Mathematical Models and Methods in Applied Sciences

STABILITY OF ACOUSTIC PROPAGATION IN A 2D FLOW DUCT: A LOW FREQUENCY APPROACH

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In this paper we derive a simplified “quasi-1D” limit model for the propagation of low frequency acoustic waves in a laminar flow filling a 2D duct. We analyze the well-posedness of this model in function of he Mach profile of the flow. This can be reduced to the study of the spectrum of a bounded non-normal operator. As a by-product of this analysis, we establish new resulta for hydrodynamic instabilities of Kelvin-Helnholtz type in compressible fluids.

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1. Introduction

The present work has been motivated by applications to computational aeroacoustics, namely the numerical modelling of the propagation of sound in a moving fluid. This problem is of very great importance from the industrial point of view, particularly in aeronautics : the question of noise reduction from airplanes is an important question from both environmental and economical issues. The mathematical models used for the numerical simulations are obtained from the linearization (acoustic perturbations are small perturbations) of the equations of fluid mechanics around a reference flow, i. e. a stationary state described through a reference velocity distribution. By this way one can derive either the well-known linearized Euler equations or or the Galbrun's equations [7], that are less popular but equivalent from the mathematical point of view. One delicate issue in computational aeroacoustics is the modeling of the interaction between acoustic waves and walls, a question that cannot be avoided, for instance, when studying the propagation of sound around an airplane. What makes the problem difficult is the presence in the fluid of a thin boundary layer in the reference flow at the neighbourhood of the walls. If one wants to avoid a dramatic mesh refinement at the neighborhood of physical boundaries, with all the technical problems that this induces, in particular in the time dependent case, a natural idea is to use an equivalent or effective boundary condition that would represent the conjugated effect of the wall itself and of the thin boundary layer. Many engineers have proposed several boundary conditions for computational aeroacoustics, to begin with the well-known Myers condition [15]. However, as this has been recently pointed out in [2], all the boundary conditions proposed in the literature lead to strongly ill-posed boundary value problems, even in the simple situation of a flat boundary and a uniform reference flow parallel to this boundary. Our interpretation is that these boundary conditions do not take into account properly the presence of the thin boundary layer. To solve this problem, it appears crucial to understand the propagation of sound in such a thin layer. In this perspective, we have chosen to reconsider the problem from a fundamental point of view and to begin with the propagation of sound between two parallel flat boundaries (a thin tube in 2D) in the presence of a strongly varying parallel flow. In this case, the reference flow can be simply described with the help of a scalar function $M(y)$ which represents, after appropriate normalisation (it is the Mach number), the lateral variations of the velocity of the reference flow. We shall call this function the Mach profile. Intuitively one would expect some "1D behaviour" of the solution, thus a 1D-like simplified model for the propagation of sound, as it is the case in the absence of flow (standard acoustic propagation in a fluid at rest).

Our objective in this paper is to derive such a simplified model using a formal asymptotic analysis with respect to the width of the tube. This can also be seen as a low frequency analysis : the small parameter should be the ratio between the width of he tube and the wavrlength. We use Galbrun's equations, which appear to be better adapted for the asymptotic analysis than linearized Euler equations. The model that we shall obtain in section 2.1, is a quasi-1D model. This model is local (differential) in x, the coordinate along the axis of the tube, but non local in y, the transversal coordinate. Most of the rest of the article is devoted to the analysis of this limit model. Because of the non-standard structure of this problem, it appears that classical results (from semi-group theory for instance) can not be applied and that an ad hoc analysis is needed. Using Fourier transform along x, the analysis can be reduced to a family of one dimensional problems in y, and then to the spectral theory of a bounded operator $A \in \mathcal{L}\big(L^2[-1,1]^2\big)$ (of non-local nature). This is the object of section 2.2. In spite of its apparent simplicity, the problem has rather surprising properties because of the non normality of A. In section 3 we relate the well-posedness of the quasi-1D model to the non-existence of non real eigenvalues. This property clearly depends on the properties of the function $M(y)$, which allows us to make the distinction between stable and unstable Mach profiles. This is illustrated at the end of section 3 by the analysis of elementary cases and through numerical calculations. Sections 4, 5 and 6 are the main sections of this paper. In section 4, we establish sufficient stability conditions for the Mach profile, using an appropriate approximation process. In section 5, we exhibit unstable profiles by studying the particular case of odd profiles (Theorems 5.3 and 5.6). Finally, the content of section 6 is a by-product of our analysis : in the case of unstable profiles, the ill-posedness of the limit model implies the exponential blow-up in time of the solution of the original problem (Theorem 6.1).

That is why this paper also provides a contribution to the analysis of hydrodynamic instabilities in laminar flows (often called Kelvin-Helmholtz instabilities). The study of the existence [4] and nature [11] (convective or absolute instabilities) of hydrodynamic instabilities is an important issue in fluid mechanics. The results are generally established on the linearized model but reveal also instabilities for the nonlinear case (see [8,9] for instance). Up to our knowledge, most of the results are established in the compressible case (let us however mention [6,13] for very particular situations in the compressible case) and that is why that we believe that our results are new, even though one can emphasize similarities between our results and results for the compressible case (see Remark 4.2). Technically, which makes the study of the incompressible case easier is the existence of a velocity potential, which leads to study a simple scalar differential equation, known as the Rayleigh's equation. There is no equivalent of this equation for the incompressible case but, in some sense, our low frequency model provides such an equation (not differential but of integral type. We think that the method we develop here is original, even though that one should notice that using a low frequency (or high frequency) asymptotic analysis has been used for the study of Rayleigh-Taylor instabilities [3,10].

2. The low frequency model: derivation and well-posedness issues

2.1. *Formal derivation of the asymptotic model*

Let us consider a thin two-dimensional duct of height $2\epsilon > 0$:

$$\Omega_\epsilon = \mathbb{R} \times \,]-\epsilon, \epsilon[\tag{2.1}$$

filled with a perfect compressible fluid which is moving. We suppose that the flow is stationary and laminar. If $(x, \mathbf{y})$ denotes a current point of Ω_ϵ, the flow is characterized by its Mach profile which is given by

$$\forall\, \mathbf{y} \in \,]-\epsilon, \epsilon[\,, \quad M_\epsilon(\mathbf{y}) = M(\frac{\mathbf{y}}{\epsilon}) \tag{2.2}$$

where $M(y),\ y \in \,]-1, 1[$ is a given fixed profile such that $M \in L^\infty(]-1, 1[)$.

Sound propagation in a duct can be modelized for instance by the following equations [7] whose unknowns $\mathbf{u}_\epsilon(x,\mathbf{y},t)$ and $\mathbf{v}_\epsilon(x,\mathbf{y},t)$ are respectively the x and the $\mathbf{y}$ components of the perturbation of Lagrangian displacement (the velocity of sound is taken equal to 1 in the following):

$$\left\{\begin{aligned} &\left(\frac{\partial}{\partial t}+M_\epsilon(\mathbf{y})\,\frac{\partial}{\partial x}\right)^2\mathbf{u}_\epsilon-\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{u}_\epsilon}{\partial x}+\frac{\partial \mathbf{v}_\epsilon}{\partial \mathbf{y}}\right)=0,\\ &\left(\frac{\partial}{\partial t}+M_\epsilon(\mathbf{y})\,\frac{\partial}{\partial x}\right)^2\mathbf{v}_\epsilon-\frac{\partial}{\partial \mathbf{y}}\left(\frac{\partial \mathbf{u}_\epsilon}{\partial x}+\frac{\partial \mathbf{v}_\epsilon}{\partial \mathbf{y}}\right)=0, \end{aligned}\right. \qquad (x,\mathbf{y})\in\Omega_\epsilon,\quad t>0. \tag{2.3}$$

These equations must be completed by the slip condition on the walls $\mathbf{y}=\pm\,\epsilon$:

$$\mathbf{v}_\epsilon(x,\pm\,\epsilon,t)=0,\quad x\in\mathbb{R},\quad t>0. \tag{2.4}$$

To derive an asymptotic model when ϵ tends to 0, we first apply classically the following scaling ($\mathbf{y}=\epsilon\; y$) in order to work in a fixed geometrical domain $\Omega=\mathbb{R}\times\,]-1,1[$. Setting

$$\mathbf{u}_\epsilon(x,\mathbf{y},t)=u_\epsilon(x,\frac{\mathbf{y}}{\epsilon},t)\ \text{ and }\ \mathbf{v}_\epsilon(x,\mathbf{y},t)=v_\epsilon(x,\frac{\mathbf{y}}{\epsilon},t), \tag{2.5}$$

Galbrun's equations can be rewritten as follows:

$$\left\{\begin{aligned} &\left(\frac{\partial}{\partial t}+M(y)\,\frac{\partial}{\partial x}\right)^2 u_\epsilon-\frac{\partial}{\partial x}\left(\frac{\partial u_\epsilon}{\partial x}+\frac{1}{\epsilon}\,\frac{\partial v_\epsilon}{\partial y}\right)=0,\\ &\left(\frac{\partial}{\partial t}+M(y)\,\frac{\partial}{\partial x}\right)^2 v_\epsilon-\frac{1}{\epsilon}\,\frac{\partial}{\partial y}\left(\frac{\partial u_\epsilon}{\partial x}+\frac{1}{\epsilon}\,\frac{\partial v_\epsilon}{\partial y}\right)=0, \end{aligned}\right. \qquad (x,y)\in\Omega,\quad t>0, \tag{2.6}$$

while the boundary condition becomes:

$$v_\epsilon(x,\pm\,1,t)=0,\quad x\in\mathbb{R},\quad t>0. \tag{2.7}$$

Postulating a formal asymptotic expansion of the form

$$\left\{\begin{aligned} u_\epsilon&=u^0+\epsilon\;u^1+\epsilon^2\;u^2+\cdots\,,\\ v_\epsilon&=v^0+\epsilon\;v^1+\epsilon^2\;v^2+\cdots\,, \end{aligned}\right. \tag{2.8}$$

where functions u^j and v^j are independent of ϵ, we will derive the limit problem satisfied by (u^0,v^0). We proceed in three steps:

(i) The term in ϵ^{-2} in the second equation of (2.6) gives:

$$\frac{\partial^2 v^0}{\partial y^2}=0.$$

Combined with the boundary condition $v^0(x,\pm\,1,t)=0$ this implies:

$$v^0(x,y,t)=0.$$

(ii) The term in ϵ^{-1} in the second equation of (2.6) gives:

$$\frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^1}{\partial y}\right)=0,$$

As a consequence there exists a function $c(x,t)$ independent of y such that

$$\Big(\frac{\partial u^0}{\partial x} + \frac{\partial v^1}{\partial y}\Big)(x,y,t) = c(x,t).$$

The expression of function $c(x,t)$ is derived by integrating with respect to y and using $v^1(x, \pm 1, t) = 0$:

$$c(x,t) = \frac{1}{2}\int_{-1}^{1} \frac{\partial u^0}{\partial x}(x,\xi,t)\, d\xi.$$

(iii) The term in ϵ^0 in the first equation of (2.6) gives:

$$\Big(\frac{\partial}{\partial t} + M(y)\,\frac{\partial}{\partial x}\Big)^2 u^0 - \frac{\partial}{\partial x}\left(\frac{\partial u^0}{\partial x} + \frac{\partial v^1}{\partial y}\right) \equiv \Big(\frac{\partial}{\partial t} + M(y)\,\frac{\partial}{\partial x}\Big)^2 u^0 - \frac{\partial c}{\partial x} = 0.$$

Using the expression of $c(x,t)$, we get finally:

$$\left(\frac{\partial}{\partial t} + M(y)\,\frac{\partial}{\partial x}\right)^2 u^0 - \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\int_{-1}^{1} u^0(x,\xi,t)\, d\xi = 0. \tag{2.9}$$

From now on, we omit the exponent 0 and denote u instead of u^0. We also introduce the following useful notation:

$$E\,u(x,t) = \frac{1}{2}\int_{-1}^{1} u(x,y,t)\, dy \tag{2.10}$$

and equation (2.9) becomes

$$\left(\frac{\partial}{\partial t} + M(y)\,\frac{\partial}{\partial x}\right)^2 u - \frac{\partial^2}{\partial x^2} Eu = 0. \tag{2.11}$$

The object of this paper is the analysis of the well-posedness of (2.11) with appropriate initial conditions, which correspomnds to the following evolution problem:

$$\begin{cases} \text{Find } u(x,y,t) : \mathbb{R}\times[-1,1]\times\mathbb{R}^+ \to \mathbb{R}, \\[2mm] (\dfrac{\partial}{\partial t} + M(y)\,\dfrac{\partial}{\partial x})^2\, u - \dfrac{\partial^2}{\partial x^2} Eu = 0, \quad (x,y)\in\mathbb{R}\times[-1,1], \quad t>0, \\[2mm] u(x,y,0) = u_0(x,y), \quad (x,y)\in\mathbb{R}\times[-1,1], \\[2mm] \dfrac{\partial u}{\partial t}(x,y,0) = u_1(x,y), \quad (x,y)\in\mathbb{R}\times[-1,1]. \end{cases} \tag{2.12}$$

where (u_0, u_1) are the initial data. Proving the stability of (2.12), which is strongly related to the behavior of the profile $M(y)$, has at least two kinds of motivation:

(1) The well-posedness of (2.12) is required, if (2.12) is used as an approximation of the initial problem (2.6).
(2) Stability properties of (2.6) (or equivalently (2.3)) can be deduced from the properties of (2.12). In particular, if (2.12) is unstable, one can guess that (2.6) will be also unstable at low frequency.

We notice finally that (2.12) is well-posed in the simple case of the uniform flow ($M(y) = M$). Indeed in this case, taking the mean value of the equation shows that $\psi = Eu$ is a solution of the convected wave equation

$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)^2 \psi - \frac{\partial^2 \psi}{\partial x^2} = 0$$

which can be easily proved to be well-posed. However, we will prove in the sequel that stability fails for several profiles, which is not so surprising since, in some sense, (2.12) can be seen as a perturbation of a weakly hyperbolic system (the squared advection equation).

2.2. *Reduction to a spectral problem*

A classical approach for the analysis of stability of a physical model which is invariant under translation in one space coordinate x, is the determination of the so-called normal modes which are the non trivial solutions of (2.11) of the form (where the function of y is abusively also denoted by u)

$$u(x, y, t) = u(y)\ \exp i(\omega t - kx), \quad k \in \mathbb{R}, \quad \omega \in \mathbb{C}. \tag{2.13}$$

The justification of such an analysis relies on the use of the Fourier transform in x. Injecting (2.13) in equation (2.11) leads to

$$-(\omega - Mk)^2 u + k^2 Eu = 0 \tag{2.14}$$

where the mean-value operator E can now be seen as a bounded operator acting on $L^2(]-1, 1[)$, which associates to a function u the constant function equal to its mean value:

$$Eu = \frac{1}{2}\int_{-1}^{1} u(y)\, dy.$$

Setting $\omega = k\lambda$ and $\dot{u} = (\lambda - M)u$, the previous equation can be written as follows:

$$\begin{cases} Mu + \dot{u} = \lambda\, u, \\ \\ Eu + M\dot{u} = \lambda\, \dot{u}. \end{cases} \tag{2.15}$$

It is an eigenvalue problem and the physical instabilities are related to the non-real eigenvalues λ.

Introducing the bounded operator A of $L^2(]-1, 1[)^2$ defined by:

$$\forall\ (u, \dot{u})^t \in L^2(]-1, 1[)^2, \quad A\begin{pmatrix} u \\ \\ \dot{u} \end{pmatrix} = \begin{pmatrix} M & I \\ \\ E & M \end{pmatrix}\begin{pmatrix} u \\ \\ \dot{u} \end{pmatrix}, \tag{2.16}$$

the system (2.15) takes the following form

$$A\begin{pmatrix} u \\ \\ \dot{u} \end{pmatrix} = \lambda \begin{pmatrix} u \\ \\ \dot{u} \end{pmatrix}. \tag{2.17}$$

Most of this paper will be devoted to the spectral analysis of the (non-selfadjoint) operator A, the main question being the existence or non-existence of non-real spectrum.

In fact, the equivalence between the well-posedness of (2.12) and the absence of non-real spectrum of A is not obvious and counter-examples exist for non-normal operators in other applications. Indeed, it is clear that the well-posedness does not hold if unstable modes (with $\lambda \notin \mathbb{R}$) exist (see [1] for a more precise statement) but the converse is not straightforward. Using a Fourier analysis, well-posedness of (2.12) can be obtained as a consequence of a uniform estimate of the semi-groups e^{iktA} for $k \in \mathbb{R}$. The difficult question is then to derive this uniform bound. A possible approach consists in using the Fourier-Laplace transform in (x, t) and complex variable techniques, as it is done in [12].

For simplicity, we will use in the sequel the following definition:

Definition 2.1. A profile $M(y)$ is said stable (resp. unstable) if $\sigma(A) \subset \mathbb{R}$ (resp. $\sigma(A) \not\subset \mathbb{R}$).

3. General properties of the spectrum

Since the function M is real, $\sigma(A)$ is symmetric with respect to the real axis:

$$\lambda \in \sigma(A) \Rightarrow \bar{\lambda} \in \sigma(A).$$

Notice also that A is non-normal ($AA^* \neq A^*A$) but one can easily check that A and A^* are similar operators:

$$A = S\, A^*\, S \text{ with } S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Both operators A and A^* have therefore the same eigenvalues and the same spectrum. This proves in particular that A has no residual spectrum.

3.1. *The continuous spectrum*

Let us first point out that

$$A = A_0 + B, \tag{3.1}$$

where

$$A_0 = \begin{pmatrix} M & I \\ 0 & M \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}. \tag{3.2}$$

The operator B has rank 1 and is a fortiori compact. The operator A_0 is non-normal but one can however easily determine its spectrum which coincides with the essential range of function M:

$$\sigma(A_0) = R(M) \ , \tag{3.3}$$

where the essential range is defined by

$$\lambda \in R(M) \quad \Longleftrightarrow \quad \forall\, \delta > 0, \quad mes\,\big\{\, y \in]-1,1[\ / \ M(y) \in]\lambda-\delta, \lambda+\delta[\,\big\} > 0. \tag{3.4}$$

Since A is a compact perturbation of A_0, we expect the perturbation B to only produce a possible discrete spectrum outside the range of M. Indeed this holds: it can be proved directly (see [1]) or deduced from non-trivial results of spectral theory for non-selfadjoint operators [17] (lemma XIII 4.3), using the fact that $\sigma(A_0)$ has an empty interior :

Theorem 3.1. *The range of the Mach profile $R(M)$ is included in $\sigma(A)$ and its complementary set in $\sigma(A)$ is the discrete spectrum of A (which is by definition the set of eigenvalues of finite multiplicity which are isolated in the spectrum).*

3.2. *The eigenvalues: general properties*

The following result gives a very useful characterization of the discrete spectrum:

Lemma 3.1. *The following equivalence holds:*

$$\lambda \in \sigma(A) \setminus R(M) \quad \Longleftrightarrow \quad F(\lambda) = 1 \tag{3.5}$$

where

$$F(\lambda) = \frac{1}{2} \int_{-1}^{1} \frac{dy}{\big(\lambda - M(y)\big)^2}, \quad \lambda \in D = \mathbb{C} \setminus R(M) \ . \tag{3.6}$$

Moreover the discrete spectrum $\sigma(A) \backslash R(M)$ contains only simple eigenvalues.

Proof. Let $\lambda \in \sigma(A) \setminus R(M)$; by theorem 3.1 λ is an eigenvalue of A. If $(u, \dot{u})^t$ denotes an associated eigenvector, one has:

$$\begin{cases} (\lambda - M)^2\, u = Eu, \\ \\ \dot{u} = (\lambda - M)\, u. \end{cases}$$

The function u cannot be equal to 0 and its mean value Eu is also not equal to 0 since $(\lambda - M)^2 u = 0$ implies $u = 0$. Therefore

$$u = \frac{Eu}{(\lambda - M)^2} \text{ and } \dot{u} = \frac{Eu}{\lambda - M} \tag{3.7}$$

which proves that the eigenvalue is simple. Integrating the first equality of (3.7) between -1 and 1 and using again $Eu \neq 0$ gives the characterization. $\square$

Corollary 3.1. *A profile $M(y)$ is stable (in the sense of definition 2.1) if and only if the equation $F(\lambda) = 1$ has no solutions λ such that $\lambda \notin \mathbb{R}$. Equivalently, a profile*

$M(y)$ is unstable (in the sense of definition 2.1) if and only if there exists $\lambda \notin \mathbb{R}$ such that $F(\lambda) = 1$.

We deduce from lemma 3.1 the existence of at most two real eigenvalues outside the range of M. Let us set:

$$M_- = \inf M(y), \quad M_+ = \sup M(y). \tag{3.8}$$

Theorem 3.2. *The operator A has at most one eigenvalue λ_- in the interval $]-\infty, M_-[$ and at most one eigenvalue λ_+ in the interval $]M_+, +\infty[$. More precisely, the eigenvalue λ_- exists if and only if*

$$1 < \frac{1}{2}\int_{-1}^{1} \frac{dy}{\big(M_- - M(y)\big)^2} \leq +\infty,$$

and it is characterized by:

$$\lambda_- < M_- \quad \text{and} \quad F(\lambda_-) = 1.$$

Likewise, the eigenvalue λ_+ exists if and only if

$$1 < \frac{1}{2}\int_{-1}^{1} \frac{dy}{\big(M_+ - M(y)\big)^2} \leq +\infty,$$

and it is characterized by:

$$\lambda_+ > M_+ \quad \text{and} \quad F(\lambda_+) = 1.$$

Proof. Let us study the function F in the real interval $]-\infty, M_-[\cup] M_+, +\infty[$ where it takes real values. Its derivative is given by:

$$F'(\lambda) = -\int_{-1}^{1} \frac{dy}{(\lambda - M(y))^3},$$

which proves that F is strictly increasing on $]-\infty, M_-[$ and strictly decreasing on $]M_+, +\infty[$. Moreover, using Lebesgue monotone convergence theorem, one has:

$$\lim_{\lambda \to \pm\infty} F(\lambda) = 0, \quad \lim_{\lambda \to M_\pm} F(\lambda) = \frac{1}{2}\int_{-1}^{1} \frac{dy}{\big(M_\pm - M(y)\big)^2} \leq +\infty.$$

The result follows easily by the intermediate value theorem. $\square$

Remark 3.1. If $M(y)$ reaches the value $M_\pm$ at a point $y_\pm$ where it is C^1 at the left or at the right, one check easily that

$$\int_{-1}^{1} \frac{dy}{\big(M_\pm - M(y)\big)^2} = +\infty,$$

which ensures the existence of $\lambda_\pm$.

From the two previous sections, we deduce that investigating the existence of non real eigenvalues of A amounts to study the possible existence of non real solutions of the characteristic equation (3.5). The rest of the paper is devoted to this question.

3.3. *Some elementary examples*

3.3.1. *The uniform profile*

The simplest case corresponds to the case of uniform profile:

Lemma 3.2. *Suppose $M(y) = M$, $\forall y \in]-1, 1[$. Then*

$$\sigma(A) = \{\lambda_-, M, \lambda_+\}$$

with $\lambda_\pm = M \pm 1$, and $M(y)$ is stable (in the sense of definition 2.1).

Proof. The lemma is a straightforward consequence of lemma 3.1. Indeed we have for a uniform flow:

$$F(\lambda) = \frac{1}{(\lambda - M)^2}$$

and $\lambda_\pm$ are the two roots of equation $F(\lambda) = 1$. □

The eigenvalues λ_- and λ_+ are associated to classical acoustic convected modes. If the flow is subsonic ($|M| < 1$), they propagate respectively downstream and upstream with phase velocities equal to $1+|M|$ and $1-|M|$. If the flow is supersonic, both of them propagate downstream.

3.3.2. *The linear profile*

Let us consider now the case of a linear profile:

Lemma 3.3. *Suppose $M(y) = My + m$, $\forall y \in]-1, 1[$. Then*

$$\sigma(A) = \{\lambda_-, \lambda_+\} \cup [-M + m, M + m]$$

with $\lambda_\pm = m \pm \sqrt{1 + M^2}$, and $M(y)$ is stable (in the sense of definition 2.1).

Proof. A simple integration gives:

$$F(\lambda) = \frac{1}{2M}\left(\frac{1}{\lambda - m - M} - \frac{1}{\lambda - m + M}\right) = \frac{1}{(\lambda - m)^2 - M^2}$$

and one can check again that $\lambda_\pm$ are the two roots of equation $F(\lambda) = 1$. □

Notice that there are two propagative modes which propagate respectively with a phase velocity equal to $\lambda_\pm$. Let us point out that the two modes propagate in opposite directions if $|m| < \sqrt{1 + M^2}$ and both propagate downstream if $|m| \geq \sqrt{1 + M^2}$ (this last condition does not impose to the flow to be everywhere supersonic).

Fig. 1. The function $F(\lambda)$ for a uniform (left) or linear (right) Mach profile.

3.3.3. *The step profile*

Let us consider finally the case of a step profile taking two different values.

Lemma 3.4. *Suppose*

$$M(y) = \begin{cases} M_1 & \text{if } -1 < y < a \\ M_2 & \text{if } a < y < 1 \end{cases}$$

with $-1 < a < 1$ *and* $M_1 \neq M_2$*, and define* $\delta(a)$ *by:*

$$\delta(a) = \frac{1}{\sqrt{2}} \left((1-a)^{1/3} + (1+a)^{1/3} \right)^{3/2}.$$

If $|M_1 - M_2| \neq \delta(a)$*, then the operator* A *has exactly four eigenvalues different from* M_1 *and* M_2 *and the following alternative holds:*

- *if* $|M_1 - M_2| > \delta(a)$*, the four eigenvalues are real,*
- *if* $|M_1 - M_2| < \delta(a)$*, two eigenvalues are real and two eigenvalues are non real and complex conjugated.*

If $|M_1 - M_2| = \delta(a)$*, then the operator* A *has exactly three real eigenvalues different from* M_1 *and* M_2*. In conclusion, the profile* $M(y)$ *is stable (in the sense of definition 2.1) if and only if* $|M_1 - M_2| \geq \delta(a)$*.*

Proof. For the step profile, the function F which has the following expression

$$F(\lambda) = \frac{1}{2} \left(\frac{1+a}{(\lambda - M_1)^2} + \frac{1-a}{(\lambda - M_2)^2} \right)$$

takes positive values everywhere. It can easily be checked that the equation $F(\lambda) = 1$, which has four complex solutions, has exactly one real solution $\lambda_- < M_- = \min(M_1, M_2)$ and one real solution $\lambda_+ > M_+ = \max(M_1, M_2)$. In the interval

Fig. 2. The function $F(\lambda)$ for a step stable (left) or unstable (right) Mach profile.

$]M_-, M_+[$, F is strictly convex and tends to infinity when $\lambda \to M_\pm$. Its minimal value in this interval is given by

$$F_{\min} = \frac{(1-a)^{1/3} + (1+a)^{1/3}}{2(M_1 - M_2)^2}.$$

The lemma follows since:

$$F_{\min} < 1 \iff |M_1 - M_2| > \delta(a). \qquad \square$$

Surprisingly, the flow is more stable if the jump is large. Notice that $\delta(a)$ takes its maximum value when the step is located in the middle of the duct ($a = 0$): in that case (which is the most unstable) $\delta(0) = 2$ so that a step subsonic flow is always unstable. When a varies from the middle to the walls of the duct, $\delta(a)$ decreases and its minimum value is $\delta(-1) = \delta(1) = 1$.

3.4. *The discretized problem*

A way to compute the spectrum for an arbitrary Mach profile could be to discretize the eigenvalue problem (2.15). We now want to illustrate the fact that a natural discretization process will produce parasitic non real eigenvalues. As a consequence, a standard discretization of (2.12) will generally be unstable, even if the continuous problem is stable.

Let $\mathbf{V}_h$ be a finite dimensional subspace of $L^2([-1,1])$, whose dimension tends to infinity as h tends to 0. We can for example set

$$\mathbf{V}_h = span[\psi_j, j = -N_h, \cdots, N_h - 1] \text{ where } N_h = \frac{1}{h} \text{ and } \psi_j = \frac{1}{\sqrt{2h}}\,\chi_{j+\frac{1}{2}},$$

$\chi_{j+\frac{1}{2}}$ being the characteristic function of the interval $[jh, (j+1)h]$. The ψ_j form an orthonormal family of $L^2([-1,1])$ and the dimension of $\mathbf{V}_h$ is $2N_h$. A Galerkin approach leads then to consider the following eigenvalue problem

$$\left\{\begin{array}{l}\text{Find } (u_h, \dot{u}_h, \lambda) \in \mathbf{V}_h^2 \times \mathbb{C},\ (u_h, \dot{u}_h) \neq (0,0),\ \text{such that } \forall (\widetilde{u}_h, \widetilde{\dot{u}}_h) \in \mathbf{V}_h^2 \\[2mm] \displaystyle\int_{-1}^{1} M(y)\, u_h(y)\, \widetilde{u}_h(y)\, dy + \int_{-1}^{1} \dot{u}_h(y)\, \widetilde{u}_h(y)\, dy = \lambda \int_{-1}^{1} u_h(y)\, \widetilde{u}_h(y)\, dy \\[2mm] \displaystyle\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} u_h(y)\, \widetilde{\dot{u}}_h(y)\, dxdy + \int_{-1}^{1} M(y)\, v_h(y)\, \widetilde{\dot{u}}_h(y)\, dy = \lambda \int_{-1}^{1} \dot{u}_h(y)\, \widetilde{\dot{u}}_h(y)\, dy \end{array}\right. \tag{3.9}$$

This leads to approximate the spectrum of operator A (which is composed of a continuous part and a possible discrete part) by the discrete eigenvalues of the $4N_h \times 4N_h$ matrix:

$$\mathbf{A}_h = \begin{pmatrix} \mathbf{M}_h & \mathbf{I} \\ & \\ \mathbf{E}_h & \mathbf{M}_h \end{pmatrix}$$

where matrices $\mathbf{M}_h$ and $\mathbf{E}_h$ are given by:

$$(\mathbf{M}_h)_{i,j} \;=\; \int_{-1}^{1} M(y)\, \psi_i(y)\, \psi_j(y) dy \text{ and } (\mathbf{E}_h)_{i,j} \;=\; \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} \psi_i(y)\, \psi_j(x)\, dxdy.$$

Let us consider for instance the case of a linear profile $M(y) = y$: we already know that the associated spectrum is given by

$$\sigma(A) = [-1, 1] \cup \{-\sqrt{2}, \sqrt{2}\}.$$

Figure 3 represents the spectrum of $\mathbf{A}_h$ in the complex plane for different values of N_h. Let us point out that the scales for the real and the imaginary parts are very different. We observe that the spectrum of $\mathbf{A}_h$ is composed of two real eigenvalues and $4N_h - 2$ non real eigenvalues. The two real eigenvalues are good approximations of the exact eigenvalues $-\sqrt{2}$ and $\sqrt{2}$. The set of the $4N_h - 2$ non real eigenvalues is symmetric with respect to the real axis, and corresponds to the discretization of the continuous spectrum $[-1, 1]$. Their imaginary part decreases as N_h increases. The same phenomena are observed with a spectral discretization [1].

These results show that is could be dangerous to use a discrete approach to conclude to the stability or instability of a given flow. We propose instead a rigorous approach which is the object of section 4 and 5.

4. Stability conditions for continuous profiles

We will now focus on the main object of the present paper, which is the question of existence of instabilities for the asymptotic model. Our methodology combines an approximation process of the Mach profile by a step-linear profile, and the application to this step-linear profile of the characterization established above.

Fig. 3. Eigenvalues of the discretized problem for a linear profile

4.1. *The methodology*

Let us now consider a general continuous profile M. The idea is to approximate M by a profile M_h such that $M_h \to M$ uniformly as $h \to 0$, and such that the study of equation $F_h(\lambda) = 1$ is simpler than the study of equation $F(\lambda) = 1$. The main point which results from spectral perturbation theory is that the profile M is necessarily stable if all M_h profiles are stable. In the following, we denote by A_h the operator defined by (2.16) where M is replaced by M_h.

Lemma 4.1. *Let $M \in L^\infty(]-1,1[)$ and let $M_h \in L^\infty(]-1,1[)$ be a family of profiles (for $0 < h < 1$) such that:*

$$\|M - M_h\|_{L^\infty(]-1,1[)} \longrightarrow 0 \text{ when } h \longrightarrow 0. \tag{4.1}$$

If $\sigma(A_h) \subset \mathbb{R}\ \forall h \in]0,1[$, then $\sigma(A) \subset \mathbb{R}$.

Proof. Suppose by contradiction that there exists $\lambda \in \sigma(A)$ such that $\lambda \notin \mathbb{R}$. By theorem 3.1, λ belongs to the discrete spectrum of A. As

$$A - A_h = \begin{pmatrix} M - M_h & 0 \\ 0 & M - M_h \end{pmatrix}$$

it is clear that:

$$\|A - A_h\| = \|M - M_h\|_{L^\infty([-1,1])}$$

and it results from classical spectral perturbation theory that any neighborhood of λ should contain, for h small enough, an eigenvalue of A_h (see theorem 3.6 chapter 4 §3.1 in [14]). This is in contradiction with the hypothesis $\sigma(A_h) \subset \mathbb{R}$. □

Remark 4.1. This lemma means that the subset of stable (resp. unstable) profiles (in the sense of definition 2.1) is closed (resp. open) in $L^\infty(]-1,1[)$.

Let us now explain how the approximated profiles M_h are chosen. A priori, the simplest solution consists in choosing a piecewise constant profile M_h. For such profile one can easily derive an explicit expression of function F_h which is a rational fraction. But this approach does not work. Indeed, proceeding as in the proof of lemma 3.4, one can check easily that such approximated profile M_h is unstable for h small enough (because the gap between two successive constant values becomes too small).

The natural choice is then to approximate M by a piecewise linear profile M_h. Again the corresponding function F_h is a rational fraction whose expression can be easily derived. Therefore the number N_r of complex solutions of equation $F_h(\lambda) = 1$ is known a priori. To prove the stability of a profile M_h, we just have to prove that the equation $F_h(\lambda) = 1$ has (at least) N_r real solutions. This can be proved under specific geometrical conditions on M_h by very simple arguments, such as the intermediate value theorem.

4.2. *The piecewise linear profiles*

Using simple integration rules, we can easily derive the expression of the function F for an arbitrary piecewise linear profile and establish the following lemma:

Lemma 4.2. *Let $M \in \mathcal{C}^0([-1,1])$ andSuppose that there exist*

$$x_0 = -1 < x_1 < \cdots < x_N = 1, \tag{4.2}$$

such that M is linear on $]x_i, x_{i+1}[$ for every $0 \le i < N$:

$$\forall\, 0 \le i \le N-1, \quad M(y) = \alpha_i y + \beta_i \ \text{in} \ [x_i, x_{i+1}] \tag{4.3}$$

with $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$ and $\alpha_i \neq \alpha_{i+1}$. Then the function $F(\lambda)$ is a rational function, it has at most $N+1$ poles located on the real axis. More precisely, if

$$M_0 < M_1 < \ldots < M_P$$

denote the P ordered values $M(x_i)$:

$$\{M_0, M_1, \ldots M_P\} = \{M(x_i);\, i = 0, \ldots N\},$$

then the rational function F has a partial fraction expansion of the following form:

$$F(\lambda) = \sum_{j=0}^{P} \frac{\gamma_j}{\lambda - M_j} + \sum_{j=0}^{P} \frac{\zeta_j}{(\lambda - M_j)^2} \tag{4.4}$$

with real coefficients γ_j and ζ_j satisfying

$$\gamma_0 < 0,\ \gamma_P > 0 \ \text{and} \ \zeta_j \ge 0, \quad \forall\, 0 \le j \le P.$$

Proof. Suppose first that $\alpha_i \neq 0 \; \forall i$. Reminding that

$$F(\lambda) = \frac{1}{2}\int_{-1}^{1} \frac{dy}{\big(\lambda - M(y)\big)^2}, \quad \lambda \in D = \mathbb{C} \setminus R(M)$$

then a simple integration provides :

$$F(\lambda) = \frac{1}{2}\left(-\frac{1}{\alpha_0}\,\frac{1}{\lambda - M(x_0)} + \sum_{i=1}^{N-1}\left[\frac{1}{\alpha_{i-1}} - \frac{1}{\alpha_i}\right]\frac{1}{\lambda - M(x_i)} + \frac{1}{\alpha_{N-1}}\,\frac{1}{\lambda - M(x_N)} \right). \tag{4.5}$$

The expression (4.4) follows with $\zeta_j = 0$ for all j and with the following expression of γ_j where we have set $\alpha_{-1} = \alpha_N = +\infty$:

$$\gamma_j = \sum_{\substack{0 \,\leq\, i \,\leq\, N \\ M(x_i) \,=\, M_j}} \frac{1}{2}\left[\frac{1}{\alpha_{i-1}} - \frac{1}{\alpha_i}\right]. \tag{4.6}$$

To prove that $\gamma_0 < 0$, notice that $M_0 = \inf M(y)$ and that:

(1) If $M_0 = M(x_0)$, the function M is necessarily increasing on $[x_0, x_1]$, so that:
$$\frac{-1}{2\alpha_0} < 0.$$
(2) If $M_0 = M(x_N)$, the function M is necessarily decreasing on $[x_{N-1}, x_N]$, so that:
$$\frac{1}{2\alpha_{N-1}} < 0.$$
(3) If $M_0 = M(x_i)$ for some i, $0 < i < N$, the function M is decreasing on $[x_{i-1}, x_i]$ and increasing on $[x_i, x_{i+1}]$, so that:
$$\frac{1}{2}\left(\frac{1}{\alpha_{i-1}} - \frac{1}{\alpha_i}\right) < 0.$$

Therefore γ_0 is negative since it is a sum of negative terms. One can prove in the same way that $\gamma_P > 0$, with $M_P = \sup M(y)$.

If some α_i vanish, the calculations can be handled similarly, leading to (4.4) with some non-vanishing positive ζ_j. The details are left to the reader. □

Using the expression of function F, we can establish sufficient conditions on the profile ensuring that all solutions of $F(\lambda) = 1$ are real. This is the object of the following lemma:

Lemma 4.3. *Consider the same hypotheses as in lemma 4.2. Then, if $\zeta_j = 0$ for all j and if there exist two integers k' and k, satisfying $0 \leq k' < P$ and $0 \leq k < P - k'$, such that:*

$$\begin{cases} \gamma_j < 0 \;\; \textit{for} \;\; 0 \leq j \leq k', \\ \gamma_j > 0 \;\; \textit{for} \;\; k' + k + 1 \leq j \leq P, \\ \gamma_j = 0 \;\; \textit{for} \;\; k' < j < k' + k + 1, \end{cases} \tag{4.7}$$

then all complex solutions λ of $F(\lambda) = 1$ are real numbers and M is stable (in the sense of definition 2.1). The same result holds if there exists $k' \le \ell \le k' + k + 1$ such that $\zeta_\ell > 0$ and $\zeta_j = 0$ for any $j \neq \ell$.

Proof. Suppose first that all ζ_j are equal to 0. Then by (4.7):

$$F(\lambda) = F^*(\lambda) := \sum_{j=0}^{k'} \frac{\gamma_j}{\lambda - M_j} + \sum_{j=k'+k+1}^{P} \frac{\gamma_j}{\lambda - M_j}$$

where all the coefficients γ_j are different from 0. As a consequence, the equation $F(\lambda) = 1$ can be written as a polynomial equation of degree $P+1-k$. To establish the lemma, we will prove that the equation $F(\lambda) = 1$ has exactly $P+1-k$ distinct real solutions.

Considering the function $\lambda \longmapsto F(\lambda)$ on the real axis, we notice that:

$$\lim_{\lambda\to-\infty} F(\lambda) = \lim_{\lambda\to+\infty} F(\lambda) = 0.$$

Moreover for $0 \le j \le k'$, $\gamma_j < 0$ implies

$$\lim_{\lambda \overset{\le}{\to} M_j} F(\lambda) = +\infty \text{ and } \lim_{\lambda \overset{\ge}{\to} M_j} F(\lambda) = -\infty$$

and for $k' + k + 1 \le j \le P$, $\gamma_j > 0$ implies

$$\lim_{\lambda \overset{\le}{\to} M_j} F(\lambda) = -\infty \text{ and } \lim_{\lambda \overset{\ge}{\to} M_j} F(\lambda) = +\infty.$$

By the intermediate value theorem, the equation $F(\lambda) = 1$ has at least one real solution in each interval $]-\infty, M_0[$, $]M_0, M_1[$,..., $]M_{k'-1}, M_k'[$ and one real solution in each interval $]M_{k'+k+1}, M_{k'+k+2}[$,..., $]M_{P-1}, M_P[$, $]M_P, +\infty[$, which gives, summing up, $P+1-k$ real and distinct solutions.

Let us now consider the case where there exists $k' \le l \le k' + k + 1$ such that $\zeta_\ell > 0$ and $\zeta_j = 0$ for ; $j \neq \ell$. In this case

$$F(\lambda) = F^*(\lambda) + \frac{\zeta_\ell}{(\lambda - M_\ell)^2} \ .$$

The reader will first remark that the same argument as above still apllies which shows the existence of $P+1-k$ distinct roots of $F(\lambda) = 1$ which are outside the interval $[M_k', M_{k'+k+1}]$. To conclude, we distinguish three cases :

(i) If $\ell = k'$, it is clear that $F(\lambda) = 1$ is a polynomial equation of degrees $P-k+2$. To conclude it suffices to prove the existence of an additionnal real solution in the interval $]M_k', M_{k'+k+1}[$ (where F is continuous). Since $\zeta_{k'} > 0$, one has:

$$\lim_{\lambda \overset{\ge}{\to} M_{k'}} F(\lambda) = +\infty \text{ while } \lim_{\lambda \overset{\le}{\to} M_{k'+k+1}} F(\lambda) = -\infty$$

which allows us to conclude easily.

Fig. 4. Illustration of lemma 4.3 (the case where $\zeta_j = 0$ for all j)

(ii) When $\ell = k' + k + 1$, one proceeds similarly as above.

(iii) When $k'+1 \leq \ell \leq k'+k$, $F(\lambda) = 1$ is a polynomial equation of degree $P+3-k$. To conclude, it suffices to prove the existence of one solution in each intervals $]M_{k'}, M_\ell[$ and $]M_\ell, M_{k'+k+1}[$ which is done again using the intermediate value theorem. Indeed since $\zeta_\ell > 0$ we have :

$$\lim_{\lambda \overset{>}{\to} M_l} F(\lambda) = \lim_{\lambda \overset{<}{\to} M_l} F(\lambda) = +\infty$$

while

$$\lim_{\lambda \overset{>}{\to} M_{k'}} F(\lambda) = \lim_{\lambda \overset{<}{\to} M_{k'+k+1}} F(\lambda) = -\infty \qquad \square$$

4.3. *The results*

Combining lemmas 4.1 and 4.3, we will now state some geometrical assumptions on the profile M ensuring its stability (in the sense of definition 2.1).

Theorem 4.1. *Let $M \in \mathcal{C}^0(]-1,1[)$. Then M is stable if one of these conditions is satisfied:*

(1) M is a convex (resp. concave) function.

(2) M is increasing (resp. decreasing) and there exists $y_S \in]-1,1[$ such that M is concave (resp. convexe) in $]-1, y_S[$ and convex (resp. concave) in $]y_S, 1[$.

Remark 4.2. The first condition is not surprising, if one reminds the well-known result by Rayleigh [16] for incompressible linearized Euler equations: an unstable profile necessarily admits inflexion points. Note also that the second condition is equivalent to Fjortoft criterium [5]:

$$M''(y_S) = 0 \text{ and } M''(y)(M(y) - M(y_S)) \geq 0, \quad \forall y \neq y_S,$$

which is a sufficient condition of stability, again in the incompressible case. We are not aware of any similar results in the compressible case.

Proof. Let $h = 1/N$ and consider a regular mesh of $[-1, 1]$ given by the points

$$x_i = i\,h, \quad i = 0, \cdots, N$$

Let $M_h(y)$ be the piecewise linear interpolation function of M on this mesh. Then:

$$\|M - M_h\|_{L^\infty([-1,1])} \leq \omega(M, h),$$

where

$$\omega(M, h) = \sup_{|y_1 - y_2| < h} |M(y_1) - M(y_2)|.$$

In particular

$$\|M - M_h\|_{L^\infty([-1,1])} \to 0 \quad \text{when} \quad h \to 0.$$

If we prove that for the profile M_h, hypothesis (4.7) holds, then the lemma follows from lemmas 4.1 and 4.3.

Let us consider a convex profile M. Clearly, M_h is also convex which implies that

$$\alpha_i > \alpha_{i-1}, \quad 0 < i < N.$$

Then reminding that $\gamma_P > 0$ and using (4.6) we can check easily that :

$$\gamma_j > 0, \quad \forall\, j > 0.$$

We can prove in the same way that

$$\gamma_j < 0, \quad \forall\, j < P,$$

for a concave profile M_h.

Finally if M is increasing and there exists $y_S \in]-1, 1[$ such that M is concave on $]-1, y_S[$ and convex in $]y_S, 1[$ then M_h is increasing and one easily sees that there exists $i_S \in]0, N[$ such that M_h is concave on $]x_0, x_{i_S}[$ and convex in $]x_{i_S}, x_N[$. As M_h is increasing we have $P = N$, $M(x_i) = M_i$ and we deduce from (4.6) that :

$$\gamma_j = \frac{1}{\alpha_{j-1}} - \frac{1}{\alpha_j}, \quad \forall\, 0 < j < P.$$

M_h is concave on $]x_0, x_{i_S}[$ so that $\alpha_i > \alpha_{i+1}$ for $i < i_S$. As a consequence, reminding that $\gamma_0 < 0$ we get from the previous equation:

$$\gamma_j < 0, \quad 0 \leq j < i_S.$$

Similarly we get :

$$\gamma_j > 0, \quad i_S < j \leq P.$$

To conclude we only have to notice that whatever happens to γ_{i_S}, (4.7) holds. $\square$

5. Instability conditions for odd continuous profiles

We consider now the particular case of an odd (and not necessarily continuous) Mach profile, i.e. a profile M such that $M(-y) = -M(y)$ for all $y \in [-1,1]$.

We will prove that the corresponding F function takes real values $F(\lambda)$ for purely imaginary λ. As a consequence, the intermediate value theorem applies to the function $\nu \longmapsto F(i\nu)$. This will allow to derive sufficient conditions on the Mach profile $M(y)$ such that the equation $F(\lambda) = 1$ has non real solutions, leading to the instability of $M(y)$.

Remark 5.1.

- Notice that, as adding a constant m to the profile ($M(y) \to M(y) + m$) results in a translation of the spectrum ($\lambda \to \lambda + m$), all results of this section can be easily extended to profiles which are odd up to an additive constant: $M(-y) + M(y) = 2m$ for all $y \in [-1,1]$. Let us point out that there is however a main difference between the cases $m = 0$ and $m \neq 0$: indeed, for $m = 0$, the instabilities which are studied in this section are "absolute" ones (as defined by [11]) while they are "convective" instabilities if $m \neq 0$.

- Thanks to remark 4.1, if an odd profile $M(y)$ is unstable, all profiles which are close enough to $M(y)$ (in $L^\infty(]-1,1[)$) are unstable. As a consequence, the following instability results still hold for almost odd profiles.

5.1. *A first instability condition*

First we have the

Lemma 5.1. *If M is odd, then:*

(1) $\forall\, \lambda \notin R(M), \quad F(\lambda) = F(-\lambda),$

(2) $\forall\, \nu \in \mathbb{R}^*, \quad F(i\nu) \in \mathbb{R}\,.$

Proof. The change of variable $y \to -y$ in (3.6) gives the first equality. Then using the obvious identity $\overline{F(\lambda)} = F(\overline{\lambda})$, we get for $\nu \in \mathbb{R}^*$:

$$F(i\nu) = F(-i\nu) = F(\overline{i\nu}) = \overline{F(i\nu)}$$

which proves that F takes real values on the imaginary axis. □

A simple calculation gives:

$$\forall \nu \in \mathbb{R}^* \qquad F(i\nu) = \frac{1}{2}\int_{-1}^{1} \frac{dy}{\big(i\nu - M(y)\big)^2} = \int_0^1 \frac{M(y)^2 - \nu^2}{\big(M(y)^2 + \nu^2\big)^2} dy\,. \tag{5.1}$$

A consequence of the previous lemma is the following corollary, which will be the main tool of this section:

Corollary 5.1. *If M is odd and if*

$$\limsup_{\nu \to 0} F(i\nu) > 1 \tag{5.2}$$

then the operator A has at least two eigenvalues $i\nu$ and $-i\nu$ with $\nu > 0$, and the profile M is unstable (in the sens of definition 2.1).

Proof. The function $\nu \longmapsto F(i\nu)$ is continuous on $\mathbb{R}^{+*}$ and tends to 0 when ν tends to $+\infty$. By the intermediate value theorem, if condition (5.2) is satisfied, there exists at least one value of ν, $\nu \neq 0$, such that $F(i\nu) = F(-i\nu) = 1$. The corollary then follows from lemma 3.1. □

The condition (5.2) is a sufficient condition for instability in the case of an odd profile. Clearly it is not in general a necessary condition for instability: indeed non real eigenvalues of the operator A are not necessarily on the imaginary axis and even if A has non real eigenvalues on the imaginary axis, condition (5.2) may not be true. However, we can establish some partial converse statements. For instance, we have the

Lemma 5.2. *If M is a continuous and odd function of y on $[-1, 1]$, and is a concave increasing function of y on $[0, 1]$, then M is unstable if and only if condition (5.2) is fulfilled.*

Proof. Suppose M is unstable and let us prove that condition (5.2) holds. We denote by λ_M a complex number satisfying $F(\lambda_M) = 1$ and $\Im(\lambda_M) > 0$.

Let $h = 1/N$ and consider a regular mesh of the interval $[-1, 1]$ given by the points $x_i = i\,h$, $i = -N, \cdots, 0, \cdots, N$. Let $M_h(y)$ be the piecewise linear interpolation of M on this mesh, so that $\|M - M_h\|_{L^\infty([-1,1])}$ tends to 0 with h. From lemma 4.1, we know that there exists λ_h such that $F_h(\lambda_h) = 1$ and $\lambda_h \to \lambda_M$ as $h \to 0$.

Clearly $M_h(y)$ is odd and continuous, and is a concave increasing function of y on $[0, 1]$. By symmetry, denoting $M(jh) = M_j$, we get from (4.5):

$$F_h(\lambda) = h \sum_{j=1}^{N} \gamma_j^h \left(\frac{1}{\lambda - M_j} - \frac{1}{\lambda + M_j} \right)$$

with

$$\gamma_j^h = -2 \frac{M_{j+1} + M_{j-1} - 2M_j}{(M_j - M_{j-1})(M_{j+1} - M_j)} \text{ for } j = 1, \cdots N-1,$$

and

$$\gamma_N^h = \frac{1}{M_N - M_{N-1}}\,.$$

By the concavity and the monotonicity of $M(y)$, we have $\gamma_j^h < 0$ for $j = 0, \cdots N-1$ and $\gamma_N^h > 0$. Hence, proceeding as in the proof of lemma 4.3, we see that equation

Fig. 5. Illustration of the proof of lemma 5.2: impossible behaviours of the function $F(i\nu)$.

$F_h(\lambda) = 1$, which has at most $2N$ different complex roots, has at least $2N-2$ different real roots. In other words, equation $F_h(\lambda) = 1$ has at most two non real solutions. By lemma 5.1, this proves that $\lambda_h \in i\mathbb{R}$, otherwise $F_h(\lambda) = 1$ would have four different roots $\lambda_h, -\lambda_h, \overline{\lambda_h}$ and $-\overline{\lambda_h}$. More precisely, $\lambda_h = i\nu_h$ where ν_h is the unique solution of $F_h(i\nu) = 1$ for $\nu > 0$. Notice that a fortiori, $\lambda_M = i\nu_M$ where ν_M is the unique solution of $F(i\nu) = 1$ for $\nu > 0$.

To go further, we must use the following result, proved in appendix :

$$\text{if } \frac{d}{d\nu} F_h(i\nu) = 0 \text{ for } \nu > 0, \text{ then } F_h(i\nu) \leq 0. \tag{5.3}$$

As a consequence, (recall that $F_h(i\nu) = 1$ tends to 0 at infinity), the behaviour represented on the left on figure 5 is not allowed, and we have:

$$F_h(i\nu) > 1 \text{ if } 0 < \nu < \nu_h \,,$$

and therefore (using the uniqueness of the solution of $F(i\nu) = 1$):

$$F(i\nu) > 1 \text{ if } 0 < \nu < \nu_M.$$

This proves that

$$\limsup_{\nu \to 0} F(i\nu) \geq 1.$$

To conclude, notice that if $\limsup_{\nu \to 0} F(i\nu) = 1$ (see figure 5 on the right), then for h small enough, there exists ν such that $0 < \nu < \nu_h$ and $\frac{d}{d\nu} F_h(i\nu) = 0$, which contradicts (5.3). □

In the following, we will exhibit various situations for which we can prove the existence and compute the value of the limit of the function $F(i\nu)$ as $\nu \to 0$.

5.2. *Application to singular profiles*

The simplest case where the limit of $F(i\nu)$ as $\nu \to 0$ can be obtained is the case where Lebesgue's dominated convergence theorem can be used:

Theorem 5.1. *Let M be an odd profile such that*

$$\int_0^1 \frac{dy}{M(y)^2} < +\infty. \tag{5.4}$$

Then the profile M is unstable (in the sense of definition 2.1) if the following condition holds:

$$\int_0^1 \frac{dy}{M(y)^2} > 1, \tag{5.5}$$

If moreover M is a continuous and odd function of y on $[-1,1]$, and a concave increasing function of y on $[0,1]$, then M is unstable if and only if (5.5) holds.

Proof. For $\nu > 0$, we have:

$$|F(i\nu)| \leq \frac{1}{2}\int_{-1}^1 \frac{dy}{\nu^2 + M(y)^2} \leq \int_0^1 \frac{dy}{M(y)^2}.$$

This proves that Lebesgue's dominated convergence theorem applies and we get:

$$\lim_{\nu \to 0} F(i\nu) = \int_0^1 \frac{dy}{M(y)^2}.$$

The theorem then follows from corollary 5.1 and lemma 5.2. □

Notice that (5.5) proves that any subsonic profile satisfying (5.4) is unstable.

The condition (5.4) is clearly satisfied for a profile M which does not vanish: as M is odd, it must then be discontinuous at $y = 0$. For instance, if M is an odd step profile, we recover a part of lemma 3.4.

Let us now consider a continuous profile. Then condition (5.4) cannot be fulfilled by a regular profile M but it can be used for example if $M(y)$ behaves like y^α with $0 < \alpha < 1/2$ for $y \overset{>}{\to} 0$:

Lemma 5.3. *The odd profile defined by $M(y) = M^+ y^\alpha$ with $0 < \alpha < 1/2$ is unstable (in the sense of definition 2.1) if and only if*

$$(1 - 2\alpha)M_+^2 < 1 \tag{5.6}$$

In other words, unstable modes exist if and only if $M_+ < M_\alpha$ with $M_\alpha = 1/(1-2\alpha)$. Notice that $M_\alpha \overset{>}{\to} 1$ when $\alpha \to 0$, again in accordance with the case of the step-profile.

One can wonder what happens if $1/2 < \alpha < 1$. We will prove now that such profile is always unstable:

Fig. 6. Stability area of the M^+y^α profiles in the (α, M^+) plane

Lemma 5.4. *The odd profile defined by $M(y) = ay^\alpha$ with $a > 0$ and $1/2 < \alpha < 1$ is unstable (in the sense of definition 2.1).*

Proof. For such profile, the function $F(i\nu)$ has the following expression:

$$F(i\nu) = \int_0^1 \frac{a^2y^{2\alpha} - \nu^2}{(a^2y^{2\alpha} + \nu^2)^2} dy.$$

A simple change of variable gives:

$$F(i\nu) = \nu^{1/\alpha-2}a^{-1/\alpha} \int_0^{(a/\nu)^{1/\alpha}} \frac{y^{2\alpha} - 1}{(y^{2\alpha} + 1)^2} dy,$$

so that:

$$\lim_{\nu \to 0} \frac{F(i\nu)}{\nu^{2-1/\alpha}} = a^{-1/\alpha} S(\alpha) \text{ with } S(\alpha) = \int_0^{+\infty} \frac{y^{2\alpha} - 1}{(y^{2\alpha} + 1)^2} dy.$$

One easily computes that for $\alpha > 1/2$:

$$S(\alpha) = \int_0^1 \frac{1 - y^{2\alpha}}{(y^{2\alpha} + 1)^2} \frac{1 - y^{2-2\alpha}}{y^{2-2\alpha}} dy$$

which shows that $S(\alpha) > 0$ for $1/2 < \alpha < 1$. Thus $\lim_{\nu \to 0} F(i\nu) = +\infty$ and the lemma results from corollary 5.1 and lemma 5.2 □

All the results concerning the y^α profiles are summarized in figure 5.2.

5.3. *A perturbation technique*

If M is an odd regular profile (say for instance $\mathcal{C}^1$), the previous results do not apply. Some results will be however deduced from the following lemma:

Lemma 5.5. *Suppose that M and $\tilde{M}$ are two odd profiles such that*

$$\int_0^1 \left| \frac{M(y) - \tilde{M}(y)}{M(y)^2 \tilde{M}(y)} \right| dy < +\infty \text{ and } \int_0^1 \left| \frac{M(y) - \tilde{M}(y)}{M(y) \tilde{M}(y)^2} \right| dy < +\infty \tag{5.7}$$

then (with an obvious definition of $\tilde{F}$), $F(i\nu) - \tilde{F}(i\nu)$ has a finite limit as ν tends to 0 given by:

$$\lim_{\nu \to 0} (F(i\nu) - \tilde{F}(i\nu)) = \frac{1}{2} \int_0^1 \frac{\tilde{M}(y)^2 - M(y)^2}{M(y)^2 \tilde{M}(y)^2} \, dy \tag{5.8}$$

Proof. For $\lambda \notin \mathbb{R}$, we have:

$$\begin{aligned}
F(\lambda) - \tilde{F}(\lambda) &= \frac{1}{2} \int_{-1}^1 \left(\frac{1}{\big(\lambda - M(y)\big)^2} - \frac{1}{\big(\lambda - \tilde{M}(y)\big)^2} \right) dy \\
&= \frac{1}{2} \int_{-1}^1 \frac{\Big[\big(\lambda - \tilde{M}(y)\big) + \big(\lambda - M(y)\big)\Big] (M(y) - \tilde{M}(y))}{\big(\lambda - M(y)\big)^2 \big(\lambda - \tilde{M}(y)\big)^2} dy \\
&= \frac{1}{2} \int_{-1}^1 \left(\frac{M(y) - \tilde{M}(y)}{\big(\lambda - M(y)\big)^2 \big(\lambda - \tilde{M}(y)\big)} + \frac{M(y) - \tilde{M}(y)}{\big(\lambda - M(y)\big)\big(\lambda - \tilde{M}(y)\big)^2} \right) dy
\end{aligned}$$

Taking $\lambda = i\nu$, thanks to hypothesis (5.7), we can apply Lebesgue's dominated convergence theorem to prove that:

$$\lim_{\nu \to 0} (F(i\nu) - \tilde{F}(i\nu)) = \frac{-1}{2} \int_{-1}^1 \left(\frac{M(y) - \tilde{M}(y)}{M(y)^2 \tilde{M}(y)} + \frac{M(y) - \tilde{M}(y)}{M(y) \tilde{M}(y)^2} \right) dy$$

which gives (5.8). □

This result is useful when $\tilde{M}$ is chosen such that $\lim_{\nu \to 0} \tilde{F}(i\nu)$ is known. In particular, if $\lim_{\nu \to 0} \tilde{F}(i\nu) = +\infty$, the previous lemma proves the instability of M. A simple application is given by the following lemma:

Lemma 5.6. *Let $1/2 < \alpha < 1$ and $M(y)$ be an odd profile such that*

$$\int_0^1 \frac{|M(y) - a y^\alpha|}{y^{3\alpha}} dy < +\infty, \tag{5.9}$$

then M is unstable (in the sense of definition 2.1).

Proof. Let us set $\tilde{M}(y) = a y^\alpha$. Then by lemma 5.4, $\lim_{\nu \to 0} \tilde{F}(i\nu) = +\infty$. On the other hand, (5.9) is equivalent to (5.7). Finally, lemma 5.5 proves that $\lim_{\nu \to 0} F(i\nu) = +\infty$. □

5.4. *Application to a regular profiles*

Let us now show how lemma 5.5 applies to a regular profile:

Theorem 5.2. *Let M be an odd profile, $\mathcal{C}^1$ in a neighborhood of 0, such that:*

(1) $M(y) \neq 0$ *for* $y \neq 0$,

(2) $M'(0) \neq 0$,

(3) $\displaystyle\int_0^1 \frac{|M(y) - M'(0)\,y|}{y^3}\,dy < +\infty$.

Then M is unstable (in the sense of definition 2.1) if the following conditions holds

$$1 + M'(0)^2 < \int_0^1 \frac{M'(0)^2 - (M(y)/y)^2}{M(y)^2}\,dy\ . \tag{5.10}$$

If moreover M is a continuous function of y on $[-1,1]$ and a concave increasing function of y on $[0,1]$, then M is unstable if and only if (5.10) holds.

Remark 5.2. Notice that the hypothesis 3) of theorem 5.2 is automatically satisfied if M is a $\mathcal{C}^2$ odd profile since $M"(0) = 0$.

Proof. Let us set $\tilde{M}(y) = M'(0)y$. Then, as we already noticed, $\tilde{F}$ has a simple expression:

$$\tilde{F}(i\nu) = \frac{-1}{\nu^2 + M'(0)^2}\ ,$$

which gives directly $\lim_{\nu\to 0} \tilde{F}(i\nu) = -1/M'(0)^2$. On the other hand, the third hypothesis is in this case equivalent to (5.7), so that lemma 5.5 applies and (5.8) reads:

$$\lim_{\nu\to 0} F(i\nu) + \frac{1}{M'(0)^2} = \frac{1}{2}\int_0^1 \frac{M'(0)^2y^2 - M(y)^2}{M(y)^2M'(0)^2y^2}dy$$

which gives (5.10). □

The reader will notice that the condition (5.10) relates the behavior at 0 (left hand side) to a global quantity (right hand side). As this condition is rather difficult to interpret, let us illustrate it on the case of more specific profiles.

Corollary 5.2. *For $a > 0$ and $\alpha > 0$, let M be the profile defined by*

$$M(y) = a\tanh(\alpha y)$$

and let $\alpha^* > 0$ be the unique solution of equation $\alpha\tanh\alpha = 1$. Then, the profile M is unstable (in the sense of definition 2.1) if and only if

$$\alpha > \alpha^* \text{ and } a < 1 - \frac{1}{\alpha\tanh\alpha}$$

Again, the result for a step profile is recovered when $\alpha \to +\infty$.

6. Back to the analysis of stability of the duct case

We will prove now that the stability properties of the asymptotic model (2.12) (obtained for $\epsilon \to 0$) can be related to the stability properties of the Cauchy problem in a duct of height $\epsilon > 0$. More precisely, we will show that, if the Mach profile $M(y)$ is unstable (in the sense of definition 2.1), then for all $\epsilon > 0$, the Cauchy problem associated to equations (2.6, 2.7) is unstable.

6.1. *The modal approach*

A classical approach for the analysis of stability is the determination of the so-called normal modes which are the nontrivial solutions of (2.6, 2.7) of the form :

$$\left\{ \begin{array}{l} u_\epsilon(x,y,t) = u(y)\; e^{-i\omega t}\; e^{ikx}, \\[2ex] v_\epsilon(x,y,t) = v(y)\; e^{-i\omega t}\; e^{ikx}, \end{array} \right. \tag{6.1}$$

with $k \in \mathbb{R}$ and $\omega \in \mathbb{C}$. Injecting (6.1) in (2.6) leads to

$$\left\{ \begin{array}{l} \left(k^2 - (\omega - k\, M(y))^2\right) u = \dfrac{ik}{\epsilon}\dfrac{dv}{dy} \,, \\[2ex] (\omega - k\, M(y))^2 \; v + \dfrac{ik}{\epsilon}\dfrac{du}{dy} + \dfrac{1}{\epsilon^2}\dfrac{d^2 v}{dy^2} = 0 \,, \end{array} \right. \tag{6.2}$$

while the boundary condition becomes :

$$v(\pm 1) = 0. \tag{6.3}$$

The stability analysis consists in looking for possible values of $\omega \in \mathbb{C}^+$ where

$$\mathbb{C}^+ = \{z \in \mathbb{C};\; \Im(z) > 0\}$$

such that (6.2) has nontrivial solutions in $L^2(]-1,1[)^2$. Notice that if (u,v) is a solution of (6.2) associated to (ω,k), then $(u,-v)$ (resp. $(\bar{u},-\bar{v})$) is a solution of (6.2) associated to $(-\omega,-k)$ (resp. $(\bar{\omega},k)$). Summing up, we can restrict the study to solutions such that $\Im(\omega) > 0$ and $k > 0$. For such values of ω and k, the first equation of (6.2) becomes:

$$u = \frac{ik}{\epsilon}\,\frac{1}{k^2 - (\omega - kM(y))^2}\,\frac{dv}{dy} \,.$$

Then, setting

$$\lambda = \frac{\omega}{k} \,, \quad \eta = \epsilon^2 k^2, \quad \gamma_\lambda(y) = M(y) - \lambda \,, \tag{6.4}$$

and eliminating u in the second equation of (6.2), we obtain :

$$\eta\, \gamma_\lambda^2(y)\big(1 - \gamma_\lambda^2(y)\big) v(y) - \frac{2\,\gamma_\lambda'(y)\;\gamma_\lambda(y)}{1 - \gamma_\lambda^2(y)}\frac{dv(y)}{dy} - \gamma_\lambda^2(y)\frac{d^2 v(y)}{dy^2} = 0 \,, \tag{6.5}$$

which can be rewritten as follows :

$$\eta\, \gamma_\lambda^2(y) v(y) - \frac{d}{dy}\left(\frac{\gamma_\lambda^2(y)}{1 - \gamma_\lambda^2(y)}\,\frac{dv(y)}{dy}\right) = 0 \,. \tag{6.6}$$

Here η can be seen as a parameter. We will say that the Cauchy problem associated to equations (2.6, 2.7) is unstable if there exists $\lambda \in \mathbb{C}^+$ such that (6.5) (or equivalently (6.6)) as a nontrivial solution. Indeed, in that case there will exist a nomal mode of the form (6.1) solution of (2.6, 2.7) with $\Im(\omega) > 0$ so that its norm grows exponentially with respect to t.

6.2. *The perturbation technique*

6.2.1. *The limit result*

We are first going to see what happens when $\eta \to 0$, which makes a link with the analysis of stability of the asymptotic model (2.12).

Lemma 6.1. *Let $\lambda \in \mathbb{C}^+$ and $\eta = 0$. Then (6.5) has non-trivial solutions $v \in H_0^1(]-1,1[)$ if and only if*

$$F(\lambda) = 1$$

where the function F is defined by (3.6), or, equivalently, if and only if the profile $M(y)$ is unstable (in the sense of definition 2.1).

Proof. For $\eta = 0$, (6.5) becomes:

$$\frac{d}{dy}\Big(\frac{\gamma_\lambda^2(y)}{1-\gamma_\lambda^2(y)}\frac{dv(y)}{dy}\Big) = 0.$$

which implies that the existence of a constant $\beta \in \mathbb{C}, \beta \neq 0$ such that:

$$\frac{dv}{dy} = \beta\,\frac{1-\gamma_\lambda^2}{\gamma_\lambda^2}$$

Using the boundary conditions $v(\pm 1) = 0$, we get finally:

$$\int_{-1}^{1} \frac{1-\gamma_\lambda^2(y)}{\gamma_\lambda^2(y)}dy = 0 \tag{6.7}$$

which gives the desired result by using the definition of γ_λ. $\square$

The idea now is to prove that if $M(y)$ is unstable (in the sense of definition 2.1), then unstable modes exist not only for $\eta = 0$ but also for any small values of η. This will be a consequence of Steinberg's theorem which we recall in the next paragraph.

6.2.2. *Steinberg's theorem*

We give here the formal statement of Steinberg's theorem [18] and of a corollary better suited for our purpose.

Proposition 6.1. *Suppose $T(\lambda, \eta)$ is a family of compact operators which are analytic in λ for fixed η and real analytic in η for fixed λ, for all $(\lambda, \eta) \in D \times \mathbb{R}$ (D is* *an open connected set of $\mathbb{C}$). If in addition, for all $\eta \in \mathbb{R}$, $I - T(\lambda,\eta)$ is invertible for some $\lambda \in D$, then*

- *$\lambda \to I - T(\lambda,\eta)$ is invertible except for isolated points in D and the function $\lambda \to \big(I - T(\lambda,\eta)\big)^{-1}$ is meromorphic.*

- *Moreover if $\lambda_0 \in D$ is a pole of $\lambda \to \big(I - T(\lambda,\eta_0)\big)^{-1}$, then there exists at least a function $\lambda(\eta)$ defined on an interval $]a,b[\subset \mathbb{R}$ containing η_0, continuous in η (and analytic in $\eta^{1/p}$ for some integer p) such that $\lambda(\eta_0) = \lambda_0$ and $\lambda(\eta)$ is a pole of $\lambda \to \big(I - T(\lambda,\eta)\big)^{-1}$.*

As a consequence of the compacity of $T(\lambda,\eta)$ the following equivalence holds:

$$\mathrm{Ker}\big[I - T\big(\lambda_p,\eta\big)\big] \neq \{0\} \quad \Leftrightarrow \quad \lambda_p \text{ is a pole of } \lambda \to (I - T(\lambda,\eta))^{-1}.$$

This allows us to refourmulate the proposition 6.1 as follows

Corollary 6.1. *Under the conditions of previous theorem, suppose there exists $\lambda^* \in D$ such that*

$$\mathrm{Ker}\big[I - T(\lambda^*,0)\big] \neq \{0\}$$

then there exist $\eta^* \in \mathbb{R}^+$ and a function $\lambda^* :]-\eta^*,\eta^*[\to \Omega$ continuous in η (and analytic in $\eta^{1/p}$ for some integer p) satisfying $\lambda^*(0) = \lambda^*$ and

$$\mathrm{Ker}\big[I - T\big(\lambda^*(\eta),\eta\big)\big] \neq \{0\}, \quad \forall\, \eta \in]-\eta^*,\eta^*[\ .$$

6.2.3. *The main result*

As a consequence of corollary 6.1, we will obtain the

Theorem 6.1. *Suppose $M(y)$ is unstable (in the sense of definition 2.1). Then, for all $\epsilon > 0$, the Cauchy problem associated to equations (2.6, 2.7) is unstable, in the sense that, for η small enough, there exists $\lambda \in \mathbb{C}^+$ such that (6.5) (or equivalently (6.6)) as a nontrivial solution.*

Proof. Let us consider the operators A and $K(\lambda,\eta)$ defined on $H^1_0(]-1,1[)$ by:

$$(Av,u)_{H^1} = a(v,u), \quad \big(K(\lambda,\eta)v,u\big)_{H^1} = k(\lambda,\eta;v,u), \quad \forall\,(u,v) \in \big(H^1_0(]-1,1[)\big)^2$$

where $a(\cdot,\cdot)$ and $k(\lambda,\eta\,;\cdot,\cdot)$ denote the two following bilinear forms :

$$\left\{\begin{aligned} &a(v,u) = \int_{-1}^{1} \frac{dv}{dy}\,\frac{du}{dy}\,dy,\\ &k(\lambda,\eta;\,v,u) = \int_{-1}^{1} \left(\eta\,(1-\gamma_\lambda^2)v\,u - \frac{2\,\gamma_\lambda'}{(1-\gamma_\lambda^2)\,\gamma_\lambda}\frac{dv}{dy}\,u\right)dy.\end{aligned}\right. \tag{6.8}$$

Using a variational formulation of (6.5,6.3) leads to prove the existence of $\eta > 0$ and $\lambda \in \mathbb{C}^+$ such that

$$\exists\, v \in \mathcal{H}^1_0(]-1,1[),\ v \neq 0 \text{ such that } \ Av + K(\lambda,\eta)v = 0$$

Since A is invertible and $K(\lambda,\eta)$ is compact (see lemma 6.2), we can equivalently show the existence of $\eta > 0$ and $\lambda \in \mathbb{C}^+$ such that $I + A^{-1}K(\lambda,\eta)$ is not invertible. Clearly $T(\lambda,\eta) = A^{-1}K(\lambda,\eta)$ is a family of compact operators which are analytic in λ for fixed η and real analytic in η for fixed λ, for all $(\lambda,\eta) \in \mathbb{C}^+ \times \mathbb{R}$. Finally, by lemma 6.3, for all $\eta \in \mathbb{R}$, there exists $\lambda \in \mathbb{C}^+$ such that $I - T(\lambda,\eta)$ is invertible.

The theorem is therefore a direct consequence of corollary 6.1 : indeed, if $M(y)$ is unstable (in the sense of definition 2.1), there exists $\lambda^* \in \mathbb{C}^+$ such that $F(\lambda^*) = 1$ and by lemma 6.1, this means that $\mathrm{Ker}\big[I - T(\lambda^*, 0)\big] \neq \{0\}$. □

We now state and prove the two lemmas used in the previous proof.

Lemma 6.2. *For all $\lambda \in \mathbb{C}$ with $\Im(\lambda) > 0$ and $\eta \in \mathbb{R}$, $K(\lambda,\eta)$ is a compact operator.*

Proof. Consider a bounded sequence (u_n) in $H^1_0(]-1,1[)$. Our goal is to prove that there exists a subsequence (u_{n_k}) such that $\big(K(\lambda,\eta)u_{n_k}\big)$ converges in $H^1_0(]-1,1[)$. The sequence $\big(K(\lambda,\eta)u_n\big)$ being bounded in $H^1_0(]-1,1[)$, we can extract a subsequence $\big(K(\lambda,\eta)u_{n_k}\big)$ that converges in $L^2(]-1,1[)$ (Rellich's theorem). Moreover, for any v in $H^1_0(\,]-1,1[\,)$ abd by definition of $K(\lambda,\eta)$:

$$||K(\lambda,\eta)v||^2_{H^1} = k(\lambda,\eta;v,K(\lambda,\eta)v) \leq C\,||v||_{H^1}||K(\lambda,\eta)v||_{L^2}. \tag{6.9}$$

Taking $v = u_{n_k} - u_{n_\ell}$ in (6.9), we obtain that $\big(K(\lambda,\eta)u_{n_k}\big)$ is a Cauchy sequence in $H^1_0(]-1,1[)$. In particular, it converges and $K(\lambda,\eta)$ is compact. □

Lemma 6.3. *For all $\eta \in \mathbb{R}$, there exists $\lambda \in \mathbb{C}^+$ such that $I + A^{-1}K(\lambda,\eta)$ is invertible.*

Proof. It is equivalent to prove that for all $\eta \in \mathbb{R}$ there exists $\lambda \in \mathbb{C}^+$ such that the only solution of (6.6,refcnd) is equal to 0. This results from the coerciveness, for some $\lambda \in \mathbb{C}^+$, of the following bilinear form:

$$b(v,u) = \int_{-1}^{1} \eta\,\gamma_\lambda^2\,v\,u + \frac{\gamma_\lambda^2}{1-\gamma_\lambda^2}\frac{dv}{dy}\,\frac{du}{dy}\,dy \quad \forall (u,v) \in \big(H^1_0(]-1,1[)\big)^2. \tag{6.10}$$

First, we notice that $\lim\limits_{|\lambda|\to\infty} -\dfrac{\gamma_\lambda^2(y)}{1-\gamma_\lambda^2(y)} = 1$ uniformly in y. So that:

$$\Re\mathrm{e}\Big(-\frac{\gamma_\lambda^2(y)}{1-\gamma_\lambda^2(y)}\Big) \geq \frac{1}{2}, \quad \forall\, y \in]-1,1[, \tag{6.11}$$

for $|\lambda|$ large enough. The coerciveness follows by choosing λ such that

$$\Re\mathrm{e}\left(-\eta\,\gamma_\lambda^2(y)\right) \geq 0, \quad \forall\, y \in]-1,1[,$$

which is always possible. □

Appendix

The object of this appendix is to prove the result (5.3) used in the proof of lemma 5.2. Using the same hypotheses and notations as in lemma 5.2, this result has the following statement:

$$\text{if } \frac{d}{d\nu}F_h(i\nu) = 0 \text{ for } \nu > 0, \text{ then } F_h(i\nu) \leq 0.$$

Proof. Taking $\lambda = i\nu$ in the expression of F_h used in lemma 5.2, we get :

$$F_h(i\nu) = -h\sum_{j=1}^{N} \gamma_j^h \frac{2M_j}{\nu^2 + M_j^2}$$

Suppose $\frac{d}{d\nu}F_h(i\nu) = 0$ for a $\nu > 0$; then we have:

$$\sum_{j=1}^{N} \gamma_j^h \frac{2M_j}{(\nu^2 + M_j^2)^2} = 0 \tag{6.12}$$

Using the previous equation, we get:

$$F_h(i\nu) = -h\sum_{j=1}^{N} \gamma_j^h \frac{2M_j(\nu^2 + M_j^2)}{(\nu^2 + M_j^2)^2} = -h\sum_{j=1}^{N} \gamma_j^h \frac{2M_j^3}{(\nu^2 + M_j^2)^2}$$

Using $\gamma_j^h < 0$ for $j = 0, \cdots N-1$, $\gamma_N^h > 0$ and $0 < M_1 < \cdots < M_N$ which are consequences of the concavity and the monotonicity of $M(y)$ (see the proof of lemma 5.2), we easily obtain from (6.12) the following inequality :

$$\gamma_N^h \frac{2M_N^3}{(\nu^2 + M_N^2)^2} > -\sum_{j=1}^{N-1} \gamma_j^h \frac{2M_j^3}{(\nu^2 + M_j^2)^2}$$

which immediately implies $F_h(i\nu) < 0$. □

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