Semi-stable reduction and maximal wild monodromy

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Notations

(K,v) is a (discretely) valued complete (or henselian) field.

O_K denotes its valuation ring.

M_K is the maximal ideal of O_K

π is a uniformizing element in the discretely valued case

$k := O_K/M_K$,the residue field, is algebraically closed of char. $p > 0$

$\lambda = \zeta - 1$ where ζ is a primitive p-th root of 1.

Abelian varieties

Theorem

(Grothendieck) Let A be an abelian variety over K. There is a finite separable extension K'/K such that the neutral component of the special fiber of the Néron model $\mathscr{A}'^0$ of $A' = A \times K'$ over $O_{K'}$ is semi-abelian (i.e. $0 \to T \to \mathscr{A}'^0 \times k \to B \to 0$ where T is a torus and B is an abelian variety over k). We say that A has semi-stable reduction over K'.

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- Let $m \geq 3$ and prime to p, if the points of m-torsion are rational over K then A has semi-stable reduction over K.
- Moreover (see [Deschamps 81]) there is a K-subscheme ${}_mE$ of the K-scheme ${}_mA$ of m division point of A such that A has semi-stable reduction over K iff the points of ${}_mE$ are K-rational (note that ${}_mE = {}_mA$ when A has good reducton over K ([Serre-Tate 68]).

Curves

Definition

A curve X/k is *semi-stable* if it is reduced and if its singularities are ordinary double points. It is *stable* if it is semi-stable, connected, projective, $p_a(X) \geq 2$ and irreducible components $\simeq \mathbb{P}^1_k$ intersect other irreducible components in at least 3 points.
A curve C/K has *semi-stable reduction* (resp. *stable reduction*) if there is a model $\mathscr{C}$ over $\operatorname{Spec} O_K$ with semi-stable (resp. stable) special fiber $\mathscr{C}_s$ over k.

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Theorem

(Deligne-Mumford 69). Let C be a smooth, projective, geometrically connected curve of genus $g \geq 2$ over K. Then there is K'/K finite separable such that $C \times K'$ has a unique stable model $\mathscr{C}$ over $O_{K'}$. The special fiber $\mathscr{C} \times k$ doesn't depend on K'/K, we refer to it as the **potential stable reduction of C.**

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C has stable reduction over K iff $\operatorname{Jac} C$ has semi-stable reduction over K.

Monodromy

Monodromy

Let X be an abelian variety or a curve over K.
There is a minimal (unique) extension K'/K such that $X \times K'$ has stable reduction. We call it the *finite monodromy extension*, its Galois group $\mathrm{Gal}(K'/K)$ is the *monodromy group* and its p-Sylow subgroup $\mathrm{Gal}(K'/K)_w$ the *wild monodromy group*.

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- The quotient group $\frac{\mathrm{Gal}(K'/K)}{\mathrm{Gal}(K'/K)_w}$ is cyclic of order e the prime to p part of $[K' : K]$. It corresponds to the tame cyclic extension $K'^t := K(\pi^{1/e}) \subset K'$ (the *tame monodromy extension*).

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- *2. For a given dimension for abelian varieties or a given genus for curves (one can also fix the type of the potential stable reduction) what are the groups in 1. which are maximal?*

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  In order to answer question 1, it is sufficient to answer question 2.

  Indeed if $G = \mathrm{Gal}(K'/K)$ for some K-curve C with genus g then any subgroup $H \subset G$ is the monodromy group of the K'^H-curve $C \times K'^H$ and K'^H is a power series field isomorphic to K.

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  If, as in the equal characteristic case, we want a fixed base field, there is a natural one $K := (\mathrm{Fr}\, W(k))^t$ (it doesn't matter if it is not discretely valued); but this time the answer to question 2 doesn't solve priori question 1.

Elliptic curves

Monodromy groups

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- $\Sigma_2(0,1) = \Sigma(0,1) \cup \{Q_8, SL_2(\mathbb{F}_3)\}$,
- $\Sigma_3(0,1) = \Sigma(0,1) \cup C_3 \rtimes C_4$
- $\Sigma_p(0,1) = \Sigma(0,1)$ for $p \geq 5$

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If E/K is an elliptic curve with non semi-stable reduction then [Serre 72], [Kraus 90], [Cali 04], the monodromy group $\mathrm{Gal}(K'/K) \in \Sigma_p(0,1)$ if the reduction is potentially good and $\in \Sigma_p(1,0)$ if the reduction is potentially multiplicative. Conversely the groups listed above occur in this way.

It follows that the wild monodromy group $\mathrm{Gal}(K'/K)_w$ belongs to

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Moreover Kraus [90] and Cali [04] (resp. Billerey [08]) , give an algorithm to calculate $\mathrm{Gal}(K'/K)$ for K an unramified (resp. a quadratic totally ramified) extension of $\mathbb{Q}_2$,

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By the functoriality of Néron model, for all prime $\ell \neq p$ there is an injection $\mathrm{Gal}(K'/K) \hookrightarrow \mathrm{Gl}_{t_{K'}-t_K}(\mathbb{Z}) \times \mathrm{Sp}_{2(a_{K'}-a_K)}(\mathbb{Q}_\ell)$

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They deduce bounds on the order (resp. the largest prime divisor of the order) of the monodromy group.

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For the wild monodromy groups their list is the set of the subgroups of:

- $\{1\}$ for $p \geq 7$.
- $\{C_5\}$ for $p = 5$
- $\{C_3 \times C_3\}$ for $p = 3$
- $\{(Q_8 \times Q_8) \rtimes C_2\}$ for $p = 2$ where C_2 exchanges the Q_8 factors.

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- 2. The description of the absolute Galois group of $k((t))$ for k an algebraically closed field of char. $p > 0$.
- 3. A cohomological argument in order to twist abelian varieties, namely:

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They ask for inequal characteristic realizations.

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In Lehr-Matignon [06], we give a proof in the case of p-cyclic covers of the projective line.

Monodromy

Let C/K a curve. From the unicity of the stable model $\mathscr{C}$ we deduce a faithful action of the monodromy group on the potential stable reduction of C:

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$\mathrm{Gal}(K'/K)$ is a semi-direct product of a cyclic group of order prime to p and a p group.

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  $p^w \leq \max\{4g, \frac{4p}{(p-1)^2} g^2\}$
- In the case of potential stable reduction with trivial toric part, one can prove using the action on ℓ torsion point of $\mathrm{Pic}^0(C)$ with $\ell \neq 2, p$ that $w \leq a + [a/p] + \ldots$, with $a = [\frac{2g}{p-1}]$,and is an optimal bound for $g \in p^{\mathbb{N}}(p-1)/2$.

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Theorem

Lehr-Matignon [05] Let (C,G) a big action. Then $\frac{|G|}{g_C^2} \geq \frac{4}{(p-1)^2}$ iff there is $\Sigma(F) \in k\{F\}$ and $f = cX + X\Sigma(F)(X) \in k[X]$ with $C \simeq C_f : W^p - W = f(X)$.

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- $\frac{|G|}{g_C^2} = \frac{4p}{(p-1)^2}$ *and $G = G_{\infty,1}(f)$ the p-Sylow subgroup of* $\mathrm{Aut}_k(C_f)$.

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Let C/k a curve of genus $g \geq 2$ and $G \subset \mathrm{Aut}_k(C)$. We say that (C,G) is *a big action* if G is a p-group and $\frac{|G|}{g} > \frac{2p}{p-1}$.

Theorem

Lehr-Matignon [05] Let (C,G) a big action. Then $\frac{|G|}{g_C^2} \geq \frac{4}{(p-1)^2}$ iff there is $\Sigma(F) \in k\{F\}$ and $f = cX + X\Sigma(F)(X) \in k[X]$ with $C \simeq C_f : W^p - W = f(X)$. Moreover there are two possibilities for G:

- $\frac{|G|}{g_C^2} = \frac{4p}{(p-1)^2}$ *and $G = G_{\infty,1}(f)$ the p-Sylow subgroup of $\mathrm{Aut}_k(C_f)$. For $\deg\Sigma(F) = s$, it is the extraspecial group of order p^{2s+1} and exponent p for $p > 2$.*

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- $\frac{|G|}{g_C^2} = \frac{4}{(p-1)^2}$ *and $G \subset G_{\infty,1}(f)$ has index p.*

Monodromy polynomial

- Let $C \longrightarrow \mathbb{P}^1_K$ be birationally given by the equation: $Z_0^p = f(X_0) = \prod_{1\leq i\leq m}(X_0 - x_i)^{n_i} \in O_K[X_0]$, $(n_i, p) = 1$ and $(n := \deg f, p) = 1$, $v(x_i - x_j) = v(x_i) = 0$ for $i \neq j$.

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- $f'(Y)/f(Y) = S_1(Y)/S_0(Y)$, $(S_0(Y), S_1(Y)) = 1$; then $\deg(S_1(Y)) = m - 1$ and $\deg(S_0(Y)) = m$.

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- $f(X+Y) = f(Y)((1 + a_1(Y)X + \ldots + a_r(Y)X^r)^p - \sum_{r+1\leq i\leq n} A_i(Y)X^i)$, where $r+1 = [n/p]$, $a_i(Y), A_i(Y) \in K(Y)$.

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- $\mathscr{L}(Y) := S_1(Y)^{p^\alpha} + pT(Y)$. This is a polynomial of degree $p^\alpha(m-1)$ which is called the *monodromy polynomial* of $f(Y)$.

Special fiber of the easy model

By easy model, we mean the O_K-model $\mathscr{C}_{O_K}$ defined by $Z_0^p = f(X_0) = \prod_{1 \leq i \leq m}(X_0 - x_i)^{n_i} \in O_K[X_0]$.

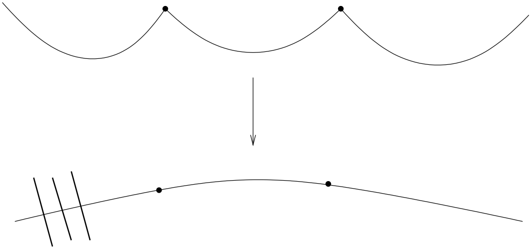

Figure: $\mathscr{C}_{O_K} \otimes_{O_K} k \longrightarrow \mathbb{P}^1_k$ with singularities and branch locus

Stable model

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- *The components with genus > 0 of the marked stable model of C correspond bijectively to the Gauss valuations v_{X_j} with $\rho_j X_j = X_0 - y_j$, where y_j is a zero of the monodromy polynomial $\mathscr{L}(Y)$*

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- *The dual graph of the special fiber of the marked stable model of* C *is an oriented tree whose ends are in bijection with the components of genus* > 0.

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$f(X+Y) := s_0 + s_1 X + s_2 X^2 + s_3 X^3 + s_4 X^4 + X^5 =$
$s_0((1 + a_1 X)^3 + A_2 X^2 + A_3 X^3 + A_4 X^4 + A_5 X^5)$

- The monodromy polynomial is the simplified numerator N_3 of A_3:
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- Finally if we write $X_0 = rT + y_i$ we get
$f(X_0) = f(y_i)((1 + a_1(y_i)rT)^3 + A_2(y_i)r^2T^2 + A_4(y_i)r^4T^4 + A_5(y_i)r^5T^5)$

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- In this way we get 9 distinct translations and as the full 3-Sylow subgroup of automorphisms of the curve $W^3 - W = 2T^4$ is the non abelian group of order 3^3 and exponent 3 it follows that we get the full 3-Sylow subgroup.

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This suggests that we can refine the problem of realizing maximal wild monodromy groups over $\mathbb{Q}_p^{\text{tame}}$ and also prescribe the branch cycle description.

Potentially good reduction with $m = 1 + p^s$

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- $p \geq 2$, $s \geq 1$, $K = \mathbb{Q}_p^{\mathrm{ur}}(p^{1/(p^s+1)}, \zeta)$, ζ *a primitive p-th root of 1. and* $C \longrightarrow \mathbb{P}^1_K$ *is birationally defined by the equation* $Z^p = f(X_0) = 1 + p^{1/(p^s+1)} X_0^{p^s} + X_0^{p^s+1}$.

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- *The monodromy group is the extraspecial group with exponent p and order* p^{2s+1} *(which is maximal for this conductor).*

Comments on the proof

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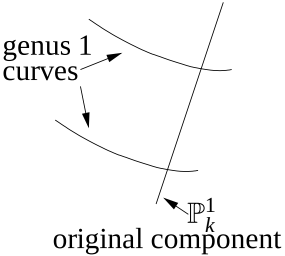

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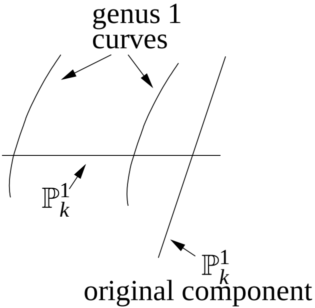

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We can give a bound for a p-Sylow subgroup $\mathrm{Syl}_p(C)$ of $\mathrm{Aut}_k(C)$.

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Then $\operatorname{Syl}_p(C_{j+1}) = \operatorname{Syl}_p(C_j) \wr \mathbb{Z}/p\mathbb{Z}$, the wreath product (i.e. the semidirect product of p copies of $\operatorname{Syl}_p(C_j)$ and $\mathbb{Z}/p\mathbb{Z}$ where this last group acts cyclically on the components). This gives equality in (1).

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