Semi-stable reduction and maximal wild monodromy

Michel Matignon

Institut of Mathematics, University Bordeaux 1

MariusFest,Groningen, April 2007

Marius and Bordeaux

Dear all,
it is a great pleasure for me to participate to Mariusfest in Groningen. To start with, I would like like to say some words about the long-standing relationships between Marius and Bordeaux. Indeed, in the eighties, as I worked in Bordeaux with Reversat under Fresnel's supervision, Marius visited us for one year on a CNRS position. Then, he gave lectures on rigid analytic geometry and organized a seminar on Drinfeld's modules and related topics, which gives birth to a book in collaboration with Jean Fresnel on rigid analytic geometry. This book met great success, as proved by its new enlarged edition in 2004. This was the opportunity for our team to learn algebraic and arithmetic geometry, which had a very strong influence on our mathematical lives. From then on, Marius got used to visit Bordeaux and Toulouse, as Reversat moved there. It proved to be a fruitful collaboration insofar as he wrote many papers with Fresnel, Liu and Reversat.

Last Thursday, I got a phone call from Reversat. He feels very sorry not to be here with us to pay tribute to Marius whom he considered, to some extent, as his maths ”guru”. My talk deals with the semi stable reduction theorem for curves, a key result in arithmetic geometry for which Marius gave a rigid analytic proof at the beginning of the eighties.

Notations

(K,v) is a (discretely) valued complete (or henselian) field.

O_K denotes its valuation ring.

M_K is the maximal ideal of O_K

π is a uniformizing element in the discretely valued case

$k := O_K/M_K$,the residue field, is algebraically closed of char. $p > 0$

$\lambda = \zeta - 1$ where ζ is a primitive p-th root of 1.

Abelian varieties

Theorem

(Grothendieck) Let A be an abelian variety over K. There is a finite separable extension K'/K such that the neutral component of the special fiber of the Néron model $\mathscr{A}'^0$ of $A' = A \times K'$ over $O_{K'}$ is semi-abelian (i.e. $0 \to T \to \mathscr{A}'^0 \times k \to B \to 0$ where T is a torus and B is an abelian variety over k). We say that A has semi-stable reduction over K'.

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- Let $m \geq 3$ and prime to p, if the points of m-torsion are rational over K then A has semi-stable reduction over K.
- Moreover (see Deschamps 81) there is a K-subscheme ${}_mE$ of the K-scheme ${}_mA$ of m division point of A such that A has semi-stable reduction over K iff the points of ${}_mE$ are K-rational (note that ${}_mE = {}_mA$ when A has good reducton over K (Serre-Tate 68)).

Curves

Definition

A curve X/k is *semi-stable* if it is reduced and if its singularities are ordinary double points. It is *stable* if it is semi-stable, connected, projective, $p_a(X) \geq 2$ and irreducible components $\simeq \mathbb{P}^1_k$ intersect other irreducible components in at least 3 points.
A curve C/K has *semi-stable reduction* (resp. *stable reduction*) if there is a model $\mathscr{C}$ over $\operatorname{Spec} O_K$ with semi-stable (resp. stable) special fiber $\mathscr{C}_s$ over k.

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Theorem

(Deligne-Mumford 69). Let C be a smooth, projective, geometrically connected curve of genus $g \geq 2$ over K. Then there is K'/K finite separable such that $C \times K'$ has a unique stable model $\mathscr{C}$ over $O_{K'}$. The special fiber $\mathscr{C} \times k$ doesn't depend on K'/K, we refer to it as the **potential stable reduction of C.**

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C has stable reduction over K iff $\operatorname{Jac} C$ has semi-stable reduction over K.

Monodromy

Monodromy

Let X be an abelian variety or a curve over K.
There is a minimal (unique) extension K'/K such that $X \times K'$ has stable reduction. We call it the *finite monodromy extension*, its Galois group $\mathrm{Gal}(K'/K)$ is the *monodromy group* and its p-Sylow subgroup $\mathrm{Gal}(K'/K)_w$ the *wild monodromy group*.

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- The quotient group $\frac{\mathrm{Gal}(K'/K)}{\mathrm{Gal}(K'/K)_w}$ is cyclic of order e the prime to p part of $[K' : K]$. It corresponds to the tame cyclic extension $K'^t := K(\pi^{1/e}) \subset K'$ (the *tame monodromy extension*).

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- *2. For a given dimension for abelian varieties or a given genus for curves what are the groups in 1. which are maximal?*

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  Indeed if $G = \mathrm{Gal}(K'/K)$ for some K-curve C with genus g then any subgroup $H \subset G$ is the monodromy group of the K'^H-curve $C \times K'^H$ and K'^H is a power series field isomorphic to K.

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  If, as in the equal characteristic case, we want a fixed base field, there is a natural one $K := (\mathrm{Fr}\, W(k))^t$ (it doesn't matter if it is not discretely valued); but this time the answer to question 2 doesn't solve question 1.

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Monodromy groups

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- $\Sigma_2(0,1) = \Sigma(0,1) \cup \{Q_8, SL_2(\mathbb{F}_3)\}$,
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- $\Sigma_p(0,1) = \Sigma(0,1)$ for $p \geq 5$

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If E/K is an elliptic curve with non semi-stable reduction then (Serre 72, Kraus 90, Cali 04), the monodromy group $\mathrm{Gal}(K'/K) \in \Sigma_p(0,1)$ if the reduction is potentially good and $\in \Sigma_p(1,0)$ if the reduction is potentially multiplicative. Conversely the groups listed above occur in this way.

It follows that the wild monodromy group $\mathrm{Gal}(K'/K)_w$ belongs to

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- $\{1\}, \{C_3\}$ for $p = 3$
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When $p = 2$, let $K = \mathbb{Q}_2$ and K^{unr} the maximal Kraus (90) has shown that the groups in this list are the monodromy groups $\mathrm{Gal}(K'K^{unr}/K^{unr}$ for elliptic curves over $K = \mathbb{Q}_2$.

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Moreover Kraus (90) and Cali (04) give an algorithm to calculate $\mathrm{Gal}(K'/K)$ for K an unramified extension of $\mathbb{Q}_2$.

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By the functoriality of Néron model, for all prime $\ell \neq p$ there is an injection $\mathrm{Gal}(K'/K) \hookrightarrow \mathrm{Gl}_{t_{K'}-t_K}(\mathbb{Z}) \times \mathrm{Sp}_{2(a_{K'}-a_K)}(\mathbb{Q}_\ell)$

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They deduce bounds on the order (resp. the largest prime divisor of the order) of the monodromy group.

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For the wild monodromy groups their list is the set of the subgroups of:

- $\{1\}$ for $p \geq 7$.
- $\{C_5\}$ for $p = 5$
- $\{C_3 \times C_3\}$ for $p = 3$
- $\{(Q_8 \times Q_8) \rtimes C_2\}$ for $p = 2$ where C_2 exchanges the Q_8 factors.

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- 2. The description of the absolute Galois group of $k((t))$ for k an algebraically closed field of char. $p > 0$.
- 3. A cohomological argument in order to twist abelian varieties, namely:

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They ask for inequal characteristic realizations.

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In Lehr-Matignon (06) we give a proof in the case of p-cyclic covers of the projective line.

Monodromy

Let C/K a curve. From the unicity of the stable model $\mathscr{C}$ we deduce a faithful action of the monodromy group on the potential stable reduction of C:

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$\mathrm{Gal}(K'/K)$ is a semi-direct product of a cyclic group of order prime to p and a p group.

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- By Stichtenoth (73), Nakajima (87) one has: $p^w \leq \max\{4g, \frac{4p}{(p-1)^2}g^2\}$
- In the case of potential stable reduction with trivial toric part, one can prove using the action on ℓ torsion point of $\mathrm{Pic}^0(C)$ with $\ell \neq 2,p$ that $w \leq a + [a/p] + \ldots$, with $a = [\frac{2g}{p-1}]$, is an optimal bound.

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Theorem

Lehr-Matignon (05) Let (C, G) a big action. Then $\frac{|G|}{g_C^2} \geq \frac{4}{(p-1)^2}$ iff there is $\Sigma(F) \in k\{F\}$ and $f = cX + X\Sigma(F)(X) \in k[X]$ with $C \simeq C_f : W^p - W = f(X)$.

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Let C/k a curve of genus $g \geq 2$ and $G \subset \mathrm{Aut}_k(C)$. We say that (C, G) is *a big action* if G is a p-group and $\frac{|G|}{g} > \frac{2p}{p-1}$.

Theorem

Lehr-Matignon (05) Let (C, G) a big action. Then $\frac{|G|}{g_C^2} \geq \frac{4}{(p-1)^2}$ iff there is $\Sigma(F) \in k\{F\}$ and $f = cX + X\Sigma(F)(X) \in k[X]$ with $C \simeq C_f : W^p - W = f(X)$. Moreover there are two possibilities for G:

- $\frac{|G|}{g_C^2} = \frac{4p}{(p-1)^2}$ *and $G = G_{\infty,1}(f)$ the p-Sylow subgroup of* $\mathrm{Aut}_k(C_f)$.

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- $\frac{|G|}{g_C^2} = \frac{4}{(p-1)^2}$ *and $G \subset G_{\infty,1}(f)$ has index p.*

Monodromy polynomial

- Let $C \longrightarrow \mathbb{P}^1_K$ be birationally given by the equation: $Z_0^p = f(X_0) = \prod_{1\le i\le m}(X_0 - x_i)^{n_i} \in O_K[X_0]$, $(n_i, p) = 1$ and $(n := \deg f, p) = 1$, $v(x_i - x_j) = v(x_i) = 0$ for $i \neq j$.

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- $f'(Y)/f(Y) = S_1(Y)/S_0(Y)$, $(S_0(Y), S_1(Y)) = 1$; then $\deg(S_1(Y)) = m - 1$ and $\deg(S_0(Y)) = m$.

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- $f(X+Y) = f(Y)((1 + a_1(Y)X + \ldots + a_r(Y)X^r)^p - \sum_{r+1\leq i\leq n} A_i(Y)X^i)$, where $r + 1 = [n/p]$, $a_i(Y), A_i(Y) \in K(Y)$.

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- $\mathscr{L}(Y) := S_1(Y)^{p^\alpha} + pT(Y)$. This is a polynomial of degree $p^\alpha(m-1)$ which is called the *monodromy polynomial* of $f(Y)$.

Special fiber of the easy model

By easy model, we mean the O_K-model $\mathscr{C}_{O_K}$ defined by $Z_0^p = f(X_0) = \prod_{1\leq i\leq m}(X_0 - x_i)^{n_i} \in O_K[X_0]$.

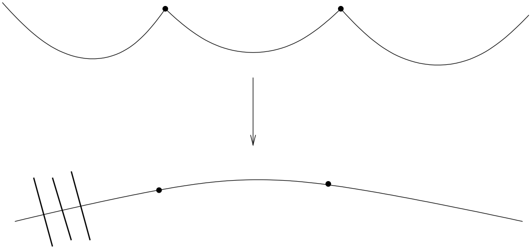

Figure: $\mathscr{C}_{O_K} \otimes_{O_K} k \longrightarrow \mathbb{P}_k^1$ with singularities and branch locus

Stable model

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- *The components with genus* > 0 *of the marked stable model of* C *correspond bijectively to the Gauss valuations* v_{X_j} *with* $\rho_j X_j = X_0 - y_j$, *where* y_j *is a zero of the monodromy polynomial* $\mathscr{L}(Y)$

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- $\rho_j \in {O_K}^{\mathrm{alg}}$ *satisfies* $v(\rho_j) = \max\{\frac{1}{i} v\left(\frac{\lambda^p}{A_i(y_j)}\right)$ *for* $r+1 \leq i \leq n\}$.

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- *The dual graph of the special fiber of the marked stable model of* C *is an oriented tree whose ends are in bijection with the components of genus* > 0.

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$f(X + Y) := s_0 + s_1X + s_2X^2 + s_3X^3 + s_4X^4 + X^5 =$
$s_0((1 + a_1X)^3 + A_2X^2 + A_3X^3 + A_4X^4 + A_5X^5)$

- The monodromy polynomial is the simplified numerator N_3 of A_3:
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- Finally if we write $X_0 = rT + y_i$ we get $f(X_0) = f(y_i)((1 + a_1(y_i)rT)^3 + A_2(y_i)r^2T^2 + A_4(y_i)r^4T^4 + A_5(y_i)r^5T^5)$

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- In this way we get 9 distinct translations and as the full 3-Sylow subgroup of automorphisms of the curve $W^3 - W = 2T^4$ is the non abelian group of order 3^3 and exponent 3 it follows that we get the full 3-Sylow subgroup.

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This suggests that we can refine the problem of realizing maximal wild monodromy groups over $\mathbb{Q}_p^{\mathrm{tame}}$ and also prescribe the branch cycle description.

Potentially good reduction with $m = 1 + p^s$

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- $p \geq 2$, $s \geq 1$, $K = \mathbb{Q}_p^{ur}(p^{1/(p^s+1)}, \zeta)$, ζ *a primitive p-th root of 1. and* $C \longrightarrow \mathbb{P}^1_K$ *is birationally defined by the equation* $Z^p = f(X_0) = 1 + p^{1/(p^s+1)} X_0^{p^s} + X_0^{p^s+1}$.

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- *Then, C has potentially good reduction with special fiber birational to the curve $w^p - w = t^{p^s+1}$ and $\mathscr{L}(Y)$ is irreducible over K.*
- *The monodromy extension K'/K is the decomposition field of $\mathscr{L}(Y)$ obtained by adjoining the p-roots $f(y)^{1/p}$, for y describing the zeroes of $\mathscr{L}(Y)$.*

Potentially good reduction with $m = 1 + p^s$

Theorem

- $p \geq 2$, $s \geq 1$, $K = \mathbb{Q}_p^{\mathrm{ur}}(p^{1/(p^s+1)}, \zeta)$, ζ *a primitive p-th root of 1. and* $C \longrightarrow \mathbb{P}^1_K$ *is birationally defined by the equation* $Z^p = f(X_0) = 1 + p^{1/(p^s+1)} X_0^{p^s} + X_0^{p^s+1}$.
- *Then, C has potentially good reduction with special fiber birational to the curve* $w^p - w = t^{p^s+1}$ *and* $\mathscr{L}(Y)$ *is irreducible over K.*
- *The monodromy extension* K'/K *is the decomposition field of* $\mathscr{L}(Y)$ *obtained by adjoining the p-roots* $f(y)^{1/p}$, *for y describing the zeroes of* $\mathscr{L}(Y)$.
- *The monodromy group is the extraspecial group with exponent p and order* p^{2s+1} *(which is maximal for this conductor).*

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$f(X+y) = (s_0(y)^{1/p} + X(a_s(y,X))^p + X^{1+p^s} \mod \lambda^p M_K$, $a_s(y,X) \in M_K[T]$

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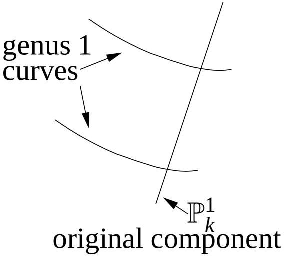

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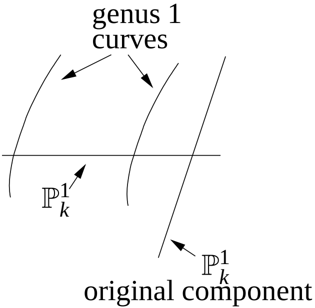

Type 2
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G

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