High order time stepping and local time stepping for first order wave problems

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Bibliography and motivation

- Case of second-order hyperbolic problems treated by :
  Jean-Charles Gilbert and Patrick Joly *Higher order time stepping for second order hyperbolic problems and optimal CFL conditions*,
  Julien Diaz and Marcus Grote, *Energy Conserving Explicit Local Time-Stepping for Second-Order Wave Equations*

- For first-order hyperbolic problems, second-order time scheme :
  Serge Piperno *Symplectic local time-stepping in non-dissipative DGTD methods applied to wave propagation problems*

Model problem

First-order hyperbolic problem :

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{d} A_i(x) \frac{\partial U}{\partial x_i} = f(x,t)$$

with $A_i(x)$ symmetric matrices.
Use of Local Discontinuous formulation with centered fluxes :

$$\int_K \frac{\partial U}{\partial t} \varphi \, dx - \int_K \sum_{i=1}^{d} A_i(x)\, U \frac{\partial \varphi}{\partial x_i} dx + \int_{\partial K} (\sum_{i=1}^{d} A_i(x) n_i) \{U\} \varphi \, dx = \int_K f(x,t) \varphi \, dx$$

Model problem

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with $A_i(x)$ symmetric matrices.
Evolution problem :

$$\frac{dU}{\partial t} + A_h U = F_h$$

With conservative boundary conditions, A_h is skew-symmetric.

Modified equation approach

Leap-frog scheme :

$$\frac{U^{n+1} - U^{n-1}}{2\Delta t} + A_h U^n = \mathcal{F}_h^n,$$

Stability condition of this scheme :

$$\Delta t ||A_h||_2 \leq 1$$

In absence of source, the exact solution is given by

$$\frac{U^{n+1} - U^{n-1}}{2} = i \sin(i \Delta t A_h) U^n$$

Taylor expansion of the sinus provide the following scheme :

$$\frac{U^{n+1} - U^{n-1}}{2} + \left[\Delta t \, A_h + \sum_{q=1}^{m} \frac{(\Delta t \, A_h)^{2q+1}}{(2q+1)!} \right] U^n = 0.$$

Stability condition of modified equation

Let us denote the polynomial :

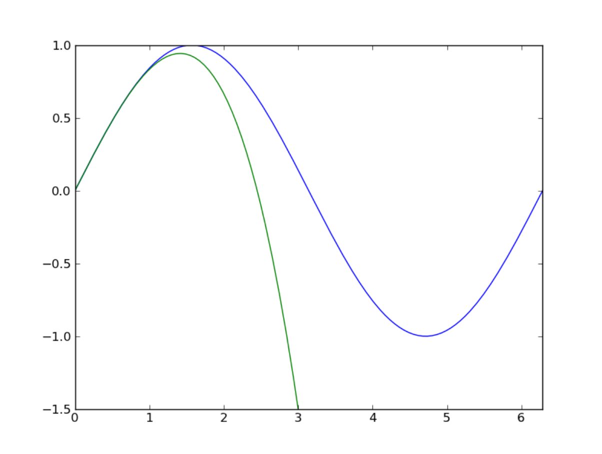
$$\tau_m(x) = x + \sum_{q=1}^{m} (-1)^q \frac{x^{2q+1}}{(2q+1)!}$$

Stability is obtained if

$$|\tau_m(x)| \leq 1 \;\Leftrightarrow\; x \in [0, \alpha_m]$$

Stability condition of modified equation

For *m* even, $\alpha_m \leq \dfrac{3\pi}{2}$



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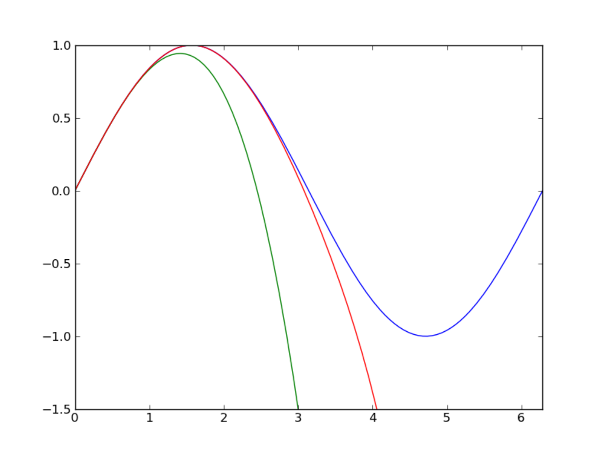

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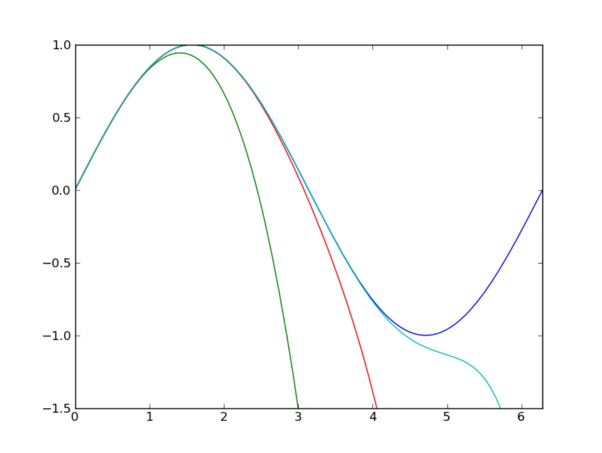

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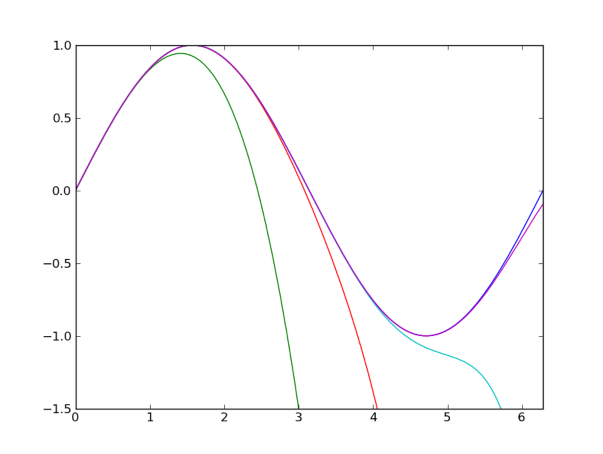

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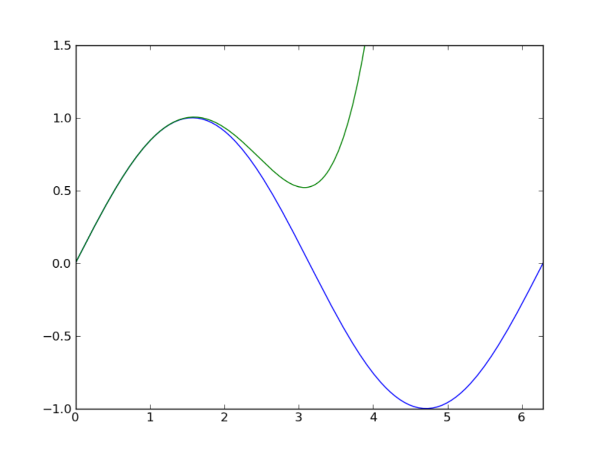

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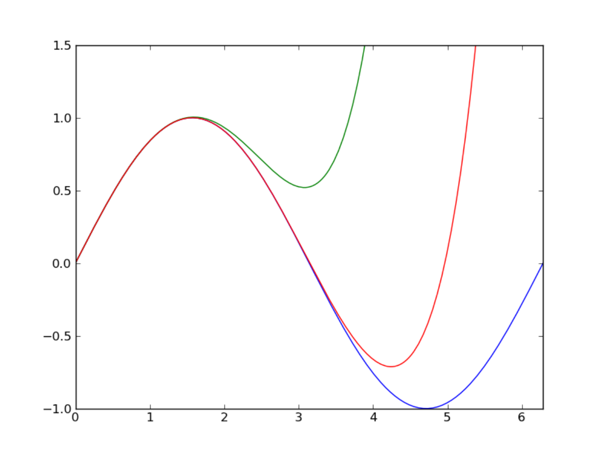

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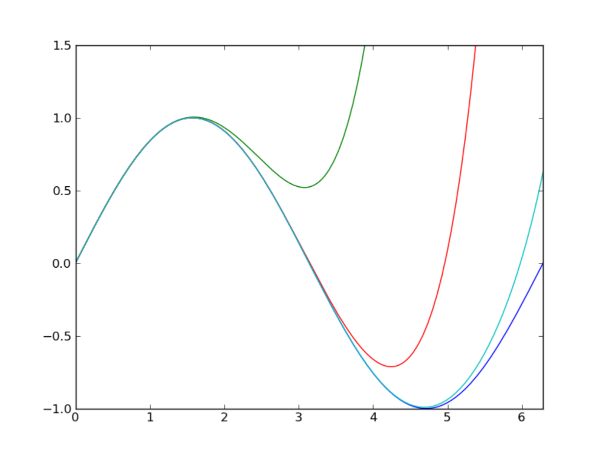

Improvement of modified equation

Higher-order terms are added

$$\frac{U^{n+1} - U^{n-1}}{2} + \Big[\sum_{q=0}^{m} \frac{(\Delta t \, A_h)^{2q+1}}{(2q+1)!} \Big] U^n + \Big[\sum_{q=m+1}^{r} \alpha_q (\Delta t A_h)^{2q+1} \Big] U^n = 0$$

This scheme is written under the form

$$U^{n+1} - U^{n-1} + 2i \, \mathcal{T}_{2r+1}(i \Delta t A_h) U^n = 0$$

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Optimal polynomial for $m = 0$, and nearly optimal for $m = 1$

$$\mathcal{T}^m_{2r+1}(x) \;=\; \frac{1}{\xi_r} \mathcal{T}^{Cheb}_{2r+1}\Big(\frac{(-1)^r \xi_r^m\, x}{(2r+1)} \Big)$$

where $\mathcal{T}^{Cheb}_{2r+1}$ are Chebyshev polynomials of the first kind and

$$\xi_r^0 \;=\; 1, \quad \xi_r^1 \;=\; \frac{2r+1}{2\sqrt{r(r+1)}},$$

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Stability condition :

$$\Delta t ||A_h||_2 \;\leq\; \frac{2r+1}{\xi^m_r}$$

with

$$\xi^1_r \;=\; 1 + O(\frac{1}{r^2})$$

Chebyshev reccurence

- Use of Horner algorithm leads to numerical instabilities for large values of r
- Use of Chebyshev reccurence leads to stable algorithms :

$$\mathcal{Q}_0 = U^n$$

$$\mathcal{Q}_1 = \frac{\xi_r}{2r+1} \Delta t A_h U^n$$

$$\mathcal{Q}_n = \frac{2\xi_r}{2r+1} \Delta t A_h \mathcal{Q}_{n-1} + \mathcal{Q}_{n-2}$$

$$\ldots$$

$$U^{n+1} = U^{n-1} - \frac{2}{\xi_r} \mathcal{Q}_{2r+1}$$

Two-level time stepping

Computational domain split into a “fine region” and a “coarse region”
P_h : projector onto the fine region

$$\frac{U^{n+1} - U^{n-1}}{2} + \Big[\sum_{q=0}^{m} \frac{(\Delta t\, A_h)^{2q+1}}{(2q+1)!}\Big] U^n$$
$$+\Big[\sum_{q=m+1}^{r} \alpha_q\, (\Delta t A_h P_h)^{2q}\Big]\, \Delta t A_h U^n \; = \; 0,$$

Presence of P_h $\Rightarrow$ terms of second sum are computed only on the “fine region”
Skew-symmetry of the matrices $A_h P_h A_h \cdots A_h P_h A_h$ $\Rightarrow$ stability of this scheme

Stable two-level algorithm

For $m = 0$, it is equivalent to the following scheme (obtained by reproducing the strategy of Diaz and Grote) :

$$\left\{ \begin{array}{l} w_h \;=\; A_h(I - P_h)U^n \\ Q_0 \;=\; U^n \\ Q_1 \;=\; -\dfrac{\Delta t}{2r+1}(w_h + A_h P_h Q_0) \\ \text{For } k = 1, 2r \\ \qquad Q_{k+1} \;=\; Q_{k-1} - \dfrac{2\Delta t}{2r+1}(A_h P_h Q_k + w\delta_k \text{ even}) \\ \text{End For} \\ U^{n+1} = U^{n-1} + 2Q_{2r+1} \end{array} \right.$$

Stable algorithm even for large values of r

Multilevel algorithm

Domain split into hierarchical subdomains

$$\Omega = \bigcup \Omega_i = \bigcup K_e$$

with

$$\Omega \supset \Omega_1 \supset \Omega_2 \cdots \supset \Omega_r$$

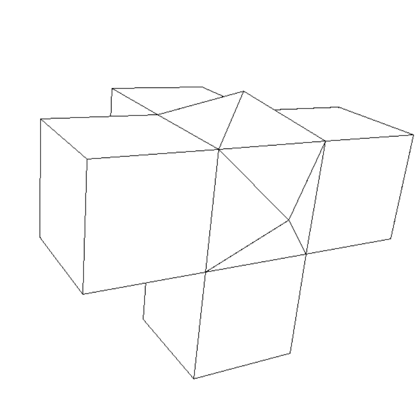
Multilevel algorithm

For each element, a nominal time step is computed

$$\Delta t_e = \frac{2r + 1}{\xi_r^m ||\mathcal{P}_e A_h \mathcal{P}_e||_2}$$

Multilevel algorithm

by considering only direct neighbors of each element



Multilevel algorithm

Global time step Δt is chosen by the user, then a level i is affected to each element with respect to the rule :

$$\text{if} \quad \Delta t_e \leq \frac{\xi_i^m \Delta t}{2i+1}, \quad \text{then} \quad K_e \in \Omega_i.$$

Multilevel algorithm

We consider the following time scheme

$$\frac{U^{n+1} - U^{n-1}}{2\Delta t} \quad + A_h U^n + \Delta t^2 A_h P_1 A_h P_1 A_h U^n$$
$$+\Delta t^4 A_h P_1 A_h P_2 A_h P_2 A_h P_1 A_h U^n$$
$$+\Delta t^6 A_h P_1 A_h P_2 A_h P_3 A_h P_3 A_h P_2 A_h P_1 A_h U^n + \cdots = 0$$

where P_k are diagonal matrices :

$$P_k = \begin{pmatrix} \beta_k^0 & \dots & & \\ \dots & \beta_k^1 & \dots & \\ & \dots & \dots & \dots \\ & & \dots & \beta_k^r \end{pmatrix}$$

with

$$\beta_k^m = 0, \ \forall m < k$$

Multilevel algorithm

If we write the expansion of optimal polynomial $\tau_{opt}^k(X)$ as :

$$\tau_{opt}^k(X) = X + \gamma_1^k X^3 + \gamma_2^k X^5 + \cdots + \gamma_k^k X^{2k+1}$$

Coefficients β_k^m are chosen to coincide with these polynomials for each level

For k = 1, r

  For m = 1, k-1

    $\beta_k^m = 0$

  End For

  For m = k, r

    $\beta_k^m = \sqrt{\gamma_k^m}$

    For n = 1, k-1

      $\beta_k^m = \beta_k^m / \beta_n^m$

    End For

  End For

End For

Multilevel algorithm

Use of Horner algorithm :

$$
\begin{aligned}
Q_0 &= \Delta t A_h U^n \\
Q_1 &= \Delta t A_h P_1 Q_0 \\
Q_2 &= \Delta t A_h P_2 Q_1 \\
&\dots \\
Q_r &= \Delta t A_h P_r Q_{r-1} \\
Q_{r-1} &= Q_{r-1} + \Delta t A_h P_r Q_r \\
Q_{r-2} &= Q_{r-2} + \Delta t A_h P_{r-1} Q_{r-1} \\
&\dots \\
Q_0 &= Q_0 + \Delta t A_h P_1 Q_1 \\
U^{n+1} &= U^{n-1} - 2Q_0
\end{aligned}
$$

unstable due to round-off errors when $r \geq 14$.

2-D numerical results

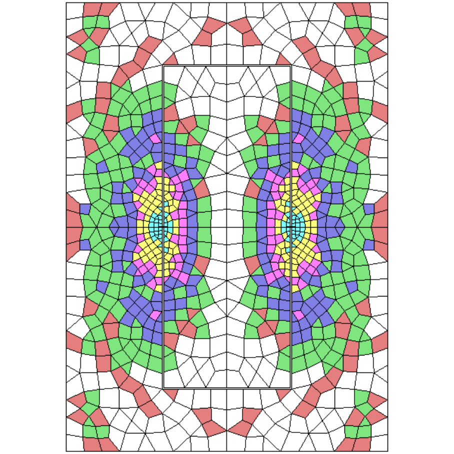
We consider wave equation

$$A_i = \begin{pmatrix} 0 & e_i^* \\ e_i & 0 \end{pmatrix}$$

and Neumann boundary conditions so that A_h is skew-symmetric

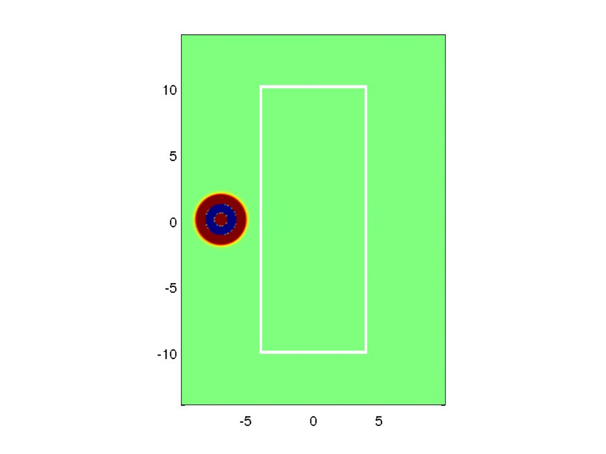
2-D numerical results

Box pierced with two small holes



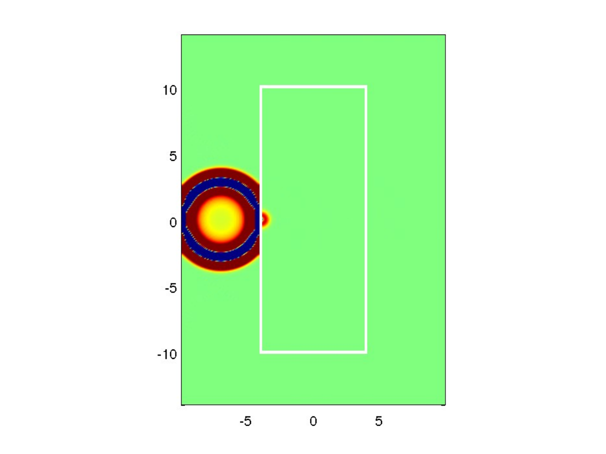
2-D numerical results

Solution obtained for $t = 2$



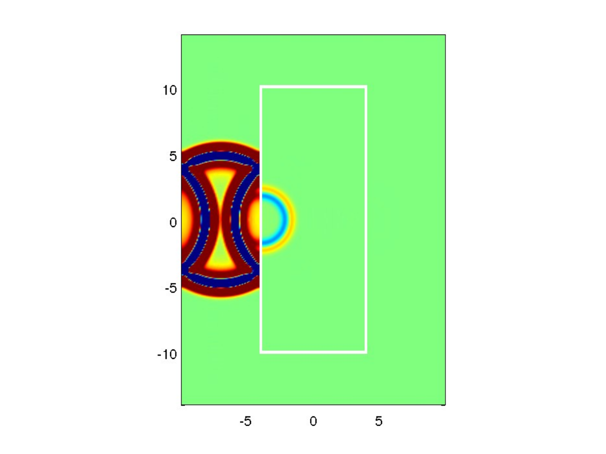
2-D numerical results

Solution obtained for $t = 4$



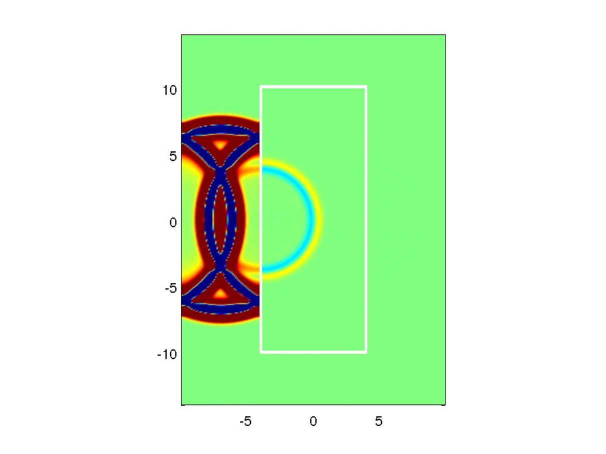
2-D numerical results

Solution obtained for $t = 6$



2-D numerical results

Solution obtained for $t = 8$



2-D numerical results

$\Delta t_{max} = 0.01036, \quad \Delta t_{min} = 0.000737$

Ratio $\dfrac{\Delta t_{max}}{\Delta t_{min}}$ = 14.1

Computational time with optimized fourth order ($\Delta t = 0.005$): 767s

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Fourth-order local time stepping with the following repartition :

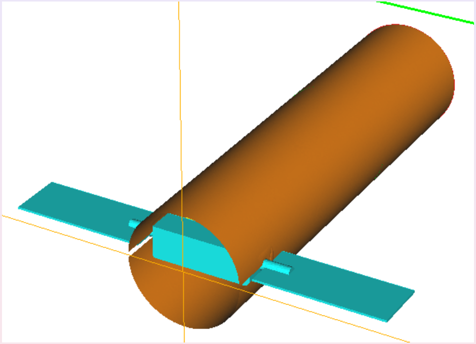
Level	1	2	3	4	5	6	7	8
Number of elements	1024	0	0	0	0	0	16	4

L^2 error for $t = 10$: 7.78e-6

Computational time ($\Delta t = 0.01$): 177s

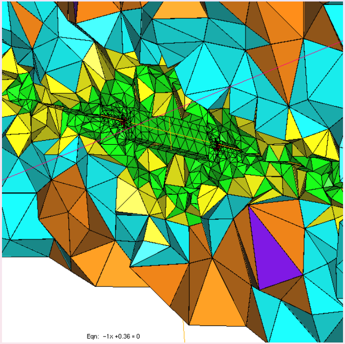
3-D numerical results

Scattering by a satellite



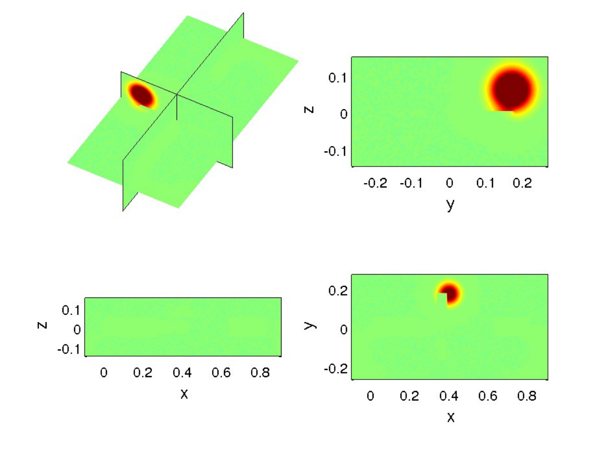
3-D numerical results

Mesh used for the simulations



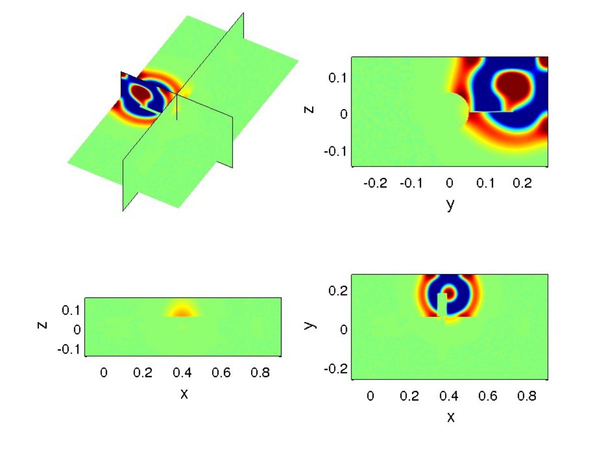
3-D numerical results

Solution obtained for $t = 0.1$

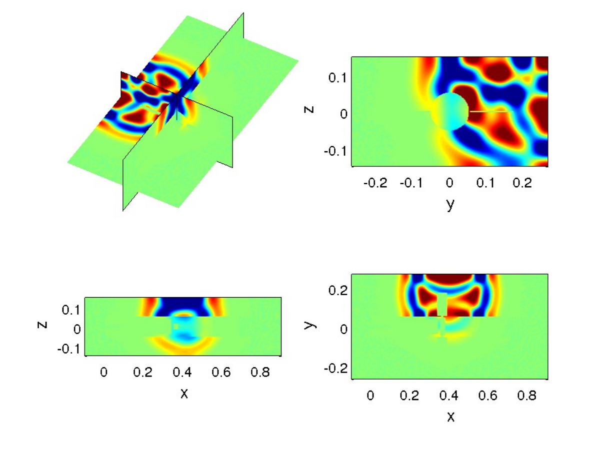


3-D numerical results

Solution obtained for $t = 0.2$

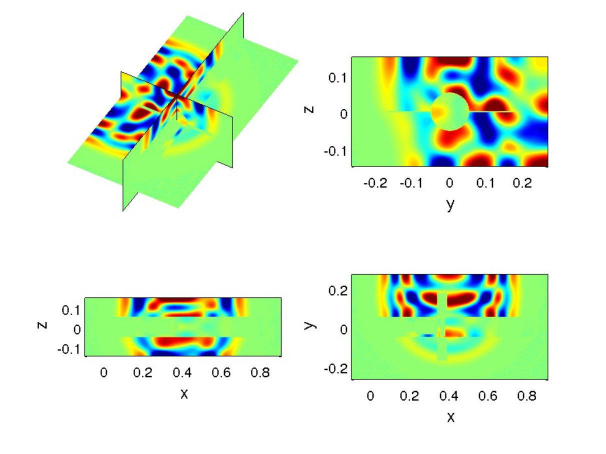


Solution obtained for $t = 0.3$



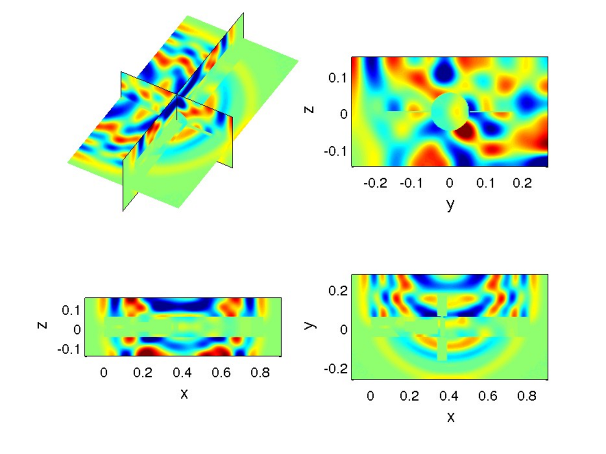
3-D numerical results

Solution obtained for $t = 0.4$



3-D numerical results

Solution obtained for $t = 0.5$



3-D numerical results

$\Delta t_{max} = 1.177e-3, \quad \Delta t_{min} = 1.442e-5$

Ratio $\frac{\Delta t_{max}}{\Delta t_{min}}$ = 81.6

Computational time with standard leap frog ($\Delta t = 1e-5$): 63.4h

3-D numerical results

$\Delta t_{max} = 1.177e-3, \quad \Delta t_{min} = 1.442e-5$

Ratio $\frac{\Delta t_{max}}{\Delta t_{min}}$ = 81.6

Computational time with standard leap frog ($\Delta t = 1e-5$): 63.4h

Second-order local time stepping with the following repartition :

Level	1	2	3	4	5	6	7	8	9	10
Number of elements	64468	7629	867	35	3	0	0	3	2	1

L^2 error for $t = 0.5$: 2.31e-3

Computational time ($\Delta t = 2.5e-4$) : 9.48h