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Regularisable and Minimal Orbits for Group Actions in Infinite Dimensions

M. Arnaudon, S. Paycha

Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7, rue René Descartes, 67084 Strasbourg Cedex, France

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Abstract: We introduce a class of regularisable infinite dimensional principal fibre bundles which includes fibre bundles arising in gauge field theories like Yang-Mills and string theory and which generalise finite dimensional Riemannian principal fibre bundles induced by an isometric action. We show that the orbits of regularisable bundles have well defined, both heat-kernel and zeta function regularised volumes. We introduce a notion of μ-minimality ($\mu \in \mathbb{R}$) for these orbits which extend the finite dimensional one. Our approach uses heat-kernel methods and yields both "heat-kernel" (obtained via heat-kernel regularisation) and "zeta function" (obtained via zeta function regularisation) minimality for specific values of the parameter μ. For each of these notions, we give an infinite dimensional version of Hsiang's theorem which extends the finite dimensional case, interpreting μ-minimal orbits as orbits with extremal (μ-regularised) volume.

0. Introduction

This article is concerned with the notions of regularisability and minimality of orbits for an isometric action of an infinite dimensional Lie group **G** on an infinite dimensional manifold $\mathcal{P}$. Our study is based on heat-kernel regularisation methods but it involves a larger class including zeta function regularisations. The ones we consider are parametrised by $\mu \in \mathbb{R}$; we recover the zeta function regularisation by setting $\mu = \gamma$, the Euler constant and the heat-kernel regularisation by setting $\mu = 0$.

Notions of regularisability and minimality for actions of infinite dimensional Lie groups on infinite dimensional manifolds have already been studied by other authors (see [KT, MRT]) in a particular context and using zeta function regularisation methods. We recover these notions for $\mu = \gamma$.

We shall introduce a class of principal fibre bundles called (resp. *pre-*)*regularisable* fibre bundles which generalise to the infinite dimensional case finite dimensional Riemannian principal fibre bundles arising from a free isometric action. We show that the fibres of these (resp. pre-)regularisable bundles have a well defined regularised (resp. preregularised) volume which is Gâteaux differentiable. This class of (pre-) regularisable fibre bundles includes some infinite dimensional principal bundles arising from gauge field theories such as Yang-Mills and string theory.

We introduce the notion of strong minimality and μ-minimality using μ- regularisation, all of which extend the finite dimensional notion and coincide in the finite dimensional case. However, μ-minimality depends on the choice of the parameter μ, in particular zeta function minimality ($\mu = \gamma$) does not in general coincide with heat-kernel minimality ($\mu = 0$).

We show that if the metric on the structure group is fixed, μ-(resp. strongly) minimal fibres of a (resp. pre-)regularisable principal fibre bundle coincide with the ones with extremal μ-(resp. pre-)regularised volume among orbits of the same type for the group action. This gives an infinite dimensional version of Hsiang's theorem on (pre-) regularisable principal fibre bundles with structure group equipped with a fixed Riemannian metric, which we extend (adding a term which reflects the variation of the metric on the structure group) to any (pre-)regularisable principal bundle.

Starting from a systematic review of the notions of μ- regularised determinants in Sect. 1, in Sect. 2 we introduce the notions of regularisable principal fibre bundle, (resp. pre-) regularisability and μ- (resp. strong) minimality of orbits, relating (resp. strong) minimality with the Gâteaux-differentiability of μ (resp. pre-)regularised determinants interpreted as volumes of fibres. In Sect. 3, we compare these notions for different values of μ. The relations we set up between the regularised mean curvature vector and the directional gradients of the regularised determinants yield an infinite dimensional version of Hsiang's theorem.

To avoid making this article any longer than it already is, we chose not to treat examples in detail here. Let us just however point out some examples the results in this article can be applied to.

When applied to the *coadjoint action of a loop group*, one recovers some results concerning regularisability and minimality of fibres studied in [KT].

The notion of minimality investigated in this article also applies to the study of orbits of a *Yang-Mills action* (see e.g [FU, KR, MV] for the corresponding geometric setting). A notion of zeta function minimality in the Yang Mills context had already been suggested in [MRT1]. Our heat-kernel approach leads to a slightly different definition; when the underlying manifold is of dimension 4, only if the irreducible connections are Yang-Mills, do the different notions of minimality (in particular zeta function and heat-kernel) coincide.

Let us stress that in both the examples mentioned here, the space $\mathcal{P}$, resp. the group **G** are modelled on a space of sections of a vector bundle $\mathcal{E}$, resp. $\mathcal{F}$ with finite dimensional fibres on a closed finite dimensional manifold M and **G** acts on $\mathcal{P}$ by isometries.

The case of *diffeomorphisms acting on metrics* which has been investigated carefully in [MRT2] is also very interesting since it relates to string theory. One could show, in a similar way to the Yang-Mills case, that the bundle $\mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathrm{Diff}_0$ (see [FT, T]) arising in bosonic string theory (where $\mathcal{M}_{-1}$ is the manifold of smooth Riemannian metrics with curvature -1 on a compact boundaryless Riemannian surface of genus greater than 1 and Diff_0 is the group of smooth diffeomorphisms of the surface which are homotopic to zero), is also a regularisable fibre bundle so that most results of this paper can be applied to this fibre bundle. Unlike the case of Yang-Mills theory, its structure group Diff_0 is not equipped with a fixed Riemannian structure but with a family of Riemannian metrics which is parametrised by $g \in \mathcal{M}_{-1}$; this example was our initial motivation when considering the general case of a structure group equipped with a family of metrics indexed by P. Investigating carefully the geometry of the orbits in this particular example leads to interesting questions concerning the geometry of some associated determinant bundles [PR].

In this particular example, minimality of the fibres is still equivalent to extremality of the volumes of the fibres since the additional term arising from the varying metric on the group (see Proposition 2.2) vanishes.

The geometric notions developed in this paper play a important role when projecting a class of semi-martingales defined on the total manifold onto the orbit space for a certain class of infinite dimensional group actions. The regularisation based on heat-kernel methods used here yields natural links between the geometric and the stochastic picture, which we investigate in [AP2]. The stochastic picture described in [AP2] leads to a stochastic interpretation of the Faddeev-Popov procedure used in gauge field theory to reduce a formal volume measure on path space to a measure on the orbit space, the formal density of which is a regularised "Faddeev–Popov" determinant.

1. Regularised Determinants

In this section, we recall some basic facts about regularised determinants, comparing different regularisation methods. Although the results presented here are in some way well known (see e.g [BGV]) and frequently used in the physics literature, the presentation we offer is maybe a little unusual, since it involves defining a one parameter family of regularised limits (parametrised by $\mu \in \mathbb{R}$). Zeta function regularisation corresponds to $\mu = \gamma$, the Euler constant, heat-kernel regularisation to $\mu = 0$.

Let us first introduce some notations. For a function $t \mapsto f(t)$, defined on an interval of $\mathbb{R}^{+*}$ containing $]0,1]$, we shall write $f(t) \simeq_0 \sum_{j=-J}^{K} a_j t^{\frac{j}{m}} + b \log t$, $a_j \in \mathbb{R}, b \in \mathbb{R}$, $J, K \in \mathbb{N}$, $m \in \mathbb{N}^*$, if

$$f(t) = \sum_{j=-J}^{K} a_j t^{\frac{j}{m}} + b \log t + O(t^{\frac{K+1}{m}}) \quad \forall \quad 0 < t < 1. \tag{1.1}$$

Let us call $\mathcal{C}$ this class of functions and $\mathcal{C}_0$ the subclass of functions $f \in \mathcal{C}$ with no logarithmic divergence at zero (i.e. $b = 0$). In the following, we shall always assume that $J \geq m$.

Functions in the class $\mathcal{C}$ arise naturally as primitives of functions in the class $\mathcal{C}_0$ as shown in the following

Lemma 1.1. *If f is a differentiable function on an interval I of $\mathbb{R}$ containing $]0,1]$ with derivative $f' \in \mathcal{C}_0$, then $f \in \mathcal{C}$. More precisely, if $f'(t) = \sum_{j=-J}^{K} a_j t^{\frac{j}{m}} + O(t^{\frac{K+1}{m}})$, then $f(t) = \sum_{j=-J+m}^{K+m} \alpha_j t^{\frac{j}{m}} + \beta \log t + O(t^{\frac{K+m+1}{m}})$ with $\alpha_j = \frac{m}{j} a_{j-m}$ for $j \neq 0$ and $\beta = a_{-m}$ and for some $\alpha_0 \in \mathbb{R}$.*

Proof. Let us set $g(t) = f(t) - \sum_{j=-J+m, j\neq 0}^{K+m} \frac{m}{j} a_{j-m} t^{\frac{j}{m}} - a_{-m} \log t$. Since $g'(t) = O(t^{\frac{K+1}{m}})$, for $0 < s < t \leq 1$ we can estimate $|g(t) - g(s)| \leq C t^{\frac{k+1}{m}} |t-s| \leq C t^{\frac{K+1}{m}}$ and this yields the existence of the limit $\lim_{t \to 0} g(t) = \alpha_0$. Using the same estimate for $s = 0$ then yields the result. $\square$

In similar way to [BGV], we set the following:

Definition. *For $f \in \mathcal{C}$, $\mu \in \mathbb{R}$, and with the notations of (1.1):*

$$Lim^{\mu}_{t \to 0} f(t) \equiv \lim_{t \to 0} \left(f(t) - \sum_{j=-J}^{K} a_j t^{\frac{j}{m}} - b log t \right) - \mu b, \tag{1.2}$$

*which we call the μ-**regularised** limit of f at point zero.*

Remark. Although the parameter μ might seem artificial at this stage, it will prove to be useful when comparing heat-kernel regularisations and zeta-function regularisations. A similar parameter μ arises in the work of Bismut and Freed on determinant bundles where similar regularisations are needed [BF]. Further analogies between the gauge orbit picture discussed here and the determinant bundles picture are discussed in [P]. Of course, for a function in the class $\mathcal{C}_0$, this limit does not depend on the parameter μ.

Let $(A_t), t \in]0, 1]$ be a one parameter family of trace-class operators on a separable Hilbert space H (in particular tr(A_1) is finite) such that $t \to \mathrm{tr} A_t$ is a function in the class $\mathcal{C}$, then for any $\mu \in \mathbb{R}$ we can define the *μ-regularised limit trace* of $\mathcal{A} \equiv (A_t, t \in]0, 1])$ by

$$tr^{\mu}_{reg} \mathcal{A} \equiv Lim^{\mu}_{t \to 0} tr A_t. \tag{1.3}$$

This regularised limit trace depends of course on the whole one parameter family $\mathcal{A}$ and on the choice of the parameter t. Whenever the context we are working in allows no ambiguity on the choice of μ, we shall sometimes leave the explicit mention of μ out.

We now introduce a family of heat-kernel operators which play a fundamental role in this paper. For this we define for $\varepsilon > 0$ a function $h_\varepsilon : \mathbb{R}^{+*} \to \mathbb{R}$ by $h_\varepsilon(\lambda) \equiv -\int_\varepsilon^\infty \frac{e^{-t\lambda}}{t} dt$. Notice that h_ε is C^∞, non decreasing and $(h_\varepsilon)'(\lambda) = \lambda^{-1} e^{-\varepsilon\lambda}$. Writing $h_\varepsilon(\lambda) - \log \varepsilon = -\int_\varepsilon^\infty \frac{e^{-t}}{t} dt - \int_\varepsilon^{\frac{\varepsilon}{\lambda}} \frac{e^{-\lambda t}}{t} dt + \int_\varepsilon^1 \frac{1}{t} dt$, we find that the function $\varepsilon \mapsto h_\varepsilon(\lambda)$ lies in $\mathcal{C}$ for fixed λ. Moreover we have:

$$Lim^{\mu}_{\varepsilon \to 0} h_\varepsilon(\lambda) = log \lambda - \mu + \gamma, \tag{1.4}$$

where $\gamma = e^{\int_0^1 \frac{1-e^{-t}}{t} dt - \int_1^\infty \frac{e^{-t}}{t} dt}$ is the Euler constant. For a strictly positive self-adjoint operator B on a Hilbert space H, we can define $h_\varepsilon(B)$ which yields a one parameter family of operators $(h_\varepsilon(B), \varepsilon > 0)$.

Definition. *Let B be a strictly positive self-adjoint operator on a separable Hilbert space. Whenever the one parameter family $\mathcal{B} \equiv (h_\varepsilon(B), \varepsilon \in]0, 1])$ has a regularized limit trace, for any $\mu \in \mathbb{R}$, we shall call μ-**regularised determinant** $det^{\mu}_{reg}(B)$ of B the expression:*

$$det^{\mu}_{reg} B \equiv e^{\mathrm{tr}^{\mu}_{reg}(\mathcal{B})} = e^{Lim^{\mu}_{\varepsilon \to 0} tr\, h_\varepsilon(B)}. \tag{1.5}$$

In the following, we give conditions under which we can define the heat-kernel regularized determinant of an operator B. But before that, let us state an easy lemma which will prove to be useful for what follows.

Lemma 1.2. *Let B be a strictly positive self-adjoint operator on a separable Hilbert space such that*

1) $e^{-\varepsilon B}$ *is trace class for any* $\varepsilon > 0$.

2) The function $\varepsilon \to tr(e^{-\varepsilon B})$ *lies in the class* $\mathcal{C}_0$ *with* $(b_j, j \geq -J)$ *as coefficients in the expansion (1.1).*

Then the operator B has a heat-kernel regularised determinant and we have for $\mu \in \mathbb{R}$

$$\begin{aligned} \det{}^{\mu}_{reg} B &= e^{Lim^{\mu}_{\varepsilon\to 0} trh_\varepsilon(B)} \\ &= e^{\left(-\sum_{j=-J, j\neq 0}^{m-1} \frac{mb_j}{j} - \int_1^\infty \mathrm{tr}\frac{e^{-tB}}{t}dt - \int_0^1 \frac{F(t)}{t}dt\right) - \mu b_0} \end{aligned} \tag{1.6}$$

with

$$F(t) = \mathrm{tr} e^{-tB} - \sum_{j=-J}^{m-1} b_j t^{\frac{j}{m}}. \tag{1.7}$$

Proof. One easily shows that $A_\varepsilon \equiv h_\varepsilon(B) = -\displaystyle\int_\varepsilon^\infty \frac{e^{-tB}}{t}dt$ is trace-class. Since all the terms involved are positive, we can exchange the integral and sum symbols so that $\mathrm{tr}A_\varepsilon = -\int_\varepsilon^\infty \mathrm{tr}\frac{e^{-tB}}{t}dt$. Let us check that the family $(A_\varepsilon, \varepsilon \in]0,1])$ has a regularized limit trace. The map $t \to \mathrm{tr}A_t$ is differentiable and from $tr(e^{-tB}) = \sum_{j=-J}^K b_j t^{\frac{j}{m}} + O(t^{\frac{K+1}{m}})$ follows that $\displaystyle\frac{d}{dt}\mathrm{tr}A_t \simeq_0 \sum_{j=-J-m}^{0} b_{j+m} t^{\frac{j}{m}}$. Applying Lemma 1.1 to $f(t) = tr(A_t)$ shows that the one parameter family $\mathcal{A} = (A_\varepsilon)$ has a finite regularised limit trace $\mathrm{tr}^\mu_{reg}(\mathcal{A}) \equiv Lim^\mu_{\varepsilon\to 0}\mathrm{tr}A_\varepsilon$. By (1.5) this in turn yields that B has a μ-regularised determinant $\det^\mu_{reg} B = e^{\,\mathrm{tr}^\mu_{reg}\mathcal{A}}$. Since $\mathrm{tr}A_1 = -\int_1^\infty \mathrm{tr}\frac{e^{-tB}}{t}dt$, integrating $\frac{F_t}{t}$ between ε and 1 yields

$$\mathrm{tr}A_\varepsilon - \sum_{j=-J, j\neq 0}^{m-1} \frac{mb_j}{j}\varepsilon^{\frac{j}{m}} - b_0 \log\varepsilon = -\sum_{j=-J, j\neq 0}^{m-1} \frac{mb_j}{j} - \int_\varepsilon^1 \frac{F(t)}{t}dt - \int_1^\infty \mathrm{tr}\frac{e^{-tB}}{t}dt. \tag{1.8}$$

Since $m \geq 1$, we have

$$\lim_{\varepsilon\to 0}(\mathrm{tr}A_\varepsilon - \sum_{j=-J}^{-1} \frac{mb_j}{j}\varepsilon^{\frac{j}{m}} - b_0\log\varepsilon) = \lim_{\varepsilon\to 0}(\mathrm{tr}A_\varepsilon - \sum_{j=-J, j\neq 0}^{m-1} \frac{mb_j}{j}\varepsilon^{\frac{j}{m}} - b_0\log\varepsilon)$$

which combined with (1.8) and using (1.5) yields (1.6). □

The following lemma gives a class of operators which fit in the framework described above.

Lemma 1.3. *Let B be a strictly positive self adjoint elliptic operator of order* $m > 0$ *on a compact boundaryless manifold. For any* $\varepsilon > 0$, $e^{-\varepsilon B}$ *is trace class and B has a well defined μ-regularised determinant.*

Proof. We shall show that the assumptions of Lemma 1.2 are fulfilled.

Condition 1) in Lemma 1.2 follows from the fact that a strictly positive s.a elliptic operator on a compact boundaryless manifold has purely discrete spectrum $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n > 0$, $\lambda_n \simeq Cn^\alpha$, for some $C > 0$, $\alpha > 0$ (see e.g [G], Lemma 1.6.3). Indeed, from this fact easily follows that $\mathrm{tr} e^{-\varepsilon B} = \sum_n e^{-\varepsilon\lambda_n}$ is finite.

Conditions 2) of Lemma 1.1 follow from the fact that for a s.a elliptic operator B of order m on a compact manifold of dimension d without boundary, $\mathrm{tr} e^{-tB} \simeq_0 \sum_{j=-d}^K a_j t^{\frac{j}{m}}$ for any $K > 0$ (this follows for example from Lemma 1.7.4 in [G]). Applying Lemma 1.1, we can therefore define the heat-kernel regularised determinant of B. □

The above definition extends to a class of positive self-adjoint operators which satisfy requirements 1) and 2) of Lemma 1.2 and have possibly non zero kernel. Requirement 1) of the lemma implies that this kernel is finite dimensional. Let P_B be the orthogonal projection onto the kernel of the operator B acting on H and let us set $H^\perp \equiv (I - P_B)H$. Let us consider the restriction $B' \equiv B/H^\perp$. It is easily seen that the operator B' satisfies requirements of Lemma 1.2 with coefficients $b'_j = b_j$ for $j \neq 0$ and $b'_0 = b_0 - dim(KerB)$. Formula (1.6) extends to B' with adapted changes in the coefficients.

Let us at this stage see how the zeta-function regularised determinant fits into this picture. We refer the reader to [BGV, G] for a precise description of the zeta-function regularisation procedure and only describe the main lines of this procedure here.

Recall that for a strictly positive self adjoint operator B acting on a separable Hilbert space with purely discrete spectrum given by the eigenvalues $(\lambda_n, n \in \mathbb{N})$ with the property $\lambda_n \geq Cn^\alpha, C > 0, \alpha > 0$ for large enough n, we can define the zeta function of B by:

$$\zeta_B(s) \equiv \sum_n \lambda_n^{-s}, \quad s \in \mathbb{C}, \quad \mathrm{Re}s > \frac{1}{\alpha}.$$

Furthermore, $\zeta_B(s)$ admits a meromorphic continuation on the whole plane (see e.g [G], Lemma 1.10.1) which is regular at $s = 0$ and one can define the zeta function regularised determinant of A by

$$\mathrm{det}_\zeta(B) = e^{-\zeta'_B(0)}. \tag{1.9}$$

Remark. From the definition, easily follows that in the finite dimensional case the zeta-function regularised and the ordinary determinants coincide.

The following lemma compares the zeta function and μ-regularisations.

Lemma 1.4. *Let B be a strictly positive self-adjoint densely defined operator on a Hilbert space H such that*

1) B has purely discrete spectrum $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \geq Cn^\alpha$,$C > 0, \alpha > 0$ for large enough n,

2) The function $\varepsilon \to tre^{-\varepsilon B}$ lies in $\mathcal{C}_0$.

Then for $\mu \in \mathbb{R}$,

$$e^{\zeta_B(0)(\gamma-\mu)} det_\zeta B = det^\mu_{reg}(B), \tag{1.10}$$

where $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n)$ is the Euler constant. In particular, $det_\zeta B = det^\gamma_{reg} B$.

Remark. A proof of this result for $\mu = 0$ and the Laplace operator on a compact Riemannian surface without boundary can be found in [AJPS].

Proof. Before starting the proof, let us recall that the function Gamma is defined by $\Gamma(z) = \int_0^\infty \frac{e^{-t}}{t} t^z dt$ for $0 < \mathrm{Re} z$. Moreover $\Gamma(z)^{-1}$ is an entire function and we have $\Gamma(z)^{-1} = ze^{\gamma z} \prod_{n=1}^{\infty}(1 + \frac{z}{n})e^{\frac{-z}{n}}$, where γ is the Euler constant. From this follows that in a neighborhood of zero, we have the asymptotic expansion $\Gamma(s)^{-1} = s + \gamma s^2 + O(s^3)$.

Using the Mellin transform of the function $\lambda^{-s} = \Gamma(s)^{-1} \int_0^{+\infty} t^{s-1} e^{-t\lambda} dt$ we can write:

$$\Gamma(s)\zeta_B(s) = \int_0^1 t^{s-1} \mathrm{tr} e^{-tB} dt + \int_1^\infty t^{s-1} \mathrm{tr} e^{-tB} dt. \tag{1.11}$$

Notice that the last expression on the r.h.s converges for $\mathrm{Re} s \leq R$, $R > 0$ for, setting $C_R = \sup_n \sup_{t \geq 1} t^{R-1} e^{-\frac{1}{2} t\lambda_n}$, we have $\int_1^\infty t^{R-1} e^{-t\lambda_n} \leq C_R \int_1^\infty e^{-\frac{1}{2} t\lambda_n} = 2C_R \lambda_n^{-1} e^{-\frac{1}{2}\lambda_n}$, which is the general term of a convergent series.

As before we set

$$F(t) \equiv \mathrm{tr} e^{-tB} - \sum_{j=-J}^{m-1} b_j t^{\frac{j}{m}}. \tag{1.12}$$

Using (1.11) and (1.12), we can write for $s \in \mathbb{C}$ with large enough real part, $\mathrm{Re} s > \frac{J}{m}$:

$$\zeta_B(s) = \Gamma(s)^{-1} \left(\sum_{j=-J}^{m-1} \frac{b_j}{\frac{j}{m} + s} + \int_1^\infty t^{s-1} \mathrm{tr} e^{-tB} dt + \int_0^1 t^{s-1} F(t) dt \right).$$

This equality then extends to an equality of meromorphic functions on $\mathrm{Re} s > 0$ with poles $s = \frac{-j}{m}$. Using the asymptotic expansion of the inverse of the Gamma function $\Gamma(s)^{-1}$ around zero, we have

$$\zeta'_B(s) = (s + \gamma s^2 + O(s^3)) \left(\sum_{j=-J}^{m-1} \frac{b_j}{\frac{j}{m} + s} + \int_1^\infty t^{s-1} \mathrm{tr} e^{-tB} dt + \int_0^1 t^{s-1} F(t) dt \right),$$

which yields $b_0 = \zeta_B(0)$. Moreover

$$\zeta'_B(s) = (1 + 2\gamma s + O(s^2)) \left(\sum_{j=-J}^{m-1} \frac{b_j}{\frac{j}{m} + s} + \int_1^\infty t^{s-1} \mathrm{tr} e^{-tB} dt + \int_0^1 t^{s-1} F(t) dt \right)$$
$$+ (s + \gamma s^2 + O(s^3)) \left(-\sum_{j=-J}^{m-1} \frac{b_j}{(\frac{j}{m} + s)^2} + \int_0^1 t^{s-1} F(t) \ln(t) dt + \int_1^\infty \ln(t) t^{s-1} \mathrm{tr} e^{-tB} dt \right).$$

Letting s tend to zero, $s > 0$, since the divergent terms $\frac{b_0}{s}$ and $-s\frac{b_0}{s^2}$ arising in each of the terms of this last sum compensate, we get

$$\zeta'_B(0) = b_0 \gamma + \left(\sum_{j=-J, j \neq 0}^{m-1} \frac{m b_j}{j} + \int_0^1 \frac{F(t)}{t} dt + \int_1^\infty \frac{\mathrm{tr} e^{-tB}}{t} dt \right).$$

Hence, comparing with the expression of $\det^{\mu}_{reg} B$ given in (1.5), for any $\mu \in \mathbb{R}$ we find $\log \det_{\zeta}(B) = -\zeta'_B(0) = -\zeta_B(0)\gamma + \log \det {}^{\mu}_{reg}(B) + \mu\zeta_B(0)$ and hence the equality of the lemma. □

Remarks. 1) In the finite dimensional case with $\dim H = d$, since $\lim_{\varepsilon\to 0} \operatorname{tr} e^{-\varepsilon B} = d = \zeta_B(0)$, from the result of Lemma 1.4 and the fact that the zeta function regularised determinant coincides with the ordinary one, it follows that for $\mu \in \mathbb{R}$:

$$\det^{\mu}_{reg} B = e^{d(\gamma-\mu)} \det_{\zeta} B = e^{d(\gamma-\mu)} \det B, \tag{1.13}$$

where detB denotes the ordinary determinant of B. For $\mu = \gamma$, $det^{\gamma}_{reg} B = det B = det_{\zeta} B$.

2) Let M be a Riemannian manifold of dimension d and B a positive self-adjoint elliptic operator with smooth coefficients acting on sections of a vector bundle V on M with finite dimensional fibres of dimension k. We know by [G] Theorem 1.7.6 (a) that $\zeta_B(0) = 0$ if n is odd. However, in general the coefficient $\zeta_B(0)$ is a complicated expression given in terms of the jets of the symbol of the operator B. In the following we shall be concerned with the dependence of $\zeta_B(0)$ on the geometric data given on that manifold.

2. Regularisable Principal Fibre Bundles

The aim of this section is to describe a class of principal fibre bundles for which we can define a notion of regularised volume of the fibres and for which these regularised volumes have differentiability properties.

Let $\mathcal{P}$ be a Hilbert manifold equipped with a (possibly weak) right invariant Riemannian structure. The scalar product induced on $T_p\mathcal{P}$ by this Riemannian structure will be denoted by $< \cdot, \cdot >_p$. We shall assume this Riemannian structure induces a Riemannian connection denoted by ∇ and an exponential map with the usual properties. In particular, for all p_0, $\exp_{p_0}$ yields a diffeomorphism of a neighborhood of 0 in the tangent space $T_{p_0}\mathcal{P}$ onto a neighborhood of p_0 in the manifold $\mathcal{P}$.

Let **G** be a Hilbert Lie group (in fact a Hilbert manifold with smooth right multiplication is enough here, see e.g. [T]) acting smoothly on $\mathcal{P}$ on the right by an isometric action

$$\begin{aligned} \Theta : \mathbf{G} \times \mathcal{P} &\to \mathcal{P}, \\ (g, p) &\to p \cdot g. \end{aligned} \tag{2.0}$$

Let for $p \in \mathcal{P}$,

$$\begin{aligned} \tau_p : \mathcal{G} &\to T_p\mathcal{P}, \\ u &\mapsto \frac{d}{dt}(p \cdot e^{tu})_{t=0}, \end{aligned} \tag{2.0bis}$$

where $\mathcal{G}$ denotes the Lie algebra of **G**.

We shall assume that the action Θ is free (so that τ_p is injective on $\mathcal{G}$) and that it induces a smooth manifold structure on the quotient space $\mathcal{P}/\mathbf{G}$ and a smooth principal fibre bundle structure given by the canonical projection $\pi : \mathcal{P} \to \mathcal{P}/\mathbf{G}$.

Let us furthermore equip the group **G** with a smooth family of equivalent (possibly weak) Ad_g invariant Riemannian metrics indexed by $p \in \mathcal{P}$. The scalar product induced on $\mathcal{G}$ by the Riemannian metric on **G** indexed by $p \in \mathcal{P}$ will be denoted by $(\cdot, \cdot)_p$. Since the metrics are all equivalent, the closure of $\mathcal{G}$ w.r.t $(\cdot,\cdot)_p$ does not depend on p and we shall denote it by H.

Since $\mathcal{G}$ is dense in H, τ_p is a densily defined operator on H and we can define its adjoint operator τ_p^* w.r. to the scalar products $(\cdot,\cdot)_p$ and $<\cdot,\cdot>_p$.

We shall assume that $\tau_p^*\tau_p$ has a self-adjoint extension on a dense domain $D(\tau_p^*\tau_p)$ of H.

Definition. *The orbit of a point p_0 is volume preregularisable if the following assumptions 1) and 2) on the operator $\tau_p^*\tau_p$ are satisfied:*

1) *Assumption on the spectral properties of $\tau_{p_0}^*\tau_{p_0}$. The operator $e^{-\varepsilon\tau_{p_0}^*\tau_{p_0}}$ is trace class for any $\varepsilon > 0$ and for any vector X at point p_0, there is a neighborhood $\mathcal{I}_0$ of p_0 on the geodesic $p_\kappa = exp_{p_0}\kappa X$ such that for all $p \in \mathcal{I}_0$, $e^{-\varepsilon\tau_p^*\tau_p}$ is trace class.*
2) *Regularity assumptions. We shall assume that the maps $p \mapsto \tau_p$ and $p \mapsto \tau_p^*\tau_p$ are Gâteaux differentiable and that for any $t > 0$, the function $p \mapsto \mathrm{tr}e^{-t\tau_p^*\tau_p}$ is Gâteaux differentiable at point p_0. We furthermore assume that the Gâteaux-differentials at point p_0 in the direction X of these operators are related as follows:*

$$\delta_X(\mathrm{tr}e^{-\varepsilon\tau_p^*\tau_p}) = -\varepsilon\mathrm{tr}(\delta_X(\tau_p^*\tau_p)e^{-\varepsilon\tau_p^*\tau_p}). \tag{2.1}$$

Moreover, for any vector X at point p_0, there are constants $C > 0$, $u > 0$ and a neighborhood I_0 of p_0 on the geodesic $p_\kappa = exp_{p_0}\kappa X$ such that for any $p \in I_0$:

$$\mathrm{tr}e^{-t\tau_p^*\tau_p} \leq Ce^{-tu} \tag{2.2}$$

and

$$M_{I_0}(t) \equiv sup_{p\in I_0}|||\delta_{\bar{X}(p)}(\tau_p^*\tau_p)e^{-t\tau_p^*\tau_p}|||_\infty \tag{2.3}$$

is finite and a decreasing function in t.

Here $|||\cdot|||_\infty$ denotes the operator norm <u>on</u> $\mathcal{G}$ induced by $(\cdot,\cdot)_p$, $\bar{X}$ is a local vector field defined in a neighborhood of p_0 by $\bar{X}(p_\kappa) = exp_{p_\kappa *}(\kappa X)(X)$.

The orbit O_{p_0} is called volume-regularisable if dim Ker$\tau_p^*\tau_p$ is constant on some neighborhood of p_0 on any geodesic containing p_0 and if the following assumption is satisfied:

3) *Assumption on the asymptotic behavior of the heat-kernel traces. Both the functions $t \mapsto tre^{-t\tau_p^*\tau_p}$ and $t \mapsto \delta_X tre^{-t\tau_p^*\tau_p}$ lie in the class $\mathcal{C}_0$ (see Sect. 1). There is an integer $m > 0$ and a family of maps $p \mapsto b_j(p), j \in \{-J,\cdots,m-1\}$ which are Gâteaux differentiable in the direction X at point p_0 such that*

$$\mathrm{tr}e^{-\varepsilon\tau_p^*\tau_p} \simeq_0 \sum_{j=-J}^{m-1} b_j(p)\varepsilon^{\frac{j}{m}} \tag{2.4}$$

in a neighborhood $\mathcal{I}_0$ of p_0 on the geodesic $p = exp_{p_0}\kappa X$, and

$$\delta_X\mathrm{tr}e^{-\varepsilon\tau_p^*\tau_p} \simeq_0 \sum_{j=-J}^{m-1} \delta_X b_j(p)\varepsilon^{\frac{j}{m}}. \tag{2.5}$$

Furthermore, setting $F_p(t) \equiv \mathrm{tr} e^{-t\tau_p^*\tau_p} - \sum_{j=-J}^{m-1} b_j(p) t^{\frac{j}{m}}$*, for any vector* X *at point* p_0*, there is a constant* $K > 0$*, and a neighborhood* I_0 *of* p_0 *on the geodesic* $\kappa \to p_\kappa = exp_{p_0} \kappa X$ *such that:*

$$sup_{p \in I_0} \|\delta_{\bar{X}(p)} F_p(t)\|_\infty \leq Kt. \tag{2.5bis}$$

A principal bundle as described above with all its orbits volume-preregularisable (resp. volume- regularisable) will be called preregularisable (resp. regularisable).

Remark. Since the Riemannian structure on $\mathcal{P}$ is right invariant and the one on **G** is Ad_g invariant, the above assumptions do not depend on the point chosen in the orbit for we have $\tau_{p \cdot g} = R_{g_*} \tau_p \mathrm{Ad}_g$.

Most fibre bundles we shall come across are not only preregularisable but also regularisable so that the notion of preregularisabiblity might seem somewhat artificial. However, in the case of the coadjoint action of loop groups mentioned in the introduction, it is sufficient to verify the conditions required for preregularisability in order to prove a certain minimality of the orbits, namely strong minimality, a notion which will be defined in the following and which implies minimality.

Natural examples of regularisable fibre bundles arise in gauge field theories (Yang-Mills, string theory). In gauge field theories, $\mathcal{P}$ and $\mathcal{G}$ are modelled on spaces of sections of vector bundles $\mathcal{E}$ and $\mathcal{F}$ based on a compact finite dimensional manifold M and the operators $\tau_p^*\tau_p$ arise as smooth families of Laplace operators on forms. As elliptic operators on a compact boundaryless manifold, they have purely discrete spectrum which satisfies condition 1) (see [G] Lemma 1.6.3) and (2.4) (see [G], Lemma 1.7.4.b)). By classical results concerning one parameter families of heat-kernel operators, they satisfy (2.1) (see [RS], Proposition 6.1) and (2.2) (see proof of Theorem 5.1 in [RS]). Since $\delta_X B_p$ is also a partial differential operator, by [G], Lemma 1.7.7, $\delta_X \mathrm{tr} e^{-\varepsilon B_p}$ satisfies (2.5). Assumptions on the Gâteaux-differentiability and assumptions (2.3), (2.5 bis) are fulfilled in applications. Indeed, the parameter p is a geometric object such as a connection, a metric on M and choosing these objects regular enough (of class H^k for k large enough) ensures that the maps $p \mapsto \tau_p$, $p \mapsto \tau_p^*\tau_p$, $p \mapsto \mathrm{tr} e^{-t\tau_p^*\tau_p}$, etc., are regular enough for they involve these geometric quantities and their derivatives, but no derivative of higher order.

Remark. In the context of gauge field theories, the underlying Riemannian structure w.r.to which the traces (arising in (2.2)-(2.5 bis)) are taken are weak L^2 Riemannian structures, the ones that also underlie the theory of elliptic operators on compact manifolds. In [AP2], we discuss how far this weak Riemannian structure could be replaced by a strong Riemannian structure, in order to set up a link between this geometric picture and a stochastic one developed in [AP2].

Notation. *We shall set with the notations of Sect. 1, for* $\varepsilon > 0$ *and* $p \in \mathcal{P}$ $det_\varepsilon(B_p) = exptr(h_\varepsilon(B_p))$.

Proposition 2.1. *Let* O_{p_0} *be a volume-preregularisable orbit such that for any geodesic containing* p_0*, there is a neighborhood of* p_0 *on this geodesic on which* $\tau_p^*\tau_p$ *is injective.*

1) $det_\varepsilon(\tau_p^*\tau_p)$ *is well defined for any* $\varepsilon > 0$ *and for* p *in a neighborhood of* p_0 *on any geodesic containing* p_0.

2) *The map $p \mapsto det_\varepsilon(\tau_p^*\tau_p)$ is Gâteaux-differentiable at point p_0, the operator $\int_\varepsilon^{+\infty} \delta_X(\tau_p^*\tau_p)e^{-t\tau_p^*\tau_p}dt$ is trace class for any p in a neighborhood of p_0 on any geodesic of p_0. For any tangent vector X at point p_0, we have:*

$$\begin{aligned}\delta_X \textit{ log det}_\varepsilon(\tau_p^*\tau_p) &= \int_\varepsilon^\infty tr\ (\delta_X\tau_p^*\tau_p)e^{-t\tau_{p_0}^*\tau_{p_0}}dt \\ &= \mathrm{tr}\int_\varepsilon^{+\infty}(\delta_X\tau_p^*\tau_p)e^{-t\tau_{p_0}^*\tau_{p_0}}dt.\end{aligned} \tag{2.6a}$$

3) *If the orbit O_{p_0} is moreover volume-regularisable, for any $\mu \in \mathbb{R}$, the map $p \mapsto det^\mu_{reg}(\tau_p^*\tau_p)$ is Gâteaux differentiable in all directions at point p_0, and for p in a geodesic neighborhood of p_0, the map $\varepsilon \mapsto \delta_X$ log det $_\varepsilon(\tau_p^*\tau_p)$ lies in the class $\mathcal{C}$. For $\mu \in \mathbb{R}$,*

$$\begin{aligned}&Lim^\mu_{\varepsilon\to 0}\delta_X \textit{ log det}_\varepsilon(\tau_p^*\tau_p) = \delta_X \textit{ log det}^\mu_{reg}\tau_p^*\tau_p \\ &= -\sum_{j=-J, j\neq 0}^{m-1}\frac{m}{j}\delta_X b_j - \int_1^\infty \delta_X\left(\mathrm{tr}\frac{e^{-t\tau_p^*\tau_p}}{t}\right)dt - \int_0^1 \frac{\delta_X F_p(t)}{t}dt - \mu\delta_X b_0.\end{aligned} \tag{2.6b}$$

Proof. We set $B_p = \tau_p^*\tau_p$ and as before, $det_\varepsilon(B_p) = exptrh_\varepsilon(B_p)$.

1) By the first assumption for volume-preregularisable orbits, we know that $e^{-\varepsilon B_p}$ is trace class so that by Lemma 1.1 so is $A^p_\varepsilon \equiv \log h_\varepsilon(B_p)$. Hence $\det_\varepsilon(B_p) = e^{\mathrm{tr}A^p_\varepsilon}$ is well defined.
2) Let us show the first equality in (2.6 a). Assumption 2) for volume-preregularisability yields that for any $p \in I_0$ and any $t > \varepsilon > 0$ $|\mathrm{tr}(\delta_{\bar{X}(p)}B_p e^{-tB_p})| \leq CM_{I_0}(\frac{t}{2})e^{-\frac{t}{2}u}$. Here, we have used the fact that $|\mathrm{tr}(UV)| \leq |||U|||\mathrm{tr}V|$ for any bounded operator U and any trace class operator V applied to $U = \delta_{\bar{X}(p)}B_p e^{-\frac{t}{2}B_{p_0}}$ and $V = e^{-\frac{t}{2}B_p}$. Hence, by the Lebesgue dominated convergence theorem, the map $p \mapsto \int_\varepsilon^\infty t^{-1}\mathrm{tr}e^{-tB_p}dt$ is Gâteaux-differentiable in the direction X at point p_0 and

$$\begin{aligned}\delta_X\int_\varepsilon^\infty t^{-1}\mathrm{tr}e^{-tB_p}dt &= \int_\varepsilon^\infty t^{-1}\delta_X\mathrm{tr}e^{-tB_p}dt \\ &= -\int_\varepsilon^\infty \mathrm{tr}((\delta_X B_p)e^{-tB_{p_0}})dt,\end{aligned}$$

using (2.2). Using the fact that $\log\det_\varepsilon(B_p) = -\int_\varepsilon^{+\infty}t^{-1}\mathrm{tr}e^{-tB_p}dt$ then yields the first equality in (2.6 a).

The second equality in (2.6 a) and the fact that we can swap the trace and the integral follow from the estimate:

$$\begin{aligned}|||\delta_X B_p e^{-tB_{p_0}}|||_1 &\leq |||\delta_X B_p e^{\frac{-\varepsilon}{2}B_{p_0}}|||_\infty|||e^{-\frac{1}{2}tB_{p_0}}|||_1 \\ &\leq C|||\delta_X B_p e^{\frac{-\varepsilon}{2}B_{p_0}}|||_\infty e^{-tu},\end{aligned} \tag{$*$}$$

valid for $t \geq \varepsilon$, using Assumption (2.2). We finally obtain by dominated convergence: $\mathrm{tr}\int_\varepsilon^{+\infty}\delta_X B_p e^{-tB_{p_0}}dt = \int_\varepsilon^{+\infty}\mathrm{tr}\delta_X B_p e^{-tB_{p_0}}dt$.

3) Let us first check that the map $p \mapsto \det^{\mu}_{reg} B_p$ is Gâteaux differentiable at point p_0 in the direction X. By (1.5), we have

$$\log \det{}_{reg} B_p = -\sum_{j=-J, j\neq 0}^{m-1} \frac{b_j(p)}{j} - \int_1^\infty \mathrm{tr} \frac{e^{-tB_p}}{t} dt - \int_0^1 \frac{F_p(t)}{t} dt.$$

The first term on the r.h.s. is Gâteaux differentiable in the direction X by the assumption on the maps $p \mapsto b_j(p)$. The second term on the r.h.s. is Gâteaux differentiable by the result (applied to $\varepsilon = 1$) of part 2 of this proposition which tells us that $p \mapsto \det_\varepsilon(B_p)$ is Gâteaux differentiable. The Gâteaux differentiability of the last term follows from the local uniform upper bound (2.5 bis).

We now check (2.6 b). The map $p \mapsto \log \det_\varepsilon(B_p) - \sum_{j=-J, j\neq 0}^{m-1} \frac{m b_j}{j} \varepsilon^{\frac{j}{m}} - b_0 \log \varepsilon$ is Gâteaux differentiable in the direction X and we can write

$$\begin{aligned}
&\delta_X \left(\log \det_\varepsilon(B_p) - \sum_{j=-J, j\neq 0}^{m-1} \frac{m b_j \varepsilon^{\frac{j}{m}}}{j} - b_0 \log \varepsilon \right) \\
&= \delta_X \left(-\int_\varepsilon^\infty \mathrm{tr} \frac{e^{-tB_p}}{t} dt - \sum_{j=-J, j\neq 0}^{m-1} \frac{m b_j \varepsilon^{\frac{j}{m}}}{j} - b_0 \log \varepsilon \right) \\
&= \delta_X \left(-\sum_{j=-J, j\neq 0}^{m-1} m \frac{b_j}{j} - \int_1^\infty \mathrm{tr} \frac{e^{-tB_p}}{t} dt - \int_\varepsilon^1 \frac{F_p(t)}{t} dt \right) \quad \text{as in (1.8)} \\
&= -\sum_{j=-J, j\neq 0}^{m-1} \delta_X b_j \frac{m}{j} - \int_1^\infty \delta_X \, \mathrm{tr} \frac{e^{-tB_p}}{t} dt - \int_\varepsilon^1 \delta_X \frac{F_p(t)}{t} dt,
\end{aligned}$$

which tends to $\delta_X \log \det_{reg} B_p$ by (1.6) and dominated convergence. Here we have used the results of point 2) of the proposition applied to $\varepsilon = 1$ to write $\delta_X \displaystyle\int_1^\infty \mathrm{tr} \frac{e^{-tB_p}}{t} dt = \int_1^\infty \delta_X \, \mathrm{tr} e^{-tB_p} dt$ and (2.5 bis) to write $\delta_X \int_\varepsilon^1 \frac{F_p(t)}{t} dt = \int_\varepsilon^1 \frac{\delta_X F_p(t)}{t} dt$. $\square$

Remark. These results extend to the case when instead of assuming that $\tau_p^* \tau_p$ is injective locally around p_0, one considers orbits of an action at points p_0 for which the dimension of the kernel of τ_p is constant on some neighborhood of p_0 on each geodesic starting at point p_0. For this, one should replace $\det_\varepsilon \tau_p^* \tau_p$ and $\det_{reg} \tau_p^* \tau_p$ by $\det'_\varepsilon \tau_p^* \tau_p$ and $\det'_{reg} \tau_p^* \tau_p$. This extension is useful for the applications mentioned in the introduction.

A naive generalisation of the finite dimensional notion of volume to volume of infinite dimensional orbits would give infinite quantities. But for volume-preregularisable or regularisable orbits, one can define a notion of preregularised or μ-regularised volume ($\mu \in \mathbb{R}$), which justifies a posteriori the term "volume-preregularisable or volume-regularisable orbits" for these orbits. Since $\tau_{p\cdot g} = R_{g*} \tau_p \mathrm{Ad}_g$ and since the metric on $\mathbf{G}$ is Ad_g and that on $\mathcal{P}$ right invariant, for any $\varepsilon > 0$, we have $\det_\varepsilon(\tau_{p\cdot g}^* \tau_{p\cdot g}) = \det_\varepsilon(\tau_p^* \tau_p)$ so that it makes sense to set the following definitions:

Definition. *1) Let O_p be a volume-preregularisable orbit, then* $\mathrm{vol}_\varepsilon(O_p) \equiv \sqrt{\det_\varepsilon(\tau_p^*\tau_p)'}$ *defines a one parameter family of preregularised volumes of O_p.*

2) Let O_p be a volume-regularisable orbit, then for $\mu \in \mathbb{R}$, $\mathrm{vol}^\mu_{reg}(O_p) = \sqrt{\det^\mu_{reg}(\tau_p^*\tau_p)'}$ *defines the μ-regularised volume of O_p.*

3) Let O_p be a volume-regularisable orbit, then $\mathrm{vol}_\zeta(O_p) = \sqrt{\det_\zeta(\tau_p^*\tau_p)'}$ *defines the zeta function regularised volume of O_p.*

From Lemma 1.4 it follows that

$$\mathrm{vol}_\zeta(O_p) = e^{\frac{1}{2}(-\gamma+\mu)b_0'(p)}\,\mathrm{vol}^\mu_{reg}(O_p), \tag{2.7}$$

where γ is the Euler constant and $b_0'(p) = \zeta_{\tau_p^*\tau_p}(0) - \dim \mathrm{Ker}(\tau_p^*\tau_p)$ is the coefficient arising from the heat-kernel asymptotic expansion of $\tau_p^*\tau_p$ given by (2.4). In finite dimensions, when $dimH = d$ and **G** is a compact Lie group equipped with the Haar measure $dvol$, this yields $vol^\mu_{reg}(O_p) = e^{(\mu-\gamma)d}|det\tau_p|\int_G |detAd_g dvol(g)|$.

As a consequence of Proposition 2.1:

Proposition 2.2. *For any $\mu \in \mathbb{R}$, the μ-(resp. pre)-regularised volume of a volume-(pre)regularisable orbit O_p is Gâteaux-differentiable at the point p.*

Let us now introduce a notion of extremality of orbits which generalises the corresponding finite dimensional notion [H].

Definition. *A* **strongly extremal orbit** *is a volume-preregularisable orbit, the preregularised volume of which is extremal, i.e. O_p is strongly extremal if* $\delta_X\,\mathrm{vol}_\varepsilon(O_p) = 0$ *for any horizontal vector X at point p and any $\varepsilon > 0$.*

For a given $\mu \in \mathbb{R}$, a μ- extremal orbit of a preregularisable bundle is an orbit, the μ-regularised volume of which is extremal, i.e. $\delta_X\,\mathrm{vol}^\mu_{reg}(O_p) = 0$ *for any horizontal vector X at point p.*

Notice that whenever $\zeta_{\tau_p^*\tau_p}(0) - dim(Ker(\tau_p^*\tau_p))$ does not depend on p, the extremality of the volume of an orbit does not depend on the parameter μ. From (2.7) it also follows that this notion generalises the finite dimensional notion of extremality of the volume of the fibre.

3. Minimal Orbits as Orbits with Extremal Volume

We shall consider a preregularisable principal fibre bundle $\mathcal{P} \to \mathcal{P}/\mathbf{G}$. By assumption, the bundle is equipped with a Riemannian connection given by a family of horizontal spaces $H_p, p \in \mathcal{P}$ such that

$$T_p\mathcal{P} = H_p \oplus V_p,$$

where V_p is the tangent space to the orbit at point p and the sum is an orthogonal one.

For a horizontal vector X at point p, we define the shape operator

$$\begin{aligned}\mathcal{H}_X : V_p &\to V_p\\ Y &\mapsto -(\nabla_Y \bar{X})^v(p),\end{aligned}$$

where the subscript v denotes the orthogonal projection onto V_p and $\bar{X}$ is a horizontal vector field with value X at p. Similarly, we define the second fundamental form:

$$S^p : V_p \times V_p \to H_p$$
$$(Y, Y') \mapsto (\nabla_{\bar{Y}} \bar{Y}')^h(p),$$

where $\bar{Y}$, $\bar{Y}'$ are vertical vector fields such that $\bar{Y}(p) = Y$, $\bar{Y}'(p) = Y'$. These definitions are independent of the choice of the extensions of X,Y and Y'.

An easy computation shows that the shape operator and the second fundamental form are related as follows:

$$< \mathcal{H}_X(Y), Y' >_p = < S^p(Y, Y'), X >_p \ . \tag{3.1}$$

Note that this explicitly shows that $\mathcal{H}_X$ only depends on X and not on the extension $\bar{X}$ of X. Since S^p is symmetric, so is $\mathcal{H}_X$.

As in the finite dimensional case, one can define the notion of totally geodesic orbit, an orbit O_p being totally geodesic whenever the second fundamental form S^p vanishes.

Definition. *The orbit O_p of a point $p \in \mathcal{P}$ will be called* **preregularisable** *if for any horizontal vector X at p, $\forall \varepsilon > 0$,*

$$\mathcal{H}_X^\varepsilon \equiv e^{-\frac{1}{2}\varepsilon\tau_p\tau_p^*} \mathcal{H}_X e^{-\frac{1}{2}\varepsilon\tau_p\tau_p^*} \tag{3.2}$$

is trace class. A preregularisable orbit O_p will be called **strongly minimal** *if moreover for any $q \in O_p$ and X a horizontal vector at point q, tr$\mathcal{H}_X^\varepsilon = 0$ $\forall \varepsilon > 0$.*

Remarks. 1) The preregularisability of the orbits (namely $\mathcal{H}_X^\varepsilon$ trace class) is automatically satisfied if the manifold $\mathcal{P}$ is equipped with a strong smooth Riemannian structure, since in that case the second fundamental form is a bounded bilinear form and its weighted trace is well defined (see also [AP2] where this is discussed in further detail).

2) Since on a preregularisable bundle, the Riemannian structure on $\mathcal{P}$ is right invariant and the one on **G** is Ad_g invariant, the notion of (pre)regularisability and (strong) minimality of the orbit does not depend on the point chosen on the orbit.

3) Notice that if $\mathcal{H}_X$ is trace class, as in the finite dimensional case, strong minimality implies that tr$\mathcal{H}_X = 0$ and hence ordinary minimality. The fact that strong minimality implies minimality in the finite dimensional case motivates the choice of the adjective “strong”.

4) This preregularised shape operator $\mathcal{H}_X^\varepsilon$ and the second fundamental form are related as follows:

$$< \mathcal{H}_X^\varepsilon(Y), Y' >_p = < S^p(e^{-\frac{1}{2}\varepsilon\tau_p\tau_p^*}Y, e^{-\frac{1}{2}\varepsilon\tau_p\tau_p^*}Y'), X >_p$$

Since $\tau_p\tau_p^*$ is an isomorphism of the tangent space to the fibre T_pO_p, $\mathcal{H}_X^\varepsilon$ vanishes whenever the second fundamental form vanishes and an orbit is totally geodesic whenever this regularised shape operator vanishes on the orbit for some $\varepsilon > 0$.

Definition. *A preregularisable orbit O_p will be said to be regularisable if furthermore, the one parameter family $\mathcal{H}_X^\varepsilon, \varepsilon \in]0, 1]$ admits a regularised limit-trace (as defined in Sect. 1). For $\mu \in \mathbb{R}$, we denote by* tr$_{reg}^\mu \mathcal{H}_X$ *its μ-regularised limite trace.*

Definition. *For a given $\mu \in \mathbb{R}$, a regularisable orbit O_p will be called $\mu-$ minimal if* $\mathrm{tr}^{\mu}_{reg}\mathcal{H}_X = 0$ *for any horizontal vector X at point p.*

Remarks. 1) As we shall see later on, for different values of μ, the notions of μ-minimality do not coincide in general.

2) In the finite dimensional case, the one parameter family $\mathcal{H}^{\varepsilon}_X$ admits a regularised limit trace given by the ordinary trace $\mathrm{tr}^{\mu}_{reg}\mathcal{H}_X = \mathrm{tr}\mathcal{H}_X$ and μ- minimality is equivalent to the finite dimensional notion of minimality.

3) A strongly minimal preregularisable orbit O_p is μ-regularisable and μ-minimal for any $\mu \in \mathbb{R}$.

The notion of minimality of orbits for group actions in the infinite dimensional case has been discussed in the literature before. King and Terng in [KT] introduced a notion of regularisability and minimality for submanifolds of path spaces using zeta-function regularisation methods. They show zeta function regularisability and minimality for the orbits of the coadjoint action of a (based) loop group on a space of loops in the corresponding Lie algebra. One can check that these orbits are also regularisable and strongly minimal (hence minimal) within our framework .

A notion of zeta function regularisability and minimality was discussed by Maeda, Rosenberg and Tondeur in [MRT1] (see also [MRT2]) in the case of orbits of the gauge action in Yang-Mills theory. In fact, it can be seen as a particular example of μ-minimality for $\mu = \gamma$, the Euler constant.

Let us introduce some notations. Let $\mathcal{P} \to \mathcal{P}/\mathbf{G}$ be a preregularisable principal fibre bundle and let $(T^p_n)_{n\in\mathbb{N}}$ be a set of eigenvectors of $\tau_p^*\tau_p$ in $\mathcal{G}$ corresponding to the eigenvalues $(\lambda^p_n)_{n\in\mathbb{N}}$ counted with multiplicity and in increasing order. Let p_0 be a fixed point in $\mathcal{P}$ and let $\mathcal{I}^p_{p_0}$ be the isometry from $(\mathcal{G}, (\cdot,\cdot)_{p_0})$ into $(\mathcal{G}, (\cdot,\cdot)_p)$ which takes the orthonormal set $(T^{p_0}_n)_n$ of eigenvectors of $\tau^*_{p_0}\tau_{p_0}$ to the orthonormal set of eigenvectors $(T^p_n)_n$ of $\tau_p^*\tau_p$. Notice that $\mathcal{I}^{p_0}_{p_0} = I$.

Lemma 3.1. *Let $\mathcal{P} \to \mathcal{P}/\mathcal{G}$ be a preregularisable principal fibre bundle. Let $p_0 \in \mathcal{P}$ be a point at which the map $p \mapsto \mathcal{I}^p_{p_0}u$ is Gâteaux-differentiable for any $u \in \mathcal{G}$. Let X be a horizontal vector at p_0. We shall consider eigenvalues λ^p_n that correspond to eigenvectors that do not belong to $\mathcal{I}^p_{p_0}Ker\tau^*_{p_0}\tau_{p_o}$.*

1) The maps $p \to \lambda^p_n$ are Gâteaux-differentiable in the direction X at point p_0, $\delta_X\lambda^p_n = (\delta_X(\tau_p^*\tau_p)T^{p_0}_n, T^{p_0}_n)_{p_0}$ and $\delta_X \log h_\varepsilon(\lambda^p_n) = \int_\varepsilon^{+\infty} (\delta_X(\tau_p^*\tau_p)e^{-t\tau^*_{p_0}\tau_{p_0}}T^{p_0}_n, T^{p_0}_n)_{p_0}dt.$

2) Furthermore,we have

$$- < \mathcal{H}^{\varepsilon}_X\tilde{U}^p_n, \tilde{U}^p_n >_{p_0} + e^{-\varepsilon\lambda^{p_0}_n}(\delta_X\mathcal{I}^p_{p_0}T^{p_0}_n, T^{p_0}_n)_{p_0} = \frac{1}{2}\delta_X \log h_\varepsilon(\lambda^p_n), \tag{3.5}$$

where we have set $\tilde{U}^p_n = \|\tau_p T^p_n\|^{-1}\tau_p T^p_n$.

3) If the Riemannian structure on $\mathcal{G}$ is fixed (independent of p), then $\delta_X\mathcal{I}^p_{p_0}$ is antisymmetric and

$$\begin{aligned}\frac{1}{2}\int_\varepsilon^{+\infty}(\delta_X(\tau_p^*\tau_p)e^{-t\tau_p^*\tau_p}T^p_n, T^p_n)_{p_0}dt &= - < \mathcal{H}^{\varepsilon}_X\tilde{U}^p_n, \tilde{U}^p_n >_{p_0} \\ = \frac{1}{2}\delta_X \log h_\varepsilon(\lambda^p_n) &= \frac{1}{2}{\lambda^{p_0}_n}^{-1}\delta_X\lambda^p_n e^{-\varepsilon\lambda^{p_0}_n}.\end{aligned} \tag{3.6}$$

Proof. As before, we shall set $B_p = \tau_p^* \tau_p$. Since p_0 is fixed, we drop the index p_0 in $\mathcal{I}_{p_0}^p$ and denote this isometry by $\mathcal{I}^p$. Notice that $\mathcal{I}^{p_0} = I$. As before, we denote by $(T_n^p)_{n\in\mathbb{N}}$ the orthonormal set of eigenvectors of $\tau_p^* \tau_p$ which correspond to the eigenvalues $(\lambda_n^p)_{n\in\mathbb{N}}$ in increasing order and counted with multiplicity. We shall set $\tilde{T}_n^p = \tau_p T_n^p$, $\bar{T}_n^p = \tau_p T_n^{p_0}$.

1) Using the relations $(\mathcal{I}^p \cdot, \mathcal{I}^p \cdot)_p = (\cdot,\cdot)_{p_0}, \mathcal{I}^p(T_n^{p_0}) = T_n^p, \mathcal{I}^{p*}\mathcal{I}^p = I$, we can write $\lambda_n^p = (B_p T_n^p, T_n^p)_p = (B_p \mathcal{I}^p T_n^{p_0}, \mathcal{I}^p T_n^{p_0})_{p_0}$ and the map $p \mapsto \lambda_n^p$ is Gâteaux differentiable in all directions at point p_0 since $p \mapsto B_p$, $p \mapsto \mathcal{I}^p$ are Gâteaux-differentiable by assumption on the bundle. Furthermore

$$\begin{aligned} \delta_X (B_p T_n^p, T_n^p)_p &= \delta_X (\mathcal{I}^{p*} B_p \mathcal{I}^p T_n^{p_0}, T_n^{p_0})_{p_0} \\ &= ((\delta_X B_p) T_n^{p_0}, T_n^{p_0})_{p_0} + (\delta_X (\mathcal{I}^{p*}) B_{p_0} T_n^{p_0}, T_n^{p_0})_{p_0} + \\ &+ (\mathcal{I}^{p_0 *} B_{p_0} (\delta_X \mathcal{I}^p) T_n^{p_0}, T_n^{p_0})_{p_0} \\ &= ((\delta_X B_p) T_n^{p_0}, T_n^{p_0})_p + \lambda_n^{p_0} ([\mathcal{I}^{p_0 *} \delta_X (\mathcal{I}^p) + (\delta_X \mathcal{I}^{p*}) \mathcal{I}^{p_0}] T_n^{p_0}, T_n^{p_0})_{p_0}. \end{aligned}$$

Since $\mathcal{I}^{p*}\mathcal{I}^p = I$, we have $\delta_X \mathcal{I}^{p*} \mathcal{I}^{p_0} + \mathcal{I}^{p_0 *} \delta_X \mathcal{I}^p = 0$ so that finally λ_n^p is Gâteaux-differentiable and $\delta_X \lambda_n^p = ((\delta_X B_p) T_n^{p_0}, T_n^{p_0})_{p_0}$.
Using the local uniform estimate (2.3), and with the same notations, we have for $t > \varepsilon$ $\|(\delta_{\bar{X}(p)}(B_p) e^{-tB_{p_0}} T_n^{p_0}, T_n^{p_0})_{p_0}\| \leq M_{I_0}(\frac{1}{2}t) e^{-\frac{1}{2}t\lambda_n^{p_0}}$ so that the map $p \mapsto \log h_\varepsilon(\lambda_n^p)$ is Gâteaux-differentiable at point p_0 in the direction X and

$$\begin{aligned} \delta_X \log h_\varepsilon(\lambda_n^p) &= -\delta_X \int_\varepsilon^\infty t^{-1} (e^{-tB_p} T_n^p, T_n^p) dt \\ &= (\int_\varepsilon^{+\infty} \delta_X(B_p) e^{-tB_{p_0}} T_n^{p_0}, T_n^{p_0})_{p_0} dt. \end{aligned}$$

2) By definition of h_ε we have:

$$\begin{aligned} \delta_X \log h_\varepsilon(\lambda_n^p) &= (\log h_\varepsilon)'(\lambda_n^p) \delta_X \lambda_n^p \\ &= (\lambda_n^{p_0})^{-1} e^{-\varepsilon \lambda_n^{p_0}} \delta_X \lambda_n^p. \end{aligned}$$

On the other hand

$$\begin{aligned} \delta_X \lambda_n^p &= \delta_X < \tilde{T}_n^p, \tilde{T}_n^p >_p = 2 < \delta_X (\tau_p \mathcal{I}^p) T_n^{p_0}, \bar{T}_n^{p_0} >_{p_0} \\ &= 2 < \delta_X \bar{T}_n^p, \bar{T}_n^{p_0} >_{p_0} + 2 < \tau_p \delta_X \mathcal{I}^p T_n^{p_0}, \bar{T}_n^{p_0} >_{p_0} \\ &= -2 < \nabla_{\bar{T}_n^{p_0}} \bar{X}, \bar{T}_n^{p_0} >_{p_0} + 2 < \tau_p \delta_X \mathcal{I}^p T_n^{p_0}, \bar{T}_n^{p_0} >_{p_0} \\ &= -2 < \nabla_{\tilde{T}_n^{p_0}} \bar{X}, \tilde{T}_n^{p_0} >_{p_0} + 2 < \tau_p \delta_X \mathcal{I}^p T_n^{p_0}, \bar{T}_n^{p_0} >_{p_0} \\ &= -2\lambda_n^{p_0} < \mathcal{H}_X \tilde{U}_n^{p_0}, \tilde{U}_n^{p_0} >_{p_0} + 2\lambda_n^{p_0} (\delta_X \mathcal{I}^p T_n^{p_0}, T_n^{p_0})_{p_0}, \end{aligned}$$

where for the third equality, we have used the fact that, $\bar{X}$ being right invariant, $[\bar{T}_n^p, \bar{X}] = 0$. Hence $\delta_X \log h_\varepsilon(\lambda_n^p) = -2e^{-\varepsilon\lambda_n^{p_0}} < \mathcal{H}_{\bar{X}(p_0)} \bar{U}_n^{p_0}, \bar{U}_n^{p_0} >_{p_0}$ $+2e^{-\varepsilon\lambda_n^{p_0}} (\delta_X \mathcal{I}^p T_n^{p_0}, T_n^{p_0})_{p_0}$, which yields 2).

3) On one hand, since the scalar product on the Lie algebra is fixed, we have $\delta_X \mathcal{I}^{p*} \subset (\delta_X \mathcal{I}^p)^*$. On the other hand, since $\mathcal{I}^{p*}\mathcal{I}^p = I$, we have $-\delta_X \mathcal{I}^p \subset \delta_X \mathcal{I}^{p*}$ so that the second term in the l.h.s of (3.5) vanishes. □

Definition. *We shall call an orbit O_{p_0} of a preregularised bundle an* **orbit of type** $(\mathcal{T})$ *whenever the following conditions are satisfied:*

1) *The map $p \mapsto \mathcal{I}^p_{p_0}$ is Gâteaux-differentiable at point p_0.*
2) *The operator $\delta_X \mathcal{I}^p_{p_0} e^{-\varepsilon\tau^*_{p_0}\tau_{p_0}}$ is trace class for any $p_0 \in \mathcal{P}$ and $\varepsilon > 0$.*
3) *For any $p \in \mathcal{P}$, $\mathrm{tr}\left(\mathcal{I}^p_{p_0} e^{-\varepsilon\tau^*_{p_0}\tau_{p_0}}\right)$ is Gâteaux-differentiable at point $p_0 \in \mathcal{P}$ and $\delta_X \mathrm{tr}(\mathcal{I}^p_{p_0} e^{-\varepsilon\tau^*_{p_0}\tau_{p_0}}) = \mathrm{tr}(\delta_X \mathcal{I}^p_{p_0} e^{-\varepsilon\tau^*_{p_0}\tau_{p_0}})$.*

Whenever the Riemannian structure on $\mathcal{G}$ is independent of p, any orbit satisfying condition 1) is of type $(\mathcal{T})$, for in that case the traces involved in 2) and 3) vanish, $\delta_X \mathcal{I}^p_{p_0}$ being an antisymmetric operator.

Proposition 3.2. *Let $\mathcal{P} \to \mathcal{P}/\mathbf{G}$ be a preregularisable principal fibre bundle. Then*

1) *Any orbit of type $(\mathcal{T})$ is preregularisable. More precisely, if O_{p_0} is an orbit of type $(\mathcal{T})$, for any horizontal vector X at point p_0, the operator $\mathcal{H}^\varepsilon_X$ is trace class, the map $p \mapsto \mathrm{vol}_\varepsilon(O_p)$ is Gâteaux differentiable in the direction X at point p_0 and*

$$\begin{aligned} \mathrm{tr}\mathcal{H}^\varepsilon_X - \delta_X \mathrm{tr}(\mathcal{I}^p_{p_0} e^{-\varepsilon\tau^*_{p_0}\tau_{p_0}}) &= -\delta_X \ \log vol'_\varepsilon(O_p) \\ &= -\frac{1}{2}\int_\varepsilon^{+\infty} \mathrm{tr}'[\delta_X(\tau^*_p\tau_p)e^{-t\tau^*_{p_0}\tau_{p_0}}]dt. \end{aligned} \tag{3.7}$$

2) *If the Riemannian structure on $\mathcal{G}$ is independent of p, the orbit of any point p_0 is a preregularisable orbit and*

$$\mathrm{tr}\mathcal{H}^\varepsilon_X = -\delta_X \ \log vol'_\varepsilon(O_p) = -\frac{1}{2}\int_\varepsilon^{+\infty} \mathrm{tr}'[\delta_X(\tau^*_p\tau_p)e^{-t\tau^*_{p_0}\tau_{p_0}}]dt, \tag{3.7bis}$$

where tr' means we have restricted to the orthogonal of the kernel of $\tau^*_{p_0}\tau_{p_0}$ and vol'_ε means that we only consider eigenvalues λ^p_n that correspond to eigenvectors that do not belong to $\mathcal{I}^p_{p_0} Ker\tau^*_{p_0}\tau_{p_o}$.

Remarks. 1) In finite dimensions, for a compact connected Lie group acting via isometries on a Riemannian manifold $\mathcal{P}$ of dimension d, we have for any $\varepsilon > 0$ and using the various definitions of the volumes, including the μ-volume, $\mu \in \mathbb{R}$:

$$\begin{aligned} \lim_{\varepsilon\to 0} \delta_X \log \mathrm{vol}_\varepsilon(O_p) &= \delta_X \ \log \mathrm{vol}^\mu_{reg} O_p \\ &= \delta_X \ \log \mathrm{vol} O_p. \end{aligned}$$

Hence going to the limit $\varepsilon \to 0$ on either side of (3.7 bis) we find:

$$\mathrm{tr}\mathcal{H}_X = -\delta_X \ \log \mathrm{vol} O_p.$$

If the Gâteaux-differentiability involved is a C^1- Gâteaux-differentiability, this yields

$$\mathrm{tr} S^p = -\mathrm{grad} \log \mathrm{vol} O_p$$

This leads to a well known result, namely (Hsiang's theorem [H]) that the orbits of G whose volume are extremal among nearby orbits is a minimal submanifold of M.

2) Equality (3.7) tells us that whenever the Riemannian structure on **G** is independent of p (as in the case of Yang-Mills theory), strongly minimal orbits of a preregularisable principal fibre bundle are pre-extremal orbits. This gives a weak (in the sense that we only get a sufficient condition for strong minimality and not for minimality) infinite dimensional version of Hsiang's [H] theorem.
3) If both the spectrum of $\tau_p^*\tau_p$ and the Riemannian structure on **G** are independent of p, as in the case of Yang-Mills theory in the abelian case (where the spectrum only depends on a fixed Riemannian structure on the manifold M), the orbits are strongly minimal (see also [MRT 1] par.5).

Proof of Proposition 3.2. We set $B_p = \tau_p^*\tau_p$. For the sake of simplicity, we assume that B_p is injective on its domain, the general case then easily follows.

1) From the preregularisability of the principal bundle follows (see Proposition 2.1) that the map $p \mapsto \det_\varepsilon(B_p)$ is Gâteaux-differentiable in the direction X at point p_0 and $\delta_X \log\det{}_\varepsilon(B_p) = \int_\varepsilon^{+\infty} dt \operatorname{tr}(\delta_X B_p e^{-tB_p})$. On the other hand, by Lemma 3.1

$$\begin{aligned}&\frac{1}{2}\langle \int_\varepsilon^{+\infty} dt(\delta_X B_p e^{-tB_p})T_n^p, T_n^p\rangle_p - e^{-\varepsilon\lambda_n^{p_0}}(\delta_X \mathcal{I}^p T_n^{p_0}, T_n^{p_0})_{p_0}\\ &= -\langle \mathcal{H}_X^\varepsilon \tilde{U}_n^p, \tilde{U}_n^p\rangle_p.\end{aligned} \qquad (*)$$

The fibre bundle being preregularisable, by the results of Proposition 2.0, the first term on the left-hand side is the general term of an absolutely convergent series. On the other hand, the orbit being of type $(\mathcal{T})$, the series with general term given by $e^{-\varepsilon\lambda_n^{p_0}}(\delta_X \mathcal{I}^p T_n^{p_0}, T_n^{p_0})_{p_0}$ is also absolutely convergent. Hence the right-hand side of (*) is absolutely convergent and $\mathcal{H}_X^\varepsilon$ is trace class since $(\tilde{U}_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis of $\operatorname{Im}\tau_p$,

$$-\int_\varepsilon^{+\infty} dt \operatorname{tr}(\delta_X B_p e^{-\varepsilon B_p}) = \operatorname{tr}\mathcal{H}_X^\varepsilon - \delta_X \log \operatorname{Vol}_\varepsilon^{p_0}(B_p) = -\delta_X \log\det{}_\varepsilon(B_p),$$

which then yields (3.7).

2) This follows from the above and point 3) of Lemma 3.1 and holds for any orbit O_p of a regularisable fibre bundle since it does not involve $\delta_X \mathcal{I}_p$.

□

The following proposition gives an interpretation of $\operatorname{tr}_{reg}^\mu H_X$ in terms of the variation of the regularised volume of the orbit.

Proposition 3.3. *The fibres of a regularisable principal fibre bundle with structure group equipped with a fixed (p-independent) Riemannian metric are regularisable.*

1) *For a given $\mu \in \mathbb{R}$, orbits are μ- minimal whenever they are μ- extremal.*

More precisely, for any point $p_0 \in \mathcal{P}$ and any horizontal vector X at point p_0, the one parameter family $\mathcal{H}_X^\varepsilon$ has a limit trace $\operatorname{tr}_{reg}^\mu \mathcal{H}_X$ and

$$\begin{aligned}\operatorname{tr}_{reg}^\mu \mathcal{H}_X &= -\delta_X \log vol_{reg}^\mu(O_p)\\ &= \frac{1}{2}\left[\sum_{j=-J, j\neq 0}^{m-1} \delta_X \frac{b_j(p)}{j} + \int_0^1 \frac{\delta_X F_p(t)}{t} dt + \int_1^\infty t^{-1}\delta_X\, tr e^{-t\tau_p^*\tau_p} dt - \mu\delta_X b_0'\right].\end{aligned} \qquad (3.8)$$

For $\mu' \in \mathbb{R}$,

$$\mathrm{tr}^{\mu}\mathcal{H}_X = \mathrm{tr}^{\mu'}\mathcal{H}_X + \gamma(\mu' - \mu)\delta_X b_0', \tag{3.9}$$

where as before $b_0' = b_0 - dimKerB$.

2) *Orbits are* μ*-minimal whenever*

$$\lim_{s\to 1} -\frac{1}{2}\left[\Gamma(s)^{-1}\int_0^\infty t^{s-1}\sum_{\lambda_n\neq 0} e^{-t\lambda_n^p}\delta_X\lambda_n^p dt + (s-1)^{-1}\delta_X b_0'(p)dt\right]$$

exists for any horizontal field X *at point* p. *Furthermore, setting* $\mu = \gamma$, *the Euler constant, we have*

$$tr_{reg}^{\gamma}\mathcal{H}_X = -\frac{1}{2}\lim_{s\to 1}\left[\Gamma(s)^{-1}\int_0^\infty t^{s-1}\sum_{\lambda_n\neq 0} e^{-t\lambda_n^p}\delta_X\lambda_n^p dt + (s-1)^{-1}\delta_X b_0'(p)\right]. \tag{3.10}$$

If moreover $\delta_X b_0' = 0$ *for any horizontal vector* X *at point* p_0, *if an orbit is* μ*-minimal for one value of* μ, *it is for any value of* μ.

Remarks. 1) From (3.9) follows that unless $\delta_X b_0 = 0$, μ-minimality depends on the choice of the parameter μ.

2) In the case of a compact connected Lie group acting via isometries on a finite dimensional Riemannian manifold $\mathcal{P}$ of dimension d, the various notions of minimality coincide since $b_0 = d$, $\mathrm{vol}_{reg}^{\mu}(O_p) = \mathrm{vol}(O_p)$ (this being the ordinary volume) and (1.10) yields:

$$\mathrm{tr}S^p = -\mathrm{grad}\ \log\ \mathrm{vol}(O_p),$$

where S^p is the second fundamental form. It tells us that the orbits of G, the volume of which are extremal among nearby orbits is a minimal submanifold of $\mathcal{P}$. This proposition therefore gives an infinite dimensional version of Hsiang's theorem [H].

2) A zeta function formulation of Hsiang's theorem in infinite dimensions was already discussed in [MRT1] in the context of Yang-Mill's theory. However, there was an obstruction due to the factor $b_0(p)$ in the zeta-function regularisation procedure which does not appear here (see also [MRT2]). A formula similar to (3.10) (but using zeta function regularisation) can be found in [GP] (see in [GP] formula (3.17) combined with formula (A.3)).

Proof of Proposition 3.3. As before, we set $B_p = \tau_p^*\tau_p$. and we shall assume for simplicity that B_p is injective; the proof then easily extends to the case when the dimension of the kernel is locally constant on each geodesic containing p_0.

1) Since the fibre bundle is regularisable, we know by Proposition 2.1 that the map $p \mapsto \mathrm{det}_{reg}(B_p)$ is Gâteaux-differentiable in the direction X. Let us now check that $\mathcal{H}_X^{\varepsilon}$ has a regularized limit trace, applying Lemma 1.1. For this, we first investigate the differentiability of the map $\varepsilon \mapsto \mathrm{tr}\mathcal{H}_X^{\varepsilon}$. By the result of Proposition 3.2, we have $\mathrm{tr}\mathcal{H}_X^{\varepsilon} = \frac{1}{2}\int_{\varepsilon}^{\infty} dt\delta_X\ \mathrm{tr}\frac{e^{-tB_p}}{t} = -\frac{1}{2}\delta_X\ \log\ \mathrm{det}_{\varepsilon}(B_p)$. The differentiability in ε easily follows from the shape of the middle expression.

Setting as before $F_p(t) = \mathrm{tr} e^{-tB_p} - \sum_{j=-J}^{m-1} b_j t^{\frac{j}{m}}$, we have furthermore

$$\frac{\partial}{\partial\varepsilon}\mathrm{tr}\mathcal{H}_X^\varepsilon = -\frac{1}{2}\varepsilon^{-1}\delta_X \ \mathrm{tr}e^{-\varepsilon B_p}$$
$$= -\frac{1}{2}\delta_X \frac{F_p(\varepsilon)}{\varepsilon} - \frac{1}{2}\sum_{j=-J}^{m-1}\delta_X b_j \varepsilon^{\frac{j-m}{m}}.$$

From the regularisability of the fibre bundle follows that $|\delta_X \frac{F_p(\varepsilon)}{\varepsilon}| \leq K$ for some $K > 0$ and $0 < \varepsilon < 1$ (see assumption (2.5 bis)) which in turn implies that

$$\frac{\partial}{\partial\varepsilon}\mathrm{tr}\mathcal{H}_X^\varepsilon \simeq_0 -\frac{1}{2}\sum_{j=-J-m}^{-1}\delta_X b_{j+m}\varepsilon^{\frac{j}{m}}.$$

Setting $f(\varepsilon) \equiv tr(\mathcal{H}_X^\varepsilon)$ in Lemma 1.1, we can define the regularised limit trace

$$\mathrm{tr}^\mu_{reg}\mathcal{H}_X + \frac{1}{2}\mu\delta_X b_0 = \lim_{\varepsilon\to 0}(\mathrm{tr}\mathcal{H}_X^\varepsilon + \frac{1}{2}\sum_{j=-J}^{-1} m\frac{\delta_X b_j}{j}\varepsilon^{\frac{j}{m}} + \frac{1}{2}\delta_X b_0 \log\varepsilon)$$
$$= \lim_{\varepsilon\to 0} -\frac{1}{2}\left(\delta_X \ \log \det{}_\varepsilon(B_p) - \sum_{j=-J}^{-1} m\delta_X \frac{b_j}{j}\varepsilon^{\frac{j}{m}} - \delta_X b_0 \log\varepsilon\right) \ \text{by (3.7 bis)}$$
$$= \lim_{\varepsilon\to 0} -\frac{1}{2}\delta_X\left(\log \det{}_\varepsilon(B_p) - \sum_{j=-J}^{-1}\frac{mb_j}{j}\varepsilon^{\frac{j}{m}} - b_0\log\varepsilon\right)$$
$$= -\frac{1}{2}\delta_X \ \log\det{}^\mu_{reg}(B_p) \quad \text{by (1.5)}$$
$$= \frac{1}{2}[\sum_{j=-J,j\neq 0}^{m-1}\frac{m\delta_X b_j}{j} + \int_0^1 dt\frac{\delta_X F_p(t)}{t} + \int_1^\infty t^{-1}\delta_X \mathrm{tr}e^{-t\tau_p^*\tau_p}dt] \ \text{by (2.6 b).}$$

2) It is well known that the expression $\Gamma(s)^{-1}\int_0^\infty t^{s-1}\sum_n e^{-t\lambda_n}$ is finite for Res large enough and that it has a meromorphic continuation to the whole plane. Since $\Gamma(s) = (s-1)\Gamma(s-1)$, we have for s with large enough real part:

$$\Gamma(s)^{-1}\int_0^\infty t^{s-1}\sum_n e^{-t\lambda_n^p}\delta_X\lambda_n^p dt = (s-1)^{-1}\frac{1}{\Gamma(s-1)}\int_0^\infty t^{s-1}\sum_n e^{-t\lambda_n^p}\delta_X\lambda_n^p dt$$
$$= -(s-1)^{-1}\frac{1}{\Gamma(s-1)}\int_0^\infty t^{s-2}\delta_X \ \mathrm{tr}e^{-tB_p}dt$$

see assumption (2.2) and Lemma 3.1

$$= -(s-1)^{-1}\frac{1}{\Gamma(s-1)}\left(\sum_{j=-J}^{m-1}\int_0^1 t^{\frac{j}{m}+s-2}\delta_X b_j dt + \right.$$

$$\left. + \int_1^\infty t^{s-2}\delta_X \operatorname{tr} e^{-tB_p}dt + \int_0^1 \delta_X F_p(t)t^{s-2}dt\right) \text{ by (2.5)}$$

$$= -(s-1)^{-1}\frac{1}{\Gamma(s-1)}\left[\sum_{j=-J}^{m-1}\frac{1}{\frac{j}{m}+s-1}\delta_X b_j\right.$$

$$\left. + \int_1^\infty t^{s-2}\delta_X \operatorname{tr} e^{-tB_p}dt + \int_0^1 t^{s-2}\delta_X F_p(t)dt\right],$$

where we have set $F_p(t) = \operatorname{tr} e^{-\varepsilon B_p} - \sum_{j=-J}^{m-1} b_j(p)t^{\frac{j}{m}}$. Hence, since $\Gamma(s)^{-1} = s + \gamma s^2 + O(s^3)$ around $s = 0$, going to the limit $s \to 1$, we find:

$$\lim_{s\to 1}[\Gamma(s)^{-1}\int_0^\infty t^{s-1}\sum_n e^{-t\lambda_n^p}\delta_X\lambda_n^p dt + (s-1)^{-1}\delta_X b_0(p)] =$$

$$= \lim_{s\to 0}(-1-\gamma s + O(s^2))\left[\sum_{j=-J, j\neq 0}^{m-1}\frac{1}{\frac{j}{m}+s}\delta_X b_j + \int_1^\infty t^{s-1}\delta_X \operatorname{tr} e^{-tB_p}dt\right.$$

$$\left. + \int_0^1 t^{s-1}\delta_X F_p(t)dt - \gamma\delta_X b_0\right]$$

$$= \delta_X \operatorname{det}^0_{reg}(B_p) - \gamma\delta_X b_0 \quad \text{by formula (1.6) (with } \mu = 0\text{) and (2.6 b)}$$

$$= -2\operatorname{tr}{}^0_{reg}\mathcal{H}_X - \gamma\delta_X b_0,$$

where $\lim_{s\to 0}\int_1^\infty t^{s-1}\delta_X \operatorname{tr} e^{-tB_p}dt = \int_1^\infty t^{-1}\delta_X \operatorname{tr} e^{-tB_p}dt$ holds using estimate (*) arising in the proof of Proposition 2.0 and $\lim_{s\to 0}\int_0^1 t^{s-1}\delta_X F_p(t)dt + s^{-1}\delta_X b_0 = \int_0^1 t^{-1}\delta_X F_p(t)dt$ by (2.5 bis) and using dominated convergence.

The rest of the assertions of 2) then easily follow.

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