Ann. Inst. Henri Poincaré,

Vol. 35, n° 6, 1999, p. 765-791. *Probabilités et Statistiques*

Manifold-valued martingales, changes of probabilities, and smoothness of finely harmonic maps

by

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Article received on 17 June 1998, revised 26 April 1999

ABSTRACT. – This paper is concerned with regularity results for starting points of continuous manifold-valued martingales with fixed terminal value under a possibly singular change of probability. In particular, if the martingales live in a small neighbourhood of a point and if the stochastic logarithm M of the change of probability varies in some Hardy space H_r for sufficiently large $r < 2$, then the starting point is differentiable at $M = 0$. As an application, our results imply that continuous finely harmonic maps between manifolds are smooth, and the differentials have stochastic representations not involving derivatives. This gives a probabilistic alternative to the coupling technique used by Kendall (1994). © Elsevier, Paris

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RÉSUMÉ. – On étudie la régularité par changement de probabilité éventuellement singulier, du point de départ d'une martingale continue à valeurs dans une variété et de valeur terminale donnée. On prouve en particulier que si la martingale est à valeurs dans un petit voisinage d'un point et si le logarithme stochastique M du changement de probabilité est dans un espace de Hardy H_r pour $r < 2$ suffisamment grand, alors le point de départ est différentiable en $M = 0$. On donne en application une nouvelle preuve du résultat suivant obtenu par Kendall (1994) avec des méthodes de couplage : les applications continues et finement harmoniques entre variétés sont C^∞. On donne une expression de leur différentielle qui ne fait pas intervenir de dérivée. © Elsevier, Paris

1. INTRODUCTION

Throughout this article $(\Omega, (\mathcal{F}_t)_{t\geqslant 0}, \mathcal{F}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions, such that all real-valued martingales have a continuous version. Examples of such filtrations include Brownian filtrations, Walsh filtrations, or filtrations $(\mathcal{F}_t)_{t\geqslant 0}$ such that there exists a continuous martingale which has the $(\mathcal{F}_t)$-predictable representation property. For simplicity we assume that the probability of elements in $\mathcal{F}_0$ is 0 or 1.

Let W be a smooth manifold and ∇ a torsion-free connection on W. For the sake of calculations we choose occasionally a Riemannian metric $g = \langle\cdot|\cdot\rangle$ on W with corresponding Riemannian distance δ. However, in general we do not assume that ∇ is a metric connection to this or any other Riemannian metric. Only when we refer explicitly to Riemannian manifolds we always work with the Levi-Civita connection and the given metric g.

Recall that a W-valued continuous semimartingale $(Y_t)_{t\geqslant 0}$ is a martingale, if for each real-valued C^2 function f on W,

$$f(Y) - f(Y_0) - \int_0^{\cdot} \nabla df(Y)\, dY \otimes dY$$

is a real-valued local martingale.

If Y is a semimartingale taking values in W, we denote by $d^\nabla Y$ its Itô differential (see [7]). There is a canonical decomposition $d^\nabla Y =$

$d^m Y + \tilde{d}^\nabla Y$ into a martingale part $d^m Y$ and a finite variation part $\tilde{d}^\nabla Y$. The latter is also called the drift. Denote by Γ^i_{jk} the Christoffel symbols of the connection. In local coordinates and in terms of the decomposition $dY^i = dN^i + dA^i$, where N^i is a local martingale and A^i a process of finite variation, we have:

$$d^\nabla Y = \left(dY^i + \frac{1}{2}\Gamma^i_{jk}(Y)\, d\langle Y^j, Y^k\rangle \right) \frac{\partial}{\partial x^i},$$

and so

$$d^m Y = dN^i \frac{\partial}{\partial x^i} \quad \text{and} \quad \tilde{d}^\nabla Y = \left(dA^i + \frac{1}{2}\Gamma^i_{jk}(Y)\, d\langle Y^j, Y^k\rangle \right) \frac{\partial}{\partial x^i}. \tag{1.1}$$

Formally, $d^\nabla Y_\bullet$, $d^m Y_\bullet$ and $\tilde{d}^\nabla Y_\bullet$ are tangent vectors at the point $Y_\bullet$.

For a real-valued local martingale M let $Y_\bullet(M)$, if it exists, be a W-valued semimartingale with drift $-dM\, dY(M)$ converging almost surely as t tends to infinity to a fixed W-valued random variable L. Here $dM\, dY(M)$ is the "vector" $d\langle M, Y(M)^i\rangle \frac{\partial}{\partial x^i}$. The principal objective of this article is to find conditions on W under which the map $M \mapsto Y_0(M)$ is Hölder continuous or differentiable. The main results (Proposition 3.3 and Theorem 3.5) show that if the processes take their values in a compact convex subset V of W with p-convex geometry, then the distance between $Y_0(0)$ and $Y_0(M)$ is less than $C\|\langle M, M\rangle_\infty^{1/2}\|_r^{1/p}$ for some constant C depending only on V and $r > 1$. Moreover, if W is sufficiently small and M varies in some Hardy space H_r for $r < 2$ sufficiently large, then $M \mapsto Y_0(M)$ is differentiable at $M \equiv 0$ and a formula for its derivative can be given in terms of the geodesic transport above $Y_\bullet(0)$.

Note that if M is a real-valued martingale, there exist stopping times T arbitrarily large in probability such that $\mathcal{E}(M)^T$ is a uniformly integrable martingale. The semimartingale $Y(M)$ stopped at T is a $\mathbb{P}^{M,T}$-martingale where $\mathbb{P}^{M,T} = \mathcal{E}(M)^T \cdot \mathbb{P}$. Hence, Proposition 3.3 and Theorem 3.5 cover regularity results for starting points of martingales under an equivalent change of probability.

The notion of p-convexity plays a fundamental role. We prove (Proposition 2.4) that for every $p > 1$ and every $x \in W$ there exist a neighbourhood of x with p-convex geometry.

In order to establish the differentiability of the map $M \mapsto Y_0(M)$, we also need (Proposition 2.7) that for every $\lambda > 0$ and every $x \in W$ there exist a neighbourhood V of x such that L^λ-norms of the inverse of the geodesic transport along any V-valued martingale are finite.

In Section 4 the results and estimates from Section 3 are applied to give an alternative proof of Kendall's result that continuous finely harmonic maps from a Riemannian manifold to a manifold with a connection, i.e., maps which send Brownian motions to local martingales, are smooth.

2. PRELIMINARIES

Let V be a subset of W. A V-valued martingale Y is said to have exponential moments of order λg (or simply of order λ when ∇ is a metric connection to g) if

$$\mathbb{E}\left[\exp\left(\lambda\int_0^\infty \langle dY \mid dY\rangle\right)\right] < \infty. \tag{2.1}$$

We use the notation $\langle Y \mid Y\rangle$ for $\int_0^\bullet \langle dY \mid dY\rangle$. By Proposition 2.1.2 of [11] and the observation that there exists locally a function with negative Hessian, we have:

LEMMA 2.1. – *Let $\lambda > 0$ and $x \in W$. There exists a neighbourhood V of x such that every V-valued martingale Y has exponential moment of order λg.*

Remark 2.2. – Another consequence of [11], Proposition 2.1.2, is that if a compact subset V of W has a neighbourhood which carries a function with positive Hessian, then there exists $\lambda > 0$ such that all V-valued martingales have exponential moments of order λg. In particular, the quadratic variation of V-valued martingales has moments of any order, which are bounded by a constant depending only on the order and on V.

DEFINITION 2.3. – (1) *Let $p \geqslant 1$. We say that W has p-convex geometry if there exist a C^2 function on W with positive Hessian, a convex function $\psi : W \times W \to \mathbb{R}_+$, smooth outside the diagonal and vanishing precisely on the diagonal, i.e., $\psi^{-1}(\{0\}) = \{(x,x),\ x \in W\}$, and a Riemannian distance δ on W such that $c\delta^p \leqslant \psi \leqslant C\delta^p$ with constants $0 < c < C$.*

A subset of W is said to have p-convex geometry if there exists an open neighbourhood of W with p-convex geometry.

(2) *A subset V of W is called convex if it has an open neighbourhood V' such that any two points x, y in V' are connected by one and only one geodesic in V', which depends smoothly on x and y, and entirely lies in V if x and y are in V.*

Note p-convex geometry implies p'-convex geometry for $p' \geqslant p$: if ψ satisfies the conditions in the definition for p-convex geometry, then $\psi^{p'/p}$ satisfies the conditions for p'-convex geometry.

Simply connected Riemannian manifolds with nonpositive curvature have 1-convex geometry. In general, a manifold does not have p-convex geometry. However, we have the following local result.

PROPOSITION 2.4. – *For every $x \in W$ and $p > 1$ there exists a neighbourhood of x with p-convex geometry.*

Proposition 2.4 is a direct corollary of a more general result on totally geodesic submanifolds (compare with [7] 4.59):

PROPOSITION 2.5. – *Let $\widetilde{W}$ be a totally geodesic submanifold of W. For every point $a \in \widetilde{W}$ and $p > 1$, there exist a neighbourhood U of a in W, a convex function f on U such that $f^{2/p}$ is smooth and constants $0 < c < C$ such that $c\delta^p(\cdot, \widetilde{W}) \leqslant f \leqslant C\delta^p(\cdot, \widetilde{W})$ on U.*

Proof. – For $p \geqslant 2$ the result is proved in [7, 4.59]. Let us assume $1 < p < 2$. As in [7, 4.59], we choose coordinates $(x^1, \ldots, x^q, y^{q+1}, \ldots, y^n)$ vanishing together with the Christoffel symbols at a such that the equation for $\widetilde{W}$ is $\{x^1 = \cdots = x^q = 0\}$. We use Latin letters for indices ranging from 1 to q and Greek letters for indices ranging from $q+1$ to n. Define $f = h^{p/2}$ where

$$h(x^1, \ldots, x^q, y^{q+1}, \ldots, y^n) = \frac{1}{2}(\varepsilon^2 + \|y\|^2)\|x\|^2.$$

Clearly f vanishes precisely on $\widetilde{W}$ and possibly by reducing U it satisfies

$$c\delta^p(\cdot, \widetilde{W}) \leqslant f \leqslant C\delta^p(\cdot, \widetilde{W})$$

for some $0 < c < C$.

It is shown in [7] that h is convex for U small and $\varepsilon > 0$ close to 0. It suffices to prove that f is convex, and since $p > 1$, it is enough to check this on $\{h > 0\}$. But on $\{h > 0\}$,

$$\nabla df = \frac{p}{2} h^{p/2-1}\left(\nabla dh - \frac{2-p}{2h} dh \otimes dh\right).$$

Hence, for f to be convex, it is sufficient to verify that on $\{h > 0\}$ the bilinear form b defined by

$$b = \nabla dh - \frac{2-p}{2h} dh \otimes dh$$

is positive. As in [7] it suffices to check the matrix

$$H = \begin{pmatrix} \dfrac{1}{\varepsilon^2} b_{ij} & \dfrac{1}{\varepsilon \|x\|} b_{i\alpha} \\ \dfrac{1}{\varepsilon \|x\|} b_{\alpha i} & \dfrac{1}{\|x\|^2} b_{\alpha\beta} \end{pmatrix}$$

to be positive on $\{h > 0\}$. But a Taylor expansion of the entries reveals

$$H = \begin{pmatrix} \delta_{ij} - (2-p)\dfrac{x^i x^j}{\|x\|^2} + \mathrm{o}(1) & \mathrm{o}(1) \\ \mathrm{o}(1) & \delta_{\alpha\beta} - \dfrac{\varepsilon^2 \sum_i \Gamma^i_{\alpha\beta} x^i}{\|x\|^2} + \mathrm{o}(1) \end{pmatrix}.$$

It is easy to see that the 0-order term of the matrix with Latin index entries is greater than $(p-1)\,\mathrm{Id}$. Regards the matrix with Greek index entries, since $\widetilde{W}$ is totally geodesic the $\Gamma^i_{\alpha\beta}$ vanish on $\widetilde{W}$, hence $|\Gamma^i_{\alpha\beta}| \leqslant C\|x\|$, and for ε sufficiently small, the 0-order term of this matrix is greater than $\varepsilon'\,\mathrm{Id}$ with $\varepsilon' > 0$. This implies that f is convex in a neighbourhood of a. $\square$

In the case of Riemannian manifolds, Picard establishes a relation between p, the radius of small geodesic balls and an upper bound for the sectional curvatures ([12], proof of Proposition 3.6): if all sectional curvatures are bounded above by $K > 0$, then a regular geodesic ball with radius smaller than $\frac{\pi}{2\sqrt{Kq}}$, $q > 1$, has p-convex geometry where p is the conjugate exponent to q, and martingales with values in this geodesic ball have exponential moments of order $Kq/2$.

The torsion-free connection ∇ on W induces a torsion-free connection ∇^c on TW called the complete lift of ∇ and characterized by the fact that its geodesics are the Jacobi fields for ∇ (see [15]), or by the fact that the ∇^c-martingales in TW are exactly the derivatives of ∇-martingales in W depending differentiably on a parameter (see [2]).

The connection ∇ induces another connection ∇^h on TW, called the horizontal lift of ∇, which in general has nonvanishing torsion and is characterized by the fact that if J is a TW-valued semimartingale with projection $X = \pi(J) \in W$, then the parallel transport $//^t_{0,t} w$ along J (with respect to ∇^h) of a vector $w = w^{\mathrm{vert}} \oplus w^{\mathrm{hor}}$ in $T_{J_0}TW = V_{J_0} \oplus H_{J_0}$ is given by

$$//^h_{0,t} w = v_{J(t)} \circ //_{0,t} \circ (v_{J_0})^{-1}(w^{\text{vert}}) \oplus h^{\nabla}_{J(t)} \circ //_{0,t} \circ \big(h^{\nabla}_{J_0}\big)^{-1}(w^{\text{hor}}),$$

where $v:\pi^*TW \to V$ and $h^\nabla:\pi^*TW \to H$ are, respectively, the vertical and horizontal lift, $//_{0,\bullet}$ the parallel transport along X with respect to ∇.

DEFINITION 2.6. – *Let Y be a semimartingale taking values in W. The geodesic transport $\Theta_{0,s}$, $0 \leqslant s$ (also called deformed parallel transport or Dohrn–Guerra parallel translation) is the linear map from $T_{Y_0}W$ to $T_{Y_s}W$ such that*

(i) *$\Theta_{0,0}$ is the identity map on $T_{Y_0}W$,*

(ii) *for $w \in T_{Y_0}W$ the Itô differential $d^{\nabla^c}\Theta_{0,\bullet}(w)$ is the horizontal lift of $d^\nabla Y$ above $\Theta_{0,\bullet}(w)$.*

We define $\Theta_{s,t} = \Theta_{0,t}\Theta_{0,s}^{-1}$ for $0 \leqslant s \leqslant t$.

Let J be a TW-valued semimartingale which projects to a semimartingale Y on W. By [2] we have

$$\begin{aligned} d\big(\Theta_{0,\bullet}^{-1}J\big) &= \Theta_{0,\bullet}^{-1}\big(//_{0,\bullet}d\big(//_{0,\bullet}^{-1}J\big) + \tfrac{1}{2}R(J,dY)\,dY\big) \\ &= \Theta_{0,\bullet}^{-1}\big(v_J^{-1}\big(d^{\nabla^h}J\big)^{\text{vert}} + \tfrac{1}{2}R(J,dY)\,dY\big), \end{aligned}$$

where R is the curvature tensor associated to ∇. Using the relation between d^{∇^c} and d^{∇^h} in Lemma 4.1 of [2], we get

$$d\big(\Theta_{0,\bullet}^{-1}J\big) = \Theta_{0,\bullet}^{-1}\big(v_J^{-1}\big(d^{\nabla^c}J\big)^{\text{vert}}\big). \tag{2.2}$$

In local coordinates, adopting the summation convention, Eq. (2.2) can be written as

$$\begin{aligned} d\big(\Theta_{0,\bullet}^{-1}\big)_i = \big(\Theta_{0,\bullet}^{-1}\big)_j\big(&\Gamma^j_{im}(Y)\,dY^m \\ &+ \tfrac{1}{2}\big(D_i\Gamma^j_{k\ell}(Y) + \Gamma^j_{im}(Y)\Gamma^m_{k\ell}(Y)\big)d\langle Y^k, Y^\ell\rangle\big). \end{aligned} \tag{2.3}$$

In the case when Y is a martingale, we are able to establish the existence of moments for the norm of $\Theta_{\bullet,\bullet}^{-1}$ along Y:

PROPOSITION 2.7. – *Let $x \in W$ and $\lambda > 0$. There exists a neighbourhood V of x such that for every V-valued martingale Y, the geodesic transport above Y satisfies*

$$\mathbb{E}\Big[\sup_{0\leqslant s\leqslant t<\infty} \big\|\Theta_{s,t}^{-1}\big\|^\lambda\Big] < \infty, \tag{2.4}$$

where the norm $\|\Theta_{s,t}^{-1}\|$ is defined via the metric g.

Proof. – Take V included in the domain of a local chart. Since Y is a martingale, we have

$$dY^m = dM^m - \tfrac{1}{2}\,\Gamma^m_{k\ell}(Y)\,d\langle Y^k, Y^\ell\rangle$$

in local coordinates, where M^m is a local martingale. Hence by (2.3),

$$d\big(\Theta^{-1}_{0,\bullet}\big)_i = \big(\Theta^{-1}_{0,\bullet}\big)_j\big(\Gamma^j_{im}(Y)\,dM^m + \tfrac{1}{2}\,D_i\Gamma^j_{k\ell}(Y)\,d\langle Y^k, Y^\ell\rangle\big). \qquad (2.5)$$

This equation, together with Lemma 2.1 and [10], Theorem 3.4.6, gives the result for an appropriately chosen V depending on λ. □

In the case of the Levi-Civita connection on a Riemannian manifold, the situation is simpler because $\|\Theta_{s,t}(w)\|^2$, $0 \leqslant s \leqslant t$, is a process of finite variation, and one can give a more quantitative result.

PROPOSITION 2.8. – *Let W be a Riemannian manifold and for $y \in W$ let $K(y) = \sup(K'(y), 0)$ (respectively $-k(y) = \inf(-k'(y), 0)$) where $K'(y)$ (respectively $-k'(y)$) is the supremum (respectively the infimum) of the sectional curvatures at y. Then, for any W-valued semimartingale Y, the geodesic transport Θ along Y can be estimated in terms of the quadratic variation $\langle Y \mid Y\rangle$ of Y as follows*:

$$\|\Theta_{s,t}\| \leqslant \exp\left(\frac{1}{2}\int_s^t k(Y)\langle dY \mid dY\rangle\right), \quad 0 \leqslant s \leqslant t, \qquad (2.6)$$

and

$$\big\|\Theta^{-1}_{s,t}\big\| \leqslant \exp\left(\frac{1}{2}\int_s^t K(Y)\langle dY \mid dY\rangle\right), \quad 0 \leqslant s \leqslant t. \qquad (2.7)$$

Proof. – By means of [2], see (4.30), we have for any $\mathcal{F}_s$-measurable random variable w in $T_{Y_s}W$

$$d\|\Theta_{s,\bullet}w\|^2 = -\big\langle \Theta_{s,\bullet}w, R(\Theta_{s,\bullet}w, dY)\,dY\big\rangle.$$

Now with the bounds for the sectional curvatures we obtain

$$k(Y)\,\|\Theta_{s,\bullet}w\|^2\langle dY \mid dY\rangle \geqslant d\|\Theta_{s,\bullet}w\|^2 \geqslant -K(Y)\|\Theta_{s,\bullet}w\|^2\langle dY \mid dY\rangle.$$

Hence

$$\|w\|^2 \exp\left(\int_s^t k(Y)\langle dY \mid dY\rangle\right) \geqslant \|\Theta_{s,t} w\|^2$$
$$\geqslant \|w\|^2 \exp\left(-\int_s^t K(Y)\langle dY \mid dY\rangle\right)$$

which gives the claim. □

When Y is Brownian motion the bounds in (2.6) and (2.7) can be given in terms of Ricci curvature as well known.

COROLLARY 2.9. – *Let W be a Riemannian manifold and V a regular geodesic ball in W with radius smaller than $\pi/(2\sqrt{Kq})$, $q > 1$, where $K > 0$ is an upper bound for the sectional curvatures. Then, with respect to the Levi-Civita connection, the geodesic transport along any V-valued martingale satisfies*

$$\sup_{0\leqslant s\leqslant t<\infty} \left\|\Theta_{s,t}^{-1}\right\| \in L^q. \tag{2.8}$$

Proof. – Just note that a V-valued martingale has exponential moments of order $Kq/2$ by [12], and use (2.7). □

3. VARIATIONS OF MARTINGALES WITH PRESCRIBED TERMINAL VALUE BY A CHANGE OF PROBABILITY

In the sequel we will say that a process has a random variable L as *terminal value* if it converges $\mathbb{P}$-a.s. to L as t tends to infinity. The aim of this section is to establish regularity results for initial values of martingales with prescribed terminal value when the probability is allowed to vary. To formulate the main result of this article we first give some definitions and lemmas.

LEMMA AND DEFINITION 3.1. – *Let M be a real-valued local martingale and Z a W-valued semimartingale. The following two conditions are equivalent*:

(i) *The semimartingale Z has drift $-dM\,dZ$ where $dM\,dZ$ is the "vector" with the components $d\langle M, Z^i\rangle$ in a system of coordinates.*

(ii) *The stopped semimartingale Z^T is a $\mathbb{Q}^T$-martingale where $\mathbb{Q}^T = \mathcal{E}(M^T)\cdot\mathbb{P}$ for every stopping time T such that the stochastic exponential $\mathcal{E}(M^T)$ is a uniformly integrable martingale.*

If one of these conditions is satisfied we say that Z is a $\mathbb{Q}$-martingale with $\mathbb{Q} = \mathcal{E}(M) \cdot \mathbb{P}$ even if there is no probability equivalent to $\mathbb{P}$ such that Z is a martingale, and the notion $\mathbb{Q} = \mathcal{E}(M) \cdot \mathbb{P}$ will mean that a probability $\mathbb{Q}$ is defined on the subalgebras $\mathcal{F}_T$ where it coincides with $\mathbb{Q}^T$.

Proof. – Since there exists a sequence $(T_n)_{n\in\mathbb{N}}$ converging almost surely to infinity such that for every $n \in \mathbb{N}$, $\mathcal{E}(M^{T_n})$ is a uniformly integrable martingale, one can assume that $\mathcal{E}(M)$ is a uniformly integrable martingale and hence that $\mathbb{Q} = \mathcal{E}(M) \cdot \mathbb{P}$ defines a probability equivalent to $\mathbb{P}$. Now, as a consequence of Girsanov's theorem, we have that the relation between the drift of Z with respect to $\mathbb{P}$ (denoted by $\tilde{d}_{\mathbb{P}}^{\nabla} Z$) and with respect to $\mathbb{Q}$ (denoted by $\tilde{d}_{\mathbb{Q}}^{\nabla} Z$) is

$$\tilde{d}_{\mathbb{Q}}^{\nabla} Z = \tilde{d}_{\mathbb{P}}^{\nabla} Z + dM\, dZ.$$

This gives the equivalence of (i) and (ii). □

LEMMA 3.2. – *Let M be a real-valued martingale such that $\langle M, M\rangle_\infty \leqslant 1$ a.s. Assume that W has convex geometry and let Z be a semimartingale with values in a compact subset V of W and with drift $-dZ\,dM$. Then, for every $r > 0$, there exists a constant $C(V,r) > 0$ such that*

$$\big\|\langle Z|Z\rangle_\infty^{1/2}\big\|_r \leqslant C(V,r). \tag{3.1}$$

Proof. – Set $G = \mathcal{E}(M)$. Then

$$\big\|\langle Z \mid Z\rangle_\infty^{1/2}\big\|_r \leqslant \mathbb{E}\left[\frac{1}{G_\infty}\right]^{1/(2r)} \mathbb{E}\big[G_\infty \langle Z \mid Z\rangle_\infty^r\big]^{1/(2r)}.$$

Now by Lemma 3.1, Z is a $G \cdot \mathbb{P}$-martingale. Since W has convex geometry one can construct a function with positive Hessian on a neighbourhood of V. Hence according to Remark 2.2, quadratic variations of martingales in V have uniformly bounded L^s norms for $s > 0$. This reveals the last term to be bounded. The second term is obviously bounded (e.g., [13, Proposition 1.15, p. 318]). □

For $r > 1$ let H_r be the set of real valued martingales M such that $M_0 = 0$ and

$$\|M\|_{H_r} := \big\|\langle M, M\rangle_\infty^{1/2}\big\|_r$$

is finite. Then $(H_r, \|\cdot\|_{H_r})$ is a Banach space.

In the sequel, if S is a real valued process and τ a stopping time, we write S_τ^* for $\sup_{s\leqslant\tau} S_s$.

PROPOSITION 3.3. – *Let V be a compact convex subset of W with p-convex geometry for some $p \geqslant 1$. Let $r \in]1,2[$ and $r' \in]1,r[$. There exists a constant $C > 0$ depending only on δ, V, r and r' such that for every $M \in H_r$, if Y is the V-valued martingale with terminal value L and Z a V-valued semimartingale with drift $-dZ\,dM$ and terminal value L, then*

$$\left\|\delta(Y,Z)_\infty^*\right\|_{r'p} \leqslant C\|M\|_{H_r}^{1/p}. \tag{3.2}$$

Proof. – Let $\psi : V \times V \to \mathbb{R}_+$ be the convex function appearing in the definition of p-convex geometry. Then $a\delta^p \leqslant \psi \leqslant A\delta^p$ on V with constants $0 < a < A$. The required estimate (3.2) is equivalent to

$$\left\|\psi(Y,Z)_\infty^*\right\|_{r'} \leqslant C\|M\|_{H_r}. \tag{3.3}$$

Let $\tau = \inf\{t > 0,\ \langle M,M\rangle_t > 1\}$ (with $\inf\emptyset = \infty$). We have

$$\begin{aligned}\left\|\psi(Y,Z)_\infty^*\right\|_{r'} &\leqslant \left\|\psi(Y,Z)_\tau^*\mathbf{1}_{\{\tau=\infty\}} + \psi(Y,Z)_\infty^*\mathbf{1}_{\{\tau<\infty\}}\right\|_{r'}\\ &\leqslant \left\|\psi(Y,Z)_\tau^*\right\|_{r'} + \sup_{V\times V}\psi\,\left\|\mathbf{1}_{\{\langle M,M\rangle_\infty>1\}}\right\|_{r'}\\ &\leqslant \left\|\psi(Y,Z)_\tau^*\right\|_{r'} + \sup_{V\times V}\psi\,\left\|M\right\|_{H_{r'}}.\end{aligned}$$

Hence we are left to bound the first term on the right. First, Itô's formula for convex functions yields

$$\begin{aligned}&\psi(Y_\tau,Z_\tau)\\ &\quad\geqslant \psi(Y_{s\wedge\tau},Z_{s\wedge\tau}) + \int_{s\wedge\tau}^{\tau}\mathbf{1}_{\{\psi(Y,Z)\neq0\}}\big\langle d\psi, d^{\nabla\otimes\nabla}(Y,Z)\big\rangle\\ &\qquad + \frac{1}{2}\int_{s\wedge\tau}^{\tau}\mathbf{1}_{\{\psi(Y,Z)\neq0\}}(\nabla\otimes\nabla)\,d\psi\,d(Y,Z)\otimes d(Y,Z).\end{aligned} \tag{3.4}$$

Since ψ is convex and the drift of (Y,Z) with respect to $\mathbb{P}$ is $(0,-dM\,dZ)$, we have

$$\begin{aligned}\psi(Y_{s\wedge\tau},Z_{s\wedge\tau}) &\leqslant \mathbb{E}\big[\psi(Y_\tau,Z_\tau)\,\big|\,\mathcal{F}_{s\wedge\tau}\big]\\ &\quad + \mathbb{E}\left[\int_{s\wedge\tau}^{\tau}\big|\big\langle d\psi,\big(0,dM\,dZ\big)\big\rangle\big|\,\Big|\,\mathcal{F}_{s\wedge\tau}\right].\end{aligned} \tag{3.5}$$

We get for the first term on the right-hand side of (3.5)

$$\begin{aligned}\left\|\mathbb{E}\left[\psi(Y_\tau, Z_\tau) \,\middle|\, \mathcal{F}_{\bullet\wedge\tau}\right]^*\right\|_{r'} &\leqslant \sup\psi\left\|\mathbb{P}\left[\langle M, M\rangle_\infty > 1 \,\middle|\, \mathcal{F}_{\bullet\wedge\tau}\right]^*\right\|_{r'}\\ &\leqslant \frac{r'}{r'-1}\sup\psi\,\mathbb{P}\left[\langle M, M\rangle_\infty > 1\right]^{1/r'}\\ &\leqslant \frac{r'}{r'-1}\sup\psi\,\mathbb{E}\left[\langle M, M\rangle_\infty^{r'/2}\right]^{1/r'}\\ &\leqslant C'\|M\|_{H_r}\end{aligned}$$

by using successively Doob's inequality and Bienaymé–Tchebichev inequality. To deal with the second term in (3.5), let

$$D_{r'} = \left\|\mathbb{E}\left[\int_{\bullet\wedge\tau}^{\tau}\langle d\psi, (0, dM\,dZ)\rangle \,\middle|\, \mathcal{F}_{\bullet\wedge\tau}\right]^*\right\|_{r'}.$$

We use successively the fact that ψ is Lipschitz, Doob's inequality and Hölder inequality. Choose $r_1 > 1$ such that $r'r_1 \leqslant r$ and let r_1' be its conjugate number. Then

$$\begin{aligned}D_{r'} &\leqslant \sup|d\psi|\left\|\mathbb{E}\left[\langle M, M\rangle_\tau^{1/2}\langle Z \mid Z\rangle_\tau^{1/2} \,\middle|\, \mathcal{F}_{\bullet\wedge\tau}\right]^*\right\|_{r'}\\ &\leqslant \frac{r'}{r'-1}\sup|d\psi|\left\|\langle M, M\rangle_\tau^{1/2}\,\langle Z \mid Z\rangle_\tau^{1/2}\right\|_{r'}\\ &\leqslant \frac{r'}{r'-1}\sup|d\psi|\left\|\langle M, M\rangle_\tau^{1/2}\right\|_{r'r_1}\left\|\langle Z \mid Z\rangle_\tau^{1/2}\right\|_{r'r_1'}.\end{aligned}$$

According to Lemma 3.2 the last term is bounded. Thus, finally we get

$$\left\|\psi(Y, Z)_\tau^*\right\|_{r'} \leqslant C\|M\|_{H_r}. \qquad \square$$

Let $M \in H_r$. By $Y(M)$ we always mean a semimartingale with drift $-dM\,dY$ and terminal value L. In the rest of this section we want to prove differentiability of the map $M \mapsto Y_0(M)$ at $M \equiv 0$ in H_r. The processes we consider live in a convex set V, and since convex sets are included in the domain of an exponential chart, we will identify V and its image in such a chart.

First we need some lemmas.

LEMMA 3.4. – *Let V be a compact convex subset of W with p-convex geometry for some $p \in \,]1, 2[$.*

(1) *Let $r \in \,]\frac{p+1}{p}, 2[$. There exists a constant $C > 0$ depending only on V and r such that for every $M \in H_r$, if Y and Z are as in Proposition* 3.3,

then

$$\big\|\langle Z-Y, Z-Y\rangle_\tau\big\|_1 \leqslant C\|M\|_{H_r}^{\frac{1}{p}+1} \tag{3.6}$$

where $\tau = \inf\{t>0,\ \langle M,M\rangle_t > 1\}$ (*with* $\inf\emptyset = \infty$).

(2) *Let* $r' \in]1, \frac{4p}{3+p}[$. *For all* $r \in]\sup(\frac{2r'}{p}, \frac{r'(3-p)}{(2-r')p}), 2[$ *there exists a constant* $C>0$ *depending only on* V, r *and* r' *such that for every* $M \in H_r$,

$$\big\|\langle Z-Y, Z-Y\rangle_\tau\big\|_{r'} \leqslant C\|M\|_{H_r}^{2/p} \tag{3.7}$$

for all Y, Z *and* τ *as in* (1).

Proof. – In the calculations below, the elimination of the brackets of Y and Z by taking smaller Hölder norms is done in the same way as in the proof of Proposition 3.3 and will not again be carried out in detail. Set $\alpha = 1/p$.

(1) Using the facts that $\phi = \delta^2$ is convex and

$$(\nabla\otimes\nabla)\,d\phi\big((A,B),(A,B)\big) \geqslant c\,\|B-A\|^2,$$

we obtain by Itô's formula (3.4)

$$\begin{aligned}
&\big\|\langle Z-Y, Z-Y\rangle_\tau\big\|_1\\
&\quad\leqslant C\mathbb{E}\left[\int_0^\tau \delta(Y,Z)\big|dM\,dZ\big|\right] + C\mathbb{E}\big[\phi(Y_\tau, Z_\tau)\big]\\
&\quad\leqslant C\mathbb{E}\Big[\sup_{s\leqslant\tau}\delta(Y_s,Z_s)\langle M,M\rangle_\tau^{1/2}\,\langle Z\mid Z\rangle_\tau^{1/2}\Big] + C\mathbb{E}\big[\langle M,M\rangle_\infty^{(1+\alpha)/2}\big]\\
&\quad\leqslant C\big\|\delta(Y,Z)_\tau^*\big\|_{r'}\,\big\|\langle M,M\rangle_\tau^{1/2}\big\|_{r''} + C\|M\|_{H_{1+\alpha}}^{1+\alpha},
\end{aligned}$$

with $r' < 2p$, $r'' < 2$ satisfying $\frac{1}{r'} + \frac{1}{r''} < 1$. This gives by (3.2)

$$\big\|\langle Z-Y, Z-Y\rangle_\tau\big\|_1 \leqslant C\|M\|_{H_r}^{\alpha+1}$$

if $\sup(\frac{1}{p}+1, \frac{2p}{2p-1}) < r < 2$. This proves the first assertion of the lemma.

(2) By Itô's formula (3.4) we have

$$\langle Z-Y, Z-Y\rangle_\tau \leqslant C\left(\phi(Y_\tau, Z_\tau) - \int_0^\tau \big\langle d\phi, \big(d^\nabla Y, d^\nabla Z\big)\big\rangle\right). \tag{3.8}$$

To obtain the next estimate it is useful to note that

$$\phi(y,z)=\phi_{ij}(y,z)(z^i-y^i)(z^j-y^j)$$

where the functions ϕ_{ij} are smooth. Splitting the second term on the right-hand side of (3.8) into its martingale and finite variation part and estimating the L^r norms using BDG inequalities gives

$$\begin{aligned}\big\|\langle Z-Y,Z-Y\rangle_\tau\big\|_{r'}\leqslant C\Big(&\big\|\phi(Y_\tau,Z_\tau)\big\|_{r'}\\ &+\Big\|\sup_{s\leqslant\tau}\delta^2(Y_s,Z_s)\big(\langle Y\mid Y\rangle_\tau^{1/2}+\langle Z\mid Z\rangle_\tau^{1/2}\big)\Big\|_{r'}\\ &+\Big\|\sup_{s\leqslant\tau}\delta(Y_s,Z_s)\langle Z-Y,Z-Y\rangle_\tau^{1/2}\big\|_{r'}\\ &+\Big\|\sup_{s\leqslant\tau}\delta(Y_s,Z_s)\langle M,M\rangle_\tau^{1/2}\langle Z\mid Z\rangle_\tau^{1/2}\Big\|_{r'}\Big),\end{aligned}$$

where the single terms may be estimated with the same method as above, using (3.2) and (3.6). For $r'<p$ and $\frac{2r'}{p}<r<2$, the first term on the right is seen to be less than $C\|M\|_{H_r}^{2\alpha}$ (the difference here with the bound on the first term on the right-hand side of (3.5) is that we use the L^r norm of the bracket to the power 2α). By means of (3.2) the second term is dominated by $C\|M\|_{H_r}^{2\alpha}$ for $r'<p$ and $\frac{2r'}{p}<r<2$, the third term can be estimated by $C\|M\|_{H_r}^{2\alpha}$ for

$$r'<\frac{4p}{3+p}\quad\text{and}\quad r>\sup\left(\frac{2r'}{p},\frac{r'}{2-r'}\frac{3-p}{p}\right)$$

(here we use (3.6), and (3.2) with $p'=\frac{2p}{3-p}$, together with the observation that p-convex implies p'-convex). Finally, the fourth term is less than $C\|M\|_{H_r}^{2\alpha}$ if

$$r'<p\quad\text{and}\quad r>\sup\left(\frac{2pr'}{(2+r')p-2r'},\frac{2r'(2-p)}{(2-r')p}\right)$$

(again by (3.2) now with $p'=\frac{p}{2-p}$). Thus, for all r,r' such that $1<r'<\frac{4p}{3+p}$ and

$$\sup\left(\frac{2r'}{p},\frac{r'(3-p)}{(2-r')p},\frac{2pr'}{(2+r')p-2r'},\frac{2r'(2-p)}{(2-r')p}\right)<r<2,$$

we have

$$\big\|\langle Z-Y, Z-Y\rangle_\tau\big\|_{r'} \leqslant C\|M\|_{H_r}^{2\alpha}. \tag{3.9}$$

We conclude with the remark that if $\frac{2r'}{p} < 2$ then

$$\sup\left(\frac{2pr'}{(2+r')p-2r'}, \frac{2r'(2-p)}{(2-r')p}\right) < \frac{2r'}{p}.$$

Note that (3.9) also holds true if p is replaced by $p' \in [p, 2[$ and α by $\alpha' = 1/p'$. $\square$

For $r > 1$, a subset V in W and an $\mathcal{F}_\infty$-measurable random variable L taking values in V, let $\mathcal{D}_L \equiv \mathcal{D}_L(V, r)$ be the set of $M \in H_r$ such that there exists a V-valued semimartingale $Y(M)$ with drift $-dM\, dY(M)$ and terminal value L. Note that for compact convex V with convex geometry, $\mathcal{D}_L$ includes all M in H_r such that $\mathcal{E}(M)$ is uniformly integrable (see [1, Theorem 7.3]).

We are now able to prove the main result.

THEOREM 3.5. – *Let $x \in W$. There exists $r_0 < 2$ such that for every $r \in]r_0, 2[$ there is a compact convex neighbourhood V of x with the following property*:

For any $\mathcal{F}_\infty$-measurable V-valued random variable L, the map $M \mapsto Y_0(M)$ from $(\mathcal{D}_L, \|\cdot\|_{H_r})$ to V is differentiable at $M \equiv 0$, and the derivative is given by

$$J_0(M) = \mathbb{E}\left[\int_0^\infty \Theta_{0,s}^{-1}\, dM_s\, dY_s(0)\right],$$

where $\Theta_{0,\bullet}$ is the geodesic transport along $Y_\bullet(0)$.

Remark 3.6. – Since a simply connected Riemannian manifold with nonpositive sectional curvatures has 1-convex geometry, any compact convex subset V of it has also 1-convex geometry and hence satisfies the conditions of Theorem 3.5.

If W is a Riemannian manifold, then we can take V to be any regular geodesic ball with radius smaller than a constant depending only on r and an upper bound for the sectional curvatures. Moreover it is possible, using Corollary 2.9, to find an explicit expression for r_0.

If W is a manifold with connection, since we used Proposition 2.7 in the proof, we cannot give an explicit expression for r_0.

Proof of Theorem 3.5. – Set $\alpha = 1/p$. Let $x \in W$, $r \in]1,2[$ and let V be a compact convex neighbourhood of x with p-convex geometry for some $p > 1$. We will use Propositions 2.7 and 2.4, and determine conditions on V via the number p and the integrability of the inverse of the geodesic transport. The conditions on r will be determined in the proof. We identify again V with its image in an exponential chart.

Let L be an $\mathcal{F}_\infty$-measurable V-valued random variable and $M \in \mathcal{D}_L \backslash \{0\}$. For simplicity we denote by $Y_\bullet = Y_\bullet(0)$ the continuous V-valued martingale with terminal value L, by $Z_\bullet \equiv Y_\bullet(M)$ the V-valued semimartingale with terminal value L and drift $-dM\,dY(M)$. Let J' be the semimartingale given by

$$J'_t = \frac{1}{\|M\|_{H_r}} \exp_{Y_t}^{-1} Z_t \equiv \frac{1}{\|M\|_{H_r}} \exp_{Y_t}^{-1} Y_t(M). \tag{3.10}$$

For $r' \in]1, r[$, if V is sufficiently small then with the help of Proposition 2.7 and Lemma 3.2 we can define

$$J_t = J_t\left(\frac{M}{\|M\|_{H_r}}\right) = \frac{1}{\|M\|_{H_r}} \mathbb{E}\left[\int_t^\infty \Theta_{t,s}^{-1}\, dM_s\, dY_s \,\middle|\, \mathcal{F}_t\right] \tag{3.11}$$

and $|J_t|$ has a $L^{r'}$ norm bounded by a finite constant depending only on V, r and r'. The process J is also the semimartingale in TW with projection Y and $J_\infty = 0$. Its drift $\tilde{d}^{\nabla^c} J$ with respect to ∇^c is identical to the vertical lift of $-\frac{1}{\|M\|_{H_r}}\, dM\, dY$:

$$\tilde{d}^{\nabla^c} J = v_{Y_\bullet}\left(-\frac{1}{\|M\|_{H_r}}\, dM\, dY\right). \tag{3.12}$$

The latter is a consequence of [2] Theorem 4.12, which says that a TW-valued semimartingale $\mathcal{J}$ is a ∇^c-martingale if and only if $\pi(\mathcal{J})$ is a ∇-martingale and $\Theta_{0,\bullet}^{-1}\mathcal{J}_\bullet$ is a local martingale. To prove the statement of the theorem it is sufficient to prove that $J'_0 - J_0$ converges to 0 as $\|M\|_{H_r}$ tends to 0.

Let $T^p\psi$ denote the function $TW \to \mathbb{R}_+$ defined by

$$T^p\psi(w) = \lim_{a \to 0,\ a > 0} \frac{1}{a^p}\, \psi\big(\pi(w), \exp a w\big),$$

where ψ is the convex function appearing in the definition of p-convex geometry. Then $T^p\psi$ is a convex function with respect to ∇^c and

$$C|w|^p \geqslant \left|T^p\psi(w)\right| \geqslant c|w|^p \tag{3.13}$$

for some constants $0 < c < C$. Hence to show that $J_0' - J_0$ converges to 0 is equivalent to show that $T^p\psi(J_0' - J_0)$ converges to 0. The idea is to use the fact that J and J' are two semimartingales with the same projection and the same terminal value, and that they have approximatively the same drift with respect to approximatively the same connection. Under certain conditions we shall be able to show that their initial values are close.

Let $\tau = \inf\{t > 0,\ \langle M, M\rangle_t > 1\}$ (with $\inf\emptyset = \infty$) as before. Itô's formula and the convexity of $T^p\psi$ yield

$$T^p\psi(J_0' - J_0) \leqslant T^p\psi\left(J_S' - J_S\right) - \int_0^S \left\langle dT^p\psi(J' - J), d^{\nabla^c}(J' - J)\right\rangle$$

for every stopping time S. If $p > 1$ is sufficiently small, then with (3.13) and again with the help of Proposition 2.7 we obtain that the random variables $T^p\psi(J_S' - J_S)$ are uniformly integrable. This gives, using an increasing sequence of stopping times converging to τ,

$$\begin{aligned} T^p\psi(J_0' - J_0) &\leqslant \mathbb{E}\left[T^p\psi(J_\tau' - J_\tau)\right] \\ &\quad + \mathbb{E}\left[\int_0^\tau \left|\left\langle dT^p\psi(J' - J), \tilde{d}^{\nabla^c}(J' - J)\right\rangle\right|\right], \end{aligned} \tag{3.14}$$

where $\tilde{d}^{\nabla^c}(J' - J)$ denotes the drift of $J' - J$ with respect to ∇^c, as defined by (1.1).

For $r' \in]1, r[$, if $1 < p < r/r'$, we have by Proposition 3.3 $\||J'|_\tau^*\|_{r'p} \leqslant C\|M\|_{H_r}^{\alpha-1}$ and as before $\|\,|J|_\tau^*\|_{r'p} \leqslant C$. We get for the first term on the right of (3.14), under the condition $p < (r+1)/2$, taking the conjugate r'' of r' and $r''' \in]p-1, r/r''[$,

$$\begin{aligned} \mathbb{E}\left[T^p\psi(J_\tau' - J_\tau)\right] &\leqslant C\mathbb{E}\left[\left(|J_\tau'|^p + |J_\tau|^p\right)\mathbf{1}_{\{\tau<\infty\}}\right] \\ &\leqslant C\left\|\left|J_\tau'\right|^p + |J_\tau|^p\right\|_{r'}\left\|\mathbf{1}_{\{\tau<\infty\}}\right\|_{r''} \\ &\leqslant C\left\|\left|J_\tau'\right|^p + |J_\tau|^p\right\|_{r'}\mathbb{E}\left[\langle M, M\rangle_\infty^{r''r'''/2}\right]^{1/r''} \\ &\leqslant C\|M\|_{H_r}^{1-p+r'''} \end{aligned}$$

which goes to 0 as $\|M\|_{H_r}$ tends to 0.

We are left to find a bound for the second term on the right-hand side of (3.14). For this purpose we need to introduce a connection ∇^ε approximating ∇^c, for which the drift of J' has a nice expression, and the canonical involution: $s: TTW \to TTW$, given by $s(\partial_1\partial_2\alpha) = \partial_2\partial_1\alpha$ for two-parameter curves $(t_1, t_2) \mapsto \alpha(t_1, t_2)$ in W.

For $\varepsilon > 0$, let ∇^ε be the connection in TW induced from the product connection $\nabla \otimes \nabla$ in $W \times W$ by the map

$$\varphi_\varepsilon : (z, y) \mapsto \frac{1}{\varepsilon} \exp_z^{-1}(y).$$

The drift $\tilde{d}^{\nabla^\varepsilon} J'$ of J' with respect to the connection ∇^ε is the vertical lift of

$$\frac{1}{\varepsilon} \left(\exp_Y^{-1}\right)_*(Z)(-dM\, dZ).$$

Take $\varepsilon = \|M\|_{H_r}$. Note that the canonical projection $\pi_1 : (TW, \nabla^\varepsilon) \to (W, \nabla)$ is affine. We deduce that $s(\tilde{d}^{\nabla^c}(J' - J))$, $s(\tilde{d}^{\nabla^c} J)$, $s(\tilde{d}^{\nabla^c} J')$, and $s(\tilde{d}^{\nabla^{\|M\|_{H_r}}} J')$ are $T_{\tilde{d}^\nabla Y} TW$-valued vectors, where $\tilde{d}^{\nabla^c} J$ denotes the drift of the Itô differential of J with respect to ∇^c. This and the equality

$$s\left(\tilde{d}^{\nabla^c}(J' - J)\right) = s\left(\tilde{d}^{\nabla^c} J'\right) - s\left(\tilde{d}^{\nabla^c} J\right)$$

yield

$$\begin{aligned}
\tilde{d}^{\nabla^c}(J' - J) &= s\Big(s\big(\tilde{d}^{\nabla^c} J'\big) - s\big(\tilde{d}^{\nabla^c} J\big)\Big) \\
&= s\Big(s\big(\tilde{d}^{\nabla^{\|M\|_{H_r}}} J'\big) - s\big(\tilde{d}^{\nabla^c} J\big) + s\big(\tilde{d}^{\nabla^c} J'\big) - s\big(\tilde{d}^{\nabla^{\|M\|_{H_r}}} J'\big)\Big) \\
&= s\Big(s\big(\tilde{d}^{\nabla^{\|M\|_{H_r}}} J'\big) - s\big(\tilde{d}^{\nabla^c} J\big)\Big) + \big(\tilde{d}^{\nabla^c} - \tilde{d}^{\nabla^{\|M\|_{H_r}}}\big) J', \qquad (3.15)
\end{aligned}$$

where the last vector has to be considered as a vertical vector above $J' - J$.

We now estimate the last term of (3.14) using (3.15). First a calculation in local coordinates shows that for vertical vectors A,

$$\left| dT^p\psi(B)(A)\right| \leqslant C|B|^{p-1}|A|.$$

Hence from Hölder's inequality, estimate (3.2), we conclude that for $r' > \frac{r}{r+1-p}$ (its conjugate r'' has to be smaller than $\frac{r}{p-1}$ so that $|J' - J|^{(p-1)r''}$ is integrable),

$$\mathbb{E}\left[\int_0^\tau \left|\left\langle dT^p\psi(J' - J), s\Big(s\big(\tilde{d}^{\nabla^{\|M\|_{H_r}}} J'\big) - s\big(\tilde{d}^{\nabla^c} J\big)\Big)\right\rangle\right|\right]$$

is bounded by

$$C\|M\|_{H_r}^{(\alpha-1)(p-1)} \times \left\| \int_0^\tau \left| \frac{1}{\|M\|_{H_r}} \Big(\big(\exp_Y^{-1}\big)_*(Z)(-dM\,dZ) - (-dM\,dY) \Big) \right| \right\|_{r'}. \quad (3.16)$$

The $L^{r'}$ norm in the last expression is less than

$$\frac{1}{\|M\|_{H_r}} \left\| \int_0^\tau \left| \big(\exp_Y^{-1}\big)_*(Z)\big(-dM\,d(Z-Y)\big) \right| \right\|_{r'} + \frac{1}{\|M\|_{H_r}} \left\| \int_0^\tau \left| \Big(\big(\exp_Y^{-1}\big)_*(Z) - \mathrm{Id} \Big)(-dM\,dY) \right| \right\|_{r'}.$$

Now, as a consequence of (3.7) (with p' close to 2) the first term is bounded by $C\|M\|_{H_r}^{1/2}$ for $r' < \frac{16}{13}$, and by (3.2) (with p replaced by $\frac{p}{2(p-1)^2}$) the second term is dominated by $C\|M\|_{H_r}^{2(1-\alpha)^2/\alpha}$ if $r' < \frac{rp}{2p^2-3p+2}$. Hence, when

$$\frac{r}{r+1-p} < r' < \frac{rp}{2p^2-3p+2},$$

(3.16) can be estimated by $C\|M\|_{H_r}^{(1-\alpha)^2/\alpha}$ for $r<2$ sufficiently large and $p>1$ sufficiently small, which goes to 0 as $\|M\|_{H_r}$ tends to 0.

Finally to estimate the term

$$\mathbb{E}\left[\int_0^\tau \left| \big\langle dT^p\psi(J'-J), \big(\tilde{d}^{\nabla^c} - \tilde{d}^{\nabla^{\|M\|_{H_r}}}\big)J' \big\rangle \right| \right],$$

we need a bound for $(\tilde{d}^{\nabla^c} - \tilde{d}^{\nabla^\varepsilon})J'$. With (1.1), we observe that

$$\big(\tilde{d}^{\nabla^c} - \tilde{d}^{\nabla^\varepsilon}\big)J' = \big(d^{\nabla^c} - d^{\nabla^\varepsilon}\big)J' = b_\varepsilon(J')(dJ', dJ'),$$

where b_ε is a smooth section of $T^*W \otimes T^*W$. Since π_1 is affine for both ∇^c and ∇^ε, $b_\varepsilon(J')(dJ', dJ')$ is vertical. Now the relations

$$\varphi_{\lambda\varepsilon}^{-1}(u/\lambda) = \big(\pi(u), \exp(\varepsilon u)\big) = \varphi_\varepsilon^{-1}(u)$$

and

$$d^{\nabla^c}\lambda J' = s\,\lambda s\,d^{\nabla^c} J', \quad \text{for } \lambda > 0,$$

yield

$$\begin{aligned} b_\varepsilon(J')(dJ',dJ') &= s\lambda s b_{\lambda\varepsilon}(\lambda^{-1}J')\big(s\lambda^{-1}s(dJ'),s\lambda^{-1}s(dJ')\big) \\ &= \lambda b_{\lambda\varepsilon}(\lambda^{-1}J')\big(s\lambda^{-1}s(dJ'),s\lambda^{-1}s(dJ')\big). \end{aligned}$$

Moreover, on compact sets we have $|b_\varepsilon| \leqslant C\varepsilon$ ([1], proof of Proposition 3.1). Take $\lambda = |J'|$ and $\varepsilon = \|M\|_{H_r}$. Then we get

$$\begin{aligned} &\big|b_\varepsilon(J')(dJ',dJ')\big| \\ &\quad \leqslant C\varepsilon\big(\varepsilon^{-2}\langle d(Z-Y) \mid d(Z-Y)\rangle + |J'|^2\langle dY \mid dY\rangle\big). \end{aligned} \tag{3.17}$$

With (3.2), (3.17) gives, for $r' > \frac{r}{r+1-p}$,

$$\begin{aligned} &\left|\mathbb{E}\left[\int_0^\tau \big\langle dT^p\psi(J'-J),\big(d^{\nabla^c} - d^{\nabla^{\|M\|_{H_r}}}\big)J'\big\rangle\right]\right| \\ &\quad \leqslant C\|M\|_{H_r}^{(\alpha-1)(p-1)+1}\Big\|\|M\|_{H_r}^{-2}\langle (Z-Y) \mid (Z-Y)\rangle_\tau \\ &\qquad\qquad + \big(|J'|^2\big)_\tau^*\langle Y \mid Y\rangle_\tau\Big\|_{r'}. \end{aligned}$$

Taking $\frac{r}{r+1-p} < r' < \frac{4p}{3+p}$ (note that $\frac{r}{r+1-p} < \frac{4p}{3+p}$ if and only if $p < \frac{3r}{4}$), the above quantity is, by formula (3.7), less than

$$C\|M\|_{H_r}^{(\alpha-1)(p-1)+2\alpha-1} = C\|M\|_{H_r}^{(\alpha^2+\alpha-1)/\alpha}$$

which goes to 0 as $\|M\|_{H_r}$ tends to 0 for p close to 1. Together with the convergence to 0 of (3.16), we conclude that for $r < 2$ sufficiently large and $p > 1$ sufficiently small, $M \mapsto Y_0(M)$ is differentiable at $M = 0$ in H_r with derivative

$$J_0(M) = \mathbb{E}\left[\int_0^\infty \Theta_{0,s}^{-1}\,dM_s\,dY_s\right]. \qquad \square$$

4. SMOOTHNESS OF CONTINUOUS FINELY HARMONIC MAPS

Let U and W be two manifolds with torsion-free connections ∇^U and ∇^W, and let $\mathcal{L}$ be a smooth second order elliptic operator on U without zero order term. We denote by g or $\langle\cdot|\cdot\rangle$ the metric generated by $\mathcal{L}$ and by $2b$ the drift of $\mathcal{L}$ with respect to ∇^U. In coordinates, $\mathcal{L}$ can be written as $g^{ij}D_{ij} + (2b^k - g^{ij}\Gamma_{ij}^k)D_k$ where (g^{ij}) is the inverse of the metric and Γ_{ij}^k are the Christoffel symbols of the connection ∇^U.

Recall that a smooth map $u : U \to W$ is $\mathcal{L}$-harmonic if in local coordinates

$$\mathcal{L}u^\gamma + g^{ij}\Gamma^\gamma_{\alpha\beta}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j} = 0,$$

where we use Latin index in U and Greek index in W (see [4]). Smooth $\mathcal{L}$-harmonic maps are particular instances of finely harmonic maps which are defined as follows:

DEFINITION 4.1. – *A map $u : U \to W$ is said to be finely $\mathcal{L}$-harmonic if $u(X)$ is a W-valued continuous martingale for every U-valued diffusion X with generator $\frac{1}{2}\mathcal{L}$.*

Note that if u is finely $\mathcal{L}$-harmonic and φ is a C^2 positive function on U then u is also finely $\varphi^2\mathcal{L}$-harmonic. This can be proved with a time change as in [14] Section 4.

In [8] it was shown how to construct continuous finely harmonic maps as solutions to small image Dirichlet problems, and in [9] the author proved via coupling techniques that continuous finely harmonic maps are in fact smooth and $\mathcal{L}$-harmonic. The aim of this section is to derive the last result, as well as an explicit formula for the derivative, via changes of probability from the methods of this paper.

First we need some constructions. Let $u : U \to W$ be a continuous finely $\mathcal{L}$-harmonic map. Fix a small open geodesic ball V' in U such that $u(V') \subset V$ where V satisfies the conclusions of Theorem 3.5 for some $r < 2$ and has p-convex geometry for some $2 > p > 1$. Let d be the dimension of U. Via an exponential chart we can identify V' with the open ball $B(0, \pi/2)$ about 0 of radius $\pi/2$ in $\mathbb{R}^d$ (note that $\pi/2$ is not assumed to be the radius of V' as a Riemannian geodesic ball). For $x \in \mathbb{R}^d$, let

$$r(x) = \sqrt{\sum_{i=1}^{d}(x^i)^2}$$

and define

$$\eta : B(0, \pi/2) \to \mathbb{R}^d, \qquad x \mapsto \frac{\tan r(x)}{r(x)}\, x,$$

with the convention $\eta(0) = 0$. The map η is a smooth diffeomorphism. For $x \in V'$, set $\varphi(x) = \cos r(x)$.

We consider a family $(X(x))_{x\in V'}$ of diffusions on V' with generator $\frac{1}{2}\varphi^2\mathcal{L}$ and $X_0(x)=x$ for $x\in V'$, constructed as solutions of the Itô SDE

$$d^{\nabla^U} X=\varphi(X)A(X)\,dB+\varphi^2(X)b(X)\,dt, \tag{4.1}$$

where $A\in\Gamma(\mathbb{R}^d\otimes TU)$ is such that $A(x):\mathbb{R}^d\to T_xU$ is invertible and $A(x)A^*(x)=g^{-1}(x)$ for each x (here we identify $\mathbb{R}^d$ and its dual space), and B is an $\mathbb{R}^d$-valued Brownian motion.

In the coordinates of the exponential chart as defined above, (4.1) is equivalent to

$$dX^i=\varphi(X)A^i(X)\,dB+\varphi^2(X)c^i(X)\,dt, \tag{4.2}$$

where $c^i(x)=b^i(x)-\Gamma^i_{jk}(x)A^j_\alpha(x))A^k_\alpha(x)$. The coefficients A^i, c^i and all their derivatives are bounded. According to [14], for all $x\in V'$ the diffusion process $X(x)$ has infinite lifetime and converges a.s. to a random variable $X_\infty(x)$ taking its values in $\partial V'$. Let $Z(z)=\eta(X(\eta^{-1}(z)))$ for $z\in\mathbb{R}^d$. This is a diffusion in $\mathbb{R}^d$ with infinite lifetime which solves

$$dZ^i=\mathcal{A}^i(Z)\,dB+\mathcal{C}^i(Z)\,dt, \tag{4.3}$$

where in terms of $z=\eta(x)$,

$$\mathcal{A}^i(z)=\varphi(x)\,\frac{\partial z^i}{\partial x^j}\,A^j(x),$$
$$\mathcal{C}^i(z)=\varphi^2(x)\left(\frac{\partial z^i}{\partial x^j}\,c^j(x)+\frac{1}{2}\,\frac{\partial^2 z^i}{\partial x^j\,\partial x^k}\,A^j_\alpha(x)A^k_\alpha(x)\right).$$

It is then a straightforward calculation to verify that there exists a constant $C>0$ such that for all α, i, z,

$$\left|\mathcal{A}^i_\alpha(z)\right|\leqslant C\bigl(r(z)+1\bigr),\qquad \left|\mathcal{C}^i_\alpha(z)\right|\leqslant C\bigl(r(z)+1\bigr),$$

all derivatives of $z\mapsto\mathcal{A}(z)$, $z\mapsto\mathcal{C}(z)$ and $z\mapsto\mathcal{A}^{-1}(z)$ of order larger or equal to 1 are bounded, and also $z\mapsto\mathcal{A}^{-1}(z)$ is bounded. Hence, using [10, Corollary 4.6.7], we obtain that for every compact subset K of $\mathbb{R}^d$, $p>0$, $t>0$ and every multiindex β,

$$\sup_{s\leqslant t}\mathbb{E}\left[\sup_{z\in K}\left|\frac{\partial^{|\beta|}}{(\partial z)^\beta}\,Z_s(z)\right|^p\right]<\infty. \tag{4.4}$$

Let $h:\mathbb{R}_+\to[0,1]$ be a smooth decreasing function such that $h(0)=1$ and $h(t)=0$ for some $t>0$. Fix $x\in V'$ and $v\in T_xV'$. We define $F_s(v)=\eta(x)+h(s)\,d\eta(v)$ and $Z_s^v=Z_s(F_s(v))$, $X_s^v=\eta^{-1}(Z_s^v)$. The process Z^v satisfies the equation

$$dZ_s^v=\mathcal{A}(Z_s^v)\,dB_s+\mathcal{C}(Z_s^v)\,ds+(T_{F_s(v)}Z)_s\big(\dot h(s)\,d\eta(v)\big)\,ds.$$

As in [2] (see also [6]) we make a change of probability using $G^v=\mathcal{E}(M^v)$ where

$$M^v=-\int_0\big\langle\mathcal{A}^{-1}\big(Z_s^v\big)(T_{F_s(v)}Z)_s\big(\dot h(s)\,d\eta(v)\big)\,\big|\,dB_s\big\rangle_{\mathbb{R}^d}. \tag{4.5}$$

Under $\mathbb{P}^v=G^v\cdot\mathbb{P}$ (defined as in 3.1 on the subalgebras $\mathcal{F}_T$ where T is a stopping time such that $(G^v)^T$ is a uniformly integrable martingale), Z^v has the same generator as Z under $\mathbb{P}$. Hence under $\mathbb{P}^v$, the process X^v has generator $\frac{1}{2}\varphi^2\mathcal{L}$. We denote by $N(v)$ the local martingale $\frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}M^{\varepsilon v}$. For $\pi(v)=x$ fixed, the map $v\mapsto N(v)$ is linear, and we have

$$N(v)=-\int_0\big\langle\mathcal{A}^{-1}\big(Z_s(z)\big)(T_zZ)_s\big(\dot h(s)\,d\eta(v)\big)\,\big|\,dB_s\big\rangle_{\mathbb{R}^d}. \tag{4.6}$$

Note that this also writes as

$$N(v)=-\int_0\frac{\dot h(s)}{\varphi(X_s(x))}\big\langle(T_xX)_s(v)\,\big|\,A\big(X_s(x)\big)\,dB_s\big\rangle. \tag{4.7}$$

LEMMA 4.2. – *For every compact subset K of V' and $r\geqslant1$, there exist $C>0$ such that for all $v\in TU$ with $\pi(v)\in K$ and norm less than* 1, *the following estimates hold*:

$$\big\|\langle M^v,M^v\rangle_\infty^{1/2}\big\|_r\leqslant C\|v\| \tag{4.8}$$

and

$$\left\|\left(\frac{M^v}{\|v\|}-N\left(\frac{v}{\|v\|}\right)\right)_\infty^*\right\|_r\leqslant C\|v\|. \tag{4.9}$$

In particular, for every $r\geqslant1$ and $x\in K$ fixed, the map $T_xU\to H_r$, $v\mapsto M^v$ is differentiable at $v=0$.

Proof. – Let K be a compact subset of V' and let K' be a compact subset of $\mathbb{R}^d$ containing all the $\eta(x)+h(s)\,d\eta(v)$ with $x\in K$, $s\geqslant0$,

$v \in T_x U$ satisfying $\|v\| \leqslant 1$. By the boundedness of $\mathcal{A}^{-1}$ and $d\eta$ on K, together with the fact that $h(t) = 0$, we get from (4.5)

$$\langle M^v, M^v \rangle_\infty \leqslant \text{const}\|v\|^2 \int_0^t \sup_{z \in K'} \|T_s Z(z)\|^2 \, ds$$

which implies (4.8) by means of (4.4).

We are left to prove (4.9). With a similar calculation, using (4.6), the boundedness of $\mathcal{A}^{-1}$ and (4.4) with $|\beta| \leqslant 2$, we can bound the L^r norm of the bracket of $(\frac{M^v}{\|v\|} - N(\frac{v}{\|v\|}))$ for $r > 0$. □

THEOREM 4.3. – *A continuous finely $\mathcal{L}$-harmonic map $u : U \to W$ is smooth.*

Proof. – We proceed in three steps. First we show that u is differentiable, secondly we show that u is C^1 and at the end we show that u is smooth.

First step, u is differentiable. With the construction above, for $x \in V'$ and $v \in T_x U$, we have $X_0^v = \eta^{-1}(\eta(x) + d\eta(v))$, the process $Y^v = u(X^v)$ is a $\mathbb{P}^v$-martingale with values in V, starting at $Y_0^v = u(X_0^v)$ and terminating at $u(X_\infty^v)$ which depends only on $\pi(v)$. Thus, differentiability of u is a consequence of Theorem 3.5 and Lemma 4.2, and we have

$$T_x u(v) = J_0(v) = \mathbb{E}\left[\int_0^\infty \Theta_{0,s}^{-1} \, dN_s(v) \, dY_s\right], \tag{4.10}$$

where Θ is the geodesic transport above Y. This recovers a known formula, see [5] and [2].

Second step, u is C^1. We still identify V' with the open subset $B(0, \pi/2)$ of $\mathbb{R}^d$. Possibly by reducing V', the image $u(V')$ is identified via a chart with an open subset of $\mathbb{R}^{d'}$ where d' is the dimension of W. Let $m \in \mathbb{N}^*$ and $r \geqslant 1$. Recall that for an $\mathbb{R}^m$-valued continuous semimartingale $Z = Z_0 + M + A$ where M is a continuous local martingale and A is a process with finite variation, the S_r norm and the H_r norm of Z are defined as

$$\|Z\|_{S_r} = \Big\| \sup_{t \geqslant 0} |Z_t| \Big\|_{L^r}$$

and

$$\|Z\|_{H_r} = \left\| |Z_0| + \sum_i \langle M^i, M^i \rangle_\infty^{1/2} + \int_0^\infty |dA_s| \right\|_{L^r}.$$

Fix $x_0 \in V'$. We have to show that $T_x u$ converges to $T_{x_0} u$ as x tends to x_0. Using local coordinates, we can compute the difference of $T_x u$ to $T_{x_0} u$ by means of (4.10). With (4.6) and (4.4) we see that the H_r norms of N converge and it is sufficient to prove that the stopped processes $Y^t(x)$ converge to $Y^t(x_0)$ in H_r for any $r > 1$, and that if $\Theta_{0,\bullet}(x)$ denotes the geodesic transport above $Y(x)$, then $(\Theta_{0,\bullet}^{-1}(x))^t$ converges to $(\Theta_{0,\bullet}^{-1}(x_0))^t$ in S_r for r sufficiently large.

Convergence of $Y^t(x)$ to $Y^t(x_0)$: From (4.8) of Lemma 4.2 and Proposition 3.3, we conclude that if V' is sufficiently small, then $|u(x) - u(y)| \leqslant C(p)|x-y|^{1/p}$ for some $p > 1$. But the stopped process $X^t(x)$ converges to $X^t(x_0)$ in S_r for every $r > 1$, hence $Y^t(x)$ converges to $Y^t(x_0)$ in S_r for every $r > 1$. To transform convergence in S_r into convergence in H_r, we use the fact that V has 2-convex geometry, and as in (3.8), we write with $\phi = \delta^2$:

$$\begin{aligned}&\langle Y(x) - Y(x_0), Y(x) - Y(x_0) \rangle_t \\ &\quad \leqslant C\left(\phi\big(Y_t(x_0), Y_t(x)\big) - \int_0^t \big\langle d\phi, \big(d^\nabla Y(x_0), d^\nabla Y(x)\big)\big\rangle \right).\end{aligned}$$

With the same calculation as in the proof of Lemma 3.4, but simpler here since $Y(x_0)$ and $Y(x)$ are martingales, we obtain by induction that $Y^t(x)$ converges to $Y^t(x_0)$ in H_r for every $r > 1$.

Convergence of $(\Theta_{0,\bullet}^{-1}(x))^t$ to $(\Theta_{0,\bullet}^{-1}(x_0))^t$: With formula (2.5), denoting by $\Theta_i^\alpha(x)$ the coordinates of $(\Theta_{0,\bullet}^{-1}(x))^t$ and

$$S_i^j(x) = \int_0 \Gamma_{im}^j(Y)\, dM^m(x) + \frac{1}{2}\int_0 D_i \Gamma_{k\ell}^j(Y(x))\, d\langle Y^k(x), Y^\ell(x) \rangle,$$

the difference $\Theta_i^\alpha(x) - \Theta_i^\alpha(x_0)$ satisfies the equation

$$\begin{aligned} d\big(\Theta_i^\alpha(x) - \Theta_i^\alpha(x_0)\big) &= \Theta_j^\alpha(x)\, d\big(S_i^j(x) - S_i^j(x_0)\big) \\ &\quad + \big(\Theta_j^\alpha(x) - \Theta_j^\alpha(x_0)\big)\, dS_i^j(x_0). \qquad (4.11)\end{aligned}$$

This equation in $\Theta_i^\alpha(x) - \Theta_i^\alpha(x_0)$ has an explicit solution in terms of the stochastic exponential of (S_i^j) (see [13], Chapter IX, Proposition 2.3, for the one-dimensional case). By Lemma 2.1, for arbitrary large r, the S_r

norm of $\Theta_j^\alpha(x)$ is bounded by a finite constant which does not depend on x, and the stopped process $(S_i^j(x) - S_i^j(x_0))^t$ converges to 0 in H_r for every $r > 1$, hence the solution of (4.11) stopped at time t converges to 0 in S_r for r as large as we want (the size of V' depends on r).

Third step, u is smooth. We proceed by iteration, i.e., by applying step one and two to Tu and exploiting the fact that on $\{v \in TU,\ v \neq 0\}$ the differential Tu transforms again an elliptic diffusion into a TW-valued $(\nabla^W)^c$-martingale, and so on. More precisely, we argue as follows: Let $X(x)_{x\in U}$, as above, be a family of $\frac{1}{2}\varphi^2\mathcal{L}$-diffusions on U constructed as solutions of the Itô SDE

$$d^{\nabla^U} X = \varphi(X)A(X)\,dB + \varphi^2(X)b(X)\,dt. \tag{4.12}$$

In terms of an independent copy B' of B let

$$\begin{aligned} d^\nabla X^\varepsilon = {} & \cos(\varepsilon)\varphi(X^\varepsilon)A(X^\varepsilon)\,dB + \sin(\varepsilon)\varphi(X^\varepsilon)A(X^\varepsilon)\,dB' \\ & + \varphi^2(X^\varepsilon)b(X^\varepsilon)\,dt, \quad \varepsilon \in \mathbb{R}, \end{aligned}$$

be a variation of (4.12). Then, in particular, $X^\varepsilon(x)$ is also an $\frac{1}{2}\varphi^2\mathcal{L}$-diffusion for each ε which depends on ε in a differentiable way and $X^\varepsilon(x)|_{\varepsilon=0} = X(x)$. For $v \in T_xM$, $v \neq 0$, let α be a curve in U such that $\dot\alpha(0) = v$. Then

$$\partial X := \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} X^\varepsilon\big(\alpha(\varepsilon)\big)$$

is a nondegenerate diffusion on TU starting from v, which is mapped under Tu to the $(\nabla^W)^c$-martingale $Tu\,\partial X = \frac{\partial}{\partial\varepsilon}|_{\varepsilon=0} u(X^\varepsilon(\alpha(\varepsilon)))$ on W. $\square$

Remark 4.4. – In the particular setting of Riemannian manifolds U, W (equipped with the Levi-Civita connections) the fact that continuous finely harmonic maps $u_0 : U \to W$ are C^1 (and actually smooth) can also be directly derived from PDE results (like the small-time existence of solutions to the nonlinear heat equation). Finely harmonic here means that u_0 maps Brownian motions on U to martingales on W. We proceed as follows:

Let $1 < p < \infty$. Let V' be a small open (relatively compact) geodesic ball in U such that $V = u_0(V')$ has p-convex geometry, say given by $\psi = \delta^p$. We let u_0 on V' develop under the heat equation, keeping the boundary conditions fixed:

$$\begin{cases} \dfrac{\partial}{\partial t} u = \frac{1}{2}\operatorname{trace}\nabla\,du, \\ u|_{t=0} = u_0,\ u_t|\partial V' = u_0|\partial V'. \end{cases}$$

Fix $t>0$ small such that there is a classical solution on V' up to time t. In particular, $u|]0,t]\times V'$ is smooth and $u|[0,t]\times \overline{V}'$ continuous. For $x\in V'$ consider the martingales

$$Y_s^1 = u_{t-s}\big(X_s(x)\big), \qquad Y_s^2 = u_0\big(X_s(x)\big), \quad 0\leqslant s\leqslant t\wedge\sigma(x),$$

where $\sigma(x)$ is the first exit time of $X(x)$ from V'. Then $\Delta := \psi(Y^1,Y^2)$ is a nonnegative bounded submartingale with $\Delta_s=0$ for $s=t\wedge\sigma(x)$. Thus $\Delta\equiv 0$, in particular $Y_0^1=Y_0^2$. This shows $u_0=u_t$ on V', with the consequence that u_0 is smooth on V'.

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