\, \phi(x) \leq f^{0,n}_k(x) \leq n \qquad \text{a.e in } \mathbb{R}^D_x, \quad \forall k \in \mathcal{K}^n, \tag{2.25}$$

$$\sup_{n\geq 0} \int_{\mathbb{R}^D} \langle (1+|x|^2+|v|^2+|\log f^{0,n}_{\mathcal{K}}|) f^{0,n}_{\mathcal{K}}\rangle_n \, dx = \Gamma_1 < +\infty, \tag{2.26}$$

where $\delta^n_0 = \frac{1}{n}\frac{1}{\Delta v_n^D} \min_k (\int_{\Lambda^n_k} \exp(-|v|^2)\, dv)$ and $\phi(x) = \exp(-|x|^2)$. The discrete-velocity approximation of (1.1) is then

$$\begin{aligned} &\partial_t f^n_k + v^n_k \cdot \nabla_x f^n_k = \mathcal{E}^n_k - f^n_k \qquad \text{in } \mathcal{D}'(]0,+\infty[\times \mathbb{R}^D_x), \quad \forall k \in \mathcal{K}^n, \\ &f^n_k(0,x) = f^{0,n}_k(x), \end{aligned} \tag{2.27}$$

where $\mathcal{E}^n_k$ is naturally defined by $(\mathcal{P}_n)$, with

$$\boldsymbol{\rho}_n = \langle \boldsymbol{m} f^n_{\mathcal{K}}\rangle_n = (\rho_n, \rho_n u_n, E_n)^T. \tag{2.28}$$

The second result proved in this paper shows an existence and uniqueness result for the discrete model (2.27). Due to the definition of the discrete equilibrium, the solution satisfies conservation and entropy properties.

Theorem 2. *Initial value problem (2.27) has a unique solution* $f_{\mathcal{K}}^n = (f_k^n)_{k\in\mathcal{K}}$ *in* $L^\infty(]0, t_{max}[\ \times \mathbb{R}_x^D)^{N_n}$*, for all* $t_{max} > 0$*. Moreover, the following conservation and entropy relations hold in a distribution sense*

$$\partial_t \langle \boldsymbol{m} f_{\mathcal{K}}^n\rangle_n + \nabla_x \cdot \langle v\boldsymbol{m} f_{\mathcal{K}}^n\rangle_n = 0, \tag{2.29}$$

$$\partial_t \langle f_{\mathcal{K}}^n \log f_{\mathcal{K}}^n\rangle_n + \nabla_x \cdot \langle v f_{\mathcal{K}}^n \log f_{\mathcal{K}}^n\rangle_n \le 0, \tag{2.30}$$

and $f_{\mathcal{K}}^n$ *satisfies the estimates*

$$\sup_n \sup_{[0,t_{max}]} \int_{\mathbb{R}^D} \langle (1 + |x|^2 + |v|^2 + |\log f_{\mathcal{K}}^n|) f_{\mathcal{K}}^n(t,x)\rangle_n \, dx \le \Gamma_1(t_{max}), \tag{2.31}$$

$$\delta_0^n \phi(x - tv_k^n)\, e^{-t} \le f_k^n(t,x) \le n\, e^{N_n t} \qquad \textit{for a.e } t, x. \tag{2.32}$$

Finally, in order to prove the convergence of this solution, we define the constant per velocity cell functions

$$f^n(t,x,v) = \sum_{k\in\mathcal{K}^n} f_k^n(t,x)\chi_k^n(v) \quad \text{and} \quad \mathcal{E}^n(t,x,v) = \sum_{k\in\mathcal{K}^n} \mathcal{E}_k^n(t,x)\chi_k^n(v), \tag{2.33}$$

where χ_k^n is the indicator function of Λ_k^n. Then (2.27) may be related to (1.1) by the equation

$$\begin{aligned} &\partial_t f^n + C^n(v)\cdot\nabla_x f^n = \mathcal{E}^n - f^n \qquad \text{in } \mathcal{D}'(]0,+\infty[\times \mathbb{R}_x^D \times \mathbb{R}_v^D), \\ &f^n(0,x,v) = f^{0,n}(x,v) = \sum_{k\in\mathcal{K}^n} f_k^{0,n}(x)\chi_k^n(v), \end{aligned} \tag{2.34}$$

where

$$C^n(v) = \sum_{k\in\mathcal{K}^n} v_k^n \chi_k^n(v). \tag{2.35}$$

We can now state our convergence result:

Theorem 3. *For all sequences* $\Delta v_n, B_n$ *satisfying (2.14), the sequence* $\{f^n\}_{n\ge 0}$ *is weakly convergent in* $L^1(]0, t_{max}[\times \mathbb{R}_x^D \times \mathbb{R}_v^D)$ $\forall t_{max} > 0$*, up to the extraction of a subsequence, to a distribution solution of BGK equation (1.1).*

3. Convergence of $\boldsymbol{\alpha}_n$ (proof of theorem 1)

Note that relations (2.21) and (2.22) can be viewed as the extremum relations $J_n'(\boldsymbol{\alpha}_n) = 0$ and $J'(\boldsymbol{\alpha}) = 0$ for the following minimization problems

$$J_n(\boldsymbol{\alpha}_n) = \min_{\mathbb{R}^{D+2}} \Big\{ J_n(\boldsymbol{\beta}) = \langle \exp(\boldsymbol{\beta}\cdot\boldsymbol{m})\rangle_n - \boldsymbol{\beta}\cdot\boldsymbol{\rho}_n \Big\}, \tag{3.36}$$

$$J(\boldsymbol{\alpha}) = \min_{\mathcal{D}} \Big\{ J(\boldsymbol{\beta}) = \langle \exp(\boldsymbol{\beta}\cdot\boldsymbol{m})\rangle - \boldsymbol{\beta}\cdot\boldsymbol{\rho} \Big\}, \tag{3.37}$$

where $\mathcal{D} = \mathbb{R}^{D+2} \cap \{\boldsymbol{\beta}, \beta^{(D+1)} < 0\}$. Here, we denote by $(\beta^{(0)}, \dots, \beta^{(D+1)})$ the components of $\boldsymbol{\beta}$. The idea of the proof is to note that $\boldsymbol{\alpha}_n$ and $\boldsymbol{\alpha}$ are in fact the unique solutions of problems (3.36) and (3.37). Then, we shall prove the convergence by studying the properties of J_n and J.

First, we need the following proposition:

Proposition 2. (P_1) *J_n is strictly convex and coercive on $\mathbb{R}^{D+2}$*
(P_2) *J is strictly convex and coercive on $\mathcal{D}$*
(P_3) *J_n is locally uniformly convergent to J on $\mathcal{D}$*
(P_4) *we have* $\sup_n J_n(\boldsymbol{\alpha}_n) < +\infty$

Proof of proposition 2. Property (P_1) is proved in [9, 10], and we do not write it. Note that this property implies the uniqueness of a solution to problem (3.36).

For (P_2), note that J is twice continuously differentiable, and that $J'' = \langle \boldsymbol{m} \otimes \boldsymbol{m} \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}) \rangle$ is clearly positive definite. Thus J'' is strictly convex in $\mathcal{D}$. The coercivity property means that $J(\boldsymbol{\beta}_p) \underset{p \to +\infty}{\longrightarrow} +\infty$ for every sequence $\{\boldsymbol{\beta}_p\} \subset \mathcal{D}$ getting close to the boundary of $\mathcal{D}$, *i.e.* such that

$$(i) \quad \beta_p^{(D+1)} \underset{p \to +\infty}{\longrightarrow} 0^- \quad \text{or} \quad (ii) \quad |\boldsymbol{\beta}_p| \underset{p \to +\infty}{\longrightarrow} +\infty.$$

For the case (i), we consider an analytic expression of J

$$\begin{aligned} J(\boldsymbol{\beta}) = &\exp\Big(\tfrac{1}{2} \sum_{i=1}^{D} |\beta^{(i)}|^2 / |\beta^{(D+1)}|\Big) \exp\Big(\beta^{(0)}\Big) \Big(\frac{2\pi}{|\beta^{(D+1)}|}\Big)^{D/2} \\ &- \beta^{(0)}\rho - \sum_{i=1}^{D} \beta^{(i)} \rho u^{(i)} - \beta^{(D+1)} E, \end{aligned} \tag{3.38}$$

where $u = (u^{(1)}, \dots, u^{(D)})$ is defined by (1.5). Then, we investigate all the limits of $\beta_p^{(0)}$ and $\beta_p^{(i=1..D)}$, and it appears that $J(\boldsymbol{\beta}_p)$ tends to $+\infty$ in any case. We summarize here the laborious study of all the possible cases. If $\beta_p^{(0)}$ is bounded below, then the exponentials of the inverse power of $\beta_p^{(D+1)}$ tend towards $+\infty$ faster than the linear part of J. If $\beta_p^{(0)} \to -\infty$, then two different cases must be considered. If $-\beta_p^{(0)}\rho - \sum_{i=1}^{D} \beta_p^{(i)} \rho u^{(i)} \to +\infty$, then $J(\boldsymbol{\beta}_p) \to +\infty$, whatever the limit of $\beta_p^{(i=1..D)}$ is. In the other case, $-\beta_p^{(0)}\rho - \sum_{i=1}^{D} \beta_p^{(i)} \rho u^{(i)}$ is bounded above, which implies that $\beta_p^{(i=1..D)}$ is not bounded; thus the first exponential grows fast enough to $+\infty$ so that we obtain $J(\boldsymbol{\beta}_p) \to +\infty$.

For the case (ii), we can assume that $\beta_p^{(D+1)} \leq c < 0 \ \forall p$. Let $R > 0$ be such that the ball $B(\boldsymbol{\alpha}, R) \subset \mathcal{D}$. Then if p is large enough (*i.e.* $p > p_0$), we have $\boldsymbol{\beta}_p \notin \bar{B}(\boldsymbol{\alpha}, R)$. Define $\boldsymbol{\gamma}_p$ in the boundary ∂B of $B(\boldsymbol{\alpha}, R)$ by $\boldsymbol{\gamma}_p = \frac{R}{|\boldsymbol{\beta}_p - \boldsymbol{\alpha}|}(\boldsymbol{\beta}_p - \boldsymbol{\alpha}) + \boldsymbol{\alpha}$.

Then $\boldsymbol{\gamma}_p = \theta_p \boldsymbol{\alpha} + (1-\theta_p)\boldsymbol{\beta}_p$ with $0 < \theta_p = 1 - \frac{R}{|\boldsymbol{\beta}_p - \boldsymbol{\alpha}|} < 1$ if $p > p_0$. Due to the strict convexity of J, we have $J(\boldsymbol{\beta}_p) > \frac{1}{1-\theta_p}(J(\boldsymbol{\gamma}_p) - \theta_p J(\boldsymbol{\alpha}))$ for $p > p_0$. Moreover, $\boldsymbol{\alpha}$ is the unique minimum of J which is continuous, and ∂B is compact, thus we have $J(\boldsymbol{\gamma}_p) - J(\boldsymbol{\alpha}) \geq m > 0$ for all $p > p_0$. Therefore $J(\boldsymbol{\beta}_p) \geq \frac{m}{1-\theta_p} + J(\boldsymbol{\alpha})$, which tends to $+\infty$ because $\theta_p \to 1^-$ as $p \to \infty$. The proof of property (P_2) is then complete. Note that this property implies the uniqueness of a solution to problem (3.37).

For property (P_3), we must prove that $e_n = |\langle \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}) \rangle_n - \langle \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}) \rangle| \to 0$ locally uniformly on $\mathcal{D}$, which is a problem of quadrature on $\mathbb{R}^D$. We split e_n into two parts

$$\begin{aligned} e_n &\leq \Big| \int_{\mathbb{R}^D} \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}(v))\, dv - \Delta v_n^D \sum_{k \in \mathbb{Z}^D} \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}(v_k^n)) \Big| + \sum_{v_k^n \notin \mathcal{V}^n} \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}(v_k^n)) \Delta v_n^D \\ &= E_1 + E_2. \end{aligned}$$

As in [15], we use the following lemma (see [18])

Lemma 1. *There exists $c > 0$ independent of Δv such that for all $g \in W^{m,1}(\mathbb{R}^D)$, $m > D$*

$$\Big| \int_{\mathbb{R}^D} g(v)\, dv - \Delta v^D \sum_{k \in \mathbb{Z}^D} g(v_k) \Big| \leq c\, \Delta v^m |g|_{m,1}, \tag{3.39}$$

where $|g|_{m,1} = \sum_{|l|=m} \|\partial^l g\|_{L^1}$.

Setting $g_{\boldsymbol{\beta}}(v) = \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}(v))$, which is in $W^{m,1}$ $\forall m \geq 0$, we deduce from this lemma that

$$E_1 \leq c\, \Delta v_n^m |g_{\boldsymbol{\beta}}|_{m,1}.$$

It may be seen that since $|g_{\boldsymbol{\beta}}|_{m,1} \leq \int |p(\boldsymbol{\beta}, v)| \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}(v))\, dv$, where p is a polynomial, then $|g_{\boldsymbol{\beta}}|_{m,1}$ is bounded on every compact subset K of $\mathcal{D}$. Then there exists a constant c_K depending only on K, such that $E_1 \leq c_K \Delta v_n^m$. For the term E_2, note that $v_k^n \notin \mathcal{V}^n$ means $|v_k^n| \geq D_n$ (where D_n is the radius of $\mathcal{V}^n$). Thus

$$E_2 \leq \frac{1}{D_n^2} \sum_{k \in \mathbb{Z}^D} |v_k^n|^2 \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}(v_k^n)) \Delta v_n^D.$$

Applying lemma 1 to $g_{\boldsymbol{\beta}}(v) = |v|^2 \exp(\boldsymbol{\beta} \cdot \boldsymbol{m}(v))$ yields $E_2 \leq \frac{1}{D_n^2} c_K$ for every compact K of $\mathcal{D}$. Therefore the bounds on E_1 and E_2 show that $e_n \to 0$, uniformly on every compact set K of $\mathcal{D}$, which completes the proof of (P_3).

For the last property (P_4), note that, by definition, $J_n(\boldsymbol{\alpha}_n) \leq J_n(\boldsymbol{\beta})$ for any $\boldsymbol{\beta} \in \mathcal{D}$. Due to (P_3), we have $J_n(\boldsymbol{\beta}) \to J(\boldsymbol{\beta})$, thus $J_n(\boldsymbol{\beta})$ is bounded, and there exists $c > 0$ such that $J_n(\boldsymbol{\alpha}_n) \leq J_n(\boldsymbol{\beta}) \leq c$ for all n. This proves (P_4), and the proof of proposition 2 is now complete. $\square$

The proof of theorem 1 consists now in proving that J_n is in fact coercive *uniformly in n*. Therefore, (P_4) insures that $\boldsymbol{\alpha}_n$ is bounded, and (P_3) implies that $\boldsymbol{\alpha}_n \to \boldsymbol{\alpha}$.

First, let S be a real number such that $S \geq J(\boldsymbol{\alpha})$. The coercivity of J implies that there exists a compact $K \subset \mathcal{D}$, such that

$$J(\boldsymbol{\beta}) \geq S + 1 \qquad \forall \boldsymbol{\beta} \in \mathcal{D} - \overset{\circ}{K}, \tag{3.40}$$

and we may assume that $\boldsymbol{\alpha} \in \overset{\circ}{K}$ (which denotes the interior of K). Property (P_3) implies that there exists $n_0(K)$ depending only on the compact set K, such that

$$|J_n(\boldsymbol{\beta}) - J(\boldsymbol{\beta})| \leq \frac{1}{4} \qquad \forall \boldsymbol{\beta} \in K, \ \forall n \geq n_0(K). \tag{3.41}$$

Therefore (3.40) and (3.41) yields

$$J_n(\boldsymbol{\beta}) \geq S + \frac{3}{4} \qquad \forall \boldsymbol{\beta} \in \partial K, \ \forall n \geq n_0(K). \tag{3.42}$$

Furthermore, the fact that $\boldsymbol{\alpha} \in \overset{\circ}{K}$ and the definition of S imply

$$J_n(\boldsymbol{\alpha}) \leq |J_n(\boldsymbol{\alpha}) - J(\boldsymbol{\alpha})| + J(\boldsymbol{\alpha}) \leq S + \frac{1}{4} < S + \frac{3}{4} \qquad \forall n \geq n_0(K). \tag{3.43}$$

We now use the following lemma:

Lemma 2. *Let ψ be a convex function on a convex open set Ω of $\mathbb{R}^D$. If there exists a convex compact subset $K \subset \Omega$ and a constant c such that*

$$\psi(x) \geq c \text{ in } \partial K \text{ and } \exists\, x_0 \in \overset{\circ}{K}, \psi(x_0) < c, \tag{3.44}$$

then $\psi(x) > c$ in $\Omega - K$.

Proof of lemma 2. Let $x \notin K$ and let y be the intersection point of ∂K and $[x_0, x]$. Then there exists $0 < \sigma < 1$ such that $y = \sigma x_0 + (1 - \sigma)x$. The convexity of ψ implies

$$\psi(x) \geq \frac{1}{1-\sigma}(\psi(y) - \sigma\psi(x_0)),$$

and hence assumption (3.44) leads to $\psi(x) > c$, which proves the lemma. □

Relations (3.43) and (3.42) show that J_n satisfies the conditions of lemma 2 for all $n \geq n_0(K)$, and thus $J_n(\boldsymbol{\beta}) \geq S + \frac{3}{4}$, $\forall n \geq n_0(K)$, $\forall \boldsymbol{\beta} \in \mathbb{R}^{D+2} - K$. Therefore, for all $S \geq J(\boldsymbol{\alpha})$, there exists a compact set K and an integer n_0 such that

$$J_n(\boldsymbol{\beta}) \geq S + \frac{3}{4} \qquad \forall n \geq n_0, \ \forall \boldsymbol{\beta} \in \mathbb{R}^{D+2} - K.$$

Thus, the functionals J_n are coercive uniformly in n. Owing to (P_4), this proves that $\boldsymbol{\alpha}_n \in K$ for all $n \geq n_0(K)$. Therefore, since K is compact, $\boldsymbol{\alpha}_n$ is convergent, up to the extraction of a subsequence, to $\bar{\boldsymbol{\alpha}} \in \mathcal{D}$. Property (P_3) and the uniqueness of $\boldsymbol{\alpha}$ imply that $\bar{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$, and thus, the whole sequence $\boldsymbol{\alpha}_n$ converges to $\boldsymbol{\alpha}$. The proof of the theorem is now complete.

4. Existence of discrete solution (proof of theorem 2)

Since this theorem is independent of n, the sub/superscript n is omitted in this section, when there is no ambiguity. In this proof, we simply use a fixed point method for the operator Φ, defined by the nonlinear problem

$$F \overset{def}{=} \Phi(G) \iff \begin{cases} \partial_t F_k + v_k \cdot \nabla_x F_k + F_k = \mathcal{E}_k[\boldsymbol{\rho}_G] \\ F_k(0,x) = f_k^0(x), \end{cases} \tag{4.45}$$

where $\boldsymbol{\rho}_G = \langle \boldsymbol{m} G \rangle_n$, and $\mathcal{E}_{\mathcal{K}}[\boldsymbol{\rho}_G]$ is the minimum of entropy on $\mathcal{X}_{\boldsymbol{\rho}_G}$ (see (2.18)). This operator is well defined if G is strictly positive in $L^\infty([0, t_{max}[\times \mathbb{R}_x^D)^N$, $\forall t_{max} > 0$.

First, we give an invariant zone for Φ. In order to be local in space and time, we consider two positive real numbers R and t_{max} and we define the domain of dependence of $\Phi(G)(t_{max}, x)$ on the ball $\overline{B}(0,R)$ by

$$\Omega_R(t_{max}) = \Big\{ (t,x);\ t \leq t_{max} \text{ and } |x| \leq R + (t_{max} - t) \max_k |v_k| \Big\}. \tag{4.46}$$

Note that since the propagation speeds of the model are bounded by $\max_k |v_k|$, this set is compact. An invariant zone is given in the following proposition:

Proposition 3. *The set*

$$\mathcal{F}_R = \Big\{ G \in L^\infty(\Omega_R(t_{max}))^N,\ \delta_0 e^{-t} \phi(x - t v_k) \leq G_k(t,x) \leq n\, e^{Nt},\ \text{a.e. } (t,x) \Big\} \tag{4.47}$$

is stable under Φ.

Proof of proposition 3. First, we verify that for $G \in \mathcal{F}_R$, the function $F = \Phi(G)$ is well defined on $\Omega_R(t_{max})$ and depends only on the values of G on this compact set, and also on the values of f^0. Using the integral representation

$$F_k(t,x) = e^{-t} f_k^0(x - t v_k) + \int_0^t e^{s-t} \mathcal{E}_k[\boldsymbol{\rho}_G(s, x - (t-s) v_k)]\, ds,$$

we obtain the expected result due to the following lemma:

Lemma 3. *Let be $(t,x) \in \Omega_R(t_{max})$. For all $0 \leq s \leq t$ and for all $k \in \mathcal{K}$, we have*

$$(s, x - (t-s) v_k) \in \Omega_R(t_{max}).$$

Proof of lemma 3. This lemma is based on the fact that, the velocities v_k being bounded, one cannot go out of $\Omega_R(t_{max})$ by following the characteristics. From definition (4.46), we have

$$t \le t_{max} \quad \text{and} \quad |x| \le R + (t_{max} - t)\max_k |v_k|,$$

and we must prove

$$s \le t_{max} \quad \text{and} \quad |x - (t-s)v_k| \le R + (t_{max} - s)\max_k |v_k|.$$

From the definition of s, it is obvious that $s \le t_{max}$. Furthermore, the triangle inequality yields

$$\begin{aligned} |x - (t-s)v_k| &\le |x| + (t-s)|v_k| \\ &\le R + (t_{max} - t)\max_k |v_k| + (t-s)\max_k |v_k| \\ &= R + (t_{max} - s)\max_k |v_k|, \end{aligned}$$

and the proof of the lemma is now complete. □

In order to conclude the proof of proposition 3, we now prove that $F = \Phi(G)$ satisfies the same estimates as G. Due to the fact that $\mathcal{V}$ is bounded and discrete, and with the definition of $\mathcal{E}_{\mathcal{K}}[\boldsymbol{\rho}_G]$, we have the following estimate on the discrete equilibrium

$$\mathcal{E}_k[\boldsymbol{\rho}_G(s,y)] \le \sum_{k'} \mathcal{E}_{k'}[\boldsymbol{\rho}_G(s,y)] = \sum_{k'} G_{k'}(s,y) \le Nne^{Ns}. \tag{4.48}$$

Then an integral representation of $F = \Phi(G)$ gives the upper bound of (4.47):

$$\begin{aligned} F_k(t,x) &= f_k^0(x - tv_k) + \int_0^t \left(\mathcal{E}_k[\boldsymbol{\rho}_G(s, x-(t-s)v_k)] - F_k(s, x-(t-s)v_k)\right) ds \\ &\le n + \int_0^t Nne^{Ns}\, ds = ne^{Nt}, \end{aligned}$$

and the lower bound is easily obtained by integrating along the characteristics

$$\begin{aligned} F_k(t,x) &= e^{-t} f_k^0(x - tv_k) + \int_0^t e^{s-t}\mathcal{E}_k[\boldsymbol{\rho}_G(s, x-(t-s)v_k)]\, ds \\ &\ge e^{-t} f_k^0(x - tv_k) \ge e^{-t}\delta_0 \phi(x - tv_k). \end{aligned}$$

□

The idea of the proof consists now in applying a fixed point theorem in the invariant zone given by proposition 3. We first prove that $G \mapsto \mathcal{E}_{\mathcal{K}}[\boldsymbol{\rho}_G]$ is Lipschitz continuous on the set $\mathcal{F}_R$. For that purpose, note that due to proposition 1, we have $\mathcal{E}_{\mathcal{K}}[\boldsymbol{\rho}_G] = \exp(\boldsymbol{\alpha}_G \cdot \boldsymbol{m}(v_k))$ for all $G \in \mathcal{F}_R$. Therefore the mapping $\boldsymbol{\rho}_G \mapsto \boldsymbol{\alpha}_G \in \mathbb{R}^{D+2}$ - defined on the set $\{\boldsymbol{\rho}_G \in \mathbb{R}^D$ strictly realizable in $\mathcal{V}\}$ - is continuously differentiable. In fact, the Jacobian matrix of the inverse mapping is $\langle \boldsymbol{m}\otimes\boldsymbol{m} \exp(\boldsymbol{\alpha}_G \cdot \boldsymbol{m})\rangle_n$ which, due to the definition of $\mathcal{V}$, is positive definite, and hence invertible (see [10]). Moreover, the elements of $\mathcal{F}_R$ are uniformly bounded and uniformly far away from 0, since

$$\delta_0 e^{-t}\phi(x - tv_k) \geq \delta_0 e^{-t_{max}} \min_{\Omega_R(t_{max})} \phi(x - tv_k) = C(t_{max}, R) > 0.$$

Therefore the operator $G \mapsto \mathcal{E}_{\mathcal{K}}[\boldsymbol{\rho}_G]$ is Lipschitz continuous on $\mathcal{F}_R$, *i.e.* there exists a positive constant $L(R, t_{max})$ that depends only on R and t_{max}, such that

$$\Big|\mathcal{E}_k[\boldsymbol{\rho}_F] - \mathcal{E}_k[\boldsymbol{\rho}_G]\Big|(t,x) \leq L(R, t_{max}) \max_k \Big|F_k - G_k\Big|(t,x)$$

for all $(t,x) \in \Omega_R(t_{max})$ and for all F and G in $\mathcal{F}_R$. A classical technique in ordinary differential equation theory then allows to prove that an iterate of Φ is a contraction mapping in $\mathcal{F}_R$. Namely, it can be deduced from the previous estimate that for any iterate $\Phi^p = \Phi \circ \Phi \ldots \circ \Phi$, we have

$$\|\Phi^p(F) - \Phi^p(G)\|_{L^\infty([0,t_{max}]\times\mathbb{R}^D_x)^N} \leq \frac{(L(R, t_{max})t_{max})^p}{p\,!} \|F - G\|_{L^\infty},$$

for all F and G in $\mathcal{F}_R$. The constant $\frac{(L(R,t_{max})t_{max})^p}{p\,!}$ is less than 1 if p is large enough (*i.e.* $p \geq p(R, t_{max})$). Consequently, Φ^p is a contraction mapping in $\mathcal{F}_R$ if $p \geq p(R, t_{max})$. Then, from a classical fixed point theorem, there exists a unique function f_R in $L^\infty(\Omega_R(t_{max}))^N$ such that $\Phi(f_R) = f_R$ almost everywhere in $\Omega_R(t_{max})$. Using a increasing sequence of R, we can thus construct a function f in $L^\infty([0, t_{max}]\times\mathbb{R}^D_x)^N$ such that $\Phi(f) = f$ almost everywhere. From the uniqueness of its restriction to $\Omega_R(t_{max})$ for all $R > 0$, this function is unique. The proof of existence and uniqueness part of the theorem is now complete. Moreover, since the bounds (2.25) are satisfied by any f_R in $\Omega_R(t_{max})$, it is clear that f satisfies these bounds.

Remark 1. We feel that it is necessary to explain why we have used a fixed point method *locally* in space. Since the lower bound of (4.47) tends to 0 as $|x| \to +\infty$, we cannot easily obtain a global Lipschitz continuity property of the operator $G \mapsto \mathcal{E}_{\mathcal{K}}[\boldsymbol{\rho}_G]$. In fact, the function G might be too close to the boundary of the set of strictly realizable moments, and thus it seems difficult to bound uniformly in x the derivative of this operator.

To obtain conservation laws (2.29), it is sufficient to multiply (2.27) by $\boldsymbol{m}(v_k)$, then to sum over $\mathcal{K}$. For local entropy dissipation relation (2.30), note that if η is a

Lipschitz continuous function, then (see [16])

$$\partial_t \eta(f_k) + v_k \cdot \nabla_x \eta(f_k) = (\mathcal{E}_k - f_k)\eta'(f_k) \qquad \text{in } \mathcal{D}'(]0, t_{max}[\times \mathbb{R}_x^D)^N. \tag{4.49}$$

But $\eta(s) = s\log s$ is not Lipschitz continuous, therefore, following Perthame [4], we bound its derivative by defining

$$\eta_R(0) = 0, \qquad \eta_R'(s) = \max(-R, \min(R, 1 + \log s))$$

on $[0, +\infty[$. This function is Lipschitz continuous and $\eta_R(s) \underset{R\to+\infty}{\longrightarrow} \eta(s)$. From (4.49), it comes

$$\eta_R(f_k^\sharp) - \eta_R(f_k^0) = \int_0^t \eta_R'(f_k^\sharp)(\mathcal{E}_k^\sharp - f_k^\sharp)\, ds \tag{4.50}$$

where $f_k^\sharp(t,x) = f_k(t, x + tv_k)$. The left-hand side of this equation converges a.e to $\eta(f_k^\sharp) - \eta(f_k^0)$ as $R \to +\infty$. We can also pass to the limit in the right-hand side, since $|\eta_R'(f_k^\sharp)|$ is bounded above by $1 + |\log f_k^\sharp|$, which is in turn bounded due to (2.25). We can therefore pass to the limit in (4.50), and this gives a formulation equivalent to (4.49). Finally, we sum (4.49) over $k \in \mathcal{K}$ and it is now classical to note that

$$\langle (\mathcal{E}_\mathcal{K} - f_\mathcal{K})\eta'(f_\mathcal{K})\rangle_n = \langle (\mathcal{E}_\mathcal{K} - f_\mathcal{K})(\log f_\mathcal{K} - \log \mathcal{E}_\mathcal{K})\rangle_n + \langle (1 + \log \mathcal{E}_\mathcal{K})(\mathcal{E}_\mathcal{K} - f_\mathcal{K})\rangle_n.$$

Due to the definition of $\mathcal{E}_\mathcal{K}$, the last term vanishes. The second one is non positive because $s \mapsto \log s$ is non decreasing. Thus, we obtain the entropy inequality (2.30).

Estimate (2.31) is now easily derived from (2.29) and (2.30), and the proof of theorem 2 is then complete.

5. Convergence of the discrete-velocity model (proof of theorem 3)

Following Perthame [4], we divide the proof into four steps.

step 1: weak convergence of f^n and $\langle \boldsymbol{m} f^n\rangle$
From (2.31), it is clear that f^n satisfies the uniform estimate

$$\sup_n \sup_{[0,t_{max}]} \int_{\mathbb{R}^{2D}} (1 + |x|^2 + |v|^2 + |\log f^n|) f^n(t,x,v)\, dxdv \leq \Gamma_2(t_{max}) \tag{5.51}$$

for all $t_{max} > 0$. We classically deduce that there exists a subsequence still denoted by $\{f^n\}_n$ such that

$$f^n \underset{n\to\infty}{\rightharpoonup} f \text{ weakly in } L^1([0, t_{max}] \times \mathbb{R}_x^D \times \mathbb{R}_v^D) \qquad \forall t_{max} > 0. \tag{5.52}$$

Moreover, it is clear that $C^n(v)$ converges pointwise to v and is locally uniformly bounded. This is sufficient with (5.52) to obtain the convergence of the left-hand side of (2.34) to $\partial_t f + v \cdot \nabla_x f$ in $\mathcal{D}'(]0, +\infty[\times \mathbb{R}_x^D \times \mathbb{R}_v^D)$.

For the convergence of the nonlinear right-hand side, we first obtain weak convergence of $\langle \boldsymbol{m} f^n \rangle$. Estimate (5.51) yields

$$\langle (1,v)^T f^n(t,x,v) \rangle \underset{n\to\infty}{\rightharpoonup} (\rho, \rho u)^T = \langle (1,v)^T f(t,x,v) \rangle \quad \forall t_{max} > 0 \tag{5.53}$$

weakly in $L^1([0,t_{max}] \times \mathbb{R}^D_x)$.

However, due to the lack of estimate of f^n for large velocities, estimate (5.51) is not sufficient to obtain weak convergence of $\langle \frac{1}{2}|v|^2 f^n \rangle$. We then use a lemma of Perthame [4] to control $|v|^3 f^n$. This lemma is based on the dispersive effect of the inversion of $\partial_t + v \cdot \nabla_x$, which also exists in the discrete-velocity case.

Lemma 4. *Let $F \in L^1(\mathbb{R}^+ \times \mathbb{R}^D_x)^{N_n}$ solve*

$$\partial_t F_k + v_k \cdot \nabla_x F_k = g_k, \qquad F_k(0,x) = 0 \quad \forall k \in \mathcal{K}^n,$$

where $g \geq 0$ satisfies

$$\int_0^{t_{max}} \int_{\mathbb{R}^D_x} \langle |v|^2 g(t,x) \rangle_n \, dxdt \leq c. \tag{5.54}$$

Then, for any bounded subset K of $\mathbb{R}^D_x$, we have

$$\int_0^{t_{max}} \int_K \langle |v|^3 F(t,x) \rangle_n \, dxdt \leq c \, diam(K).$$

The proof of this lemma is exactly the same as the one given in [4] and we do not write it here. Note that the discrete equilibrium $\mathcal{E}^n_{\mathcal{K}}$ has the same energy as $f^n_{\mathcal{K}}$, thus it satisfies estimate (5.54) of lemma 4. Therefore from this lemma, we have

$$\sup_n \int_0^{t_{max}} \int_K \sum_{|v^n_k| > R} |v^n_k|^2 f^n_k(t,x) \Delta v^D_n \, dtdx \leq \frac{c(K)}{R}, \tag{5.55}$$

for any compact set K of $\mathbb{R}^D_x$. This yields

$$\sup_n \int_0^{t_{max}} \int_K \int_{|v| \geq 2R} |v|^2 f^n(t,x,v) \, dtdxdv \leq \frac{c(K)}{R}, \tag{5.56}$$

and thus we have

$$|v|^2 f^n \rightharpoonup |v|^2 f \qquad \text{weakly in } L^1([0,t_{max}] \times K \times \mathbb{R}^D_v), \tag{5.57}$$

$$\langle \tfrac{1}{2}|v|^2 f^n \rangle \rightharpoonup E = \langle \tfrac{1}{2}|v|^2 f \rangle \qquad \text{weakly in } L^1([0,t_{max}] \times K), \tag{5.58}$$

for any compact subset K of $\mathbb{R}^D_x$ and $t_{max} > 0$. Therefore, we have proved that $\langle \boldsymbol{m} f^n \rangle \rightharpoonup \boldsymbol{\rho} = \langle \boldsymbol{m} f \rangle$ weakly in $L^1([0,t_{max}] \times K)$ for any compact K.

step 2: Weak convergence of $\mathcal{E}^n$

We need the following lemma.

Lemma 5. *For all $t_{max} > 0$ there exists a constant $c(t_{max})$ such that*

$$\sup_n \sup_{[0,t_{max}]} \int_{\mathbb{R}^{2D}} (1 + |x|^2 + |v|^2 + |\log \mathcal{E}^n|)\mathcal{E}^n(t, x, v)\, dxdv \leq c(t_{max}). \tag{5.59}$$

Proof. The bounds on $(1+|x|^2+|v|^2)\mathcal{E}^n$ directly follow from the definition of $\mathcal{E}^n$ and from estimate (5.51). For $\mathcal{E}^n|\log \mathcal{E}^n|$, note that due to the definition of the discrete equilibrium $\mathcal{E}^n_{\mathcal{K}}$, we have

$$\int_{\mathbb{R}^D} \mathcal{E}^n \log \mathcal{E}^n\, dv \leq \int_{\mathbb{R}^D} f^n \log f^n\, dv.$$

Then a classical manipulation allows to pass from $\mathcal{E}^n \log \mathcal{E}^n$ to $\mathcal{E}^n|\log \mathcal{E}^n|$ (see [4]) and yields

$$\int_{\mathbb{R}^{2D}} \mathcal{E}^n|\log \mathcal{E}^n|\, dxdv \leq c(t_{max}).$$

□

This lemma shows that $\mathcal{E}^n$ is weakly compact in $L^1([0, t_{max}] \times \mathbb{R}^{2D})$. Thus there exists a function $\mathcal{M}$ such that $\mathcal{E}^n \rightharpoonup \mathcal{M}$ weakly in $L^1([0, t_{max}] \times \mathbb{R}^D_x \times \mathbb{R}^D_v)$. Therefore, we can deduce from step 1 and step 2 that the weak limit f of f^n satisfies the equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{M} - f \qquad \text{in } \mathcal{D}'. \tag{5.60}$$

The following steps are devoted to the proof of $\mathcal{M} = M[\boldsymbol{\rho}]$.

<u>step 3:</u> Strong convergence of $\boldsymbol{\rho}_n$

The extension to discrete-velocity frame of the averaging lemma obtained by Mischler [14] implies that velocity averages of f^n on bounded sets are in fact strongly compact. Consequently, up to the extraction of a subsequence, we have

$$\int_{|v|<R} \boldsymbol{m}(v) f^n(t, x, v)\, dv \to \int_{|v|<R} \boldsymbol{m}(v) f(t, x, v)\, dv$$

strongly in $L^1([0, t_{max}] \times \mathbb{R}^D_x)$ for any $R > 0$. From the uniform estimates (5.51) and (5.56) we thus obtain

$$\langle \boldsymbol{m} f^n \rangle \to \boldsymbol{\rho} = \langle \boldsymbol{m} f \rangle \qquad \text{strongly in } L^1([0, t_{max}] \times K)$$

for every compact K. Furthermore, since $\langle (1, v)^T f^n \rangle = (\rho_n, \rho_n u_n)^T$ and $\langle \frac{1}{2}|v|^2 f^n \rangle = E_n + \rho_n \frac{D}{24} \Delta v_n^2$, then it is clear that $\boldsymbol{\rho}_n$ and $\langle \boldsymbol{m} f^n \rangle$ are asymptotically equivalent. Therefore, we conclude that $\boldsymbol{\rho}_n \to \boldsymbol{\rho}$ strongly in $L^1([0, t_{max}] \times K)$, for every compact K.

step 4: Passing to the limit
Extracting again a subsequence, we have $\boldsymbol{\rho}_n(t,x) \to \boldsymbol{\rho}(t,x)$ a.e in $[0,t_{max}] \times \mathbb{R}^D_x$. Then theorem 1 implies that on the set $\Omega = \{(t,x),\ \rho(t,x) \text{ and } \theta(t,x) > 0\}$ we have

$$\boldsymbol{\alpha}_n(t,x) \to \boldsymbol{\alpha}(t,x) \qquad \text{a.e.}$$

Therefore

$$\mathcal{E}^n \to M[\boldsymbol{\rho}] \qquad \text{a.e in } \Omega \times \mathbb{R}^D_v,$$

and since

$$\rho_n(t,x) = \|\mathcal{E}^n(t,x,.)\|_{L^1} \to \rho(t,x) = 0 \qquad \text{a.e in } \Omega^c,$$

then

$$\mathcal{E}^n(t,x) \to 0 = M[\boldsymbol{\rho}](t,x) \qquad \text{a.e in } \Omega^c \times \mathbb{R}^D_v.$$

This proves that $\mathcal{E}^n$ converges pointwise to $M[\boldsymbol{\rho}]$. Combining this result with that of step 2 proves that $\mathcal{M} = M[\boldsymbol{\rho}]$. Therefore the right-hand side of (2.27) converges toward $(M[\boldsymbol{\rho}] - f)$ weakly in L^1, and we can conclude that f is solution of BGK equation (1.1).

Remark 2. If we have sufficient regularity on the initial condition f^0 (say BV), so that $f^{0,n}$ tends to f^0 in L^1, then the convergence in Theorem 3 is in fact strong in L^1. Namely one can prove by similar arguments as Lions [19] that $\log(1+f^n)$ tends weakly towards $\log(1+f)$.

Remark 3. If we assume the assumptions of Mischler [7] on the initial condition, the solution of the BGK equation is unique. In that case, the whole sequence f^n converges to f.

References

1. K. Aoki, K. Kanba, and S. Takata. Numerical Analysis of a Supersonic Rarefied Gas Flow Past a Flat Plate. *Phys. Fluids*, 9(4), 1997.
2. J.Y. Yang and J.C. Huang. Rarefied Flow Computations Using Nonlinear Model Boltzmann Equations. *Journal of Computational Physics*, 120:323–339, 1995.
3. E.P. Gross P.L. Bathnagar and M. Krook. *Phys. Rev.*, 94:511, 1954.
4. B. Perthame. Global existence to the BGK Model of Boltzmann Equation. *J. Diff. Eq.*, 82:191–205, 1989.
5. E. Ringeisen. *Conributions à l'étude mathématique des équations cinétiques.* Thèse, Université Paris VII, 1991.
6. B. Perthame and M. Pulvirenti. Weighted L^∞ bounds and uniqueness for the Boltzmann BGK model. *Arch. Rat. Mech. Anal.*, 125:289–295, 1993.

7. S. Mischler. Uniqueness for the BGK equation in $\mathbb{R}^n$ and rate of convergence for a semi-discrete scheme. *Diff. and Int. Equations*, 9, 1996.

8. D. Issautier. Convergence of a Weighted Particle Method for Solving The Boltzmann (B.G.K.) Equation. *SIAM J. Numer. Anal.*, 33(6):2099–2119, 1996.

9. P. Charrier, B. Dubroca, J.-L. Feugeas, and L. Mieussens. Discrete-Velocity models for Kinetic Nonequilibrium Flows. *C.R Acad.Sci.*, Serie I, 326(11):1347–1352, june 1998. Paris.

10. L. Mieussens. Discrete Velocity Model and Implicit Scheme for the BGK Equation of Rarefied Gas Dynamics. to appear in Math. Models and Meth. in Applied Sciences.

11. F. Rogier and J. Schneider. A Direct Method For Solving the Boltzmann Equation. *Transp. Th. Stat. Phys.*, 23(1-3):313–338, 1994.

12. C. Buet. A Discrete-Velocity Scheme for the Boltzmann Operator of Rarefied Gas Dynamics. *Transp. Th. Stat. Phys.*, 25(1):33–60, 1996.

13. A. Heintz and V. Panferov. A new consistent discrete-velocity model for the Bolzmann equation. preprint 1999:11, Chalmers, Göteborg University.

14. S. Mischler. Convergence of discrete velocity schemes for the Boltzmann equation. *Arch. Rat. Mech. Anal.*, 140:53–77, 1997.

15. A.V. Bobylev, A. Palczewski, and J. Schneider. A Consistency Result for a Discrete-Velocity Model of the Boltzmann Equation. *Siam J. Numer. Anal.*, 34(5):1865–1883, 1997.

16. R.J. DiPerna and P.L. Lions. On the Cauchy problem for Boltzmann equations: Global existence and weak stability. *Ann. Math*, **130**:321–366, 1989.

17. L. Mieussens. Convergence of a discrete-velocity model for the Boltzmann-BGK equation. *C.R Acad.Sci.*, Serie I, 328:1231–1236, 1999. Paris.

18. P.A. Raviart. An Analysis of Particle Methods. In F. Brezzi, editor, *Numerical Methods in Fluid Dynamics*, volume 1127 of *Lecture Notes in Mathematics*, pages 243–324. Spinger-Verlag, Berlin, Heidelberg, New York, 1984.

19. P.L. Lions. Compactness in Boltzmann's Equation via Fourier Integral Operators and Applications. II. *J. Math. Kyoto Univ*, 34(2):429–461, 1994.

20. C. Cercignani. *The Boltzmann Equation and Its Applications*, volume 68. Springer-Verlag, Lectures Series in Mathematics, 1988.

21. R. Gatignol. *Théorie Cinétique des Gaz à Répartition Discrète de Vitesses*, volume 36 of *Lecture Notes in Physics*. Springer-Verlag, 1975.